REGULARIZATION OF SINGULAR STURM-LIOUVILLE EQUATIONS

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ABSTRACT. The paper deals with the singular Sturm-Liouville expressions

$$l(y) = -(py')' + qy$$

with the coefficients

$$q = Q', \quad 1/p, Q/p, Q^2/p \in L_1,$$

where the derivative of the function Q is understood in the sense of distributions. Due to a new regularization, the corresponding operators are correctly defined as quasi-differentials. Their resolvent approximation is investigated and all self-adjoint and maximal dissipative extensions and generalized resolvents are described in terms of homogeneous boundary conditions of the canonical form.

1. INTRODUCTION

This paper studies operators generated by the differential expressions

$$(1) \qquad l(y) = -(py')'(t) + q(t)y(t), \quad t \in \mathcal{J}$$

on a finite interval $\mathcal{J} := (a, b)$.

If the coefficients in (1) are real-valued and

$$(2) \qquad q \in C(\overline{\mathcal{J}}), \quad 0 < p \in C^1(\overline{\mathcal{J}}),$$

then the equation $l(y) = f$ is a differential Sturm-Liouville equation that has been investigated quite comprehensively. A modern exposition of the classical Sturm-Liouville theory may be found in many studies. Principal statements of this theory remain true under the weaker assumptions

$$(3) \qquad q, 1/p \in L_1(\mathcal{J}, \mathbb{C}) =: L_1,$$

see [1] and references therein. This is achieved through a regularization of the expression $l(y)$ applying Shin-Zettl quasi-derivatives. They were introduced in [2] and later generalized in [3], see also [4].

A further essential development of that approach was achieved in the paper [5]. It was proved there that if $p(t) \equiv 1$, then the condition on q may be significantly weakened. Namely, it is sufficient to suppose that

$$(4) \qquad p(t) \equiv 1, \quad q = Q', \quad Q \in L_2(\mathcal{J}, \mathbb{C}) =: L_2,$$

where the derivative of the function Q is understood in the sense of distributions. Note that the one-dimension Schrödinger operators with potentials that are Radon measures were introduced and investigated long before that by physicists applying operator theory methods (see [6] and references therein).

The main goal of this paper is to define and investigate Sturm-Liouville operators on a finite interval $\mathcal{J}$ under the assumptions more general than those in (3) and (4),

(5) $$q = Q', \quad 1/p, Q/p, Q^2/p \in L_1.$$

To achieve this goal, in Section 2, we propose a new regularization of the formal differential expression (1) under assumptions (5) by means of Shin-Zettl quasi-derivatives. We also define the corresponding maximal and minimal operators on the Hilbert space L_2. If conditions (3) hold, then these operators coincide with the classical ones and, under assumptions (4), they are identical to the operators introduced in [5].

Section 3 shows that, in the case of two-point boundary conditions, resolvents of the constructed operators may be approximated in the sense of the norm with resolvents of other Sturm-Liouville operators; for instance, ones that have more regular coefficients.

In Section 4, the minimal operator is supposed to be symmetric and all its self-adjoint extensions are described in terms of the homogeneous boundary conditions of the canonical form.

In addition, in Section 5, all maximal dissipative extensions and generalized resolvents of the minimal symmetric operator are described in the same form.

Extensions in Sections 4 and 5 are described by applying the boundary triplet theory (see [7] and references therein). They are parametrized by certain classes of operators on $\mathbb{C}^2$, and this parametrization is bijective and continuous. Also, separated boundary conditions are singled out.

Note that in the case where $p(t) \equiv 1$, the results of Sections 3 and 4 improve the corresponding results of [5] where stronger conditions are required for the approximation, and self-adjoint extensions are described on the basis of the Glazman-Krein-Naimark theory. Results of Section 5 deal with the questions not considered in [5].

2. Regularization of singular expression

Consider the formal differential expression (1), assuming that conditions (5) hold. We introduce the quasi-derivatives

$$D^{[0]}y = y,$$
$$D^{[1]}y = py' - Qy,$$
$$D^{[2]}y = (D^{[1]}y)' + \frac{Q}{p}D^{[1]}y + \frac{Q^2}{p}y.$$

Then expression (1) is defined to be the quasi-differential expression

$$l[y] := -D^{[2]}y.$$

Definition 1. A solution of the Cauchy problem for the resolvent equation

(6) $$l[y] - \lambda y = f \in L_2, \quad y(c) = \alpha_1, \quad (D^{[1]}y)(c) = \alpha_2,$$

where $c \in \overline{\mathcal{J}}$ and α_1, α_2 are arbitrary complex numbers, is defined to be the first component of the solution of the Cauchy problem for the correspondent system of the first order differential equations

(7) $$w'(t) = A_\lambda(t)w(t) + \varphi(t), \quad w(c) = (\alpha_1, \alpha_2),$$

where $w(t) = (y(t), D^{[1]}y(t))$, the matrix-valued function is

$$A_\lambda(t) := \begin{pmatrix} \frac{Q}{p} & \frac{1}{p} \\ -\frac{Q^2}{p} - \lambda & -\frac{Q}{p} \end{pmatrix} \in L_1^{2\times 2},$$

and $\varphi(t) := (0, -f(t))$.

