

Decay of correlations and uniqueness of Gibbs lattice systems with non-quadratic interaction.¹

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Abstract. We aim this paper to develop the classical lattice models with unbounded spin to the case of non-quadratic polynomial interaction.

We demonstrate that the distinct relation between the growths of potentials leads to the uniqueness and the fast decay of correlations for Gibbs measure.

There is an approach initiated in the papers [9, 10, 23] to the description of the probability measures on infinite dimensional spaces in the terms of conditional distributions. This approach has already found its non-trivial applications to the natural construction of the different models in the Quantum Field Theory, Mathematical and Statistical Physics [18, 26, 32, 33].

There were obtained the effective criteria on the existence and uniqueness of such systems, see Dobrushin's criterion [9, 10, 11], Dobrushin–Shlosman mixing condition [12, 13]. In the essence of the Dobrushin's type criteria lie the keen variational estimates on the one-point conditional measures, which admit iteration and application of the fixed point arguments. Moreover, such estimates were used in the applications to the lattice spin systems of the statistical physics to the study of decay of correlations, differentiability of pressure and the connected questions [7, 14, 16, 17, 21, 22, 33].

In the noncompact spin case the check of Dobrushin's conditions is rather complicated by principal unboundedness of interaction potentials. The results in this direction were mainly centered around the regular interactions [3, 7, 20, 24, 28, 29, 30], i.e. when the many-point potentials in the Hamiltonian

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admit the quadratic domination, for example with the quadratic two-point potentials

$$H(x) = \sum_{k \in \mathbb{Z}^d} F(x_k) + \lambda \sum_{k,j \in \mathbb{Z}^d} b_{k-j}(x_k - x_j)^2$$

On the other hand, a wide class of models with nonregular interaction, associated with massless free lattice field, perturbed by $(\nabla\varphi)^4$

$$H(x) = \sum_{|k-j|=1} (x_k - x_j)^2 + \lambda \sum_{|k-j|=1} (x_k - x_j)^4,$$

has already obtained a detail investigation through various techniques [4, 6, 15, 25, 27]. In particular, it was shown that the exponential decay of correlations for such systems does not occur for all $\lambda > 0$ [4], i.e. the Dobrushin uniqueness technique does not work for such Hamiltonians.

In this paper we demonstrate that there is a wide class of the Gibbs lattice systems, which *do not fulfill* the regularity assumption but have the fast decay of correlations. We aim this paper to show that the application of the Dobrushin's uniqueness technique for the Hamiltonian

$$H(x) = \sum_{k \in \mathbb{Z}^d} F(x_k) + \lambda \sum_{k,j \in \mathbb{Z}^d} G_{k-j}(x_k - x_j)$$

with *polynomials* $\{G_j\}$ requires the distinct correlations between the growths of the interaction potentials $\{G_j\}$ and selfaction $\{F\}$. This gives us possibility to treat the problem on the existence, uniqueness and the *exponentially* fast decay of correlations in the case of *non-quadratic* polynomial interaction. We base our investigation on the scheme of papers [9, 10, 11, 14, 16, 22] and apply Brascamp-Lieb inequality [5] to obtain estimates on the distance in variations.

Consider \mathbb{Z}^d to be d - dimensional integer lattice, to each point of which corresponds the linear spin space \mathbb{R}^1 . Let $\mathcal{G}\{\mu_\Lambda\}$ denote the set of Gibbs measures [9, 10, 23] on the product σ -algebra on $\mathbb{R}^{\mathbb{Z}^d}$. It means that the corresponding conditional measures $\{\mu_\Lambda\}$ in the finite volumes of the lattice $\Lambda \subset \mathbb{Z}^d$ are defined by

$$d\mu_\Lambda = \frac{1}{Z_\Lambda} \exp\left\{-\lambda \sum_{\{k,j\} \cup \Lambda \neq \emptyset} G_{k-j}(x_k - x_j)\right\} \times_{k \in \Lambda} e^{-F(x_k)} dx_k \quad (1)$$

i.e. for all cylinder bounded functions $f \in C_{b,cyl}(\mathbb{R}^{\mathbb{Z}^d})$ we have $\mu(\mu_\Lambda(f)) = \mu(f)$, where $\mu(f)$ denotes the expectation and Z_Λ is a normalization factor.

We put the following conditions on the interactive potentials $\{F, G_j\}$ in the Gibbs measure (1).

