Correlation functions evolution for the Glauber dynamics in continuum

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Abstract

We construct a correlation functions evolution corresponding to the Glauber dynamics in continuum. Existence of the corresponding strongly continuous contraction semigroup in a proper Banach space is shown. Additionally we prove the existence of the evolution of states and study their ergodic properties.

1 Introduction

Among all birth-and-death Markov processes on configuration spaces in continuum the Glauber type stochastic dynamics are objects of particular interest. The reversible states for these dynamics are grand canonical Gibbs measures. This fact gives a standard way to construct properly associated stationary Markov processes using the corresponding (non-local) Dirichlet forms related to the considered Markov generators and Gibbs measures. These processes describe the equilibrium Glauber dynamics which preserve the initial Gibbs state in the time evolution, see, e.g., [6, 13, 14], and [15]. Note that, in applications, the time evolution of initial state is the subject of the primary interest. Therefore, we understand the considered stochastic (non-equilibrium) dynamics as the evolution of initial distributions for the system. Actually, the Markov process (provided it exists) itself gives a general technical equipment to study this problem. However, we note that the transition from the micro-state evolution corresponding to the given initial configuration to the macro-state dynamics is the well developed concept in the theory of infinite particle systems. This point of view appeared initially in the framework of the Hamiltonian dynamics of classical gases, see, e.g., [2].

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The study of the non-equilibrium Glauber dynamics needs construction of the time evolution for a wider class of initial measures. The lack of the general Markov processes techniques for the considered systems makes necessary to develop alternative approaches to study the state evolutions in the Glauber dynamics. The approach realized in [5, 11, 12] is probably the only known at the present time. The description of the time evolutions for measures on configuration spaces in terms of an infinite system of evolutional equations for the corresponding correlation functions was used there. The latter system is a Glauber evolution's analog of the famous BBGKY-hierarchy for the Hamiltonian dynamics.

Here we extend constructive approach developed in [5] to correlation function evolution of the Glauber dynamics in continuum. We describe a reasonable Banach space where the evolution problem can be solved. Moreover, we construct an explicit approximation by bounded operators of the corresponding evolutional semigroup. We prove that functions in this evolution stay correlation functions of some measures (states) on configuration spaces; this means that we show the existence of states evolution. At the end we obtain the ergodic properties of the state evolution.

2 Basic facts and notation

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$, $d \geq 1$, and let $\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ denote the system of all bounded sets in $\mathcal{B}(\mathbb{R}^d)$.

The configuration space over $\mathbb{R}^d$ consists of all locally finite subsets (configurations) of $\mathbb{R}^d$. Namely,

$$\Gamma = \Gamma_{\mathbb{R}^d} := \Big\{ \gamma \subset \mathbb{R}^d \ \Big| \ |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d) \Big\}. \tag{2.1}$$

It is equipped with the vague topology, i.e., the minimal topology for which all mappings $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x) \in \mathbb{R}$ are continuous for any continuous function f on $\mathbb{R}^d$ with compact support. We note that the summation in $\sum_{x \in \gamma} f(x)$ is taken over only finitely many points of γ which belong to the support of f. It is worth pointing out that Γ with the vague topology may be metrizable and it becomes a Polish space (i.e., complete separable metric space), see, e.g., [10]. The Borel σ-algebra $\mathcal{B}(\Gamma)$ corresponding to this topology is the smallest σ-algebra for which all mappings $\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are measurable for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$. Here $\gamma_\Lambda := \gamma \cap \Lambda$, and $|\cdot|$ means the cardinality of a finite set.

The space of n-point configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_Y^{(n)} := \Big\{ \eta \subset Y \ \Big| \ |\eta| = n \Big\}, \quad n \in \mathbb{N}; \quad \Gamma_Y^{(0)} := \{\emptyset\}.$$

As a set, $\Gamma_Y^{(n)}$ may be identified with the symmetrization of

$$\widetilde{Y^n} = \Big\{ (x_1, \ldots, x_n) \in Y^n \ \Big| \ x_k \neq x_l \text{ if } k \neq l \Big\}.$$

Hence, one can introduce the corresponding Borel σ-algebra, which we denote by $\mathcal{B}(\Gamma_Y^{(n)})$. The space of finite configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_{0,Y} := \bigsqcup_{n\in\mathbb{N}_0} \Gamma_Y^{(n)}.$$

This space is equipped with the topology of disjoint unions. Therefore, one can introduce the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_{0,Y})$. In the case of $Y = \mathbb{R}^d$ we will omit the index Y in the notation, namely, $\Gamma_0 := \Gamma_{0,\mathbb{R}^d}$, $\Gamma^{(n)} := \Gamma_{\mathbb{R}^d}^{(n)}$.

The restriction of the Lebesgue product measure $(dx)^n$ to $\big(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)})\big)$ we denote by $m^{(n)}$. We set $m^{(0)} := \delta_{\{\emptyset\}}$. The Lebesgue–Poisson measure λ on Γ_0 is defined by

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}. \tag{2.2}$$

For any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the restriction of λ to $\Gamma_\Lambda := \Gamma_{0,\Lambda}$ will also be denoted by λ. The space $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ is the projective limits of the family of spaces $\big\{(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))\big\}_{\Lambda\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$. The Poisson measure π on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ is given as the projective limit of the family of measures $\{\pi^\Lambda\}_{\Lambda\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$, where $\pi^\Lambda := e^{-m(\Lambda)}\lambda$ is a probability measure on $\big(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda)\big)$ and $m(\Lambda)$ is the Lebesgue measure of $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$.

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we define a *Lebesgue–Poisson exponent*

$$e_\lambda(f, \eta) := \prod_{x\in\eta} f(x), \quad \eta \in \Gamma_0; \qquad e_\lambda(f, \emptyset) := 1. \tag{2.3}$$

According to (2.2), we have $e_\lambda(f) \in L^1(\Gamma_0, d\lambda)$ for any $f \in L^1(\mathbb{R}^d, dx)$. Moreover,

$$\int_{\Gamma_0} e_\lambda(f, \eta) d\lambda(\eta) = \exp\bigg\{ \int_{\mathbb{R}^d} f(x)dx \bigg\}. \tag{2.4}$$

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exist $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. We will use the following classes of functions on Γ_0: $L_{\mathrm{ls}}^0(\Gamma_0)$ is the set of all measurable functions on Γ_0 which have local support, i.e. $G \in L_{\mathrm{ls}}^0(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $G \upharpoonright_{\Gamma_0\setminus\Gamma_\Lambda} = 0$; $B_{\mathrm{bs}}(\Gamma_0)$ is the set of bounded measurable functions with bounded support, i.e. $G \upharpoonright_{\Gamma_0\setminus B} = 0$ for some bounded $B \in \mathcal{B}(\Gamma_0)$.

Any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0, in fact, is a sequence of functions $\big\{G^{(n)}\big\}_{n\in\mathbb{N}_0}$, where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$.

On Γ we consider the set of cylinder functions $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$. These functions are characterized by the following relation: $F(\gamma) = F \upharpoonright_{\Gamma_\Lambda} (\gamma_\Lambda)$.

We consider the following mapping from $L_{\mathrm{ls}}^0(\Gamma_0)$ into $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$, which plays the key role in our further considerations:

$$KG(\gamma) := \sum_{\eta\Subset\gamma} G(\eta), \quad \gamma \in \Gamma, \tag{2.5}$$

where $G \in L^0_{\mathrm{ls}}(\Gamma_0)$, see, e.g., [9, 16, 17]. The summation in (2.5) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible with

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{2.6}$$

Here and subsequently inclusions like $\xi \subset \eta$ also hold for $\xi = \emptyset$ as well as for $\xi = \eta$. We denote the restriction of K onto functions on Γ_0 by K_0.

