THE GRADED RING OF QUANTUM THETA FUNCTIONS FOR NONCOMMUTATIVE TORUS WITH REAL MULTIPLICATION

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ABSTRACT. For the quantum torus generated by unitaries $UV = e(\theta)VU$ there exist nontrivial strong Morita autoequivalences in the case when θ is a real quadratic irrationality. A.Polishchuk introduced and studied the graded ring of holomorphic sections of powers of the respective bimodule (depending on the choice of a complex structure). We consider a Segre square of this ring whose graded components are spanned by Rieffel scalar products of Polishchuk's holomorphic vectors as in [5] and [8]. These graded components are linear spaces of quantum theta functions in the sense of Yu. Manin.

INTRODUCTION

A quantum torus A_θ with an irrational parameter $\theta \in \mathbb{R}\backslash\mathbb{Q}$ is a transformation group C^*-algebra $C^*(\theta\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ for the group action of $\theta\mathbb{Z}$ on $\mathbb{R}/\mathbb{Z}$ or, equivalently, a universal C^*-algebra generated by two unitaries $U, V \in A_\theta$ satisfying relation $UV = e(\theta)VU$. Here $e(x) = exp(2\pi ix)$.

Definition 1. *A_θ is a quantum torus with real multiplication if θ is a real quadratic irrationality, i.e. a real irrational root of a quadratic equation with rational coefficients.*

Let k be a real quadratic field. In [1] it is proposed to use quantum tori with real multiplication A_θ, $\theta \in k\backslash\mathbb{Q}$ as geometric objects associated to k. This should be compared to consideration of elliptic curves with complex multiplication $E_\tau = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in k'\backslash\mathbb{Q}$ for complex quadratic field k'. Any endomorphism $\alpha : E_\tau \to E_\tau$ is a linear map on the universal covering $\mathbb{C}$, so $End(E_\tau)$ is identified with the ring of multipliers of the lattice Γ, that is $\{\alpha \in \mathbb{C}|\alpha\Gamma \subset \Gamma\}$. We say that E_τ is an elliptic curve with complex multiplication if $End(E_\tau)$ is larger then $\mathbb{Z}$, which happens precisely when τ is a complex quadratic number.

Real multiplication of quantum tori has similar interpretation when we consider morphisms in the sense of noncommutative geometry: every element of $End(A_\theta)$ is by definition an (isomorphism class of) A_θ-A_θ-bimodule, finitely generated and projective as left and right module at the same time. Every such isomorphism class $[M] \in End(A_\theta)$ defines an endomorphism $\phi_{[M]}$ of K_0-group of A_θ via $[P] \mapsto [P \underset{A_\theta}{\otimes} M]$ for finitely generated projective right A_θ-modules P. It is shown in [1] that when $K_0(A_\theta)$ is identified with the lattice $\Gamma = \mathbb{Z} + \theta\mathbb{Z}$ via the trace map, then $\phi_{[M]}$ becomes a multiplication by a real number. Moreover, this map

$$K_0 : End(A_\theta) \to \{\alpha \in \mathbb{R}|\alpha\Gamma \subset \Gamma\}, \ K_0([M]) = \phi_{[M]}$$

is surjective. So, A_θ is a quantum tori with real multiplication if and only if $K_0(End(A_\theta))$ is larger then $\mathbb{Z}$.

In this paper we construct the graded ring of quantum theta functions $R = \oplus_{n\geq 0}R_n$ for a quantum torus with real multiplication A_θ. The construction is described in Section 6, were we also prove that $\theta \in k\backslash\mathbb{Q}$ can be chosen such that the ring R is Koszul (Theorem 5). In our ring $R_0 = \mathbb{C}$ and all R_n are finite

dimensional $\mathbb{C}$-vector spaces. So, the Koszul property means in particular that R is a finitely generated quadratic algebra. We sketch the definition of R below.

First, we need simple facts from number theory. One can prove that $\{\alpha \in \mathbb{R} | \alpha\Gamma \subset \Gamma\} = \mathbb{Z} + fO_k$ for some integer $f \geq 1$, where O_k is the ring of integers of the real quadratic field $k = \mathbb{Q}(\theta)$. Thus there are units of infinite order of O_k in $\mathbb{Z} + fO_k$, and we take one of them $\varepsilon \in (\mathbb{Z} + fO_k) \cap O_k^{\times}$. Then there exists a bimodule M_ε with $K_0([M_\varepsilon]) = \varepsilon$, which is an $A_\theta - A_\theta$-imprimitivity bimodule. Bimodules of such a kind were studied in [2],[3],[1], and we describe them in Section 5. Now one can consider the graded ring $\oplus_{n \geq 0} M_\varepsilon^{\otimes n}$ with tensor product over A_θ as multiplication. ($M_\varepsilon^{\otimes 0}$ means just $\mathbb{C}$, i.e. we adjoin a unity to the ring.)

M_ε is an infinite dimensional $\mathbb{C}$-vector space, but one can take finite dimensional subspaces $E_n \subset M_\varepsilon^{\otimes n}$ of so-called "holomorphic" vectors, and they are compatible with the tensor product: $E_n \otimes_{A_\theta} E_m \subset E_{n+m}$ ([2],[3]). So, we obtain a graded ring $E = \oplus_n E_n$ with multiplication defined via tensor product over A_θ. This ring was studied in [4]. The choice of "holomorphic" vectors depends on a complex parameter $\tau \in \mathbb{C}$, formally defining "holomorphic" structure on A_θ.

In fact, $M_\varepsilon^{\otimes n}$ are nontrivial $A_\theta - A_\theta$-imprimitivity bimodules. Imprimitivity bimodules were introduced by M.Rieffel, we recall the definition in Section 1. Sections 2 and 3 are devoted to relation between biprojective bimodules and imprimitivity bimodules. In Section 4 we introduce a natural structure of imprimitivity bimodule on the tensor product of imprimitivity bimodules. For two C^*-algebras A and B an $A - B$-imprimitivity bimodule is endowed with two pre-inner products — $_A\langle\cdot,\cdot\rangle : M \times M \to A$ and $\langle\cdot,\cdot\rangle_B : M \times M \to B$. We define $R_n = \text{Im }_{A_\theta}\langle\cdot,\cdot\rangle\big|_{E_n}$ — the vector space of finite sums of values of the left inner product on pairs of vectors from E_n. We check that $\dim_{\mathbb{C}} R_n = \dim_{\mathbb{C}} E_n^2$ (Proposition 6.2), so $R_n \cong E_n\bar{\otimes}E_n$. Here $\bar{\otimes}$ means that $(\alpha a)\bar{\otimes}b = a\bar{\otimes}(\bar{\alpha}b)$ for $\alpha \in \mathbb{C}$. This fact allows us to identify the graded space $R = \oplus_{n \geq 0} R_n$ with a kind of Segre square of the ring $E = \oplus_{n \geq 0} E_n$ — the subspace in $E\bar{\otimes}E$ generated by elements $a\bar{\otimes}b$ with $a, b \in E_n$ for some n. Thus we have defined the ring R.