- A. Self-action potentials $F \in C^2(\mathbb{R}^1)$, fulfill $F(0) = 0$, $\exists \varepsilon > 0 \inf_{x \in \mathbb{R}} F''(x) \geq \varepsilon$ and have no more than the exponential growth on the infinity $\exists c, a: \forall x |F(x)| \leq ce^{a|x|}$;
- B. Interaction potentials $\{G_j \in C^2(\mathbb{R}^1)\}_{j \in \mathbb{Z}^d \setminus \{0\}}$, fulfill $G_j(0) = 0$, $\forall j \in \mathbb{Z}^d \setminus \{0\} \forall x \in \mathbb{R}^1 : G_j''(x) \geq 0$ and $\exists r_0 \forall j : |j| > r_0 \Rightarrow G_j \equiv 0$
- C. Growth condition $\forall k \in \mathbb{Z}^d |k| \leq r_0 \sup_{x_k, x_0 \in \mathbb{R}^1} \frac{|G_k''(x_k - x_0)|}{\sqrt{F''(x_k)}\sqrt{F''(x_0)}} < \infty$

Immediately remark that the condition C states the domination of the one-point potentials over the interaction. It always holds for the quadratic and less than quadratic interaction due to $\sup |G''| \leq \text{const}$. Actually condition C permits to consider the interaction $\{G_j\}$ to be of polynomial type.

The following theorem states the uniqueness and the exponentially fast decay of correlations for the Gibbs measure (1). The existence of such measure and finiteness of its moments is shown in Theorem 2.

Theorem 1. *Suppose conditions A-C hold and the set of measures $\mu \in \mathcal{G}\{\mu_\Lambda\}$, which satisfy*

$$m_\mu = \sup_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^{\mathbb{Z}^d}} \rho^2(x_k, 0) d\mu < \infty, \quad \rho(x, y) = \left| \int_y^x \sqrt{F''(s)} ds \right|, \quad (2)$$

is nonempty. Denote $\gamma_d = \sum_{k \in \mathbb{Z}^d} e^{d(k,0)} \sup_{x_k, x_0 \in \mathbb{R}^1} \frac{|G''(x_k - x_0)|}{\sqrt{F''(x_k)}\sqrt{F''(x_0)}}$ for some transitional invariant semimetric $d(k, j)$ on the lattice \mathbb{Z}^d .

Then $\forall \lambda \in [0, 1/\gamma_d)$ measure $\tilde{\mu} \in \mathcal{G}\{\mu_\Lambda\}$, $m_{\tilde{\mu}} < \infty$ is unique and has exponentially fast decay of correlations, i.e.

$$\sum_{k \in \mathbb{Z}^d} e^{d(k,0)} |\text{cov}_{\tilde{\mu}}(f, \tau_k g)| \leq \frac{1}{1 - \lambda \gamma_d} \left(\sum_{k \in \mathbb{Z}^d} e^{d(k,0)} \delta_k(f) \right) \left(\sum_{j \in \mathbb{Z}^d} e^{d(j,0)} \delta_j(g) \right) \quad (3)$$

Above τ_k is a shift operator on vector $k \in \mathbb{Z}^d$,

$$\delta_k(f) = \sup_{x \in \mathbb{R}^{\mathbb{Z}^d}} \left| \frac{\partial_k f(x)}{\sqrt{F''(x_k)}} \right|, \quad \partial_k f(x) = \frac{\partial f(x)}{\partial x_k}, \quad x = \{x_k\}_{k \in \mathbb{Z}^d} \quad (4)$$

Inequality (3) is understood on the cylinder bounded differentiable functions $f, g \in C_{b, \text{cyl}}^1(\mathbb{R}^{\mathbb{Z}^d})$ such that $\sum_{j \in \mathbb{Z}^d} e^{d(j,0)} \delta_j(f) < \infty$.

Proof. We discuss the main tool, which enables us to deal with the polynomial interaction in the Gibbs measure. First note that the usual estimate on the covariance [2, 8]

$$\text{cov}_\mu(f, f) \equiv \int_{\mathbb{R}^1} (f - \int_{\mathbb{R}^1} f d\mu)^2 d\mu \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^1} \left| \frac{\partial f}{\partial x} \right|^2 d\mu \quad (5)$$

for the probability measure μ , $d\mu = e^{-F(x)} dx$ on the line \mathbb{R}^1 , holds for arbitrary function $F \in C^2(\mathbb{R})$ such that $F''(x) \geq \varepsilon > 0$ for all $x \in \mathbb{R}^1$. Actually the above inequality (5) is not optimal and in the paper [5, Th.4.1] it was found that the next weighted generalization is true

$$\text{cov}_\mu(f, f) \leq \int_{\mathbb{R}^1} \frac{1}{F''(x)} \left| \frac{\partial f}{\partial x} \right|^2 d\mu \quad (6)$$

with the weight $1/F''$, which in the cases when F'' grows on the infinity improves inequality (5).