Lemma 1. *Problem (6), with assumptions (5), has only a unique solution defined on $\overline{\mathcal{J}}$.*

Proof of Lemma 1. Problem (7) with $A_\lambda(\cdot) \in L_1^{2\times 2}$ has only a unique solution for any $c \in \overline{\mathcal{J}}$ and $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ due to Theorem 1.2.1 [1]. This implies the statement of Lemma 1 by Definition 1. □

The quasi-differential expression $l[y]$ gives rise to the *maximal* quasi-differential operator

$$L_{\max} : y \to l[y], \quad \mathrm{Dom}(L_{\max}) := \left\{ y \in L_2 \,\middle|\, y, D^{[1]}y \in AC(\overline{\mathcal{J}}, \mathbb{C}), D^{[2]}y \in L_2 \right\}$$

on the Hilbert space L_2 (see [3, 4]). The *minimal* quasi-differential operator is defined as a restriction of the operator $L_{\max}$ onto the set

$$\mathrm{Dom}(L_{\min}) := \left\{ y \in \mathrm{Dom}(L_{\max}) \,\middle|\, D^{[k]}y(a) = D^{[k]}y(b) = 0, k = 0, 1 \right\}.$$

Remark 1. One can easily see that if Q is replaced with $\widetilde{Q} := Q + c$, $c \in \mathbb{C}$, then the operators $L_{\max}, L_{\min}$ do not change.

If the coefficients in (1) satisfy (3), then the operators $L_{\max}, L_{\min}$ introduced above coincide with the usual maximal and minimal Sturm-Liouville operators [1].

Consider the expression

$$l^+(y) = -(\overline{p}y')'(t) + \overline{q}(t)y(t),$$

formally adjoint to (1), where the bar denotes complex conjugation. Denote by $L_{\max}^+$ and $L_{\min}^+$ the maximal and the minimal operators generated by this expression on the space L_2. Then results of this section, together with results of [4] for general quasi-differential expressions, yield following theorem.

Theorem 1. *The operators $L_{\min}$, $L_{\min}^+$, $L_{\max}$, $L_{\max}^+$ are closed and densely defined on the space L_2,*

$$L_{\min}^* = L_{\max}^+, \quad L_{\max}^* = L_{\min}^+.$$

In the case where p and Q are real-valued, the operator $L_{\min} = L_{\min}^+$ is symmetric with the deficiency index $(2, 2)$, and

$$L_{\min}^* = L_{\max}, \quad L_{\max}^* = L_{\min}.$$

3. APPROXIMATION OF RESOLVENT

Consider the class of quasi-differential expressions $l_\varepsilon[y] = -D_\varepsilon^{[2]}y$ with the coefficients

$$p_\varepsilon, q_\varepsilon = Q_\varepsilon', \quad \varepsilon \in [0, \varepsilon_0].$$

On the Hilbert space L_2 with norm $\|\cdot\|_2$, these expressions generate the operators $L_{\min}^\varepsilon$, $L_{\max}^\varepsilon$ for every ε. Let $\alpha(\varepsilon), \beta(\varepsilon) \in \mathbb{C}^{2\times 2}$ be matrices and consider the vectors

$$\mathcal{Y}_\varepsilon(a) := \left\{ y(a), D_\varepsilon^{[1]}y(a) \right\}, \quad \mathcal{Y}_\varepsilon(b) := \left\{ y(b), D_\varepsilon^{[1]}y(b) \right\} \in \mathbb{C}^2.$$

Consider the quasi-differential operators

$$L_\varepsilon y = l_\varepsilon[y], \quad \mathrm{Dom}(L_\varepsilon) = \left\{ y \in \mathrm{Dom}\,(L_{\max}^\varepsilon) | \, \alpha(\varepsilon)\mathcal{Y}_\varepsilon(a) + \beta(\varepsilon)\mathcal{Y}_\varepsilon(b) = 0 \right\}.$$

It is evident that $L_{\min}^\varepsilon \subset L_\varepsilon \subset L_{\max}^\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$.

We denote by $\rho(L)$ the resolvent set of the operator L. Recall that the operators L_ε converge to the operator L_0 in the sense of the norm resolvent convergence, $L_\varepsilon \xrightarrow{R} L_0$, if the there is a number $\mu \in \mathbb{C}$ such that $\mu \in \rho(L_0)$ and $\mu \in \rho(L_\varepsilon)$ (for all sufficiently small ε), and

$$\|(L_\varepsilon - \mu)^{-1} - (L_0 - \mu)^{-1}\| \to 0, \quad \varepsilon \to 0 + .$$

This definition does not depend on the choice of the point $\mu \in \rho(L_0)$ [8].