But by construction elements of R_n for $n \geq 1$ are in A_θ. In fact, they are quantum theta functions. It was already noticed in [5] that operator-valued theta functions appear from imprimitivity bimodules over quantum tori. We use the definition of quantum theta functions given in [6] and [7]. Let us briefly recall it. Consider a Heisenberg group G_θ

$$1 \to \mathbb{C}^{\times} \to G_\theta \to \mathbb{C}^2 \times \mathbb{Z}^2 \to 0$$

acting on elements of the quantum torus A_θ by

$$(\alpha; \vec{x}; \vec{m}) \sum_{\vec{n} \in \mathbb{Z}^2} a_{\vec{n}} U^{n_1} V^{n_2} = \alpha \sum_{\vec{n} \in \mathbb{Z}^2} e(n_1 x_1 + n_2 x_2) a_{\vec{n}} U^{m_1} V^{m_2} U^{n_1} V^{n_2}.$$

A multiplier $\mathcal{L}$ is any free subgroup of rank 2 in G_θ, which is a lift of a free subgroup of rank 2 in $\mathbb{C}^2 \times \mathbb{Z}^2$. We denote by $\Gamma(\mathcal{L}) \subset A_\theta$ the vector space of elements fixed by $\mathcal{L}$. All elements of $\Gamma(\mathcal{L})$ are called quantum theta functions with multiplier $\mathcal{L}$. For example, take a lattice $L = \mathbb{Z}\vec{s} + \mathbb{Z}\vec{r} \subset \mathbb{Z}^2$, and a matrix $\Omega \in \mathcal{M}_2\mathbb{C}$, symmetric $\Omega = \Omega^t$ and with positive imaginary part $\Im\Omega > 0$. Then $(e(\frac{1}{2}\vec{s}^t A^t\vec{s}); A\vec{s}; \vec{s})$ and $(e(\frac{1}{2}\vec{r}^t A^t\vec{r}); A\vec{r}; \vec{r})$ generate a multiplier, where $A = \frac{\theta}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \Omega$. Let us denote this multiplier by $\mathcal{L} = \mathcal{L}(L, \Omega)$. Then $\Gamma(\mathcal{L})$ is a $\#(\mathbb{Z}^2/L)$-dimensional $\mathbb{C}$-vector space of elements of the form

$$\Theta[f](\Omega) = \sum_{\vec{m} \in \mathbb{Z}^2} f(\vec{m}) e(\frac{1}{2}\vec{m}^t\Omega\vec{m}) e(-\frac{\theta}{2}m_1 m_2) U^{m_1} V^{m_2}$$

for $f : \mathbb{Z}^2/L \to \mathbb{C}$.

In Section 6 (Proposition 6.2) we show that $R_n = \Gamma(\mathcal{L}(c_n\mathbb{Z}^2, \Omega_n)) \subset A_\theta$. Now we explain what are c_n and Ω_n. Recall we have chosen a unit $\varepsilon \in O_k$ and a complex parameter $\tau \in \mathbb{C}$. Now we need them to satisfy some technical conditions, especially $\Im\tau > 0$ and $\varepsilon = c\theta + d > 0$, $\varepsilon\theta = a\theta + b$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $c > 0$. Existence of such an ε one can get for example from Section 6. Then c_n are defined by $\varepsilon^n = c_n\theta + d_n$ with integer c_n, d_n, or, equivalently, by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, or by $\sum_n c_n t^n = \frac{c}{t^2 - (a+d)t + 1}$. In particular, the last expression shows that $\{c_n\}$ is an increasing sequence of positive integers, since $a + d = \varepsilon + \frac{1}{\varepsilon} \geq 2$. Now $\Omega_n = \frac{1}{c_n\varepsilon^n}\Omega$ where

$$\Omega = \frac{i}{2\Im\tau} \begin{pmatrix} |\tau|^2 & -\Re\tau \\ -\Re\tau & 1 \end{pmatrix}.$$

So, we get a ring R whose elements are formally elements of quantum tori A_θ, but the multiplication law is different from the one in A_θ. It was already noticed in [7] that the ordinary product of two quantum theta functions in A_θ is not a quantum theta function as a rule. As we mentioned, R is isomorphic to a kind of Segre square of E. Both R and E encapsulate the structure of real multiplication and use arithmetical data to be constructed. But the following question still remains unanswered: whether we can use such rings to obtain arithmetical invariants of the real quadratic field $k = \mathbb{Q}(\theta)$?

1. Strong Morita equivalence

Let A be a pre-C^*-algebra, i.e. a $\mathbb{C}$-algebra with involution and norm satisfying $||x||^2 = ||x^*x||$ and $||x|| = 0$ if and only if $x = 0$ for $x \in A$. If A has a unit element $1 \in A$ it is assumed that $||1|| = 1$. An A-valued pre-inner product on linear space M is an A-valued sesquilinear form $\langle \cdot, \cdot \rangle$ (here it does not matter in which variable it is conjugate linear) such that $\langle x, x \rangle \geq 0$ in the completion of A and $\langle x, y \rangle^* = \langle y, x \rangle$ for $x, y \in M$. We denote $\mathrm{Im}\,\langle \cdot, \cdot \rangle \subset A$ the set of finite sums of elements of the form $\langle x, y \rangle$ for $x, y \in M$.

The following definitions were introduced in [9].

Definition 1.1. *A left A-module M is called a left A-rigged space if it is endowed with an A-valued pre-inner product $\,_A\langle \cdot, \cdot \rangle : M \times M \mapsto A$, linear in the first argument and conjugate linear in the second, such that $\,_A\langle ax, y \rangle = a\,_A\langle x, y \rangle$ for $x, y \in M, a \in A$ and the two-sided ideal $\mathrm{Im}\,_A\langle \cdot, \cdot \rangle$ is dense in A.*

Note that $\,_A\langle ax, y \rangle = a\,_A\langle x, y \rangle$ for an A-valued inner product imply also $\,_A\langle x, ay \rangle = \,_A\langle x, y \rangle a^*$. Hence $\mathrm{Im}\,_A\langle \cdot, \cdot \rangle$ is a two-sided ideal as mentioned. we obtain the definition of a right rigged space by simple reflection from the left to the right:

Definition 1.2. *A right A-module M is called a right A-rigged space if it is endowed with a pre-inner product $\langle \cdot, \cdot \rangle_A : M \times M \mapsto A$, conjugate linear in the first argument and linear in the second, such that $\langle x, ya \rangle_A = \langle x, y \rangle_A a$ for $x, y \in M, a \in A$ and the ideal $\mathrm{Im}\,\langle \cdot, \cdot \rangle_A$ is dense in A.*

Let now A, B be pre-C^*-algebras.

Definition 1.3. *An $A - B$ bimodule M is called an imprimitivity bimodule if*

(1) M is a left-A-right-B-rigged space;

(2) $_A \langle x, y \rangle z = x \langle y, z \rangle_B$;

(3) $\langle ax, ax \rangle_B \leq ||a||_A^2 \langle x, x \rangle_B$ and $_A \langle xb, xb \rangle \leq ||b||_B^2 {}_A \langle x, x \rangle$

for $x, y, z \in M$ and $a \in A, b \in B$.

Note that in an imprimitivity bimodule we also have a relation $_A \langle x, yb \rangle = {}_A \langle xb^*, y \rangle$ for any $x, y \in M$ and $b \in B$. Indeed, for $b \in \mathrm{Im} \langle \cdot, \cdot \rangle_B$ it is a consequence of the relation (2) in the definition above. Let us check continuity. Suppose $||b_n|| \to 0$. Then $|| {}_A \langle yb_n, yb_n \rangle || \leq ||b_n||^2 || {}_A \langle y, y \rangle || \to 0$ by (3). Now by the Proposition 2.9 in [9] we have $|| {}_A \langle x, yb_n \rangle || \leq || {}_A \langle x, x \rangle ||^{\frac{1}{2}} || {}_A \langle yb_n, yb_n \rangle ||^{\frac{1}{2}} \to 0$, and analogously $|| {}_A \langle xb_n^*, y \rangle || \to 0$. Evidently, the dual relation $\langle ax, y \rangle_B = \langle x, a^* y \rangle_B$ for any $x, y \in M$ and $a \in A$ also holds.