Introduce the family of one-point conditional measures $\{\mu_k\}_{k \in \mathbb{Z}^d}$

$$d\mu_k = \frac{1}{Z_k} \exp\left\{-\lambda \sum_{j: j \neq k} G_{k-j}(x_k - x_j)\right\} e^{-F(x_k)} dx_k \quad (7)$$

where Z_k is a normalization factor. Below we also understand the measure μ_k as the operator of conditional expectation

$$\mu_k : C_{b,cyl}^1(\mathbb{R}^{\mathbb{Z}^d}) \ni f \longrightarrow \mu_k(f) \stackrel{def}{=} \int_{\mathbb{R}_k^1} f d\mu_k \in C_{b,cyl}^1(\mathbb{R}^{\mathbb{Z}^d})$$

The next identity for $j, k \in \mathbb{Z}^d$, $j \neq k$

$$\partial_j \mu_k(f) = \mu(\partial_j f) - \lambda cov_{\mu_k}(f, \partial_j G_{k-j}(x_k - x_j))$$

leads to

$$\begin{aligned} \delta_j(\mu_k(f)) &= \sup \left| \frac{\partial_j(\mu_k(f))}{\sqrt{F''}} \right| = \\ &= \sup \left| \mu_k \left(\frac{\partial_j f}{\sqrt{F''(x_j)}} \right) - \lambda cov_{\mu_k} \left(f, \frac{\partial_j G_{k-j}(x_k - x_j)}{\sqrt{F''(x_j)}} \right) \right| \leq \\ &\leq \delta_j(f) + \lambda \sup \left| cov_{\mu_k} \left(f, \frac{\partial_j G_{k-j}(x_k - x_j)}{\sqrt{F''(x_j)}} \right) \right| \end{aligned} \quad (8)$$

Using the convexness of G_j we obtain the following consequence of the weighted inequality (6)

$$\begin{aligned} cov_{\mu_k}(f, f) &\leq \int_{\mathbb{R}^1} \frac{|\partial_k f|^2}{F''(x_k) + \sum_{j \neq k} G''_{k-j}(x_k - x_j)} d\mu_k \leq \\ &\leq \int_{\mathbb{R}^1} \frac{|\partial_k f|^2}{F''(x_k)} d\mu_k \leq [\delta_k(f)]^2 \end{aligned} \quad (9)$$

Inequality (9) enables us to estimate the second term in (8)

$$\begin{aligned} \sup \left| cov_{\mu_k} \left(f, \frac{\partial_j G_{k-j}(x_k - x_j)}{\sqrt{F''(x_j)}} \right) \right| &\leq \\ &\leq \sup cov_{\mu_k}^{1/2}(f, f) cov_{\mu_k}^{1/2} \left(\frac{\partial_j G_{k-j}(x_k - x_j)}{\sqrt{F''(x_j)}}, \frac{\partial_j G_{k-j}(x_k - x_j)}{\sqrt{F''(x_j)}} \right) \leq \\ &\leq \delta_k(f) \left(\int_{\mathbb{R}^1} \frac{|\partial_k \partial_j G(x_k - x_j)|^2}{F''(x_k) F''(x_j)} d\mu_k \right)^{1/2} \leq \delta_k(f) \sup \frac{|G''_{k-j}(x_k - x_j)|}{\sqrt{F''(x_k)} \sqrt{F''(x_j)}} \end{aligned}$$

Finally from (8) we obtain that

$$\delta_j(\mu_k(f)) \leq \delta_j(f) + \lambda C_{kj} \delta_k(f) \quad (10)$$

with

$$C_{kj} = \sup_{x_k, x_j \in \mathbb{R}^1} \frac{|G''_{k-j}(x_k - x_j)|}{\sqrt{F''(x_k)} \sqrt{F''(x_j)}}$$

The estimate (10) is a key point of the Dobrushin's uniqueness technique and the special structure of the *covariance* matrix C_{kj} permits the polynomiality of $\{G_j\}$ in the interaction.

Below we follow scheme of [14, 16, 22]. The principal modification lies in the use of weighted inequality (6) and weighted estimate on covariances (10).

