Academy of Sciences of Ukraine

Institute of Mathematics

Preprint 93.33

A. Val. Antoniouk & A. Vict. Antoniouk

Weighted Spectral Gap and Logarithmic Sobolev Inequalities and their Applications.

Kiev 1993

Weighted Spectral Gap and Logarithmic Sobolev Inequalities and their Applications*/Antoniouk A.Val.†,Antoniouk A.Vict.† Kiev, 1993.- 68 p.- (Prepr./ Acad.Sci.Ukr.Inst.Mathematics: 93.33)

The conditions on smooth measure on Riemannian manifold when it admits weighted Spectral Gap and Log-Sobolev inequality are obtained.

In application of these conditions to polynomial measures on line the correspondence between weighted and classic Log-Sobolev inequalities is studied.

The problem of existence, uniqueness and decay of correlations for lattice Gibbs measures with convex pair interaction is solved via the technique of weighted Spectral Gap.

*) This work has been supported by the Fund for Fundamential research of the State Commetee of Ukraine on Science and Technology, the project 1/238 "Operator".

†) Department of Functional Analysis, Institute of Mathematics Acad.Sci Ukraine, Tereschenkovskaja str. 3, Kiev - 4, GSP, 252601, Ukraine.

Fax: Ukraine (044) 224-32-43

E-mail: mathkiev@sovam.com

The print is approved by the Academic Council of Institute of Mathematics of Ukrainean Acad.Sci.

1 Introduction.

In 1975 L.Gross [29] proved the equivalence between the Logarithmic Sobolev inequality

$$\int\limits_M |f|^2 \ln |f|^2 d\mu - \|f\|^2 \ln \|f\|^2 \leq \frac{2}{\lambda} \int\limits_M |\nabla f|^2 d\mu \tag{1.1 }$$

and the important hypercontractive property for the semigroup of operator H_μ associated with quadratic Dirichlet form

$$(H_\mu f, f)_{L_2(\mu)} = \frac{1}{2} \int\limits_M |\nabla f|^2 d\mu$$

The hypercontractivity means that $\exists t > 0$ such that $\| \exp(-tH_\mu)\|_{L_2 \to L_4(\mu)} = 1$. The further investigation of contractive properties of semigroups via Log-Sobolev inequalities was developed in the papers of J.Rosen [44], E.B.Davies and B.Simon [20], S.Kusuoka and D.W.Stroock [38] where it were obtained conditions on super and ultra contractivity. It was also noted by O.S.Rothaus that Log-Sobolev inequality (1.1) has close connection with isoperimetric inequalities for measure μ [45].

In the paper of D.Bakry and M.Emery [8] it was proposed the next general condition for measure μ on Riemannian manifold M when the Logarithmic Sobolev inequality (1.1) holds:

$$\exists \lambda > 0 \ \forall x \in M \ \ R_\mu(x) = Ric(x) + \nabla\nabla\Phi(x) \geq \lambda \tag{1.2 }$$

Here Ric denotes Ricci curvature tensor and $\nabla\nabla\Phi$ denotes the second covariant derivative of density of measure μ: $d\mu = \exp(-\Phi)d\sigma$ with respect to Riemannian volume σ.

The interesting applications such inequalities found in the study of infinite-dimensional lattice systems. The Stochastic Ising models via Log-Sobolev technique were investigated by E.A.Carlen, R.Holley and D.W.Stroock [31, 13, 32, 33]. In papers of J.-D.Deushel, D.W.Stroock [21] and B.Zegarlinski [54] it was proved that inequality (1.1) for Gibbs measure ensures Dobrushin uniqueness criterion [22], and in papers [50], [51] it was obtained the equivalence between Log-Sobolev inequalities and Dobrushin-Shlosman mixing condition [25].

For unbounded spin systems such inequalities found their applications to the study of ergodic properties of Dirichlet operators semigroups in papers of S.Albeverio, Yu.G.Kondratiev and M.Röckner [3, 2]. The case of smooth measure on rigged Hilbert space was discussed in [6].