Definition 1.4. ([10]) *Two pre-C^*-algebras A, B are said to be strongly Morita equivalent if there exist an $A - B$-imprimitivity bimodule.*

Example 1.5. *Let A be a pre-C^*-algebra with 1. Consider $M = A^n$ — free right A- module of rank n. Then $End_A M$ is the ring $\mathcal{M}_n A$ of $n \times n$ matrices with entries in A, which is a unital pre-C^*-algebra again. Then M is $\mathcal{M}_n A - A$-imprimitivity bimodule with inner products*

$$\langle x, y \rangle_A = x^* y = \sum_i x_i^* y_i$$

$$_{\mathcal{M}_n A} \langle x, y \rangle = xy^* = (x_i y_j^*)_{i,j=1}^n$$

Example 1.6. *Let G be a locally compact group, and let H and K be closed subgroups of G. Let $A = C^*(K, G/H)$, $B = C^*(H, K \backslash G)$ be transformation group C^*-algebras for the left action of K on G/H and the right action of H on $K \backslash G$ correspondingly. It is shown in [10] that there is a natural $A - B$-imprimitivity bimodule M which is the completion of the space $C_c(G)$ of $\mathbb{C}$-valued continuous functions with compact support on G with respect to an appropriate norm, with inner products given on $f, g \in C_c(G)$ by:*

$$_A \langle f, g \rangle (k, x) = \beta(k) \int_H f(\tilde{x} h) g^*(h^{-1} \tilde{x}^{-1} k) dh$$

where $\tilde{x} \in G$ is any representative of the class x, i.e. $x = \tilde{x} H$,

$$\langle f, g \rangle_B (h, y) = \gamma(h) \int_K f^*(\tilde{y}^- 1 k) g(k^{-1} \tilde{y} h) dk$$

where $y = K \tilde{y}$. Here $\beta(\cdot) = \left(\frac{\delta_G(\cdot)}{\delta_K(\cdot)} \right)^{\frac{1}{2}}$, $\gamma(\cdot) = \left(\frac{\delta_G(\cdot)}{\delta_H(\cdot)} \right)^{\frac{1}{2}}$, δ_G, δ_H, δ_K are the modular functions of locally compact groups G, H, K correspondingly, the involution is defined on $C_c(G)$ by $g \mapsto g^*(z) = \delta_G(z^{-1}) \bar{g}(z^{-1})$, and all integrals above are taken w.r.t. left Haar measures.

We will see in Sections 2,3 below that this two examples are quite similar.

Strong Morita equivalence implies Morita equivalence, i.e. equivalence of categories of hermitian representations([9]). It is not obvious from definitions that the strong Morita equivalence is indeed an equivalence relation. In [9] an inverse imprimitivity bimodule is constructed, showing that this relation is symmetric. In Section 4 we define a natural structure of imprimitivity bimodule on the tensor product of imprimitivity bimodules for unital C^*-algebras. In particular it makes evident the transitivity of the strong Morita equivalence for unital C^*-algebras.

2. Inner products for projective module

We generalize Example 1.5 in current section. Pre-inner products on a projective module which satisfy all algebraic relations from Definition 1.3 were introduced in [1]. We are going to check the condition of density of images for these inner products now.

Let A be a C^*-algebra with 1, let $p \in \mathcal{M}_n A$ be a projection, i.e. $p = p^* = p^2$. Consider a submodule $M = pA^n$ of the right A-module A^n consisting of such columns which are invariant under left multiplication by p. Then $End_A M = p\mathcal{M}_n Ap$, where matrices act by multiplication from the left. $p\mathcal{M}_n Ap$ is a C^*-algebra with norm restricted from $\mathcal{M}_n A$, and since $||p|| = 1$ this is a unital C^*-algebra. We consider two inner products on M which are restrictions of inner products from Example 1.5:

$$\langle x, y \rangle_A = x^* y = \sum_i x_i^* y_i$$

$$_{p\mathcal{M}_n Ap} \langle x, y \rangle = xy^* = (x_i y_j^*)_{i,j=1}^n$$

Then $\mathrm{Im}\; _{p\mathcal{M}Ap} \langle \cdot, \cdot \rangle = p\mathcal{M}_n Ap$, and $\mathrm{Im} \langle \cdot, \cdot \rangle_A = \sum_{i,j} Ap_{i,j}A$ — the ideal in A generated by matrix entries of p.

Proposition 2.1. *pA^n with inner products defined above is $p\mathcal{M}_n Ap - A$-imprimitivity bimodule if and only if* $\mathrm{Im} \langle \cdot, \cdot \rangle_A = A$.

Proof. In unital C^*-algebra A there is no dense ideal except A. So, condition $\mathrm{Im} \langle \cdot, \cdot \rangle_A = A$ is necessary for pA^n to be right A-rigged space. We show it is sufficient. Indeed, all necessary identities for $\langle \cdot, \cdot \rangle_A$ and $_{p\mathcal{M}nAp} \langle \cdot, \cdot \rangle$ are satisfied as they are satisfied in Example 1.5, and we already mentioned that $\mathrm{Im}\; _{p\mathcal{M}nAp} \langle \cdot, \cdot \rangle = p\mathcal{M}_n Ap$. $\square$

In the next section we show that any imprimitivity bimodule between unital C^*-algebras is of this form. This idea also comes from works of M. Rieffel — one may compare Theorem 1 below to Proposition 2.1 in [11].

3. Imprimitivity bimodule for C^*-algebras with 1

Theorem 1. *Let A, B be two strongly Morita equivalent C^*-algebras with 1, and M be a $B - A$-imprimitivity bimodule. Then*
(1) $B = End_A M$;
(2) there exist $n \in \mathbb{Z}$, a projection $p \in \mathcal{M}_n A$ and an isomorphism of right A-modules $\Psi : M \to pA^\infty$ such that for $u, v \in M$:

$$\langle u, v \rangle_A = \Psi(u)^* \Psi(v)$$

$$_B \langle u, v \rangle = \Psi^{-1} \circ \Psi(u)\Psi(v)^* \circ \Psi$$

Proof. As B is a unital C^*-algebra, any dense ideal in it is B. Then there exist an integer n and $x_1, \ldots, x_n, y_1, \ldots, y_n \in M$ such that

$$1_B = \sum_i {}_B \langle x_i, y_i \rangle.$$

Consider the unital C^*-algebra $C = \mathcal{M}_n A$ and the $B - C$-bimodule $N = M^n$ consisting of columns of elements of M. We define inner products on N by

$$_B \langle m, n \rangle = \sum_i {}_B \langle m_i, n_i \rangle$$

$$\langle m, n \rangle_C = (\langle m_i, n_j \rangle_A)_{i,j=1}^n$$

One can check N is a $B-C$-imprimitivity bimodule. Now ${}_B\langle x,y\rangle = 1_B$. Consider $z = x\langle y,y\rangle_C^{1/2}$. Then

$$ {}_B\langle z,z\rangle = {}_B\langle x, x\langle y,y\rangle_C\rangle = {}_B\langle x, {}_B\langle x,y\rangle\, y\rangle = {}_B\langle y,x\rangle\, {}_B\langle x,y\rangle = 1_B, $$

and $p = \langle z,z\rangle_C$ is a projection. Indeed, obviously $p^* = p$ and

$$ \langle z,z\rangle_C\langle z,z\rangle_C = \langle z, z\langle z,z\rangle_C\rangle_C = \langle z, {}_B\langle z,z\rangle\, z\rangle_C = \langle z,z\rangle_C. $$

Consider a homomorphism of right A-modules $\Psi: M \to pA^n$, $\Psi(m) = (\langle z_i, m\rangle_A)$, and a unital homomorphism of rings $\Phi: B \to p\mathcal{M}_n Ap$, $\Phi(b) = \langle z, bz\rangle_C$. We now prove they are both correctly defined and are in fact isomorphisms.