For any fixed $C > 1$ we consider the following space of $\mathcal{B}(\Gamma_0)$-measurable functions

$$\mathcal{L}_C := \left\{ G : \Gamma_0 \to \mathbb{R} \;\middle|\; \|G\|_C := \int_{\Gamma_0} |G(\eta)| C^{|\eta|} d\lambda(\eta) < \infty \right\}. \tag{2.7}$$

In the sequel, $\mathcal{L}^{\mathrm{ls}}_C$ denotes the set $L^0_{\mathrm{ls}}(\Gamma_0) \cap \mathcal{L}_C$. The space $\mathcal{L}_C$ can be made into a Banach space in a standard way; one simply takes the quotient space with respect to the kernel of $\|\cdot\|_C$. To simplify notations, we use the same symbol $\mathcal{L}_C$ for the corresponding Banach space.

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous with respect to (w.r.t. for short) Poisson measure π if for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the projection of μ onto Γ_Λ is absolutely continuous w.r.t. the projection of π onto Γ_Λ. In this case, there exists a *correlation functional* $k_\mu : \Gamma_0 \to \mathbb{R}_+$ (see, e.g., [9]) such that for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ the following equality holds

$$\int_\Gamma (KG)(\gamma) d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) d\lambda(\eta). \tag{2.8}$$

The functions $k^{(0)}_\mu := 1$ and

$$k^{(n)}_\mu : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+, \quad n \in \mathbb{N}, \tag{2.9}$$

$$k^{(n)}_\mu(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\}), & \text{if } (x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

are called *correlation functions of* μ.

We recall now without a proof the partial case of the well-known technical lemma (cf., [15]), which plays very important role in our calculations.

Lemma 2.1. *For any measurable function* $H : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \to \mathbb{R}$

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi, \eta)\, d\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi)\, d\lambda(\xi)\, d\lambda(\eta) \tag{2.10}$$

if only both sides of the equality make sense.

3 Non-equilibrium Glauber dynamics in continuum

Let $\phi : \mathbb{R}^d \to \mathbb{R}_+ := [0; +\infty)$ be an even non-negative function which satisfies the following integrability condition

$$C_\phi := \int_{\mathbb{R}^d} \big(1 - e^{-\phi(x)}\big) dx < +\infty \tag{3.1}$$

For any $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$ we define

$$E^\phi(x,\gamma) := \sum_{y\in\gamma} \phi(x-y) \in [0;\infty]. \tag{3.2}$$

Let us introduce the (pre-)generator of the Glauber dynamics: for any $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ we set

$$\begin{aligned}(LF)(\gamma) :=& \sum_{x\in\gamma} \big[F(\gamma\setminus x) - F(\gamma)\big] \\ &+ z\int_{\mathbb{R}^d} \big[F(\gamma\cup x) - F(\gamma)\big] \exp\big\{-E^\phi(x,\gamma)\big\} dx, \qquad \gamma \in \Gamma.\end{aligned} \tag{3.3}$$

Here $z > 0$ is the *activity* parameter. Note that for any $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ there exists $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ such that $F(\gamma\setminus x) = F(\gamma)$ for all $x \in \gamma_{\Lambda^c}$ and $F(\gamma \cup x) = F(\gamma)$ for all $x \in \Lambda^c$. On account of $\exp\big\{-E^\phi(x,\gamma)\big\} \le 1$, we conclude that sum and integral in (3.3) are finite.

In [5], it was shown that the mapping $\hat{L} := K^{-1}LK$ given on $B_{\mathrm{bs}}(\Gamma_0)$ by the following expression

$$\begin{aligned}(\hat{L}G)(\eta) =& -|\eta|G(\eta) \\ &+ z\sum_{\xi\subset\eta} \int_{\mathbb{R}^d} e^{-E^\phi(x,\xi)} G(\xi\cup x) e_\lambda(e^{-\phi(x-\cdot)} - 1, \eta\setminus\xi) dx\end{aligned} \tag{3.4}$$

is a linear operator on $\mathcal{L}_C$ with the dense domain $D(\hat{L}) = \mathcal{L}_{2C} \subset \mathcal{L}_C$. If, additionally,

$$z \le \min\big\{Ce^{-CC_\phi}; 2Ce^{-2CC_\phi}\big\}, \tag{3.5}$$

then $\big(\hat{L}, D(\hat{L})\big)$ is a closable linear operator in $\mathcal{L}_C$ and its closure (which we also denote by $\hat{L}$) generates a strongly continuous contraction semigroup $\hat{T}(t)$ on $\mathcal{L}_C$.

Let us define $d\lambda_C := C^{|\cdot|} d\lambda$. The topologically dual space to $\mathcal{L}_C$ is the space $(\mathcal{L}_C)' = \big(L^1(\Gamma_0, d\lambda_C)\big)' = L^\infty(\Gamma_0, d\lambda_C)$. The space $L^\infty(\Gamma_0, d\lambda_C)$ is isometrically isomorphic to the Banach space

$$\mathcal{K}_C := \Big\{k : \Gamma_0 \to \mathbb{R} \;\Big|\; k\cdot C^{-|\cdot|} \in L^\infty(\Gamma_0,\lambda)\Big\}$$

with the norm $\|k\|_{\mathcal{K}_C} := \left\|C^{-|\cdot|}k(\cdot)\right\|_{L^\infty(\Gamma_0,\lambda)}$. The latter isomorphism is given by the following isometry R_C

$$(\mathcal{L}_C)' \ni k \longmapsto R_C k := k \cdot C^{|\cdot|} \in \mathcal{K}_C. \tag{3.6}$$

In fact, the duality between Banach spaces $\mathcal{L}_C$ and $\mathcal{K}_C$ can be expressed clearly in the following way

$$\langle\!\langle G,\, k\rangle\!\rangle := \int_{\Gamma_0} G \cdot k\, d\lambda, \quad G \in \mathcal{L}_C,\ k \in \mathcal{K}_C \tag{3.7}$$

with

$$|\langle\!\langle G, k\rangle\!\rangle| \le \|G\|_C \cdot \|k\|_{\mathcal{K}_C}. \tag{3.8}$$

It is obvious that for any $k \in \mathcal{K}_C$

$$|k(\eta)| \le \|k\|_{\mathcal{K}_C}\, C^{|\eta|} \quad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0. \tag{3.9}$$

Let $\big(\hat{L}', D(\hat{L}')\big)$ be an operator in $(\mathcal{L}_C)'$ which is adjoint to the closed operator $\big(\hat{L}, D(\hat{L})\big)$. We consider its image in $\mathcal{K}_C$ under isometry R_C. Namely, let $\hat{L}^* = R_C \hat{L}' R_{C^{-1}}$ with a domain $D(\hat{L}^*) = R_C D(\hat{L}')$. Then, for any $G \in D(\hat{L})$ and $k \in D(\hat{L}^*)$

$$\begin{aligned}\int_{\Gamma_0} G \cdot \hat{L}^* k d\lambda &= \int_{\Gamma_0} G \cdot R_C \hat{L}' R_{C^{-1}} k d\lambda = \int_{\Gamma_0} G \cdot \hat{L}' R_{C^{-1}} k d\lambda_C \\ &= \int_{\Gamma_0} \hat{L}G \cdot R_{C^{-1}} k d\lambda_C = \int_{\Gamma_0} \hat{L}G \cdot k d\lambda.\end{aligned}$$

Therefore, $\hat{L}^*$ is the dual operator to $\hat{L}$ w.r.t. the duality (3.7). By [7], we have the precise form of $\hat{L}^*$ on $D(\hat{L}^*)$:

$$\begin{aligned}(\hat{L}^* k)(\eta) =& -|\eta| k(\eta) \\ &+ z \sum_{x\in\eta} e^{-E^\phi(x,\eta\setminus x)} \int_{\Gamma_0} e_\lambda(e^{-\phi(x-\cdot)} - 1, \xi) k((\eta \setminus x) \cup \xi)\, d\lambda(\xi).\end{aligned} \tag{3.10}$$