In the case where the matrices $\alpha(\varepsilon), \beta(\varepsilon)$ do not depend on ε and $p_\varepsilon(t) \equiv 1$, it is shown in [5] that if $\|Q_\varepsilon - Q_0\|_2 \to 0$ for $\varepsilon \to 0+$ and the resolvent set of the operator L_0 is not empty, then $L_\varepsilon \xrightarrow{R} L_0$. The following theorem generalizes this result.

Theorem 2. *Suppose $\rho(L_0)$ is not empty and, for $\varepsilon \to 0+$, the following conditions hold:*

$$(1) \; \|1/p_\varepsilon - 1/p_0\|_1 \to 0,$$
$$(2) \; \|Q_\varepsilon/p_\varepsilon - Q_0/p_0\|_1 \to 0,$$
$$(3) \; \|Q_\varepsilon^2/p_\varepsilon - Q_0^2/p_0\|_1 \to 0,$$
$$(4) \; \alpha(\varepsilon) \to \alpha(0), \quad \beta(\varepsilon) \to \beta(0),$$

where $\|\cdot\|_1$ is the norm in the space $L_1(\mathcal{J}, \mathbb{C})$. Then $L_\varepsilon \xrightarrow{R} L_0$.

Remark 2. In the case where $p_\varepsilon(t) \equiv 1$, condition (1) is automatically fulfilled and conditions (2) and (3) are weaker than the assumption that $\|Q_\varepsilon - Q_0\|_2 \to 0$.

To prove Theorem 2 we will need some auxiliary results.

We start by introducing the following definition ([9, 10]).

Definition 2. Denote by $\mathcal{M}^m(\mathcal{J}) =: \mathcal{M}^m$, $m \in \mathbb{N}$, the class of matrix-valued functions

$$R(\cdot; \varepsilon) : [0, \varepsilon_0] \to L_1^{m \times m}$$

parametrized by ε such that the solution of the Cauchy problem

$$Z'(t; \varepsilon) = R(t; \varepsilon)Z(t; \varepsilon), \quad Z(a; \varepsilon) = I_m,$$

satisfies the limit condition

$$\lim_{\varepsilon \to 0+} \|Z(\cdot; \varepsilon) - I_m\|_C = 0,$$

where $\|\cdot\|_C$ is the sup-norm.

In paper [10], the following general result is established:

Theorem 3. *Suppose that the vector boundary-value problem*

$$(8) \qquad y'(t; \varepsilon) = A(t; \varepsilon)y(t; \varepsilon) + f(t; \varepsilon), \quad t \in \mathcal{J}, \quad \varepsilon \in [0, \varepsilon_0],$$

$$(9) \qquad\qquad\qquad U_\varepsilon y(\cdot; \varepsilon) = 0,$$

where the matrix-valued functions $A(\cdot, \varepsilon) \in L_1^{m \times m}$, the vector-valued functions $f(\cdot, \varepsilon) \in L_1^m$, and the linear continuous operators

$$U_\varepsilon : C(\overline{\mathcal{J}}; \mathbb{C}^m) \to \mathbb{C}^m, \quad m \in \mathbb{N},$$

satisfy the following conditions.

1) *The homogeneous limit boundary-value problem (8), (9) with $\varepsilon = 0$ and $f(\cdot; 0) \equiv 0$ has only a trivial solution;*

2) *$A(\cdot; \varepsilon) - A(\cdot; 0) \in \mathcal{M}^m$;*

3) *$\|U_\varepsilon - U_0\| \to 0, \quad \varepsilon \to 0+$.*

Then, for a small enough ε, there exist Green matrices $G(t, s; \varepsilon)$ for problems (8), (9) and, on the square $\mathcal{J} \times \mathcal{J}$,

$$(10) \qquad\qquad \|G(\cdot, \cdot; \varepsilon) - G(\cdot, \cdot; 0)\|_\infty \to 0, \quad \varepsilon \to 0+,$$

where $\|\cdot\|_\infty$ is the norm in the space L_∞.