For Ψ consider $j: M \to N$ given by $j(m) = (m\delta_{1i})_{i=1}^n$. Then columns of $\langle z, j(m)\rangle_C$ are $\Psi(m), 0, \ldots, 0$. Since $P\langle z,j(m)\rangle_C = \langle z\langle z,z\rangle_C, j(m)\rangle_C$ $= \langle\langle z,z\rangle_C z, j(m)\rangle_C = \langle z, j(m)\rangle_C$, $\Psi(m) \in pA^n$. Injectivity of Ψ follows from $z\langle z,n\rangle_C = {}_B\langle z,z\rangle\, n = n \neq 0$ for nonzero $n \in N$. Surjectivity follows from the fact that $\operatorname{Im}\langle z,\cdot\rangle_C = pC$.

$\Phi(b)$ is obviously invariant under left ant right multiplication by p. To prove injectivity we note that $b = {}_B\langle y,x\rangle\, b\, {}_B\langle x,y\rangle = {}_B\langle bz,z\rangle$, so $bz \neq 0$ if $b \neq 0$. Thus also $\langle z,bz\rangle_C \neq 0$. Surjectivity follows from fact that pCp is spanned by $p\langle m,n\rangle_C\, p$ and equality $p\langle m,n\rangle_C\, p = \Phi({}_B\langle m,z\rangle\, {}_B\langle n,z\rangle)$. Also $\Phi(b_1 b_2) = \Phi(b_1)\Phi(b_2)$.

To prove the statement it remains to check that $\langle u,v\rangle_A = \Psi(u)^*\Psi(v)$, $\Phi({}_B\langle u,v\rangle) = \Psi(u)\Psi(v)^*$ and $\Phi({}_B\langle u,v\rangle)\Psi(t) = \Psi({}_B\langle u,v\rangle\, t)$. Indeed,

$$ \Psi(u)^*\Psi(v) = \sum_i \langle u, z_i\rangle_A\langle z_i, v\rangle_A = \langle u, {}_B\langle z,z\rangle\, v\rangle_A = \langle u,v\rangle_A. $$

Next, we compare the (i,j)'th matrix entry for $\Phi({}_B\langle u,v\rangle) = \langle z, {}_B\langle u,v\rangle\, z\rangle_C$ and $\Psi(u)\Psi(v)^*$:

$$ \langle z_i, {}_B\langle u,v\rangle\, z_j\rangle_A = \langle z_i, u\rangle_A\langle v, z_j\rangle_A. $$

Now we compare the i'th coordinate in $\Phi({}_B\langle u,v\rangle)\Psi(t)$ and $\Psi({}_B\langle u,v\rangle\, t)$:

$$ (\Psi(u)\Psi(v)^*\Psi(t))_i = \Psi(u)_i\langle v,t\rangle_A = \langle z_i, u\rangle_A\langle v,t\rangle_A = \langle z_i, {}_B\langle u,v\rangle\, t\rangle_A. $$

$\square$

Corollary 3.1. *Suppose there are two structures of a $B-A$-imprimitivity bimodule on a bimodule M: ${}_B\langle\cdot,\cdot\rangle^i$ and $\langle\cdot,\cdot\rangle_A^i$ for $i = 1, 2$. If $\langle\cdot,\cdot\rangle_A^1 = \langle\cdot,\cdot\rangle_A^2$ then also ${}_B\langle\cdot,\cdot\rangle^1 = {}_B\langle\cdot,\cdot\rangle^2$, and vice versa.*

Proof. Due to the theorem above it is sufficient to check the statement in case $M = pA^\infty$ and $\langle x,y\rangle_A^1 = \langle x,y\rangle_A^2 = x^*y$. Then for any $z \in pA^\infty$ we have ${}_B\langle x,y\rangle\, z = x\langle y,z\rangle_A = xy^*z$. Taking $z = p_k$ for all columns of $p = (p_k)$ we get ${}_B\langle x,y\rangle = {}_B\langle x,y\rangle\, p = xy^*p = xy^*$, so the second inner product is defined by the first one. $\square$

4. COMPOSITION OF STRONG MORITA MORPHISMS

Evidently the choice of inner products for an $A-B$-imprimitivity bimodule M is not unique. For example, we can multiply them both by a positive number and with such new inner products M will be again an $A-B$-imprimitivity bimodule. Anyway, the following theorem gives one natural choice of the imprimitivity bimodule structure on the tensor product of two imprimitivity bimodules.

Theorem 2. *Let A, B, C be unital C^*-algebras, M, N be $A-B$ and $B-C$-imprimitivity bimodules correspondingly. Then $M \underset{B}{\otimes} N$ with inner products defined by*

$$ \langle x\otimes z, y\otimes t\rangle_C = \langle z, \langle x,y\rangle_B\, t\rangle_C $$
$$ {}_A\langle x\otimes z, y\otimes t\rangle = {}_A\langle x\, {}_B\langle z,t\rangle, y\rangle $$

is an $A-C$-imprimitivity bimodule.

Proof. Let $K = M \underset{B}{\otimes} N$. We check that K is a right C-rigged space. First, let us see that $\langle \cdot, \cdot \rangle_C$ on K is a well-defined C-valued inner product antilinear in the first variable. Indeed, for $b \in B$

$$\langle xb \otimes z, y \otimes t \rangle_C = \langle z, \langle xb, y \rangle_B t \rangle_C = \langle z, b^* \langle x, y \rangle_B t \rangle_C$$
$$= \langle bz, \langle x, y \rangle_B t \rangle_C = \langle x \otimes bz, y \otimes t \rangle_C .$$

Taking $b \in \mathbb{C}1$ we see that $\langle \cdot, \cdot \rangle_C$ is antilinear in the first variable. Analogously $\langle x \otimes z, yb \otimes t \rangle_C = \langle x \otimes z, y \otimes bt \rangle_C$ and $\langle \cdot, \cdot \rangle_C$ is linear in the second variable. To show the positivity of $\langle \sum_{i=1}^n x_i \otimes z_i, \sum_i x_i \otimes z_i \rangle_C = \sum_{i,j} \langle z_i, \langle x_i, x_j \rangle_B z_j \rangle_C$, we recall that M^n is an $A - \mathcal{M}_n B$-imprimitivity bimodule. So the matrix $H = \langle x_i, x_j \rangle_B$ is a positive element of $\mathcal{M}_n B$. Thus $\langle y, Hy \rangle_C = \sum_{i,j} \langle y_i, h_{i,j} y_j \rangle_C \geq 0$ as N^n is an $\mathcal{M}_n B - C$-imprimitivity bimodule.