Under condition (3.5), we consider the adjoint semigroup $\hat{T}'(t)$ in $(\mathcal{L}_C)'$ and its image $\hat{T}^*(t)$ in $\mathcal{K}_C$. Now, we may apply general results about adjoint semigroups (see, e.g., [3]) to the semigroup $\hat{T}^*(t)$. The last semigroup will be weak*-continuous, moreover, weak*-differentiable at 0 and $\hat{L}^*$ will be weak*-generator of $\hat{T}^*(t)$. Here and below we mean "weak*-properties" w.r.t. the duality (3.7). Let

$$\mathring{\mathcal{K}}_C = \Big\{ k \in \mathcal{K}_C \,\Big|\, \exists \lim_{t\downarrow 0} \|\hat{T}^*(t)k - k\|_{\mathcal{K}_C} = 0 \Big\}. \tag{3.11}$$

Then, $\mathring{\mathcal{K}}_C$ is a closed, weak*-dense, $\hat{T}^*(t)$-invariant linear subspace of $\mathcal{K}_C$. Moreover, $\mathring{\mathcal{K}}_C = \overline{D(\hat{L}^*)}$ (the closure is in the norm of $\mathcal{K}_C$). Let $\hat{T}^\odot(t)$ denote the restriction of $\hat{T}^*(t)$ to the Banach space $\mathring{\mathcal{K}}_C$. Then, $\hat{T}^\odot(t)$ is a C_0-semigroup on $\mathring{\mathcal{K}}_C$ and its generator $\hat{L}^\odot$ will be part of $\hat{L}^*$, namely,

$$D(\hat{L}^\odot) = \left\{ k \in D(\hat{L}^*) \;\middle|\; \hat{L}^* k \in \overline{D(\hat{L}^*)} \right\}$$

and $\hat{L}^* k = \hat{L}^\odot k$ for any $k \in D(\hat{L}^\odot)$.

Our next goal is to construct another $\hat{T}^*(t)$-invariant subspace which can be described precisely. We first introduce a useful subspace in $D(\hat{L}^*)$.

Proposition 3.1. *For any $\alpha \in (0;1)$ the following inclusions hold $\mathcal{K}_{\alpha C} \subset D(\hat{L}^*) \subset \overline{D(\hat{L}^*)} \subset \mathcal{K}_C$.*

Proof. Let $\alpha \in (0;1)$ and $k \in \mathcal{K}_{\alpha C}$. Then, using (2.4) and (3.9), for λ-a.a. $\eta \in \Gamma_0$ we have

$$\begin{aligned}
&C^{-|\eta|} |\eta| \left|k(\eta)\right| + \sum_{x\in\eta} \int_{\Gamma_0} e_\lambda \left(1 - e^{-\phi(x-\cdot)}, \xi\right) C^{-|\eta|} \left|k\left((\eta\setminus x)\cup\xi\right)\right| d\lambda(\xi) \\
&\le C^{-|\eta|} |\eta| \, \|k\|_{\mathcal{K}_{\alpha C}} (\alpha C)^{|\eta|} \\
&\quad + \sum_{x\in\eta} \int_{\Gamma_0} e_\lambda \left(1 - e^{-\phi(x-\cdot)}, \xi\right) C^{-|\eta|} \|k\|_{\mathcal{K}_{\alpha C}} (\alpha C)^{|(\eta\setminus x)\cup\xi|} d\lambda(\xi) \\
&= \alpha^{|\eta|} |\eta| \, \|k\|_{\mathcal{K}_{\alpha C}} + \frac{1}{\alpha C} \|k\|_{\mathcal{K}_{\alpha C}} \alpha^{|\eta|} \sum_{x\in\eta} \int_{\Gamma_0} e_\lambda \left(\alpha C \left(1 - e^{-\phi(x-\cdot)}\right), \xi\right) d\lambda(\xi) \\
&= \alpha^{|\eta|} |\eta| \, \|k\|_{\mathcal{K}_{\alpha C}} + \frac{1}{\alpha C} \|k\|_{\mathcal{K}_{\alpha C}} \alpha^{|\eta|} |\eta| \exp\{\alpha C C_\phi\} \\
&\le \|k\|_{\mathcal{K}_{\alpha C}} \frac{-1}{e \ln \alpha} \left(1 + \frac{1}{\alpha C} \exp\{\alpha C C_\phi\}\right),
\end{aligned}$$

since $x\alpha^x \le -\dfrac{1}{e\ln\alpha}$ for any $\alpha \in (0;1)$ and $x \ge 0$. Using the definition of $D(\hat{L}^*)$ and Lemma 2.1 we get immediately the statement of the proposition. □

Remark 3.2. By the same arguments, the set of all functions $k \in \mathcal{K}_C$ such that

$$|k(\eta)| \le \text{const} \cdot \frac{1}{|\eta|} C^{|\eta|}, \quad \eta \in \Gamma_0 \setminus \{\emptyset\}$$

is a subset of $D(\hat{L}^*)$. Due to the elementary inequality $\alpha^x < \text{const} \cdot x^{-1}$ for any $\alpha \in (0;1)$, $x > 0$, it follows that the above introduced set contains $\mathcal{K}_{\alpha C}$. However, the set $\mathcal{K}_{\alpha C}$ will be more useful for our further calculations.

Proposition 3.3. *Suppose that (3.5) is satisfied. Furthermore, we additionally assume that*

$$z < Ce^{-CC_\phi}, \quad \text{if} \quad CC_\phi \le \ln 2. \tag{3.12}$$

Then there exists $\alpha_0 = \alpha_0(z,\phi,C) \in (0;1)$ such that for any $\alpha \in (\alpha_0;1)$ the set $\mathcal{K}_{\alpha C}$ is the $\hat{T}^*(t)$-invariant linear subspace of $\mathcal{K}_C$.

Proof. Let us consider function $f(x) := xe^{-x}$, $x \geq 0$. It has the following properties: f is increasing on $[0;1]$ from 0 to e^{-1} and it is asymptotically decreasing on $[1;+\infty)$ from e^{-1} to 0; $f(x) < f(2x)$ for $x \in (0, \ln 2)$; $x = \ln 2$ is the only non-zero solution to $f(x) = f(2x)$.

By assumption (3.5), it follows that $zC_\phi \leq \min\{CC_\phi e^{-CC_\phi}, 2CC_\phi e^{-2CC_\phi}\}$. Therefore, if $CC_\phi e^{-CC_\phi} \neq 2CC_\phi e^{-2CC_\phi}$ then (3.5) with necessity implies

$$zC_\phi < e^{-1}. \tag{3.13}$$

This inequality remains also true if $CC_\phi = \ln 2$ because of (3.12). Under condition (3.13), the equation $f(x) = zC_\phi$ has exactly two roots, say, $0 < x_1 < 1 < x_2 < +\infty$. Then, (3.12) implies $x_1 < CC_\phi < 2CC_\phi \leq x_2$.
If $CC_\phi > 1$ then we set $\alpha_0 := \max\left\{\frac{1}{2}; \frac{1}{CC_\phi}; \frac{1}{C}\right\} < 1$. This yields $2\alpha CC_\phi > CC_\phi$ and $\alpha CC_\phi > 1 > x_1$. If $x_1 < CC_\phi \leq 1$ then we set $\alpha_0 := \max\left\{\frac{1}{2}; \frac{x_1}{CC_\phi}; \frac{1}{C}\right\} < 1$ that gives $2\alpha CC_\phi > CC_\phi$ and $\alpha CC_\phi > x_1$.
As a result,

$$x_1 < \alpha CC_\phi < CC_\phi < 2\alpha CC_\phi < 2CC_\phi \leq x_2 \tag{3.14}$$

and $1 < \alpha C < C < 2\alpha C < 2C$. The last inequality shows that $\mathcal{L}_{2C} \subset \mathcal{L}_{2\alpha C} \subset \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$. Moreover, by (3.14), we may prove that the operator $(\hat{L}, \mathcal{L}_{2\alpha C})$ is closable in $\mathcal{L}_{\alpha C}$ and its closure is a generator of a contraction semigroup $\hat{T}_\alpha(t)$ on $\mathcal{L}_{\alpha C}$. The proof is identical to the one introduced in [5].