1. *Uniqueness of the Gibbs measure.* Like in [14] we say that the vector $\{a_j\}_{j \in \mathbb{Z}^d}$ is an estimate for probability measures μ, ν if $\forall f \in C_{b,cyl}^\infty(\mathbb{R}^{\mathbb{Z}^d})$: $\sum_{k \in \mathbb{Z}^d} \delta_k(f) < \infty$ we have

$$\left| \int_{\mathbb{R}^{\mathbb{Z}^d}} f d\mu - \int_{\mathbb{R}^{\mathbb{Z}^d}} f d\nu \right| \leq \sum_{j \in \mathbb{Z}^d} a_j \delta_j(f) \quad (11)$$

For any two measures $\mu_1, \mu_2 \in \mathcal{G}\{\mu_\Lambda\}$ with property (2) there is an estimate $\tilde{a} = \{\tilde{a}_j \equiv m_0 \equiv const\}_{j \in \mathbb{Z}^d}$ with $m_0 = m_{\mu_1}^{1/2} + m_{\mu_2}^{1/2}$. To show this, note first that for $f \in C_{b,cyl}^1(\mathbb{R}^{\mathbb{Z}^d})$ with $\sum_{k \in \mathbb{Z}^d} \delta_k(f) < \infty$ we have

$$|f(x) - f(y)| \leq \sum_{i \in \mathbb{Z}^d} \delta_i(f) \rho(x_i, y_i)$$

and therefore

$$\begin{aligned} \left| \int_{\mathbb{R}^{\mathbb{Z}^d}} f d\mu_1 - \int_{\mathbb{R}^{\mathbb{Z}^d}} f d\mu_2 \right| &= \left| \int_{\mathbb{R}^{\mathbb{Z}^d}} (f(x) - f(0)) d\mu_1 - \int_{\mathbb{R}^{\mathbb{Z}^d}} (f(x) - f(0)) d\mu_2 \right| \leq \\ &\leq \sum_{k \in \mathbb{Z}^d} \delta_k(f) \int_{\mathbb{R}^{\mathbb{Z}^d}} \rho(x_k, 0) \{d\mu_1(x) + d\mu_2(x)\} \leq m_0 \sum_{k \in \mathbb{Z}^d} \delta_k(f) \end{aligned} \quad (12)$$

By (10) the operator $f \rightarrow \mu_k(f)$ preserves the class of functions $\{f \in C_{b,cyl}^1(\mathbb{R}^{\mathbb{Z}^d}) : \sum_{k \in \mathbb{Z}^d} \delta_k(f) < \infty\}$. From (10) and (12) we have

$$\begin{aligned} |\mu_1(f) - \mu_2(f)| &= |(\mu_1 - \mu_2)(\mu_k(f))| \leq \\ &\leq \sum_{j \in \mathbb{Z}^d} \tilde{a}_j \delta_j(\mu_k(f)) \leq \sum_{j: j \neq k} \tilde{a}_j \delta_j(f) + \lambda \delta_k(f) \sum_{i: i \neq k} \tilde{a}_i C_{ki} \end{aligned} \quad (13)$$

Iterating the above estimate by choosing some enumeration k_1, \dots, k_n, \dots of the points of lattice \mathbb{Z}^d one can in a purely algebraic way achieve the following estimate, see [14, Lemma 2.3]

$$|\mu_1(f) - \mu_2(f)| \leq \lambda \sum_{k \in \mathbb{Z}^d} \delta_k(f) \left(\sum_{j \in \mathbb{Z}^d} \tilde{a}_j C_{kj} \right)$$

which gives

$$|\mu_1(f) - \mu_2(f)| \leq \sum_{k \in \mathbb{Z}^d} (\tilde{a}(\lambda C)^n)_k \delta_k(f)$$

for all $n \geq 0$.

Due to

$$\begin{aligned} \|\tilde{a}(\lambda C)^n\|_{\ell_\infty(\mathbb{Z}^d)} &= m_0 \sup_{k \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} \{(\lambda C)^n\}_{kj} = \\ &= m_0 \sup_{k \in \mathbb{Z}^d} \lambda^n \sum_{j(1) \in \mathbb{Z}^d} \dots \sum_{j(n-1) \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} C_{kj(1)} \dots C_{j(n-1)j} \leq \\ &\leq m_0 \left(\sup_{k \in \mathbb{Z}^d} \lambda \sum_{j \in \mathbb{Z}^d} C_{kj} \right)^n \leq m_0 (\lambda \gamma_d)^n \rightarrow 0, \quad n \rightarrow \infty, \end{aligned} \quad (14)$$

we obtain the uniqueness of the Gibbs measure.