Consider elements x_i, y_i in M such that $\sum_i \langle x_i, y_i \rangle_B = 1$. Then for any $z, t \in N$

$$\sum_i \langle x_i \otimes z, y_i \otimes t \rangle_C = \langle z, t \rangle_C ,$$

so $\operatorname{Im} \langle \cdot, \cdot \rangle_C$ on N is a subset of $\operatorname{Im} \langle \cdot, \cdot \rangle_C$ on K. Thus $\operatorname{Im} \langle \cdot, \cdot \rangle_C$ on K is dense in C.

For $c \in C$ we obviously have relation

$$\langle x \otimes z, y \otimes tc \rangle_C = \langle z, \langle x, y \rangle_B tc \rangle_C = \langle x \otimes z, y \otimes t \rangle_C c,$$

so we proved K is a right C-rigged space. Analogously K is a left A-rigged space.

Now we check condition (2) in Definition 1.3 :

$$v \otimes w \langle x \otimes z, y \otimes t \rangle_C = v \otimes w \langle \langle y, x \rangle_B z, t \rangle_C = v \otimes {}_B \langle w, \langle y, x \rangle_B z \rangle t$$
$$= v {}_B \langle w, \langle y, x \rangle_B z \rangle \otimes t = v {}_B \langle w, z \rangle \langle x, y \rangle_B \otimes t = {}_A \langle v {}_B \langle w, z \rangle, x \rangle y \otimes t$$
$$= {}_A \langle v \otimes w, x \otimes z \rangle y \otimes t.$$

For condition (3) consider $a \in A$ and

$$\left\langle a \sum_{i=1}^n x_i \otimes z_i, a \sum_i x_i \otimes z_i \right\rangle_C = \sum_{i,j} \langle z_i, \langle ax_i, ax_j \rangle_B z_j \rangle_C$$

$$\leq \sum_{i,j} \langle z_i, ||a||^2 \langle x_i, x_j \rangle_B z_j \rangle_C = ||a||^2 \left\langle \sum_i x_i \otimes z_i, \sum_i x_i \otimes z_i \right\rangle_C$$

as M^n is an $A - \mathcal{M}_n B$-imprimitivity bimodule. Analogously we can check condition (3) for ${}_A \langle \cdot, \cdot \rangle$. $\square$

We remark that this statement is also true in the case of unital pre-C^*-algebras. The proof is almost the same but we need additional continuity arguments to prove the density of images of pre-inner products.

5. Morita bimodules over quantum tori

Recall that a quantum torus A_θ for $\theta \in \mathbb{R} \backslash \mathbb{Q}$ is a transformation group C^*-algebra $C^*(\theta \mathbb{Z}, \mathbb{R}/\mathbb{Z})$. It is known that A_θ is a universal C^*-algebra generated by two unitaries $U, V \in A_\theta$ satisfying relation $UV = e(\theta)VU$. The choice of such unitaries is not unique. If $U, V \in A_\theta$ are chosen we call them a frame.

From Example 1.6 we see that $A_\theta = C^*(\theta \mathbb{Z}, \mathbb{R}/\mathbb{Z})$ is strongly Morita equivalent to $C^*(\mathbb{Z}, \mathbb{R}/\theta \mathbb{Z}) \cong C^*(\frac{1}{\theta}\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = A_{\frac{1}{\theta}}$. Obviously $A_{\theta+1} = A_\theta$, since the relation $UV = e(\theta)VU$ is invariant under the transformation $\theta \mapsto \theta + 1$. Also $A_\theta \cong A_{-\theta}$ as we can map U to V' and V to U' for any frames $U, V \in A_\theta$, $U', V' \in A_\theta$. Recall that $GL_2(\mathbb{Z})$ acts on complex numbers by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \theta = \frac{a\theta + b}{c\theta + d}.$$

So we see that A_θ is strongly Morita equivalent to $A_{g\theta}$ for any $g \in GL_2(\mathbb{Z})$. Indeed, as $GL_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, its orbit is generated by transformations $\theta \mapsto \theta + 1$ and $\theta \mapsto \frac{1}{\theta}$. Conversely, it is shown in [11] that A_θ and $A_{\theta'}$ are not strongly Morita equivalent if θ and θ' don't lie in the same orbit of $GL_2(\mathbb{Z})$.

Below we recall an explicit construction of $A_{g\theta} - A_\theta$-imprimitivity bimodule $E(g, \theta)$ for $g \in SL_2(\mathbb{Z})$, $\theta \in \mathbb{R}\backslash\mathbb{Q}$ ([2], [3],[1]). (Bimodules for $g \in GL_2(\mathbb{Z})$ can be easily obtained from these by a composition with homomorphism $U \mapsto V'$,$V \mapsto U'$ of quantum tori on the left.) It is proven in [2] that $E(hg, \theta) \cong E(h, g\theta) \underset{A_{g\theta}}{\otimes} E(g, \theta)$ as a bimodule, and we claim in the theorem below that inner products satisfy relations of Theorem 2.

To construct our bimodules we need to fix a frame in A_θ for each $\theta \in \mathbb{R}\backslash\mathbb{Q}$. (If $\theta = \theta'$ modulo $\mathbb{Z}$ then the frames should coincide.) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. If $c = 0$ we put $E(g, \theta) = A_\theta$ with the action of $A_{g\theta} = A_\theta$ via multiplication from the left and action of A_θ by right multiplication, and define inner products by $_{A_{g\theta}}\langle a, b \rangle = ab^*$ and $\langle a, b \rangle_{A_\theta} = a^*b$ as in Example 1.5. If $c \neq 0$, we consider the space $E^0(g, \theta) = S(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z})$ with following actions of generators U, V of A_θ and U', V' of $A_{g\theta}$ on $f \in E^0(g, \theta)$:

$$(fU)(x, \alpha) = f(x - \frac{c\theta + d}{c}, \alpha - 1)$$

$$(fV)(x, \alpha) = e(x - \alpha\frac{d}{c})f(x, \alpha)$$

$$(U'f)(x, \alpha) = f(x - \frac{1}{c}, \alpha - a)$$

$$(V'f)(x, \alpha) = e(\frac{x}{c\theta + d} - \frac{\alpha}{c})f(x, \alpha)$$

We define for $f, s \in E^0(g, \theta)$ inner products:

$$_{A_{g\theta}}\langle f, s \rangle = \sum_{n \in \mathbb{Z}^2} \left\langle f, U'^{n_1}V'^{n_2}s \right\rangle_{L_2} U'^{n_1}V'^{n_2}$$

$$\langle f, s \rangle_{A_\theta} = \frac{1}{c\theta + d} \sum_{n \in \mathbb{Z}^2} \langle s, fU^{n_1}V^{n_2} \rangle_{L_2} U^{n_1}V^{n_2}$$

Let $E(g, \theta)$ be the completion of $E^0(g, \theta)$ with the norm $||f|| = ||\,_{A_{g\theta}}\langle f, f \rangle||^{\frac{1}{2}}$. Then $E(g, \theta)$ is an $A_{g\theta} - A_\theta$-imprimitivity bimodule (Theorem 3.2 in [1]).

In [2],[3] bimodule isomorphisms $t_{h,g} : E(h, g\theta) \underset{A_{g\theta}}{\otimes} E(g, \theta) \to E(hg, \theta)$ are constructed for $h, g \in SL_2(\mathbb{Z})$.