It is easily seen that $\hat{T}_\alpha(t)G = \hat{T}(t)G$ for any $G \in \mathcal{L}_C$. Indeed, from the construction of the semigroup $\hat{T}(t)$ (see [5]) and analogous construction for the semigroup $\hat{T}_\alpha(t)$, it follows that there exists family of mappings $\hat{P}_\delta$, $\delta > 0$ independent of α and C, namely,

$$\begin{aligned}\left(\hat{P}_\delta G\right)(\eta) := &\sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|} G(\xi \cup \omega) \\ &\times \prod_{y \in \xi} e^{-E^\phi(y,\omega)} \prod_{y \in \eta \setminus \xi} \left(e^{-E^\phi(y,\omega)} - 1\right) d\lambda(\omega), \quad \eta \in \Gamma_0.\end{aligned} \tag{3.15}$$

such that $\hat{P}_\delta^{\left[\frac{t}{\delta}\right]}$ for any $t \geq 0$ strongly converges to $\hat{T}(t)$ and $\hat{T}_\alpha(t)$ as $\delta \to 0$ in $\mathcal{L}_C$ and $\mathcal{L}_{\alpha C}$, correspondingly. Here and below $[\,\cdot\,]$ means the entire part of a number. Then for any $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$ we see that $\hat{T}(t)G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$, $\hat{T}_\alpha(t)G \in \mathcal{L}_{\alpha C}$ and

$$\begin{aligned}\|\hat{T}(t)G - \hat{T}_\alpha(t)G\|_{\alpha C} &\leq \left\|\hat{T}(t)G - \hat{P}_\delta^{\left[\frac{t}{\delta}\right]}G\right\|_{\alpha C} + \left\|\hat{T}_\alpha(t)G - \hat{P}_\delta^{\left[\frac{t}{\delta}\right]}G\right\|_{\alpha C} \\ &\leq \left\|\hat{T}(t)G - \hat{P}_\delta^{\left[\frac{t}{\delta}\right]}G\right\|_{C} + \left\|\hat{T}_\alpha(t)G - \hat{P}_\delta^{\left[\frac{t}{\delta}\right]}G\right\|_{\alpha C} \to 0,\end{aligned}$$

as $\delta \to 0$. Therefore, $\hat{T}(t)G = \hat{T}_\alpha(t)G$ in $\mathcal{L}_{\alpha C}$ (recall that $G \in \mathcal{L}_C$) that yields $\hat{T}(t)G(\eta) = \hat{T}_\alpha(t)G(\eta)$ for λ-a.a. $\eta \in \Gamma_0$ and, hence, $\hat{T}(t)G = \hat{T}_\alpha(t)G$ in $\mathcal{L}_C$.

Note that for any $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$ and for any $k \in \mathcal{K}_{\alpha C} \subset \mathcal{K}_C$ we have $\hat{T}_\alpha(t)G \in \mathcal{L}_{\alpha C}$ and

$$\left\langle\!\!\left\langle \hat{T}_\alpha(t)G, k \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle G, \hat{T}^*_\alpha(t)k \right\rangle\!\!\right\rangle,$$

where, by construction, $\hat{T}^*_\alpha(t)k \in \mathcal{K}_{\alpha C}$. But $G \in \mathcal{L}_C$, $k \in \mathcal{K}_C$ implies

$$\left\langle\!\!\left\langle \hat{T}_\alpha(t)G, k \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \hat{T}(t)G, k \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle G, \hat{T}^*(t)k \right\rangle\!\!\right\rangle.$$

Hence, $\hat{T}^*(t)k = \hat{T}^*_\alpha(t)k \in \mathcal{K}_{\alpha C}$, $k \in \mathcal{K}_{\alpha C}$ that proves the statement. □

We have thus proved the following result.

Theorem 3.4. *Suppose that assumptions of Proposition 3.3 are satisfied. Then, $\{\hat{T}^\odot(t),\, t \geq 0\}$ is a C_0-semigroup on $\mathcal{K}_{\alpha C}$. Hence, for every $k_0 \in \mathcal{K}_{\alpha C}$, the orbit map*

$$k : t \mapsto k_t := \hat{T}^\odot(t)k_0$$

is the unique mild solution of the associated Cauchy problem in $\mathcal{K}_{\alpha C}$:

$$\begin{cases} \dfrac{\partial}{\partial t} k_t = \hat{L}^* k_t \\ k_t\big|_{t=0} = k_0. \end{cases} \tag{3.16}$$

Remark 3.5. It is worth noting, that (3.5) implies that for any $k_0 \in \overline{D(\hat{L}^*)}$ the Cauchy problem (3.16) in $\mathcal{K}_C$ has a unique mild solution: $k_t = \hat{T}^*(t)k_0 = \hat{T}^\odot(t)k_0 \in \overline{D(\hat{L}^*)}$.

Remark 3.6. The Cauchy problem (3.16) is well-posed in $\mathring{\mathcal{K}}_C = \overline{D(\hat{L}^*)}$, i.e., for every $k_0 \in D(\hat{L}^\odot)$ there exists a unique solution $k_t \in \mathring{\mathcal{K}}_C$ of (3.16).

Let (3.5) and (3.12) be satisfied and let α_0 be chosen as in the proof of Proposition 3.3. Suppose that $\alpha \in (\alpha_0; 1)$. Then, Propositions 3.1 and 3.3 imply $\overline{\mathcal{K}_{\alpha C}} \subset \overline{D(\hat{L}^*)}$ and the Banach subspace $\overline{\mathcal{K}_{\alpha C}}$ is $\hat{T}^*(t)$- and, hence, $\hat{T}^\odot(t)$-invariant due to the continuity of these operators.

Let $\hat{T}^{\odot\alpha}(t)$ be the restriction of the strongly continuous semigroup $\hat{T}^\odot(t)$ to the closed linear subspace $\overline{\mathcal{K}_{\alpha C}}$. It is immediate (see, e.g., [3]) that $\hat{T}^{\odot\alpha}(t)$ is a strongly continuous semigroups on $\overline{\mathcal{K}_{\alpha C}}$ with the generator $\hat{L}^{\odot\alpha}$ which is the restriction of the operator $\hat{L}^\odot$. Namely,

$$D(\hat{L}^{\odot\alpha}) = \left\{ k \in \overline{\mathcal{K}_{\alpha C}} \;\middle|\; \hat{L}^* k \in \overline{\mathcal{K}_{\alpha C}} \right\}, \tag{3.17}$$

and

$$\hat{L}^{\odot\alpha}k = \hat{L}^\odot k = \hat{L}^* k, \qquad k \in D(\hat{L}^{\odot\alpha}) \tag{3.18}$$

Since $\hat{T}(t)$ is a contraction semigroup on $\mathcal{L}_C$, it follows that $\hat{T}'(t)$ is also a contraction semigroup on $(\mathcal{L}_C)'$. Hence, $\hat{T}^*(t)$ is a contraction semigroup on $\mathcal{K}_C$, due to the fact that the isomorphism (3.6) is isometrical. As a result, its restriction $\hat{T}^{\odot\alpha}(t)$ is a contraction semigroup on $\overline{\mathcal{K}_{\alpha C}}$. It is worth noting that according to (3.17),

$$D_{\alpha C} := \left\{ k \in \mathcal{K}_{\alpha C} \;\middle|\; \hat{L}^* k \in \overline{\mathcal{K}_{\alpha C}} \right\}$$

is a core for $\hat{L}^{\odot\alpha}$ in $\overline{\mathcal{K}_{\alpha C}}$.