Remark 3. Condition 3) in Theorem 3 cannot be replaced with the weaker condition on the operator U_ε to strongly converge, $U_\varepsilon \xrightarrow{s} U_0$ [10]. However, one can easily see that, for the two-point boundary operators

$$U_\varepsilon y := B_1(\varepsilon)y(a) + B_2(\varepsilon)y(b), \quad B_k(\varepsilon) \in \mathbb{C}^{m \times m}, \quad k = 1, 2,$$

both the strong convergence and the norm convergence conditions are equivalent to

$$\|B_k(\varepsilon) - B_k(0)\| \to 0, \quad \varepsilon \to 0+, \quad k = 1, 2.$$

There are different sufficient conditions for the matrix-valued function $R(\cdot; \varepsilon)$ to belong to $\mathcal{M}^m$. In particular, the results of [11] give that conditions (1), (2), (3) of Theorem 2 imply

$$A(\cdot; \varepsilon) - A(\cdot; 0) \in \mathcal{M}^2,$$

where the matrix-valued function $A(\cdot; \varepsilon)$ is given by the formula

(11) $$A(\cdot; \varepsilon) := \left(\begin{array}{cc} Q_\varepsilon/p_\varepsilon & 1/p_\varepsilon \\ -Q_\varepsilon^2/p_\varepsilon & -Q_\varepsilon/p_\varepsilon \end{array} \right) \in L_1^{2\times 2}.$$

Before proving Theorem 2, we will need the following two lemmas require to reduce Theorem 2 to Theorem 3.

Lemma 2. *The function $y(t)$ is a solution of the boundary-value problem*

(12) $$l_\varepsilon[y](t) = f(t; \varepsilon) \in L_2, \quad \varepsilon \in [0, \varepsilon_0],$$

(13) $$\alpha(\varepsilon)\mathcal{Y}_\varepsilon(a) + \beta(\varepsilon)\mathcal{Y}_\varepsilon(b) = 0,$$

if and only if the vector-valued function $w(t) = (y(t), D_\varepsilon^{[1]}y(t))$ is a solution of the boundary-value problem

(14) $$w'(t) = A(t; \varepsilon)w(t) + \varphi(t; \varepsilon),$$

(15) $$\alpha(\varepsilon)w(a) + \beta(\varepsilon)w(b) = 0,$$

where the matrix-valued function $A(\cdot; \varepsilon)$ is given by (11) and $\varphi(\cdot; \varepsilon) := (0, -f(\cdot; \varepsilon))$.

Proof of Lemma 2. Consider the system of equations

$$\begin{cases} (D_\varepsilon^{[0]}y(t))' = \dfrac{Q_\varepsilon(t)}{p_\varepsilon(t)} D_\varepsilon^{[0]}y(t) + \dfrac{1}{p_\varepsilon(t)} D_\varepsilon^{[1]}y(t), \\[3mm] (D_\varepsilon^{[1]}y(t))' = -\dfrac{Q_\varepsilon^2(t)}{p_\varepsilon(t)} D_\varepsilon^{[0]}y(t) - \dfrac{Q_\varepsilon(t)}{p_\varepsilon(t)} D_\varepsilon^{[1]}y(t) - f(t; \varepsilon). \end{cases}$$

Let $y(\cdot)$ be a solution of (12), then the definition of a quasi-derivative implies that $y(\cdot)$ is a solution of this system. On the other hand, denoting $w(t) = (D_\varepsilon^{[0]}y(t), D_\varepsilon^{[1]}y(t))$ and $\varphi(t; \varepsilon) = (0, -f(t; \varepsilon))$, we rewrite this system in the form of equation (14).

Taking into account that $\mathcal{Y}_\varepsilon(a) = w(a)$, $\mathcal{Y}_\varepsilon(b) = w(b)$, one can see that the boundary conditions (13) are equivalent to the boundary conditions (15). $\qquad \square$

Due to Lemma 2, that statement that

($\mathcal{U}$) the homogeneous boundary-value problem $l_0[y](t) = 0$, $\quad \alpha(0)\mathcal{Y}_0(a) + \beta(0)\mathcal{Y}_0(b) = 0$, has only a trivial solution

implies that the homogeneous boundary-value problem

$$w'(t) = A(t; 0)w(t), \quad \alpha(0)w(a) + \beta(0)w(b) = 0$$

has only a trivial solution.

Lemma 3. *Let a Green matrix*

$$G(t, s, \varepsilon) = (g_{ij}(t, s))_{i,j=1}^2 \in L_\infty^{2\times 2}$$

exist for the problem (14), (15) for small enough ε. Then there exists a Green function $\Gamma(t, s; \varepsilon)$ for the semi-homogeneous boundary-value problem (12), (13) and

$$\Gamma(t, s; \varepsilon) = -g_{12}(t, s; \varepsilon) \quad a.e.$$

Proof of Lemma 3. According to the definition of a Green matrix, a unique solution of problem (14), (15) can be written in the form

$$w_\varepsilon(t) = \int\limits_a^b G(t, s; \varepsilon)\varphi(s; \varepsilon)ds, \quad t \in \mathcal{J}.$$