Theorem 3. *For $h, g \in SL_2(\mathbb{Z})$, $f_1, s_1 \in E(h, g\theta)$ and $f_2, s_2 \in E(g, \theta)$*

$$_{A_{hg\theta}}\langle t_{h,g}(f_1 \otimes f_2), t_{h,g}(s_1 \otimes s_2) \rangle = \,_{A_{hg\theta}}\left\langle f_1 \,_{A_{g\theta}}\langle f_2, s_2 \rangle, s_1 \right\rangle$$

$$\langle t_{h,g}(f_1 \otimes f_2), t_{h,g}(s_1 \otimes s_2) \rangle_{A_\theta} = \left\langle f_2, \langle f_1, s_1 \rangle_{A_{g\theta}} s_2 \right\rangle_{A_\theta}$$

Proof. First, due to Theorem 2 and Corollary 3.1 it is enough to check only one of two statements of the theorem. We prefer the second one.

As maps $t_{h,g}$ are associative (Proposition 1.2 in [3]) it is enough to check the statement only for generators of $SL_2(\mathbb{Z})$ at place of h. Indeed, suppose the statement is true for $E(h_1, g\theta) \otimes E(g, \theta)$, $E(h_2, h_1 g\theta) \otimes E(h_1 g, \theta)$ and $E(h_2, h_1 g\theta) \otimes E(h_1, g\theta)$. Then it is true for $E(h_2 h_1, g\theta) \otimes E(g, \theta)$ due to the associativity relation

$$t_{h_2 h_1, g} \circ (t_{h_2, h_1} \otimes id) = t_{h_2, h_1 g} \circ (id \otimes t_{h_1, g}).$$

Take $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $f_1, s_1 \in A_{g\theta}$, $\langle f_1, s_1 \rangle_{A_{g\theta}} = f_1^* s_1$, $t_{h,g}(f_1, f_2) = f_1 f_2$ (in the sense of the left action) and similarly $t_{h,g}(s_1, s_2) = s_1 s_2$. As $h \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$ we have no changes in formulas for the action of quantum tori, so $E(hg, \theta) = E(g, \theta)$ and

$$\langle f_1 f_2, s_1 s_2 \rangle_{A_\theta} = \langle f_2, f_1^* s_1 s_2 \rangle_{A_\theta}$$

as $E(g, \theta)$ is an $A_{g\theta} - A_\theta$-imprimitivity bimodule. Indeed, for an $A - B$-imprimitivity bimodule M we have $\langle ax, y \rangle_B = \langle x, a^* y \rangle_B$ for $a \in A$, $x, y \in M$.

Now take $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $hg = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$. Let us consider the case $g \neq h$, $c \neq 0$. Cases $g = h$ and $c = 0$ can be done analogously. Obviously we can restrict to the dense set of Schwartz functions $f_1, s_1 \in E^0(h, g\theta)$, $f_2, s_2 \in E^0(g, \theta)$. $E^0(h, g\theta) = S(\mathbb{R})$ with

$$\langle f_1, s_1 \rangle_{A_{g\theta}} = \frac{1}{g\theta} \sum_{n \in \mathbb{Z}^2} \int s_1(y) e(-yn_2) \bar{f}_1(y - n_1 \theta) dy \, U^{n_1} V^{n_2}$$

where $U, V \in A_{g\theta}$. Let U', V' be generators of A_θ. Comparing coefficients at $U'^{m_1} V'^{m_2}$ in the identity, which we need to prove, we see that it is equivalent to

$$\frac{1}{a\theta + b} \left\langle t_{h,g}(s_1 \otimes s_2), t_{h,g}(f_1 \otimes f_2 U'^{m_1} V'^{m_2}) \right\rangle_{L_2} = \frac{1}{c\theta + d} \left\langle \langle f_1, s_1 \rangle_{A_{g\theta}} s_2, f_2 U'^{m_1} V'^{m_2} \right\rangle_{L_2}$$

Substituting f_2 instead of $f_2 U'^{m_1} V'^{m_2}$, we need to prove for arbitrary $f_1, s_1 \in S(\mathbb{R})$, $f_2, s_2 \in S(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z})$

$$\langle t_{h,g}(s_1 \otimes s_2), t_{h,g}(f_1 \otimes f_2) \rangle_{L_2} = \sum_{n \in \mathbb{Z}^2} \int s_1(y) e(-yn_2) \bar{f}_1(y - n_1 \theta) dy \, \langle U^{n_1} V^{n_2} s_2, f_2 \rangle_{L_2}$$

This is a routine computation using Poisson summation formula. We use abbreviations LHS (RHS) for left-(right)-hand side of this identity correspondingly. By the explicit formula for $t_{h,g}$ (Proposition 1.2 in [3])

$$t_{h,g}(s_1 \otimes s_2)(x, \alpha) = \sum_{n \in \mathbb{Z}} s_1 \left(\frac{x}{c\theta + d} + g\theta \left(\frac{cb}{a} \alpha - n \right) \right) s_2 \left(x - \frac{b}{a} \alpha + \frac{n}{c}, an \right),$$

and analogously for $t_{h,g}(f_1 \otimes f_2)$. Now

$$LHS = \sum_{n,m \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}/a\mathbb{Z}} \int s_1(z) s_2 \left(y - \frac{m-n}{c}, an \right) \bar{f}_1(z - g\theta(m-n)) \bar{f}_2(y, am) dy$$

where $z = \frac{x}{c\theta+d} + g\theta\left(\frac{cb}{a}\alpha - n\right)$ and $y = x - \frac{b}{a}\alpha + \frac{m}{c}$. Let us represent $m = dm_1 + cm_2$ with $m_1 \in \mathbb{Z}/c\mathbb{Z}$ and $m_2 \in \mathbb{Z}$. Then $am = m_1$ and $an = m_1 - a(m-n)$ modulo c. Introducing a new variable $n_1 = m - n$ we proceed:

$$= \sum_{m_1 \in \mathbb{Z}/c\mathbb{Z}} \int \sum_{n_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}, \alpha \in \mathbb{Z}/a\mathbb{Z}} s_1(z) \bar{f}_1(z - g\theta n_1)(U^{n_1} s_2)(y, m_1) \bar{f}_2(y, m_1) dy$$

Let us express z via y, m_1, m_2, n_1 and α:

$$z = \frac{1}{c\theta + d} \left(y + \frac{b}{a}\alpha - \frac{m}{c} \right) + g\theta\left(\frac{cb}{a}\alpha - n\right)$$

$$= \frac{1}{c\theta + d} \left(y + \frac{b}{a}\alpha - m_2 - \frac{d}{c}m_1 \right) + \frac{a\theta + b}{c\theta + d}\left(\frac{cb}{a}\alpha - cm_2 - dm_1 + n_1\right)$$

$$= (b\alpha - m_2 a) - \frac{ad}{c}m_1 + \frac{1}{c\theta + d}(y + (a\theta + b)n_1)$$

Denote $n_2 = b\alpha - m_2 a$, and $z_0 = z - n_2$. Then by the Poisson summation formula

$$\sum_{n_2 \in \mathbb{Z}} s_1(n_2 + z_0)\bar{f}_1(n_2 + z_0 - g\theta n_1) = \sum_{n_2 \in \mathbb{Z}} e(z_0)^{n_2} \int e(-tn_2)s_1(t)\bar{f}_1(t - g\theta n_1)dt.$$