By (3.15), for any $k \in \mathcal{K}_{\alpha C}$ and $G \in B_{\mathrm{bs}}(\Gamma_0)$ we have

$$\begin{aligned}
&\int_{\Gamma_0} (\hat{P}_\delta G)(\eta)\, k(\eta)\, d\lambda(\eta) \\
&= \int_{\Gamma_0} \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|}\, G(\xi \cup \omega) \prod_{y \in \xi} e^{-E^\phi(y,\omega)} \\
&\quad \times \prod_{y \in \eta \setminus \xi} \left(e^{-E^\phi(y,\omega)} - 1 \right) d\lambda(\omega)\, k(\eta)\, d\lambda(\eta) \\
&= \int_{\Gamma_0} \int_{\Gamma_0} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|}\, G(\xi \cup \omega) \prod_{y \in \xi} e^{-E^\phi(y,\omega)} \\
&\quad \times \prod_{y \in \eta} \left(e^{-E^\phi(y,\omega)} - 1 \right) d\lambda(\omega)\, k(\eta \cup \xi)\, d\lambda(\xi)\, d\lambda(\eta) \\
&= \int_{\Gamma_0} \int_{\Gamma_0} \sum_{\omega \subset \xi} (1-\delta)^{|\xi \setminus \omega|} (z\delta)^{|\omega|}\, G(\xi) \prod_{y \in \xi \setminus \omega} e^{-E^\phi(y,\omega)} \\
&\quad \times \prod_{y \in \eta} \left(e^{-E^\phi(y,\omega)} - 1 \right) k(\eta \cup \xi \setminus \omega)\, d\lambda(\xi)\, d\lambda(\eta).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\hat{P}_\delta^* k)(\eta) = &\sum_{\omega \subset \eta} (1-\delta)^{|\eta \setminus \omega|} (z\delta)^{|\omega|} \prod_{y \in \eta \setminus \omega} e^{-E^\phi(y,\omega)} \\
&\times \int_{\Gamma_0} \prod_{y \in \xi} \left(e^{-E^\phi(y,\omega)} - 1 \right) k(\xi \cup \eta \setminus \omega)\, d\lambda(\xi).
\end{aligned} \tag{3.19}$$

Proposition 3.7. *Suppose that* (3.5) *and* (3.12) *are fulfilled. Then, for any* $k \in D_{\alpha C}$ *and* $\alpha \in (\alpha_0, 1)$*, where* α_0 *is chosen as in the proof of Proposition 3.3,*

$$\lim_{\delta \to 0} \left\| \frac{1}{\delta} (\hat{P}_\delta^* - 1\!\!1) k - \hat{L}^{\odot\alpha} k \right\|_{\mathcal{K}_C} = 0. \tag{3.20}$$

Proof. According to the definition (3.10) we set

$$\begin{aligned}
(\hat{P}_\delta^{*,(0)} k)(\eta) &= (1-\delta)^{|\eta|} k(\eta); \\
(\hat{P}_\delta^{*,(1)} k)(\eta) &= z\delta \sum_{x \in \eta} (1-\delta)^{|\eta|-1}\, e_\lambda\left(e^{-\phi(x-\cdot)}, \eta \setminus x \right) \\
&\quad \times \int_{\Gamma_0} e_\lambda\left(e^{-\phi(x-\cdot)} - 1, \xi \right) k(\xi \cup \eta \setminus x)\, d\lambda(\xi);
\end{aligned}$$

and $\hat{P}_\delta^{*,(\geq 2)} = \hat{P}_\delta^{*} - \hat{P}_\delta^{*,(0)} - \hat{P}_\delta^{*,(1)}$.

We will use the following elementary inequality, which is valid for all $n \in \mathbb{N} \cup \{0\}$ and any $\delta \in (0;1)$

$$0 \leq n - \frac{1-(1-\delta)^n}{\delta} \leq \delta \frac{n(n-1)}{2}.$$

As a result, for any $k \in \mathcal{K}_{\alpha C}$ and λ-a.a. $\eta \in \Gamma_0$, $\eta \neq \emptyset$

$$\begin{aligned} &C^{-|\eta|} \left| \frac{1}{\delta}(\hat{P}_{\delta,\varepsilon}^{*,(0)} - 1\!\!1)k(\eta) + |\eta| k(\eta) \right| \\ &\leq \|k\|_{\mathcal{K}_{\alpha C}} \alpha^{|\eta|} \left| |\eta| - \frac{1-(1-\delta)^{|\eta|}}{\delta} \right| \leq \frac{\delta}{2} \|k\|_{\mathcal{K}_{\alpha C}} \alpha^{|\eta|} |\eta|(|\eta|-1). \end{aligned} \tag{3.21}$$

We note that the function $\alpha^x x(x-1)$ is bounded for $x \geq 1$, $\alpha \in (0;1)$. Now, for any $k \in \mathcal{K}_{\alpha C}$ and λ-a.a. $\eta \in \Gamma_0$, $\eta \neq \emptyset$

$$\begin{aligned} &C^{-|\eta|} \left| \frac{1}{\delta} \hat{P}_\delta^{*,(1)} k(\eta) - z \sum_{x\in\eta} \int_{\Gamma_0} e_\lambda \left(e^{-\phi(x-\cdot)}, \eta \setminus x \right) \right. \\ &\qquad \left. \times\, e_\lambda \left(e^{-\phi(x-\cdot)} - 1, \xi \right) k\left(\xi \cup \eta \setminus x\right) d\lambda(\xi) \right| \\ &\leq \|k\|_{\mathcal{K}_{\alpha C}} \frac{z}{\alpha C} \alpha^{|\eta|} \sum_{x\in\eta} \bigl(1-(1-\delta)^{|\eta|-1}\bigr) \int_{\Gamma_0} e_\lambda \left(\alpha C \bigl(e^{-\phi(x-\cdot)} - 1\bigr), \xi \right) d\lambda\left(\xi\right) \\ &\leq \|k\|_{\mathcal{K}_{\alpha C}} \frac{z}{\alpha C} \alpha^{|\eta|} \sum_{x\in\eta} \bigl(1-(1-\delta)^{|\eta|-1}\bigr) \exp\left\{\alpha C C_\phi\right\} \\ &\leq \|k\|_{\mathcal{K}_{\alpha C}} \frac{z}{\alpha C} \alpha^{|\eta|} \delta |\eta|(|\eta|-1) \exp\left\{\alpha C C_\phi\right\}, \end{aligned} \tag{3.22}$$

which is small for small δ uniformly in $|\eta|$. Next, using inequality

$$1 - e^{-E^\phi(y,\omega)} = 1 - \prod_{x\in\omega} e^{-\phi(x-y)} \leq \sum_{x\in\omega} \left(1 - e^{-\phi(x-y)}\right),$$

we obtain

$$\begin{aligned} &\frac{1}{\delta} C^{-|\eta|} \sum_{\substack{\omega\subset\eta \\ |\omega|\geq 2}} (1-\delta)^{|\eta\setminus\omega|} (z\delta)^{|\omega|} e_\lambda \left(e^{-E^\phi(\cdot,\omega)}, \eta \setminus \omega \right) \\ &\qquad \times \int_{\Gamma_0} e_\lambda \left(\left| e^{-E^\phi(\cdot,\omega)} - 1 \right|, \xi \right) |k(\xi \cup \eta \setminus \omega)| d\lambda\left(\xi\right) \\ &= \|k\|_{\mathcal{K}_{\alpha C}} \alpha^{|\eta|} \frac{1}{\delta} \sum_{\substack{\omega\subset\eta \\ |\omega|\geq 2}} (1-\delta)^{|\eta\setminus\omega|} \left(\frac{z\delta}{\alpha C} \exp\left\{\alpha C C_\phi\right\} \right)^{|\omega|}. \end{aligned}$$