Due to Lemma 2, the latter equality can be rewritten in the form

$$\begin{cases} D_\varepsilon^{[0]} y_\varepsilon(t) = \int_a^b g_{12}(t,s;\varepsilon)(-f(s;\varepsilon))ds, \\ D_\varepsilon^{[1]} y_\varepsilon(t) = \int_a^b g_{22}(t,s;\varepsilon)(-f(s;\varepsilon))ds, \end{cases}$$

where $y_\varepsilon(\cdot)$ is a unique solution of the problem (12), (13). This implies the statement of Lemma 3. $\qquad\square$

Proof of Theorem 2. Note that, due to the equality

$$(Q_\varepsilon + \mu)^2/p_\varepsilon - (Q_0 + \mu)^2/p_0 = (Q_\varepsilon^2/p_\varepsilon - Q_0^2/p_0) + 2\mu(Q_\varepsilon/p_\varepsilon - Q_0/p_0) + \mu^2(1/p_\varepsilon - 1/p_0),$$

where $\mu \in \mathbb{C}$, conditions (1)–(3) of Theorem 2 imply that we can assume without loss of generality that $0 \in \rho(L_0)$.

We need to show that $\sup\limits_{\|f\|_2=1} \|L_\varepsilon^{-1}f - L_0^{-1}f\| \to 0$, $\varepsilon \to 0+$.

The equation $L_\varepsilon^{-1}f = y_\varepsilon$ is equivalent to $L_\varepsilon y_\varepsilon = f$, i. e., y_ε is a solution of problem (12), (13). Also the statement ($\mathcal{U}$) is verified due to $0 \in \rho(L_0)$. From the conditions 1)–3) of Theorem 2, it follows that $A(\cdot;\varepsilon) - A(\cdot;0) \in \mathcal{M}^2$, where $A(\cdot;\varepsilon)$ is given by formula (11). Thus statement of Theorem 2 implies that the problem (14), (15) satisfies conditions of Theorem 3. This means that Green matrices $G(t,s;\varepsilon)$ of the problems (14), (15) exist and the limit relation (10) is satisfied. Taking into account Lemma 3 this yields limit equality

$$\|\Gamma(\cdot,\cdot;\varepsilon) - \Gamma(\cdot,\cdot;0)\|_\infty \to 0, \quad \varepsilon \to 0 + .$$

Then

$$\|L_\varepsilon^{-1} - L_0^{-1}\| = \sup_{\|f\|_2=1} \| \int_a^b [\Gamma(t,s;\varepsilon) - \Gamma(t,s;0)] f(s)\, ds\|_2$$

$$\le (b-a)^{1/2} \sup_{\|f\|_2=1} \| \int_a^b |\Gamma(t,s;\varepsilon) - \Gamma(t,s;0)|\, |f(s)|\, ds\|_C$$

$$\le (b-a)\|\Gamma(\cdot,\cdot;\varepsilon) - \Gamma(\cdot,\cdot;0)\|_\infty \to 0, \quad \varepsilon \to 0+,$$

which proves Theorem 2. $\qquad\square$

For the case $p_\varepsilon(t) \equiv 1$, a statement stronger than Theorem 2 was proved in [12].

4. Self-adjoint boundary conditions

In what follows we will require the functions p, Q and, consequently, the distribution $q = Q'$ to be real-valued. In this case, the expression $l[y]$ is formally self-adjoint [4] and, according to Theorem 1, the minimal operator $L_{\min}$ is symmetric. So one may pose a problem of describing (in terms of homogeneous boundary conditions) all extensions of the operator $L_{\min}$ that are self-adjoint in the space L_2. To give an answer to this question, we will apply the concept of the boundary triplet.

Let us recall following definition.

Definition 3. Let L be a closed densely defined symmetric operator in a Hilbert space $\mathcal{H}$ with equal (finite or infinite) deficient indices. The triplet (H, Γ_1, Γ_2), where H is an auxiliary Hilbert space and Γ_1, Γ_2 are the linear mappings of $\mathrm{Dom}(L^*)$ onto H, is called a *boundary triplet* of the symmetric operator L, if

(1) for any $f, g \in \mathrm{Dom}(L^*)$,

$$(L^*f, g)_\mathcal{H} - (f, L^*g)_\mathcal{H} = (\Gamma_1 f, \Gamma_2 g)_H - (\Gamma_2 f, \Gamma_1 g)_H,$$

(2) for any $f_1, f_2 \in H$ there is a vector $f \in \mathrm{Dom}(L^*)$ such that $\Gamma_1 f = f_1$, $\Gamma_2 f = f_2$.

The definition of a boundary triplet implies that $f \in \text{Dom}(L)$ if and only if $\Gamma_1 f = \Gamma_2 f = 0$. A boundary triplet exists for any symmetric operator with equal non-zero deficient indices (see [7] and references therein). It is not unique.

The following result is crucial for the rest of the paper.