2. *Decay of correlations.* Fix function $g \in C_{b,cyl}^1(\mathbb{R}^{\mathbb{Z}^d})$ such that $\int_{\mathbb{R}^{\mathbb{Z}^d}} g d\mu = 1$, $g > 0$ and $\sum_{k \in \mathbb{Z}^d} e^{d(k,0)} \delta_k(g) < \infty$. Then measure $d\nu = g d\mu$ for the unique measure $\mu \in \mathcal{G}\{\mu_\Lambda\}$ with property (2) has the same property

$$\sup_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^{\mathbb{Z}^d}} \rho(x_k, 0) d\nu(x) \leq \|g\|_{C_b} m_\mu^{1/2} < \infty$$

In analog to (12) this gives the estimate $\tilde{a} = \{\tilde{a}_j \equiv m_\mu^{1/2}(\|g\|_{C_b} + 1)\}_{j \in \mathbb{Z}^d}$ on measures μ and ν

$$|\mu(f) - \nu(f)| \leq \sum_{k \in \mathbb{Z}^d} \tilde{a}_k \delta_k(f) = m_\mu^{1/2}(\|g\|_{C_b} + 1) \sum_{k \in \mathbb{Z}^d} \delta_k(f)$$

Now we prove that if $\{a_j\}_{j \in \mathbb{Z}^d}$ is an estimate, then $\{\sum_{j \in \mathbb{Z}^d} a_j C_{jk} + b_k\}_{k \in \mathbb{Z}^d}$ for $b_k = \delta_k(g)$ is an estimate too. Indeed

$$\begin{aligned} |\mu(f) - \nu(f)| &\leq |(\mu - \nu)_y \{ \int_{\mathbb{R}_k} f(\cdot|y) d\mu_k(\cdot|y) \}| + \\ &+ |\nu_y \{ \int_{\mathbb{R}_k} f(\cdot|y) d\mu_k(\cdot|y) - \int_{\mathbb{R}_k} f(\cdot|y) d\nu_k(\cdot|y) \}| = \\ &= \sum_{j \in \mathbb{Z}^d} \tilde{a}_j \delta_j(\mu_k(f)) + |\nu_y \{ \int_{\mathbb{R}_k} f(\cdot|y) d\mu_k(\cdot|y) - \int_{\mathbb{R}_k} f(\cdot|y) d\nu_k(\cdot|y) \}| \end{aligned} \quad (15)$$

Using (10) the first term in (15) can be estimated by

$$\sum_{j \neq k} \tilde{a}_j \delta_j(f) + \lambda \delta_k(f) \left\{ \sum_{i \neq k} \tilde{a}_i C_{ik} \right\}$$

We apply inequality (6) to the second term. We use that $d\nu = g d\mu$, so $d\nu_k = \frac{g}{\mu_k(g)} d\mu_k$ and obtain

$$\begin{aligned} & |\nu_y\{\int f d\mu_k(\cdot|y) - \int f d\nu_k(\cdot|y)\}| = \\ & = |\nu_y\{\int [f - \mu_k(f)](d\mu_k - \frac{g}{\mu_k(g)} d\mu_k)\}| = \\ & = |\mu_y\{\frac{g}{\mu_k(g)} \int (f - \mu_k(f))(g - \mu_k(g)) d\mu_k\}| \end{aligned}$$

The result of integration on \mathbb{R}_k doesn't depend on variable $x_k \in \mathbb{R}_k$, therefore we continue

$$\begin{aligned} & |\mu_y\{\frac{g}{\mu_k(g)} \int (f - \mu_k(f))(g - \mu_k(g)) d\mu_k\}| = \\ & = |\mu\{\int_{\mathbb{R}_k} (f - \mu_k(f))(g - \mu_k(g)) d\mu_k\}| \leq \\ & \leq \sup cov_{\mu_k}^{1/2}(f, f) cov_{\mu_k}^{1/2}(g, g) \leq \delta_k(f) \delta_k(g) = b_k \delta_k(f) \end{aligned}$$

Finally we have obtained the estimate on (15)

$$|\mu(f) - \nu(f)| \leq \sum_{j \neq k} \tilde{a}_j \delta_j(f) + \delta_k(f) \{\sum_{i \neq k} \tilde{a}_i \lambda C_{ik} + b_k\} \quad (16)$$

By iteration of (16) like in [14, 16, 22] one achieves that $(\tilde{a} \lambda C + b)$ is an estimate too

$$|\mu(f) - \nu(f)| \leq \sum_{k \in \mathbb{Z}^d} \{\tilde{a}_i \lambda C_{ik} + b_k\} \delta_k(f) \quad (17)$$