Recall that $\alpha > \alpha_0$, and consequently $z\exp\{\alpha C C_\phi\} \le \alpha C$. Hence, the latter expression can be estimated by

$$
\begin{aligned}
&\|k\|_{\mathcal{K}_{\alpha C}}\alpha^{|\eta|}\frac{1}{\delta}\sum_{\substack{\omega\subset\eta\\|\omega|\ge 2}}(1-\delta)^{|\eta\setminus\omega|}\,\delta^{|\omega|}\\
&=\|k\|_{\mathcal{K}_{\alpha C}}\delta\alpha^{|\eta|}\sum_{k=2}^{|\eta|}\frac{|\eta|!}{k!\,(|\eta|-k)!}(1-\delta)^{|\eta|-k}\,\delta^{k-2}\\
&=\|k\|_{\mathcal{K}_{\alpha C}}\delta\alpha^{|\eta|}\sum_{k=0}^{|\eta|-2}\frac{|\eta|!}{(k+2)!\,(|\eta|-k-2)!}(1-\delta)^{|\eta|-k-2}\,\delta^{k}\\
&=\|k\|_{\mathcal{K}_{\alpha C}}\delta\alpha^{|\eta|}\,|\eta|\,(|\eta|-1)\sum_{k=0}^{|\eta|-2}\frac{(|\eta|-2)!}{(k+2)!\,(|\eta|-k-2)!}(1-\delta)^{|\eta|-2-k}\,\delta^{k}\\
&\le\|k\|_{\mathcal{K}_{\alpha C}}\delta\alpha^{|\eta|}\,|\eta|\,(|\eta|-1)\sum_{k=0}^{|\eta|-2}\frac{(|\eta|-2)!}{k!\,(|\eta|-k-2)!}(1-\delta)^{|\eta|-2-k}\,\delta^{k}\\
&=\|k\|_{\mathcal{K}_{\alpha C}}\delta\alpha^{|\eta|}\,|\eta|\,(|\eta|-1)\,. \qquad (3.23)
\end{aligned}
$$

Combining inequalities (3.21)–(3.23) we obtain (3.20). □

For the convenience of the reader we mention below the well-known approximation result (cf., e.g., [4, Theorem 6.5]).

Lemma 3.8. *Let L, L_n, $n \in \mathbb{N}$ be Banach spaces, and $p_n : L \to L_n$ be bounded linear transformation, such that $\sup_n \|p_n\| < \infty$. For any $n \in \mathbb{N}$, let T_n be a linear contraction on L_n, let $\varepsilon_n > 0$ be such that $\lim_{n\to\infty}\varepsilon_n = 0$, and put $A_n = \varepsilon_n^{-1}(T_n - 1\!\!1)$. Let T_t be a strongly continuous contraction semigroup on L with generator A and let D be a core for A. Then the following are equivalent:*

1. *For each $f \in L$, $||T_n^{[t/\varepsilon_n]}p_n f - p_n T_t f||_{L_n} \to 0$, $n \to \infty$ for all $t \ge 0$ uniformly on bounded intervals. Here and below $[\cdot\,]$ mean the entire part of a real number.*
2. *For each $f \in D$, there exists $f_n \in L_n$ for each $n \in \mathbb{N}$ such that*

   $||f_n - p_n f||_{L_n} \to 0$ and $||A_n f_n - p_n A f||_{L_n} \to 0$, $n \to \infty$.

We are now in a position to claim the following result.

Theorem 3.9. *Let α_0 be chosen as in the proof of Proposition 3.3 and let $\alpha \in (\alpha_0; 1)$, $k \in \overline{\mathcal{K}_{\alpha C}}$ be arbitrary and fixed. Then*

$$(\hat{P}^*_\delta)^{[t/\delta]}k \to \hat{T}^{\odot\alpha}(t)k, \quad \delta \to 0$$

in the space $\overline{\mathcal{K}_{\alpha C}}$ with norm $\|\cdot\|_{\mathcal{K}_C}$ for all $t \ge 0$ uniformly on bounded intervals.

Proof. The statement will be proved once we verify the conditions of Lemma 3.8. For this purpose we apply Proposition 3.7 in the case $L_n = L = \overline{\mathcal{L}_{\alpha C}}$, $p_n = 1\!\!1$, $f_n = f = k$, $\varepsilon_n = \delta \to 0$, $n \in \mathbb{N}$. □

4 Positive definiteness

Definition 4.1. A measurable function $k : \Gamma_0 \to \mathbb{R}$ is called a positive defined function (cf. [16, 17]) if for any $G \in \mathcal{L}_C^{\mathrm{ls}}$ such that $KG \geq 0$ the following inequality holds

$$\int_{\Gamma_0} G(\eta)\, k(\eta)\, d\lambda(\eta) \geq 0.$$

In [16, 17], it was shown that if k is a positive defined function and

$$|k(\eta)| \leq C^{|\eta|}(|\eta|!)^2, \quad \eta \in \Gamma_0,$$

then there exists a unique measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ such that $k = k_\mu$, i.e., k is a correlation functional of μ in the sense of (2.8). Our aim is to show that the evolution $k \mapsto \hat{T}_t^{\odot} k$ preserves the property of the positive definiteness.

Theorem 4.2. *Let* (3.5) *hold and let* $k \in \overline{D(\hat{L}^*)} \subset \mathcal{K}_C$ *be a positive defined function. Then,* $k_t := \hat{T}_t^{\odot} k \in \overline{D(\hat{L}^*)} \subset \mathcal{K}_C$ *will be a positive defined function for any* $t \geq 0$.

Proof. Let $C > 1$ be arbitrary and fixed. For any $G \in \mathcal{L}_C^{\mathrm{ls}}$ we have

$$\int_{\Gamma_0} G(\eta)\, k_t(\eta)\, d\lambda(\eta) = \int_{\Gamma_0} (\hat{T}_t G)(\eta)\, k(\eta)\, d\lambda(\eta). \tag{4.1}$$

According to [5, Proposition 3.10] and condition (3.5), we have

$$\lim_{n\to 0} \int_{\Gamma_{\Lambda_n}} \left| T_n^{[nt]} 1\!\!1_{\Gamma_{\Lambda_n}} G(\eta) - 1\!\!1_{\Gamma_{\Lambda_n}}(\eta)(\hat{T}_t G)(\eta) \right| C^{|\eta|} d\lambda(\eta) = 0,$$

where

$$T_n = \hat{P}^{\Lambda_n}_{\frac{1}{n}}, \quad \text{for} \quad n \geq 2$$

and $\Lambda_n \nearrow \mathbb{R}^d$. By the dominated convergence theorem,

$$\begin{aligned} \int_{\Gamma_0} (\hat{T}_t G)(\eta)\, k(\eta)\, d\lambda(\eta) &= \lim_{n\to\infty} \int_{\Gamma_0} 1\!\!1_{\Gamma_{\Lambda_n}}(\eta)\, (\hat{T}_t G)(\eta)\, k(\eta)\, d\lambda(\eta) \\ &= \lim_{n\to\infty} \int_{\Gamma_{\Lambda_n}} (\hat{T}_t G)(\eta)\, k(\eta)\, d\lambda(\eta). \end{aligned}$$

Next,

$$\begin{aligned} &\left| \int_{\Gamma_{\Lambda_n}} (\hat{T}_t G)(\eta)\, k(\eta)\, d\lambda(\eta) - \int_{\Gamma_{\Lambda_n}} T_n^{[nt]} 1\!\!1_{\Gamma_{\Lambda_n}} G(\eta)\, k(\eta)\, d\lambda(\eta) \right| \\ &\leq \int_{\Gamma_{\Lambda_n}} \left| T_n^{[nt]} 1\!\!1_{\Gamma_{\Lambda_n}} G(\eta) - 1\!\!1_{\Gamma_{\Lambda_n}}(\eta)(\hat{T}_t G)(\eta) \right| k(\eta)\, d\lambda(\eta) \\ &\leq \|k\|_{\mathcal{K}_C} \int_{\Gamma_{\Lambda_n}} \left| T_n^{[nt]} 1\!\!1_{\Gamma_{\Lambda_n}} G(\eta) - 1\!\!1_{\Gamma_{\Lambda_n}}(\eta)(\hat{T}_t G)(\eta) \right| C^{|\eta|} d\lambda(\eta) \to 0, \quad n \to \infty. \end{aligned}$$