In this paper we investigate conditions when the next inequality holds

$$\begin{gathered} \int\limits_M \Psi(v) d\mu - \Psi(\int\limits_M v \, d\mu) \leq \\ \leq \frac{1}{2\lambda} \int\limits_M \Psi''(v) \prec A\nabla v, \nabla v \succ d\mu \end{gathered} \tag{1.3 }$$

with weight $A = A(x)$ which behaves like R_μ^{-1} (1.2) on the infinity of manifold M.

For example, in Section 2 we prove the next weighted Spectral Gap inequality (Theorem 2.2)

$$\frac{1}{2}\int\limits_M |f - \int\limits_M f\, d\mu|^2 d\mu \le \int\limits_M \prec R_\mu^{-1}\nabla f, \nabla f \succ d\mu \tag{1.4 }$$

with exact weight $A = R_\mu^{-1}$.

In Section 3 we develop Bakry-Emery scheme and obtain sufficient condition (Theorem 3.1)

$$\exists \lambda > 0\ \forall x \in M \quad AR_\mu A + H_\mu^A A \ge \lambda A \tag{1.5 }$$

for the weighted gradient estimate (1.3).

Here $H_\mu^A = \frac{1}{2}\{-div(A\nabla) + \prec A\nabla\Phi, \nabla \succ\}$ is operator associated with quadratic form

$$(H_\mu^A u, v)_{L_2} = \frac{1}{2}\int\limits_M \prec A\nabla u, \nabla v \succ d\mu$$

and coefficient operator A is a smooth $T^{1,1}M$ - valued field.

If $\Psi(v) = v\ln v$ and $v = f^2$ then inequality (1.3) transforms to the weighted Log-Sobolev inequality (see also Appendix A when $A(x) = S(x)Id_x$ on $T_x^{1,0}M$ for function S)

$$\begin{aligned} &\int\limits_M |f|^2 \ln |f|^2 d\mu - \|f\|^2 \ln \|f\|^2 \le \\ &\le \tfrac{2}{\lambda} \int\limits_M \prec A(x)\nabla f(x), \nabla f(x) \succ d\mu(x) \end{aligned} \tag{1.6 }$$

In Section 4 we provide some examples of measures when weighted inequalities (1.3), (1.6) hold. For instance, it is shown in Example 4.1 that choice $A = R_\mu^{-1}$ is possible at all.

We demonstrate in Example 4.2 that condition (1.5) works even in the situations when R_μ-object (1.2) is negative at some region on M.

In this section we also prove the set of weighted Log-Sobolev inequalities for polynomial log-concave measures on $\mathcal{R}$ (Theorem 4.3). Note that as a consequence, we obtain classic Log-Sobolev inequality (1.1) as a special limit of weighted ones.

The Sections 5 and 6 are dedicated to the application of the weighted spectral gap inequality (1.4) to the study of uniqueness and decay of correlations for the lattice Gibbs measures with *non-Gaussian interaction.*

There is an approach proposed by R.L.Dobrushin, O.E.Lanford and D.Ruelle [22, 40] to the defining of measure on countable product of metric spaces in terms of fixed family of local conditional measures. This approach gave possibility to obtain effective criterions on uniqueness and existence for such measures (see Dobrishin's criterion [22], Dobrushin-Shlosman mixing condition [25] and their applications [42, 49]).

In the situation of compact metric spaces the conditions on existence and uniqueness of Gibbs measures were found by R.L.Dobrushin [22, 23, 24]. In the papers of L.Gross and B.Simon [30, 48] it was shown that in the essence of Dobrushin type criterions lie special variational estimates on one-point conditional measures, which admit iteration and application of fixed points theorems. Moreover such estimates were used by H.Föllmer, L.Gross, D.Klein, H.Künsch [30, 37, 26, 34] to the study of decay of correlations and connected questions.