Basic Lemma. *Triplet* $(\mathbb{C}^2, \Gamma_1, \Gamma_2)$, *where* Γ_1, Γ_2 *are the linear mappings*

$$(16) \qquad \Gamma_1 y := \left(D^{[1]} y(a), -D^{[1]} y(b) \right), \ \Gamma_2 y := (y(a), y(b)),$$

from $\text{Dom}(L_{\max})$ *onto* $\mathbb{C}^2$ *is a boundary triplet for the operator* $L_{\min}$.

For convenience, we introduce the following notation.

Definition 4. Denote by L_K the restriction of the operator $L_{\max}$ onto the set of functions $y(t) \in \text{Dom}(L_{\max})$ satisfying the homogeneous boundary condition in the canonical form

$$(17) \qquad (K - I) \, \Gamma_1 y + i \, (K + I) \, \Gamma_2 y = 0,$$

where K is any bounded operator on the space $\mathbb{C}^2$.

Basic Lemma together with results of [7, Ch. 3] gives the following description of all self-adjoint extensions of $L_{\min}$.

Theorem 4. *Every* L_K, *with* K *being a unitary operator on the space* $\mathbb{C}^2$, *is a self-adjoint extension of the operator* $L_{\min}$. *Conversely, for any self-adjoint extension* $\widetilde{L}$ *of the operator* $L_{\min}$ *there is a unitary operator* K *such that* $\widetilde{L} = L_K$. *This correspondence between unitary operators* $\{K\}$ *and self-adjoint extensions* $\{\widetilde{L}\}$ *is bijective.*

We start a proof of the Basic Lemma with the following two lemmas that are special cases of the corresponding results for general quasi-differential expressions (see [4]).

Lemma 4. *Suppose* $y, z \in \text{Dom}(L_{\max})$. *Then*

$$\int_a^b \left(D^{[2]} y \cdot \overline{z} - y \cdot \overline{D^{[2]} z} \right) dt = \left(-D^{[0]} y \cdot \overline{D^{[1]} z} + D^{[1]} y \cdot \overline{D^{[0]} z} \right) \Big|_a^b.$$

Lemma 5. *Suppose that* $\{\alpha_0, \alpha_1\}$, $\{\beta_0, \beta_1\}$ *are arbitrary sets of complex numbers. Then there is a function* $y \in \text{Dom}(L_{\max})$ *such that*

$$D^{[k]} y(a) = \alpha_k, \quad D^{[k]} y(b) = \beta_k, \quad k = 0, 1.$$

Proof of the Basic Lemma. To prove the Basic Lemma, we need to prove that the triplet $(\mathbb{C}^2, \Gamma_1, \Gamma_2)$ satisfies conditions 1) and 2) in the definition of the boundary triplet for the operator $L_{\min}$. According to Theorem 1, $L_{\min}^* = L_{\max}$. Due to Lemma 4,

$$(L_{\max} y, z) - (y, L_{\max} z) = \left(D^{[0]} y \cdot \overline{D^{[1]} z} - D^{[1]} y \cdot \overline{D^{[0]} z} \right) \Big|_a^b.$$

But

$$(\Gamma_1 y, \Gamma_2 z) = D^{[1]} y(a) \cdot \overline{D^{[0]} z(a)} - D^{[1]} y(b) \cdot \overline{D^{[0]} z(b)},$$

$$(\Gamma_2 y, \Gamma_1 z) = D^{[0]} y(a) \cdot \overline{D^{[1]} z(a)} - D^{[0]} y(b) \cdot \overline{D^{[1]} z(b)}.$$

This means that condition 1) is fulfilled. Condition 2) is true due to Lemma 5. $\qquad \square$

Proof of Theorem 4. The claim in Theorem 4 follows from the Basic Lemma and Theorem 1.6 Ch. 3 [7] for the boundary triplet of an abstract symmetric operator. $\qquad \square$

Remark 4. Theorem 2, together with Theorem 4, implies that the mapping $K \to L_K$ is not only bijective but also continuous. More accurately, if unitary operators K_n converge to an operator K, then

$$\left\| (L_K - \lambda)^{-1} - (L_{K_n} - \lambda)^{-1} \right\| \to 0, \quad n \to \infty, \quad \text{Im } \lambda \neq 0.$$

The converse is also true, because the set of unitary operators in the space $\mathbb{C}^2$ is a compact set. This means that the mapping

$$K \to (L_K - \lambda)^{-1}, \quad \operatorname{Im}\lambda \neq 0,$$

is a homeomorphism for any fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Now we pass to a description of separated self-adjoint boundary conditions for expression (1).

Denote by $\mathbf{f_a}$ the germ of a continuous function f at the point a.