We put this into LHS, and note that $e(z_0)^{n_2}(U^{n_1}s_2)(y, m_1) = (U^{n_1}V^{n_2}s_2)(y, m_1)$. So LHS =

$$\sum_{m_1 \in \mathbb{Z}/c\mathbb{Z}} \int \sum_{n_1, n_2 \in \mathbb{Z}} \int e(-tn_2)s_1(t)\bar{f}_1(t - g\theta n_1)dt(U^{n_1}V^{n_2}s_2)(y, m_1)\bar{f}_2(y, m_1)dy$$

$$= \sum_{n_1, n_2} \int e(-tn_2)s_1(t)\bar{f}_1(t - g\theta n_1)dt \, \langle U^{n_1}V^{n_2}s_2, f_2 \rangle_{L_2} = RHS$$

$$\square$$

6. Real multiplication

An irrational number $\theta \in \mathbb{R}\backslash\mathbb{Q}$ is a root of a quadratic equation if and only if there exists a matrix $g \in SL_2(\mathbb{Z})$, $g \neq \pm 1$ such that $g\theta = \theta$. Let us fix such g and θ. It follows from Section 5 that there are nontrivial $A_\theta - A_\theta$-imprimitivity bimodules exactly in this case. Next we are going to construct a graded ring $R = R(g, \theta) = \underset{n \geq 1}{\oplus} R_n$ using tensor products and inner products in these imprimitivity bimodules. We start with a construction of another graded ring due to Polishchuk [4], which uses only tensor products.

We consider the set of bimodules $E(g^n, \theta)$, $n \geq 1$ defined in previous section, and have a family of isomorphisms

$$t_{g^m, g^n} : E(g^n, \theta) \otimes E(g^m, \theta) \widetilde{\to} E(g^{n+m}, \theta).$$

Let $\mathbb{H}_k = \{M \in \mathcal{M}_k\mathbb{C} | M = M^t \text{ and } \Im(M) > 0\}$ be the so-called Siegel upper half-plane. So, $\mathbb{H}_1$ is just the upper half of the complex plane $\mathbb{C}$, and we fix $\tau \in \mathbb{H}_1$. Denote matrix entries of g^n by $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Denote $\mu_n = \tau\frac{c_n}{c_n\theta + d_n}$. Note that $c_n\theta + d_n$ is an eigenvalue of g^n, so it is nonzero. Also $c_n \neq 0$ as g^n is a nontrivial matrix stabilizing θ. Thus $\mu_n \neq 0$. Denote

$$E_n = \begin{cases} \left\{\phi_f(x, \alpha) = e(\mu_n\frac{x^2}{2})f(\alpha)\Big| f : \mathbb{Z}/c_n\mathbb{Z} \to \mathbb{C}\right\}, & \frac{c_n}{c_n\theta + d_n} > 0 \\ \{0\}, & \frac{c_n}{c_n\theta + d_n} < 0 \end{cases} \subset E(g^n, \theta).$$

E_n is either 0 or a $|c_n|$-dimensional vector space. In fact we have either $E_n = \{0\}$ for all n or $E_n \neq \{0\}$ for all n. Indeed, we see that the definition of E_n is the same for $E(g, \theta)$ and $E(-g, \theta)$. Thus taking either g or $-g$ instead of g we can suppose that $c_1\theta + d_1 > 0$. $c_1\theta + d_1$ is an eigenvalue of g, so g has positive eigenvalues. Now it follows from $\sum_{n=1}^{\infty} c_n t^n = \frac{ct}{t^2 - tr(g)t + 1}$ that all c_n have the same sign, as all coefficients of the power series for $\frac{1}{t^2 - tr(g)t + 1}$ are positive. All $c_n\theta + d_n$ are eigenvalues of g^n, so they are also positive.

Consider the set

$$S_\theta := \left\{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\Big| g \neq \pm 1, g\theta = \theta, tr(g) > 0 \text{ \& } c > 0\right\}$$

It is always nonempty: we already showed how to satisfy first three conditions, then if the fourth is not satisfied we can take g^{-1} instead of g.

Further we suppose $g \in S_\theta$. Then all E_n are nonzero vector spaces. It was noticed already in [2] that vector spaces E_n are preserved under tensor products of bimodules. The following can be checked by a direct computation:

Proposition 6.1. *For* $f : \mathbb{Z}/c_n\mathbb{Z} \to \mathbb{C}$, $g : \mathbb{Z}/c_m\mathbb{Z} \to \mathbb{C}$ *we have* $t_{g^n,g^m}(\phi_f \otimes \phi_g) = \phi_{f \underset{n,m}{\star} g}$ *where*

$$f \underset{n,m}{\star} g(\alpha) = \sum_{q \in \mathbb{Z}} e\left(\frac{\tau}{2} \frac{c_{n+m}}{c_n c_m} \left(q - \frac{c_m d_{n+m}}{c_{n+m}} \alpha \right)^2 \right) f(a_n d_{n+m}\alpha - q) g(a_m q)$$

is a function on $\mathbb{Z}/c_{n+m}\mathbb{Z}$.

Let E_0 be $\mathbb{C}$, and E_n for $n > 0$ be the $\mathbb{C}$-vector spaces defined above. Consider the graded ring with unit element $E = \oplus_{n \geq 0} E_n$ where the multiplication law is given by $\phi_f * \phi_g := \phi_{t \underset{n,m}{\star} g} \in E_{n+m}$ for $\phi_f \in E_n$, $\phi_g \in E_m$, $n, m > 0$. Associativity of this multiplication follows from the identity

$$t_{g^{n+m},g^k} \circ (t_{g^n,g^m} \otimes id) = t_{g^n,g^{m+k}} \circ (id \otimes t_{g^m,g^k}) : E_n \otimes E_m \otimes E_k \to E_{n+m+k}$$

stated in Proposition 1.2 in [3]. Note that if we choose for basis in E_n functions of the form ϕ_f with characters $f \in (\mathbb{Z}/c_n\mathbb{Z})^*$, we get a multiplication table consisting of values at rational points of various theta functions with rational characters $\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\gamma, \delta\tau)$ where $\alpha, \beta, \gamma, \delta \in \mathbb{Q}$ (see, e.g. [12]). For example,

$$1 \underset{n,m}{\star} 1(\alpha) = \theta \begin{bmatrix} \dfrac{c_m d_{n+m}}{c_{n+m}} \alpha \\ 0 \end{bmatrix} \left(0, \frac{\tau}{2} \frac{c_{n+m}}{c_n c_m} \right).$$

In [4] (Theorem 2.4) criterions whether the ring E is generated over $\mathbb{C}$ by E_1, is quadratic and is Koszul are established. Using them we state a criterion whether there exist $g \in S_\theta$ such that E have these good properties:

Theorem 4. *Let* $\theta \in \mathbb{R}\backslash\mathbb{Q}$ *be a quadratic irrationality, and* θ' *be its Galois conjugate. Then the following conditions are equivalent:*
(1) $|\theta - \theta'| < 1$;
(2) there exist $g \in S_\theta$ *such that the ring* E *is generated by* E_1 *over* $\mathbb{C}$;
(3) there exist $g \in S_\theta$ *such that the ring* E *is quadratic;*
(4) there exist $g \in S_\theta$ *such that the ring* E *is Koszul.*

Proof. First we show (2),(3) and (4) imply (1). Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with given properties exist. As $g \in S_\theta$, it satisfies conditions of Theorem 2.4 in [4]. This implies $c \geq a + d + \varepsilon$, where $\varepsilon = 0$ for (2), $\varepsilon = 1$ for (3), $\varepsilon = 2$ for (4). Then, as $c\theta^2 + (d - a)\theta - b = 0$,

$$|\theta - \theta'|^2 = \frac{(d-a)^2 + 4bc}{c^2} = \frac{(d+a)^2 - 4}{c^2} \leq \frac{(d+a)^2 - 4}{(d+a)^2} < 1.$$

Let us prove that (1) implies (2),(3) and (4). Namely, we are going to show that (i) implies that for every $\varepsilon \leq 2$ there exist $g \in S_\theta$ such that $c > a + d + \varepsilon$. This will imply (2) for $\varepsilon = 1$, (3) and (4) for $\varepsilon = 2$ due to Theorem 2.4 in [4].