In non-compact situation the most optimal (known to authors) conditions could be found in paper of M.Cassandro, E.Olivieri, A.Pelegrinotti and E.Presutti [14], where it was studied the case of Gaussian measure perturbed by one-point potentials (see also [41, Ch.4,§8, Ch.7,§2]).

In this paper we consider the class of Gibbs measures on lattice product of non-compact Riemannian manifolds when non-quadratic pair interaction is dominated by one-point potentials. We work in the spirit of scheme proposed by L.Gross [30] and H.Föllmer [26]. The weighted Spectral Gap inequality (1.4) enables us to omit Lanford-Vasserstein supremum type estimates [52, 39] on one point conditional measures.

In Section 5 we consider conditions on uniqueness and decay of correlations for Gibbs measures with convex pair interaction (Theorems 5.7 and 5.9).

In Section 6 we apply results of Section 5 to the study of linear lattice systems.

2 Weighted Spectral Gap inequality for smooth measures on Riemannian manifolds.

Consider finite-dimensional complete connected smooth Riemannian manifold M, $dim\ M = m$, with metric tensor g_{ij}. Let $T^{p,q}M$ denote the tangent bundle of p-times contravariant and q-times covariant tensor fields on M, $p, q \geq 0$. Correspondingly let $C^k(M, T^{p,q}M)$ denote the set of k-times continuously differentiable (p, q) - tensor fields [46], [11]. By $C_b^k(M, T^{p,q}M)$ or $C_0^k(M, T^{p,q}M)$ we denote the spaces of k - times continuously differentiable tensor fields bounded with derivatives or with finite support.

The invariant Riemannian volume on M we denote by σ, $d\sigma = (det\, g_{ij})^{1/2}dx$ in local coordinates. The operator ∇ of covariant differentiation acting on tensor fields is defined by the next expression in local coordinates

$$\nabla_k u^{i_1..i_p}_{j_1..j_q} = \frac{\partial u^{i_1..i_p}_{j_1..j_q}}{\partial x_k} +$$

$$+\sum_{\alpha=1}^{p} u^{i_1..h..i_p}_{j_1...j_q}\Gamma_{h\ k}^{\ i_\alpha} - \sum_{\beta=1}^{q} u^{i_1...i_p}_{j_1..h..j_q}\Gamma_{k\ j_\beta}^{\ h}$$

where $\Gamma_{i\ k}^{\ h}$ are Cristoffel symbols of second order and u is p - times contravariant and q - times covariant tensor field. Hereforth we use the summation convention on coinciding indexes.

The innner scalar product of tensors u and v on manifold M is the next expression in local coordinates

$$\prec u, v \succ_{T^{p,q}M} = \mathop{\times}_{s=1}^{p} g_{i_s j_s} \mathop{\times}_{t=1}^{q} g^{r_t \ell_t} u^{i_1..i_p}_{r_1..r_q} \overline{v}^{j_1..j_p}_{\ell_1..\ell_q}$$

where $\{g^{ij}\}$ denotes the inverse to metric tensor. The same way it is defined the norm of tensor field $|u| = (\prec u, u \succ)^{1/2}$.

Consider probability measure μ which is absolutely continuous with respect to Riemannian volume σ with density $\Phi \in C^3(M)$

$$d\mu = \frac{e^{-\Phi}d\sigma}{\int\limits_M e^{-\Phi}d\sigma} \tag{2.1 }$$

Then the next integration by parts formula is valid [11], [15]

$$\int\limits_M \prec \nabla u, v \succ_{T^{p,q+1}M} d\mu = \int\limits_M \prec u, \nabla^*_\mu v \succ_{T^{p,q}M} d\mu \tag{2.2 }$$

where u,v are smooth finite tensor fields on M and $\nabla^*_\mu v$ is the next expression in local coordinates

$$(\nabla^*_\mu v)^{i_1..i_p}_{j_1..j_q} = -g^{kj_0}\nabla_k v^{i_1..i_p}_{j_0 j_1..j_q} + g^{kj_0}(\nabla_k \Phi) v^{i_1..i_p}_{j_0 j_1..j_q}$$