Definition 5. The boundary conditions that define the operator $L \subset L_{\max}$ are called *separated* if for arbitrary functions $y \in \operatorname{Dom}(L)$ and $g, h \in \operatorname{Dom}(L_{\max})$,

$$g, h \in \operatorname{Dom}(L) \quad \text{if} \quad \mathbf{g_a} = \mathbf{y_a}, \quad \mathbf{g_b} = 0, \quad \mathbf{h_a} = 0, \quad \mathbf{h_b} = \mathbf{y_b}.$$

Theorem 5. *Self-adjoint boundary conditions (17) are separated if and only if the matrix K is of the form (18), where $K_a, K_b \in \mathbb{C}$ and $|K_a| = |K_b| = 1$.*

A proof of Theorem 5 is based on the following lemma.

Lemma 6. *Boundary conditions of the form (17), with K being any $\mathbb{C}^{2\times 2}$-matrix are separated if and only if*

(18)
$$K = \left(\begin{array}{cc} K_a & 0 \\ 0 & K_b \end{array} \right),$$

where $K_a, K_b \in \mathbb{C}$.

Proof of Lemma 6. It is evident that $\mathbf{y_c} = \mathbf{g_c}$ implies

(19)
$$y(c) = g(c), \quad (D^{[1]}y)(c) = (D^{[1]}g)(c), \quad c \in [a, b].$$

Let the matrix K have the form (18) in boundary condition (17). Then conditions (17) can be written in the form of a system,

$$\begin{cases} (K_a - 1)D^{[1]}y(a) + i(K_a + 1)y(a) = 0, \\ -(K_b - 1)D^{[1]}y(b) + i(K_b + 1)y(b) = 0. \end{cases}$$

It is evident that these boundary conditions are separated.

Inversely, suppose that the boundary conditions (17) are separated. The matrix $K \in \mathbb{C}^{2\times 2}$ can be written in the form

$$K = \left(\begin{array}{cc} K_{11} & K_{12} \\ K_{21} & K_{22} \end{array} \right).$$

We need to prove $K_{12} = K_{21} = 0$.

Let us rewrite the boundary conditions (17) in the form of the system

$$\begin{cases} (K_{11} - 1)D^{[1]}y(a) - K_{12}D^{[1]}y(b) + i(K_{11} + 1)y(a) + iK_{12}y(b) = 0, \\ K_{21}D^{[1]}y(a) - (K_{22} - 1)D^{[1]}y(b) + iK_{21}y(a) + i(K_{22} + 1)y(b) = 0. \end{cases}$$

The fact that the boundary conditions are separated implies that a function g such that $\mathbf{g_a} = \mathbf{y_a}, \mathbf{g_b} = 0$ also satisfies this system. Due to equalities (19) this gives

$$\begin{cases} K_{11}\left[D^{[1]}y(a) + iy(a)\right] = D^{[1]}y(a) - iy(a), \\ K_{21}\left[D^{[1]}y(a) + iy(a)\right] = 0 \end{cases}$$

for any $y \in \operatorname{Dom}(L_K)$.

This means that either $K_{21} = 0$ or $D^{[1]}y(a) + iy(a) = 0$ for any $y \in \operatorname{Dom}(L_K)$. Suppose $K_{21} \neq 0$.

Let us return to the boundary conditions (17). For any $F = (F_1, F_2) \in \mathbb{C}^2$, consider the vectors $-i(K + I)F$ and $(K - I)F$. Due to the Basic Lemma and the definition of the boundary triplet, there exists a function $y_F \in \mathrm{Dom}(L_{\max})$ such that

(20)
$$\begin{cases} -i(K + I)F = \Gamma_1 y_F, \\ \quad (K - I)F = \Gamma_2 y_F. \end{cases}$$

A simple calculation shows that y_F satisfies the boundary conditions (17) and, therefore, $y_F \in \mathrm{Dom}(L_K)$. We can rewrite (20) in the form of the system

$$\begin{cases} -i(K_{11} + 1)F_1 - iK_{12}F_2 = D^{[1]}y_F(a), \\ -iK_{21}F_1 - i(K_{22} + 1)F_2 = -D^{[1]}y_F(b), \\ \quad (K_{11} - 1)F_1 + K_{12}F_2 = y_F(a), \\ \quad K_{21}F_1 + (K_{22} - 1)F_2 = y_F(b). \end{cases}$$

The first and the third equations of the system above imply that $0 = D^{[1]}y_F(a) + iy_F(a) = -2iF_1$ for any $F_1 \in \mathbb{C}$. We arrived at a contradiction, therefore, $K_{21} = 0$.

Similarly one may prove $K_{12} = 0$. $\qquad\square$

Proof of Theorem 5. Due to Lemma 6, we only need to remark that a matrix of the form (18) is unitary if and only if $|K_a| = |K_b| = 1$. $\qquad\square$

5. Non-self-adjoint boundary conditions and generalized resolvents

Recall the following definition.

Definition 6. A densely defined linear operator L on a complex Hilbert space $\mathcal{H}$ is called *dissipative* if

$$\mathrm{Im}\,(Lf, f)_{\mathcal{H}} \geq 0, \quad f \in \mathrm{Dom}(L)$$

and it is called *maximal dissipative* if, besides this, L has no nontrivial dissipative extensions on the space $\mathcal{H}$.