The vector $b \sum_{n=0}^{\infty} (\lambda C)^n$ is also an estimate because of the following convergence in $\ell_{\infty}(\mathbb{Z}^d)$

$$b \sum_{n=0}^N (\lambda C)^n + \tilde{a} (\lambda C)^{N+1} \rightarrow b \sum_{n=0}^{\infty} (\lambda C)^n, \quad N \rightarrow \infty$$

Thus we achieve estimate [14, 16, 22]

$$|cov_{\mu}(f, g)| = \left| \int_{\mathbb{R}^{\mathbb{Z}^d}} f d\nu - \int_{\mathbb{R}^{\mathbb{Z}^d}} f d\mu \right| \leq \sum_{k, j \in \mathbb{Z}^d} D_{kj} \delta_k(f) \delta_j(g) \quad (18)$$

for $D = \sum_{n=0}^{\infty} (\lambda C)^n$. Therefore

$$|cov_{\mu}(f, \tau_i g)| e^{d(i,0)} \leq \sum_{k, j \in \mathbb{Z}^d} e^{d(j,k)} D_{jk} e^{d(k,0)} \delta_k(f) e^{d(i,j)} \delta_{j-i}(g)$$

Summing up on $i \in \mathbb{Z}^d$ we have the required decay of correlations for $g > 0$.

The case of arbitrary $g \in C_{b, cyl}^1(\mathbb{R}^{\mathbb{Z}^d})$ with $\sum_{k \in \mathbb{Z}^d} e^{d(k,0)} \delta_k(g) < \infty$ is obvious due to the identity $cov_{\mu}(f, c_1 g + c_2) = c_1 cov_{\mu}(f, g)$ ■

Theorem 2. *Under conditions A–C the set of Gibbs measures $\mathcal{G}\{\mu_{\Lambda}\}$ with condition*

$$m_{\mu} = \sup_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^{\mathbb{Z}^d}} \rho^2(x_k, 0) d\mu(x) < \infty \quad (19)$$

is nonempty.

Moreover, at the coupling interaction constant $\lambda \in [0, 1/\gamma_d)$, the Gibbs measure $\tilde{\mu}$ of Theorem 1 fulfills estimate

$$\sup_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^{\mathbb{Z}^d}} \exp\{ax_k^2\} d\tilde{\mu} \leq \exp\left(\frac{a}{\varepsilon - 2a}\right) \quad (20)$$

for all $a \in [0, \varepsilon/2)$.

Proof. Let

$$\mathcal{U}_\Lambda = \sum_{k \in \Lambda} F(x_k) + \lambda \sum_{\{k,j\} \subset \Lambda} G_{k-j}(x_k - x_j)$$

and consider the family of Gibbs measures $\{\mu_\Lambda\}$ with the free boundary conditions in the finite volumes $\Lambda \subset \mathbb{Z}^d$

$$d\mu_\Lambda^0 = \frac{1}{Z} e^{-\mathcal{U}_\Lambda} dx_\Lambda$$

The potentials $(\mathcal{U}_\Lambda)'' \geq \varepsilon I$ are convex, so the measures μ_Λ^0 satisfy inequality (24) in form [2, 8]

$$\text{cov}_{\mu_\Lambda^0}(f, f) \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^\Lambda} \sum_{k \in \Lambda} |\partial_k f|^2 d\mu_\Lambda^0$$

Substituting $f = x_k$ and using that $\int_{\mathbb{R}^\Lambda} x_k d\mu_\Lambda^0 \equiv 0$ by the symmetry of μ_Λ^0 we have that uniformly on Λ and $k \in \Lambda$

$$\sup_{\Lambda \subset \mathbb{Z}^d, k \in \Lambda} \int_{\mathbb{R}^\Lambda} x_k^2 d\mu_\Lambda^0 \leq 1/\varepsilon \quad (21)$$

The convexness of the potentials \mathcal{U}_Λ also imply the Log-Sobolev inequality for the measures $\{\mu_\Lambda^0\}$ [2]

$$\int_{\mathbb{R}^\Lambda} f^2 \ln f^2 d\mu_\Lambda^0 - \int_{\mathbb{R}^\Lambda} f^2 d\mu_\Lambda^0 \ln \int_{\mathbb{R}^\Lambda} f^2 d\mu_\Lambda^0 \leq \frac{2}{\varepsilon} \int_{\mathbb{R}^\Lambda} \sum_{k \in \Lambda} |\partial_k f(x_\Lambda)|^2 d\mu_\Lambda^0(x_\Lambda) \quad (22)$$

Fix $\Lambda \subset \mathbb{Z}^d$ and $k \in \Lambda$. Consider increasing on $n \geq 1$ sequence of functions

$$f_n = \begin{cases} -n, & x_k < -n \\ x_k, & |x_k| \leq n \\ n, & x_k > n \end{cases}$$