The Dirichlet form [1, 28, 4, 5] of measure μ with coefficient operator A is defined by

$$a^A_\mu(u, v) = \frac{1}{2}\int\limits_M \prec A(x)\nabla u(x), \nabla v(x) \succ d\mu(x) =$$
$$= \frac{1}{2}\int\limits_M A^{ij}(x)\frac{\partial u(x)}{\partial x^i}\overline{\frac{\partial v(x)}{\partial x^j}} d\mu(x)$$

on smooth functions with compact support $u, v \in C_0^\infty(M)$. Here coefficient operator A is a continuous $T^{1,1}M$ - valued field on M such that $\forall x \in M:\ A(x) > 0$ on $T^{1,0}M_x$.

Due to the integration by parts formula (2.2) the corresponding Dirichlet operator H^A_μ, defined by identity

$$(H^A_\mu u, v)_{L_2(\mu)} = a^A_\mu(u, v)$$

on $u, v \in C_0^\infty(M)$, admits representation

$$H^A_\mu u = \frac{1}{2}\nabla^*_\mu A \nabla u$$

The next lemma provides simple conditions when this operator is essentially self-adjoint in $L_2(M, \mu)$ with domain $C_0^\infty(M)$.

Lemma 2.1. *Suppose that density $\Phi \in C^3(M)$ and coefficient operator $A \in C^3 \cap C_b(M, T^{1,1}M)$. Then operator H_μ^A is essentially self-adjoint in $L_2(M,\mu)$.*

Proof is obtained as a direct application of Wienholtz scheme [53] in the Simader form [47] and is contained in Appendix B to the paper.

The further investigations are based on the powerful technique of energy identities for the elliptic differential operators (see, for example, monographies [9, 17, 10, 16] and papers [19, 35, 18, 36]).

Below we need coercitive identity for operator ∇_μ^* [18, 16] acting on smooth vector fields $v \in C_0^1(M, T^{1,0}M)$

$$\int_M |\nabla_\mu^* v|^2 d\mu = \int_M tr[\nabla v(\nabla v)^*] d\mu + \\ + \int_M \prec R_\mu v, v \succ d\mu \tag{2.3 }$$

Here $tr[\nabla v(\nabla v)^*] = g^{ij} g^{k\ell} \nabla_i v_\ell \nabla_j \overline{v}_k$ and

$$R_\mu(x) = Ric(x) + \nabla\nabla\Phi(x) \tag{2.4 }$$

where Ric denotes the Ricci curvature tensor on M. This identity simply follows from (2.2) and $[\nabla, \nabla_\mu^*]v = R_\mu v$ on vector field v.

The next theorem provides conditions on the weighted spectral gap. Its proof is based on the modification of Bochner' scheme for Laplace-Beltrami operator [12], [11, Ch.1, §J], [7].

Theorem 2.2. *Let probability measure μ (2.1) have density $\Phi \in C^3(M)$. Suppose that measure μ satisfies property*

$$\exists \varepsilon > 0 \ \forall x \in M \ \ R_\mu(x) \geq \varepsilon \ on \ T_x^{1,0}M$$

Then we have inequality

$$\frac{1}{2} \int_M |f - \int_M f \, d\mu|^2 d\mu \leq \int_M \prec R_\mu^{-1} \nabla f, \nabla f \succ d\mu \tag{2.5 }$$

for $f \in C_0^\infty(M)$.