For instance, every symmetric operator is dissipative and every self-adjoint operator is a maximal dissipative one. Thus, if the minimal operator $L_{\min}$ is symmetric, then one can state the problem of describing its maximal dissipative extensions. According to Phillips' Theorem [7, 13], *every maximal dissipative extension of a symmetric operator is a restriction of its adjoint operator.* Therefore, every maximal dissipative extension of the operator $L_{\min}$ is a restriction of operator $L_{\max}$.

Parametric bijective description of the class of maximal dissipative extensions of the symmetric quasi-differential operator $L_{\min}$ is given by the following theorem.

Theorem 6. *Every L_K, with K being a contracting operator on the space $\mathbb{C}^2$, is a maximal dissipative extension of the operator $L_{\min}$. Conversely, for any maximal dissipative extension $\widetilde{L}$ of the operator $L_{\min}$ there exists a contracting operator K such that $\widetilde{L} = L_K$. This correspondence between contracting operators $\{K\}$ and the maximal dissipative extensions $\{\widetilde{L}\}$ is bijective.*

Proof of Theorem 6. Theorem 6 is a direct consequence of Basic Lemma and Theorem 1.6 Ch. 3 [7] for the boundary triplet of an abstract symmetric operator. $\qquad\square$

Remark 5. The mapping

$$K \to (L_K - \lambda)^{-1}, \quad \mathrm{Im}\,\lambda < 0,$$

for any fixed λ is a homeomorphism (see Remark 4).

Theorem 7. *Dissipative boundary conditions (17) are separated if and only if the matrix K is of the form (18), where $|K_a| \leq 1$, $|K_b| \leq 1$.*

Proof of Theorem 7. As in the proof of Theorem 5, due to Lemma 6, we only need to remark that the matrix K of the form (18) is a contracting operator on $\mathbb{C}^2$ if and only if $|K_a| \leq 1$, $|K_b| \leq 1$. $\qquad\square$

Recall the following definition.

Definition 7. A *generalized resolvent* of a closed symmetric operator L is the operator-valued function R_λ of the complex parameter $\lambda \in \mathbb{C}\backslash\mathbb{R}$ which can be represented in the form
$$R_\lambda f = P^+ \left(L^+ - \lambda I^+\right)^{-1} f, \quad f \in \mathcal{H},$$
where L^+ is a self-adjoint extension of the operator L, generally, on the space $\mathcal{H}^+$ which is wider than $\mathcal{H}$, I^+ is the identity operator on $\mathcal{H}^+$, and P^+ is the orthogonal projection operator from $\mathcal{H}^+$ onto $\mathcal{H}$.

The operator-valued function R_λ (Im $\lambda \neq 0$) is a generalized resolvent of a symmetric operator L if and only if
$$(R_\lambda f, g)_{\mathcal{H}} = \int_{-\infty}^{+\infty} \frac{d\left(F_\mu f, g\right)}{\mu - \lambda}, \quad f, g \in \mathcal{H},$$
where F_μ is the generalized spectral function of the operator L. In other words, the operator-valued function F_μ should have following properties [14]:

1^0. For $\mu_2 > \mu_1$, the difference $F_{\mu_2} - F_{\mu_1}$ is a bounded non-negative operator,

2^0. $F_{\mu+} = F_\mu$ for any real μ,

3^0. for any $x \in \mathcal{H}$,
$$\lim_{\mu \to -\infty} ||F_\mu x||_{\mathcal{H}} = 0, \quad \lim_{\mu \to +\infty} ||F_\mu x - x||_{\mathcal{H}} = 0.$$

A parametric inner description of all generalized resolvents of the operator $L_{\min}$ is given by the following theorem.

Theorem 8. *There is a one-to-one correspondence between the generalized resolvents of the operator $L_{\min}$ and the boundary-value problems*
$$l[y] = \lambda y + h,$$
$$\left(K(\lambda) - I\right)\Gamma_1 y + i\left(K(\lambda) + I\right)\Gamma_2 y = 0,$$
where $\lambda \in \mathbb{C}$, Im $\lambda < 0$, $h(x) \in L_2$, and $K(\lambda)$ is an operator-valued function into the space $\mathbb{C}^2$, regular in the lower half-plane, such that $||K(\lambda)|| \leq 1$. This correspondence is given by the identity
$$R_\lambda h = y, \quad \text{Im } \lambda < 0.$$

Proof of Theorem 8. Due to Basic Lemma Theorem, 8 is a consequence of Theorem 1 of the paper [15]. $\qquad\square$

For general quasi-differential operators of even and odd orders, respectively, the assertions of Theorems 4, 6 and 8 are announced without proofs in [16, 17].

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