Like in [8] introduce sequence of functions $h_n(a) = \int_{\mathbb{R}^\Lambda} \exp(af_n^2) d\mu_\Lambda^0 \geq 1$ on half-line $a \in [0, \infty)$, increasing on both a and n with all derivatives $h_n^{(k)}(a) > 0$, $a > 0$. Then for $g_n = \exp(af_n^2/2)$ we apply Log-Sobolev inequality (22)

$$\begin{aligned} ah'_n(a) &= \int_{\mathbb{R}^\Lambda} af_n^2 \exp(af_n^2) d\mu_\Lambda^0 = \int_{\mathbb{R}^\Lambda} g_n^2 \ln g_n^2 d\mu_\Lambda^0 \leq \\ &\leq \frac{2}{\varepsilon} \int_{\mathbb{R}^\Lambda} \sum_{j \in \Lambda} |\partial_j g_n|^2 d\mu_\Lambda^0 + h_n(a) \ln h_n(a) \leq \\ &\leq \frac{2}{\varepsilon} a^2 \int_{\mathbb{R}^\Lambda} f_n^2 \exp(af_n^2) d\mu_\Lambda^0 + h_n(a) \ln h_n(a) \end{aligned}$$

Therefore the family $h_n(a)$, increasing on both n and $a \geq 0$, $h_n(0) = 1$, satisfy inequality $a(1 - \frac{2a}{\varepsilon})h'_n(a) \leq h_n(a) \ln h_n(a)$. To find the major function we must set $h(0) = 1$ and take the highest growth of its derivative, so $a(1 - \frac{2a}{\varepsilon})h'(a) = h(a) \ln h(a)$ and $h(a) = \exp(\frac{aD}{1 - 2a/\varepsilon})$ for some D . The restriction on D we obtain from the highest growth of h_n at zero

$$h'_n(0) = \int_{\mathbb{R}^\Lambda} f_n^2 d\mu_\Lambda^0 \leq \int_{\mathbb{R}^\Lambda} x_k^2 d\mu_\Lambda^0(x) = D < \infty$$

and achieve estimate $h_n(a) \leq \exp\{\frac{a}{1 - 2a/\varepsilon} \int_{\mathbb{R}^\Lambda} x_k^2 d\mu_\Lambda^0(x)\}$. Tending $n \rightarrow \infty$ we obtain the estimate of the next form at $a \in [0, \varepsilon/2)$

$$\int_{\mathbb{R}^\Lambda} \exp(ax_k^2) d\mu_\Lambda^0 \leq \exp(\frac{a}{1 - 2a/\varepsilon} \int_{\mathbb{R}^\Lambda} x_k^2 d\mu_\Lambda^0)$$

which by (21) gives

$$\forall a \in [0, \varepsilon/2) \quad \sup_{\Lambda \subset \mathbb{Z}^d, k \in \Lambda} \int_{\mathbb{R}^\Lambda} \exp(ax_k^2) d\mu_\Lambda^0 < \exp(\frac{a}{\varepsilon - 2a}) \quad (23)$$

Compactness of the function $\exp(ax_k^2)$ leads by the Prochorov's theorem [33] to the existence of the weak local limit $\tilde{\mu}$

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \int_{\mathbb{R}^\Lambda} f(x_\Lambda) d\mu_\Lambda^0 = \int_{\mathbb{R}^{\mathbb{Z}^d}} f(x) d\tilde{\mu}$$

on any cylinder function $f \in C_{b,cyl}(\mathbb{R}^{\mathbb{Z}^d})$.

Due to the finiteness of the interaction radius B the limit measure $\tilde{\mu}$ has the conditional measures $\{\mu_\Lambda\}$ in the finite volumes, i.e. $\tilde{\mu} \in \mathcal{G}\{\mu_\Lambda\}$ and the set of the Gibbs measures is non-empty. From (23) we also have that the measure $\tilde{\mu}$ is tempered, i.e.

$$\sup_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^{\mathbb{Z}^d}} \exp(ax_k^2) d\tilde{\mu} < \exp\left(\frac{a}{\varepsilon - 2a}\right), \quad a \in [0, \varepsilon/2)$$

which obviously gives the statement (20). ■

Model 1. Let the potentials be defined by

$$F(x_k) = (1 + x_k^2)^{2n+1} \quad \& \quad G_{k-j}(x_k - x_j) = b_{k-j}(x_k - x_j)^{2n+2}$$

and assume that the coefficients $\{b_j\}_{j \in \mathbb{Z}^d}$ satisfy

$$\forall j \in \mathbb{Z}^d \quad b_j \geq 0 \quad \& \quad \exists r_0 \quad \forall |j| > r_0 : b_j = 0$$

Then for

$$0 \leq \lambda < \frac{1}{(n+1)2^{2n+1}\|b\|_d}, \quad \|b\|_d = \sum_{j \in \mathbb{Z}^d} b_j e^{d(j,0)} < \infty$$

the statements of Theorems 1,2 are valid.