Therefore,

$$\int_{\Gamma_0} (\hat{T}_t G)(\eta)\, k(\eta)\, d\lambda(\eta) = \lim_{n\to\infty} \int_{\Gamma_{\Lambda_n}} T_n^{[nt]} 1\!\!1_{\Gamma_{\Lambda_n}} G(\eta)\, k(\eta)\, d\lambda(\eta)\,. \tag{4.2}$$

Our next goal is to show that for any $G \in \mathcal{L}_C^{\mathrm{ls}}$ the inequality $KG \geq 0$ implies

$$\int_{\Gamma_0} G(\eta)\, k_t(\eta)\, d\lambda(\eta) \geq 0.$$

By (4.1) and (4.2), it is enough to show that for any $m \in \mathbb{N}$ and for any $G \in \mathcal{L}_C^{\mathrm{ls}}$ such that $KG \geq 0$ the following inequality holds

$$\int_{\Gamma_0} 1\!\!1_{\Gamma_{\Lambda_n}} T_n^m 1\!\!1_{\Gamma_{\Lambda_n}} G(\eta)\, k(\eta)\, d\lambda(\eta) \geq 0, \quad m \in \mathbb{N}_0. \tag{4.3}$$

The inequality (4.3) is fulfilled if only

$$K 1\!\!1_{\Gamma_{\Lambda_n}} T_n^m G_n \geq 0, \tag{4.4}$$

where $G_n := 1\!\!1_{\Gamma_{\Lambda_n}} G$. We note that

$$\begin{aligned} \left(K 1\!\!1_{\Gamma_{\Lambda_n}} T_n^m G_n\right)(\gamma) &= \sum_{\eta \Subset \gamma} 1\!\!1_{\Gamma_{\Lambda_n}}(\eta)\, (T_n^m G_n)(\eta) \\ &= \sum_{\eta \subset \gamma_{\Lambda_n}} (T_n^m G_n)(\eta) = (K T_n^m G_n)(\gamma_{\Lambda_n}) \end{aligned} \tag{4.5}$$

for any $m \in \mathbb{N}_0$. In particular,

$$(KG_n)(\gamma) = \left(K 1\!\!1_{\Gamma_{\Lambda_n}} G\right)(\gamma) = (KG)(\gamma_{\Lambda_n}) \geq 0. \tag{4.6}$$

Let us now consider any $\tilde{G} \in \mathcal{L}_C^{\mathrm{ls}}$ ($\tilde{G}$ is not necessary equal to 0 outside of Γ_{Λ_n}) and suppose that $\left(K\tilde{G}\right)(\gamma) \geq 0$ for any $\gamma \in \Gamma_{\Lambda_n}$. Then

$$\begin{aligned} \left(K T_n \tilde{G}\right)(\gamma_{\Lambda_n}) &= \left(K \hat{P}_{\frac{1}{n}}^{\Lambda_n} \tilde{G}\right)(\gamma_{\Lambda_n}) = \left(P_{\frac{1}{n}}^{\Lambda_n} K \tilde{G}\right)(\gamma_{\Lambda_n}) \\ =& \left(\Xi_{\frac{1}{n}}^{\Lambda_n}(\gamma_{\Lambda_n})\right)^{-1} \sum_{\eta \subset \gamma_{\Lambda_n}} \left(\frac{1}{n}\right)^{|\eta|} \left(1 - \frac{1}{n}\right)^{|\gamma \setminus \eta|} \\ &\times \int_{\Gamma_{\Lambda_n}} \left(\frac{z}{n}\right)^{|\omega|} \prod_{y \in \omega} e^{-E^{\phi}(y,\gamma)} \left(K\tilde{G}\right)\left((\gamma_{\Lambda_n} \setminus \eta) \cup \omega\right) d\lambda(\omega) \geq 0. \end{aligned} \tag{4.7}$$

From (4.6) and (4.7), it follows that $KT_n G_n \geq 0$ for $\tilde{G} = G_n \in \mathcal{L}_C^{\mathrm{ls}}$. Substituting $\tilde{G} = T_n G_n \in \mathcal{L}_C^{\mathrm{ls}}$ into (4.7) we get $KT_n^2 G_n \geq 0$. By induction we have

$$(K T_n^m G_n)(\gamma_{\Lambda_n}) \geq 0, \quad m \in \mathbb{N}_0.$$

Thus, by (4.4) and (4.5), we get (4.3). This finishes the proof. □

5 Ergodicity

Let $k \in \overline{\mathcal{K}_{\alpha C}}$ be such that $k(\emptyset) = 0$. Then, by (3.19), $(\hat{P}^*_\delta k)(\emptyset) = 0$. The class of all such functions we denote by $\mathcal{K}^0_\alpha$.

Proposition 5.1. *Assume that there exists $\nu \in (0;1)$ such that*

$$z \leq \min\Big\{\nu C e^{-CC_\phi};\ 2Ce^{-2CC_\phi}\Big\}. \tag{5.1}$$

Let, additionally, $\alpha \in (\alpha_0;1)$, where α_0 is chosen as in the proof of Proposition 3.3. Then, for any $\delta \in (0;1)$ the following estimate holds

$$\left\| \hat{P}^*_\delta \restriction_{\mathcal{K}^0_\alpha} \right\| \leq 1 - (1-\nu)\delta. \tag{5.2}$$

Proof. It is easily seen that for any $k \in \mathcal{K}^0_\alpha$ the following inequality holds

$$|k(\eta)| \leq 1_{|\eta|>0} \|k\|_{\mathcal{K}_C} C^{|\eta|}, \quad \lambda\text{-a.a. } \eta \in \Gamma_0.$$