Proof. Let coefficient operator $A \in C_b \cap C^3(M, T^{1,1}M)$ be such that

$$\forall \, x \in M \quad R_\mu^{-1}(x) \leq A(x)$$

1) We use here variant of coercitive identity (2.3) for $v = A\nabla u$, $u \in C_0^\infty(M)$

$$4 \int_M |(H_\mu^A - \alpha)u|^2 d\mu = \int_M |\nabla_\mu^* v|^2 d\mu - \\ -4\alpha \int_M \prec A\nabla u, \nabla u \succ d\mu + 4\alpha^2 \|u\|_{L_2(\mu)}^2 = \\ = \int_M tr(\nabla v[\nabla v]^*) d\mu + \int_M \prec (AR_\mu A - 4\alpha A)\nabla u, \nabla u \succ d\mu + \\ +4\alpha^2 \|u\|_{L_2(\mu)}^2 \geq 4\alpha^2 \|u\|_{L_2(\mu)}^2$$

when $\alpha \in (0, 1/4)$. Therefore $\alpha \in (0, 1/4)$ is a regular point for operator H^A_μ and essential self-adjointness of H^A_μ implies that there is no spectrum of H^A_μ on interval $(0, 1/4)$.
2) Consider function $u \in L_2(M, \mu)$ such that $H^A_\mu u = 0$. Then $u \in \mathcal{D}(\,(H^A_\mu)^{1/2})$ and we have that $\int\limits_M \prec A\nabla u, \nabla u \succ d\mu = 0$. From $A > 0$ follows $\prec A\nabla u, \nabla u \succ = 0 (mod\ \mu)$ and condition $\Phi \in C^3(M)$ implies that $|\nabla u| \equiv 0$. So $u \equiv const$ and we have that

$$H^A_\mu \geq 1/4 \tag{2.6 }$$

on the orthocomplement to constant in $L_2(M, \mu)$.

Inequality (2.6) written in the terms of quadratic forms is

$$\frac{1}{2} \int\limits_M |f - \int\limits_M f\, d\mu|^2 d\mu \leq \int\limits_M \prec A\nabla f, \nabla f \succ d\mu \tag{2.7 }$$

From arbitrarity of A such that $A \geq R^{-1}_\mu$, taking the infimum on A in (2.7) we obtain inequality (2.5). ■

Remark. *We note that in the case of function $\Phi \in C^5(M)$ and R_μ strictly positive measure μ we have $R^{-1}_\mu \in C^3(M)$ and so we can directly take $A = R^{-1}_\mu$ in the Theorem 2.2.*

3 Weighted Log-Sobolev inequalities and Orlicz space estimates.

Here we provide set of weighted estimates on Orlicz space norms of function by the special weighted norm of its gradient. In particular case when $A = Id_x$ Theorem 3.1 gives the result of Bakry-Emery [8].

Theorem 3.1 *Let probability measure μ (2.1) have density $\Phi \in C^3(M)$. Suppose that the coefficient operator $A \in C^3 \cap C_b(M, T^{1,1}M)$, $A > 0$ poinwisely, and*

$$\forall\, x \in M \quad AR_\mu A + H^A_\mu A \geq \lambda A \tag{3.1 }$$

with constant $\lambda > 0$.

Then for all $\Psi \in C^4([0, \infty)\,)$ such that $\Psi'' > 0$ and $(1/\Psi'')'' \leq 0$ we have

$$\begin{aligned} &\int\limits_M \Psi(f) d\mu - \Psi(\int\limits_M f\, d\mu) \leq \\ &\leq \tfrac{1}{2\lambda} \int\limits_M \Psi''(f) \prec A\nabla f, \nabla f \succ d\mu \end{aligned} \tag{3.2 }$$

for $f \in C^2_0(M)$, $f > 0$.

Proof. We partially follow the scheme of [7, 6] in 1.-4.
1. First we show correctness of all expressions appearing in the proof.

Lemma 2.1 gives self-adjointness $H^A_{\mu,min} = (H^A_{\mu,min})^*$ for the closure $H^A_{\mu,min}$ of operator H^A_μ.