Take any $g \in S_\theta$. Now, as g stabilizes θ, we have for the norm and the trace

$$N(\theta) = -\frac{b}{c} = \frac{1 - ad}{c^2}, \ Tr(\theta) = \frac{a - d}{c},$$

and

$$(a + d)^2 = (a - d)^2 + 4ad = c^2(Tr(\theta)^2 - 4N(\theta)) + 4 = c^2|\theta - \theta'|^2 + 4.$$

So, as $|\theta - \theta'| < 1$ we have $(a + d)^2 < (c - \varepsilon)^2$ if c is large enough, and $a + d < c - \varepsilon$, because $a + d > 2$ and $\varepsilon \leq 2$ and $c > 0$. To make c large enough one can take g^n instead of g, since $g^n \in S_\theta$. $\square$

Now we are going to construct another ring, which also uses inner products in imprimitivity bimodules $E(g^n, \theta)$. We will use left A_θ-valued inner products, but the same construction can be done for the right ones. Let $n > 0$. We put $R_n = \mathrm{Im}\ _{A_\theta}\langle\cdot,\cdot\rangle\big|_{E_n}$ — the vector space of finite sums of values of left inner product on pairs of vectors from $E_n \subset E(g^n, \theta)$. In Introduction we defined for a matrix $\Omega \in \mathbb{H}_2$ and a function $f : \mathbb{Z}^2 \to \mathbb{C}$ periodic w.r.t. some cofinite lattice in $\mathbb{Z}^2$ an element

$$\Theta[f](\Omega) = \sum_{\vec{m}\in\mathbb{Z}^2} f(\vec{m})e(\frac{1}{2}\vec{m}^t\Omega\vec{m})e(-\frac{\theta}{2}m_1 m_2)U^{m_1}V^{m_2} \in A_\theta.$$

Proposition 6.2. $R_n = \left\{\Theta[f](\frac{1}{c_n(c_n\theta+d_n)}\Omega)\big| f : \mathbb{Z}^2/c_n\mathbb{Z}^2 \to \mathbb{C}\right\}$ where

$$\Omega = \frac{i}{2\Im\tau}\begin{pmatrix}|\tau|^2 & -\Re\tau \\ -\Re\tau & 1\end{pmatrix} \in \mathbb{H}_2.$$

Proof. By a routine computation we get

$$_{A_\theta}\langle\phi_f, \phi_g\rangle = \frac{1}{2(\Im\mu_n)}\sum_{\vec{m}\in\mathbb{Z}^2} Q(\vec{m})e\left(\frac{1}{2}\vec{m}^t\frac{\Omega}{c_n(c_n\theta+d_n)}\vec{m}\right)e(-\frac{\theta}{2}m_1 m_2)U^{m_1}V^{m_2}$$

where

$$Q(\vec{m}) = Q_{f,g}(\vec{m}) = \sum_{\alpha\in\mathbb{Z}/c_n\mathbb{Z}} f(\alpha + a_n m_1)\bar{g}(\alpha)e(\frac{\alpha}{c_n}m_2).$$

The function $Q_{f,g}$ is periodic w.r.t. the lattice $c_n\mathbb{Z}^2$. Obviously we can arrange a basis in the space of periodic functions $\mathbb{Z}^2/c_n\mathbb{Z}^2 \to \mathbb{C}$ from such functions $Q_{f,g}$ by taking various functions $f, g : \mathbb{Z}/c_n\mathbb{Z} \to \mathbb{C}$, so the statement of the proposition follows. We have $\frac{1}{c_n(c_n\theta+d_n)}\Omega \in \mathbb{H}_2$ since $\Omega \in \mathbb{H}_2$ and $c_n(c_n\theta + d_n) > 0$ for $g \in S_\theta$. $\qquad\square$

Note, that R_n is a vector space. Moreover, $\dim R_n = c_n^2 = (\dim E_n)^2$, so that there are no linear relations among $_{A_\theta}\langle\phi_{f_i}, \phi_{f_j}\rangle$ for any basis $\{f_i\}$ in the space of functions on $\mathbb{Z}/c_n\mathbb{Z}$.

Let us define an operation $\underset{n,m}{\star} : R_n \otimes R_m \to R_{n+m}$. For $\Theta_1 = \sum_i \ _{A_\theta}\langle x_i, y_i\rangle$ and $\Theta_2 = \sum_j \ _{A_\theta}\langle z_j, t_j\rangle$ we put

$$\Theta_1 \underset{n,m}{\star} \Theta_2 := \sum_{i,j} \ _{A_\theta}\langle x_i * z_j, y_i * t_j\rangle.$$

This operation is well defined. Indeed, every element of R_n can be uniquely represented as a linear combination of $_{A_\theta}\langle\phi_{f_i}, \phi_{f_j}\rangle$ as we remarked above. Note that we have another way of calculating the product $\Theta_1 \underset{n,m}{\star} \Theta_2$, which doesn't involve multiplication in the ring E: due to Theorem 3

$$\Theta_1 \underset{n,m}{\star} \Theta_2 = \sum_i \ _{A_\theta}\langle x_i\Theta_2, y_i\rangle.$$

Consider the graded ring with unit element $R = \oplus_{n\geq 0}R_n$ where $R_0 = \mathbb{C}$ and the multiplication law is given by $\phi * \psi := \phi \underset{n,m}{\star} \psi \in R_{n+m}$ for $\phi \in R_n, \psi \in R_m, n, m > 0$.

This multiplication is obviously associative, because it is associative in the ring E defined above. Even more, due to our remark on dimensions, $R_n \cong E_n\bar{\otimes}E_n$. Here $\bar{\otimes}$ means that $(\alpha a)\bar{\otimes}b = a\bar{\otimes}(\bar{\alpha}b)$ for $\alpha \in \mathbb{C}$. Then $R = \oplus_{n\geq 0}R_n$ is just a kind of Segre square of $E = \oplus_{n\geq 0}E_n$ — the subspace in $E\bar{\otimes}E$ generated by elements $a\bar{\otimes}b$ with $a, b \in E_n$ for some n. Analogously to Theorem 4 we have:

Theorem 5. *Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$ be a quadratic irrationality, θ' be its Galois conjugate and $|\theta - \theta'| < 1$. Then there exist $g \in S_\theta$ such that the graded ring $R = R(g, \theta)$ is Koszul.*

Proof. By Theorem 4 we can find $g \in S_\theta$ such that $E = E(g, \theta)$ is Koszul. Let $\bar{E}$ be E considered as a $\mathbb{C}$-algebra via complex conjugation, i.e. $\alpha \mapsto \bar{\alpha}1 \in \bar{E}$. Then R is a Segre product of E and $\bar{E}$. But obviously $\bar{E}$ is Koszul $\mathbb{C}$-algebra, and it is known that a Segre product of two Koszul algebras is Koszul (see e.g. [13]). $\qquad\square$

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