Model 2. *Lattice spin system on Riemannian manifold.*

Denote $M = M_k$, $k \in \mathbb{Z}^d$ a noncompact Riemannian manifold with covariant derivative ∂_k and Ricci curvature tensor Ric_k .

Let potentials $F_k(x_k)$, $G_{kj}(x_k, x_j)$ satisfy

- 1) $F_k \in C^2(M)$, $\exists \varepsilon > 0 \quad \forall x_k \in M_k \quad Ric_k + \partial_k \partial_k F(x_k) \geq \varepsilon$
- 2) $G_{kj} \in C^2(M \times M)$, $\exists \alpha \in \mathbb{R}^1 \quad \partial_k \partial_k G_{kj}(x_k, x_j) \geq -\alpha$, $k, j \in \mathbb{Z}^d$
and $G_{kj} \equiv 0$, for $|k - j| \geq r_0$.
- 3) $\alpha_{k,j} = \sup_{x \in M^{\mathbb{Z}^d}} \|B^{-1/2}(x_k) B^{-1/2}(x_j) \partial_k \partial_j G_{kj}(x_k, x_j)\|_{TM_k \times TM_j} < \infty$

where $B(x_k) = Ric_k(x_k) + \partial_k \partial_k F(x_k)$ and $\|\cdot\|_{TM_k \times TM_j}$ is a standard Hilbert norm on tangent space to $M_k \times M_j$.

Then for $\lambda \in [0, \min(\varepsilon/\alpha(2r_0 + 1)^d, 1/\gamma_d))$ the lattice system, described by Hamiltonian

$$H = \sum_{k \in \mathbb{Z}^d} F_k(x_k) + \lambda \sum_{|k-j| \leq r_0} G_{kj}(x_k, x_j)$$

has exponentially fast decay of correlations and Gibbs measure is unique [1]. Above $\gamma_d = \sup_{k \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} e^{d(k,j)} \alpha_{kj}$.

This result is achieved by the scheme of this paper. One needs to consider

$$\delta_k(f) = \sup_{x \in M^{\mathbb{Z}^d}} \|(Ric_k + \partial_k \partial_k F(x_k))^{-1/2} \partial_k f(x)\|_{TM_k}$$

and apply in corresponding places the following generalization of Brascamp-Lieb inequality (6) to the case of arbitrary Riemannian manifold [1]: under condition $\exists \varepsilon > 0 \quad Ric + \partial \partial F \geq \varepsilon$ we have

$$cov_\mu(f, f) \leq \int_M \langle (Ric + \partial \partial F)^{-1} \partial f, \partial f \rangle d\mu, \quad f \in C_b^1(M) \quad (24)$$

with probability measure $d\mu = e^{-F} d\sigma$ (σ denotes Riemannian volume on manifold M) and Riemannian pairing $\langle \cdot, \cdot \rangle$ on tangent space to manifold.

Developing idea of Helffer [19, 27] we can shortly explain inequality (24) next way. Take $u = \partial H_\mu^{-1}(g - \int g d\mu)$ for $H_\mu = \partial_\mu^* \partial$ with dual gradient $\partial_\mu^* v = -\text{div } v + \langle \partial F, v \rangle$, then $\partial_\mu^* u = H_\mu H_\mu^{-1}(g - \int g d\mu) = g - \int g d\mu$ and we have

$$\begin{aligned} \text{cov}_\mu(g, g) &= \int g(g - \int g d\mu) d\mu = \int \langle u, \partial g \rangle d\mu = \\ &= \int \langle (H_\mu + \text{Ric} + \partial \partial F)^{-1} \partial g, \partial g \rangle d\mu \leq \int_M \langle (\text{Ric} + \partial \partial F)^{-1} \partial g, \partial g \rangle d\mu \end{aligned}$$

where we used positivity of H_μ and $u = (H_\mu + \text{Ric} + \partial \partial F)^{-1} \partial g$ by a simple commutation $\partial g = \partial \partial_\mu^* u = (\partial_\mu^* \partial + \text{Ric} + \partial \partial F)u$.

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