In the next modification of coercitive identity (2.3)

$$4\int_M |H_\mu^A u|^2 d\mu = \int_M tr(\nabla(A\nabla u)[\nabla(A\nabla u)]^*)d\mu +$$
$$+\int_M \prec AR_\mu A\nabla u, \nabla u \succ d\mu =$$
$$= \frac{1}{2}\int_M tr(\nabla(A\nabla u)[\nabla(A\nabla u)]^*)d\mu +$$
$$+\frac{1}{2}\int_M \prec (A\otimes A)\nabla\nabla u, \nabla\nabla u \succ d\mu +$$
$$+\frac{1}{2}\int_M tr\{(\nabla.A^{\cdot i})\nabla_i u[(\nabla.A^{\cdot j})\nabla_j u]^*\}d\mu + \tag{3.3 }$$
$$+\frac{1}{2}\int_M \prec (A\nabla)_1 A^{23}, \nabla_1(\nabla_2 u\otimes\nabla_3 u) \succ d\mu +$$
$$+\int_M \prec AR_\mu A\nabla u, \nabla u \succ d\mu$$

we can integrate by parts in term with ∇_1 (see (2.2))

$$\frac{1}{2}\int_M \prec (A\nabla)_1 A^{23}, \nabla_1(\nabla_2 u\otimes\nabla_3 u) \succ d\mu =$$
$$= \int_M \prec (H_\mu^A A)\nabla u, \nabla u \succ d\mu$$

This leads to estimate

$$4\int_M |H_\mu^A u|^2 d\mu \geq \int_M \prec (AR_\mu A + H_\mu^A A)\nabla u, \nabla u \succ d\mu \tag{3.4 }$$

and condition (3.1) imply like in Theorem 2.2 the presence of spectral gap

$$H_\mu^A \geq \lambda/4 \tag{3.5 }$$

on the orthocomplement to constant in $L_2(M,\mu)$.

Here and below for the sake of simplicity we adopt the next convention: the coinciding integer numbers at tensor fields and ∇ - symbols show the coordinates on which summation runs. For example

$$\prec \nabla_2(A\nabla u)_1, \nabla_1(A\nabla)_2 \succ = tr(\nabla(A\nabla u)[\nabla(A\nabla u)]^*) =$$

$$= \nabla_i(A^{jk}\nabla_k u)\cdot\nabla_j(A^{i\ell}\nabla_\ell)$$

$$\prec (\nabla_1 A_{23})\nabla_3 u, (\nabla_2 A_{14})\nabla_4 u \succ =$$

$$= tr\{(\nabla.A^{\cdot i})\nabla_i u[(\nabla.A^{\cdot j})\nabla_j u]^*\} = (\nabla_i A^{jk})\nabla_k u\cdot(\nabla_j A^{i\ell})\nabla_\ell u$$

$$\prec A_{13}, \prec A\nabla u, \nabla \succ (\nabla_1 u\otimes\nabla_3 u) \succ =$$

$$= A_{13}A_{24}\nabla_4 u\cdot\nabla_2(\nabla_1 u\cdot\nabla_3 u)$$

As a consequence of (3.3) and (3.4) we have characterization of domain of operator $H_{\mu,min}^A$:

$f \in \mathcal{D}(H^A_{\mu,min}) \quad \Leftrightarrow \quad f \in L_2(M,\mu)$ is twice weakly differentiable and the next integrals are finite:

$$\begin{aligned}
&\int_M \prec (A\otimes A)\nabla\nabla f, \nabla\nabla f \succ d\mu < \infty \\
&\int_M tr(\nabla(A\nabla f)[\nabla(A\nabla f)]^*) d\mu < \infty \\
&\int_M tr\{(\nabla . A^{\cdot i})\nabla_i f[(\nabla . A^{\cdot j})\nabla_j f]^*\} d\mu < \infty \\
&\int_M \prec (AR_\mu A + H^A_\mu A + A)\nabla f, \nabla f \succ d\mu < \infty
\end{aligned} \tag{3.6 }$$

For function $u \in C_0^2(M)$ we have that

$$P_t u = \exp(-tH^A_{\mu,min})u \in \mathcal{D}(\,(H^A_{\mu,min})^2)$$

because $H^A_\mu u \in L_2(M,\mu)$ and

$$H^A_{\mu,min} P_t u = P_t H^A_\mu u \in \mathcal{D}(H^A_{\mu,min})$$

This implies that integrals (3.6) are finite for $f = P_t u$ or $f = H^A_{\mu,min} P_t u$.