Thus, using (3.19), we have

$$\begin{aligned}
&C^{-|\eta|} \left|(\hat{P}^*_\delta k)(\eta)\right| \\
\leq& C^{-|\eta|} \sum_{\omega \subset \eta} (1-\delta)^{|\eta\setminus\omega|} (z\delta)^{|\omega|} \int_{\Gamma_0} \prod_{y\in\xi} \left(1 - e^{-E^\phi(y,\omega)}\right) |k(\xi \cup \eta \setminus \omega)| \, d\lambda(\xi) \\
\leq& \|k\|_{\mathcal{K}_C} \sum_{\omega \subset \eta} (1-\delta)^{|\eta\setminus\omega|} \left(\frac{z\delta}{C}\right)^{|\omega|} \int_{\Gamma_0} \prod_{y\in\xi} \left(1 - e^{-E^\phi(y,\omega)}\right) C^{|\xi|} 1\!\!1_{|\xi|+|\eta\setminus\omega|>0} d\lambda(\xi) \\
=& \|k\|_{\mathcal{K}_C} \sum_{\omega \subsetneq \eta} (1-\delta)^{|\eta\setminus\omega|} \left(\frac{z\delta}{C}\right)^{|\omega|} \int_{\Gamma_0} \prod_{y\in\xi} \left(1 - e^{-E^\phi(y,\omega)}\right) C^{|\xi|} d\lambda(\xi) \\
&+ \|k\|_{\mathcal{K}_C} \left(\frac{z\delta}{C}\right)^{|\eta|} \int_{\Gamma_0} \prod_{y\in\xi} \left(1 - e^{-E^\phi(y,\omega)}\right) C^{|\xi|} 1\!\!1_{|\xi|>0} d\lambda(\xi) \\
=& \|k\|_{\mathcal{K}_C} \sum_{\omega \subsetneq \eta} (1-\delta)^{|\eta\setminus\omega|} \left(\frac{z\delta}{C}\right)^{|\omega|} \int_{\Gamma_0} \prod_{y\in\xi} \left(1 - e^{-E^\phi(y,\omega)}\right) C^{|\xi|} d\lambda(\xi) \\
&+ \|k\|_{\mathcal{K}_C} \left(\frac{z\delta}{C}\right)^{|\eta|} \int_{\Gamma_0} \prod_{y\in\xi} \left(1 - e^{-E^\phi(y,\omega)}\right) C^{|\xi|} d\lambda(\xi) - \|k\|_{\mathcal{K}_C} \left(\frac{z\delta}{C}\right)^{|\eta|} \\
=& \|k\|_{\mathcal{K}_C} \sum_{\omega \subset \eta} (1-\delta)^{|\eta\setminus\omega|} \left(\frac{z\delta}{C}\right)^{|\omega|} \int_{\Gamma_0} \prod_{y\in\xi} \left(1 - e^{-E^\phi(y,\omega)}\right) C^{|\xi|} d\lambda(\xi) \\
&- \|k\|_{\mathcal{K}_C} \left(\frac{z\delta}{C}\right)^{|\eta|} \\
=& \|k\|_{\mathcal{K}_C} \sum_{\omega \subset \eta} (1-\delta)^{|\eta\setminus\omega|} \left(\frac{z\delta}{C}\right)^{|\omega|} \exp\left\{C \int_{\mathbb{R}^d} \left(1 - e^{-E^\phi(y,\omega)}\right) dy\right\}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \|k\|_{\mathcal{K}_C} \left(\frac{z\delta}{C}\right)^{|\eta|} \\
&\le \|k\|_{\mathcal{K}_C} \sum_{\omega\subset\eta} (1-\delta)^{|\eta\setminus\omega|} \left(\frac{z\delta}{C}\right)^{|\omega|} \exp\left\{CC_\beta\left|\omega\right|\right\} - \|k\|_{\mathcal{K}_C} \left(\frac{z\delta}{C}\right)^{|\eta|} \\
&\le \|k\|_{\mathcal{K}_C} \sum_{\omega\subset\eta} (1-\delta)^{|\eta\setminus\omega|} (\nu\delta)^{|\omega|} - \|k\|_{\mathcal{K}_C} \left(\frac{z\delta}{C}\right)^{|\eta|} \\
&= \|k\|_{\mathcal{K}_C} \left((1-(1-\nu)\,\delta)^{|\eta|} - \left(\frac{z\delta}{C}\right)^{|\eta|} \right) \\
&= \|k\|_{\mathcal{K}_C} \left(1-(1-\nu)\,\delta - \frac{z\delta}{C} \right) \sum_{j=0}^{|\eta|-1} (1-(1-\nu)\,\delta)^{|\eta|-1-|j|} \left(\frac{z\delta}{C}\right)^{j} \\
&\le \|k\|_{\mathcal{K}_C} \left(1-(1-\nu)\,\delta - \frac{z\delta}{C} \right) \sum_{j=0}^{|\eta|-1} \left(\frac{z\delta}{C}\right)^{j} \\
&= \|k\|_{\mathcal{K}_C} \left(1-(1-\nu)\,\delta - \frac{z\delta}{C} \right) \frac{1-\left(\frac{z\delta}{C}\right)^{|\eta|}}{1-\frac{z\delta}{C}} \\
&\le \|k\|_{\mathcal{K}_C} \left(1-(1-\nu)\,\delta - \frac{z\delta}{C} \right) \frac{1}{1-\frac{z\delta}{C}} \\
&= \|k\|_{\mathcal{K}_C} \left(1 - \frac{(1-\nu)\,\delta}{1-\frac{z\delta}{C}} \right) \le \|k\|_{\mathcal{K}_C} \bigl(1-(1-\nu)\,\delta\bigr),
\end{aligned}
$$

where we have used the trivial bound

$$z < \nu C < C.$$

This completes the proof. □

Remark 5.2. Condition (5.1) is equivalent to (3.5) and (3.12).

Suppose that (cf. (3.13))

$$zC_\phi < (2e)^{-1}. \tag{5.3}$$

Then (see, e.g., [6]) there exists a Gibbs measure μ on $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ corresponding to the potential $\phi \ge 0$ and the activity parameter z. We denote the corresponding correlation function by k_μ. The measure μ is reversible (symmetrizing) for the operator defined by (3.3) (see, e.g., [6], [13]). Hence, for any $F \in KB_{\mathrm{bs}}(\Gamma_0)$

$$\int_\Gamma LF(\gamma)d\mu(\gamma) = 0. \tag{5.4}$$

Theorem 5.3. *Let* (5.3) *and* (5.1) *hold and let* $\alpha \in (\alpha_0; 1)$*, where* α_0 *is chosen as in the proof of Proposition 3.3. Let* $k_0 \in \overline{\mathcal{K}_{\alpha C}}$ *and* $k_t = \hat{T}^{\odot\alpha}(t)k_0$*. Then, for any* $t \ge 0$

$$\|k_t - k_\mu\|_{\mathcal{K}_C} \le e^{-(1-\nu)t} \|k_0 - k_\mu\|_{\mathcal{K}_C}. \tag{5.5}$$

Proof. First, note that for any $\alpha \in (\alpha_0; 1)$ the inequality (3.14) implies $z \leq \alpha C \exp\{-\alpha C C_\phi\}$. Hence, $k_\mu(\eta) \leq (\alpha C)^{|\eta|}$, $\eta \in \Gamma_0$. Thus, $k_\mu \in \mathcal{K}_{\alpha C} \subset \overline{\mathcal{K}_{\alpha C}} \cap D(\hat{L}^*)$. By (5.4), for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ we have $\langle\!\langle \hat{L}G, k_\mu \rangle\!\rangle = 0$. It means that $\hat{L}^* k_\mu = 0$. Therefore, $\hat{L}^{\odot\alpha} k_\mu = 0$. As a result, $\hat{T}^{\odot\alpha}(t) k_\mu = k_\mu$. Let $r_0 = k_0 - k_\mu \in \overline{\mathcal{K}_{\alpha C}}$. Then $r_0 \in \mathcal{K}_a^0$ and

$$
\begin{aligned}
\|k_t - k_\mu\|_{\mathcal{K}_C} &= \left\|\hat{T}^{\odot\alpha}(t) r_0\right\|_{\mathcal{K}_C} \\
&\leq \left\| \left(\hat{P}_\delta^*\right)^{\left[\frac{t}{\delta}\right]} r_0 \right\|_{\mathcal{K}_C} + \left\| \hat{T}^{\odot\alpha}(t) r_0 - \left(\hat{P}_\delta^*\right)^{\left[\frac{t}{\delta}\right]} r_0 \right\|_{\mathcal{K}_C} \\
&\leq \left\| \hat{P}_\delta^* \restriction_{\mathcal{K}_\alpha^0} \right\|^{\left[\frac{t}{\delta}\right]} \cdot \|r_0\|_{\mathcal{K}_C} + \left\| \hat{T}^{\odot\alpha}(t) r_0 - \left(\hat{P}_\delta^*\right)^{\left[\frac{t}{\delta}\right]} r_0 \right\|_{\mathcal{K}_C} \\
&\leq \left(1 - (1-\nu)\delta\right)^{\frac{t}{\delta} - 1} \|r_0\|_{\mathcal{K}_C} + \left\| \hat{T}^{\odot\alpha}(t) r_0 - \left(\hat{P}_\delta^*\right)^{\left[\frac{t}{\delta}\right]} r_0 \right\|_{\mathcal{K}_C},
\end{aligned}
$$

since $0 < 1 - (1-\nu)\delta < 1$ and $\frac{t}{\delta} < \left[\frac{t}{\delta}\right] + 1$. Taking the limit as $\delta \downarrow 0$ in the right hand side of this inequality we obtain (5.5). □

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