As $H^A_{\mu,min}$ coincide with Friedrichtz' extension of operator H^A_μ we obtain that semigroup P_t preserves positivity [28], [5], i.e.

$$\forall v \in C_0^2(M),\ v \geq 0 \ \Rightarrow\ P_t v \geq 0 \ \ pointwisely$$

This implies that P_t is $L_p(M,\mu)$ – contractive, $p \in [1,\infty]$ [43, Ch.X, Th.X.55] and for $\Psi \in C([0,\infty)\,)$ we can write the next estimate

$$\begin{aligned}
&\|\Psi(P_t u)\|_p \leq ess \sup_{x\in M} |\Psi(P_t u)(x)| \leq \\
&\leq \max_{x\in M} |\Psi(u)(x)| \leq \max_{|t|\leq \|u\|_{C_b(M)}} |\Psi(t)|
\end{aligned} \tag{3.7 }$$

for $u \in C_0^2(M)$, $p \in [1,\infty]$.

2. For every $\Psi \in C^2(\mathcal{R}_+)$ and $f \in C_0^2(M)$ consider function $g(t) = \int_M \Psi(P_t f) d\mu$. Then

$$\begin{aligned}
&\int_M \Psi(f) d\mu - \Psi(\int_M f\ d\mu) = -g(t)\ |_0^\infty = \\
&= -\int_0^\infty g'(t) dt = \int_0^\infty \int_M \Psi'(P_t f) H^A_\mu P_t f\ d\mu\, dt = \\
&= \frac{1}{2}\int_0^\infty \int_M \Psi''(P_t f) \prec A\nabla P_t f, \nabla P_t f \succ d\mu\, dt
\end{aligned} \tag{3.8 }$$

The limits $g(t)\ |_0^\infty$ we substitute using the Lagrange formula for function Ψ on $[0,\infty)$. For example, for $t = \infty$ we have

$$\begin{aligned}
&|\int_M \Psi(P_t f) d\mu - \Psi(\int_M f\, d\mu)| = \\
&= |\int_M \Psi'(\theta)\{P_t f - \int_M f\, d\mu\} d\mu| \leq
\end{aligned} \tag{3.9 }$$

$$\leq (\int_M |\Psi'(\theta)|^2 d\mu)^{1/2} \ \|P_t f - \int_M f\, d\mu\|_{L_2(\mu)}$$

Here $\theta = \theta(x) \geq 0$ is some point between $(P_t f)(x)$ and $\int_M f\, d\mu$, $x \in M$.

The first integral in (3.9) is simply estimated by

$$(\int_M |\Psi'(\theta)|^2 d\mu)^{1/2} \leq \sup_{|t|\leq \|f\|_{C_b(M)}} |\Psi'(t)|$$

like in (3.7) and the second one tends to zero at $t \to \infty$ due to the next estimate

$$\|P_t f - \int_M f\, d\mu\|^2_{L_2(\mu)} \leq e^{-\lambda t/2} \|f - \int_M f\, d\mu\|^2_{L_2(\mu)}$$

which follows from the spectral gap (3.5).

The limit $\lim\limits_{t\to 0+}$ we substitute using the strong continuity of semigroup P_t, $t > 0$:

$$\forall f \in L_2(M,\mu) \ \ \|P_t f - f\|_{L_2(\mu)} \to 0, \ t \to 0+$$

3. We only have to prove inequality (3.2) for strictly positive functions

$$f_\varepsilon = f + \varepsilon, \ f \in C_0^2(M), \ \varepsilon > 0$$

Indeed, tending $\varepsilon \to 0$ in inequality

$$\int_M \Psi(f+\varepsilon) d\mu - \Psi(\int_M f\, d\mu + \varepsilon) \leq$$

$$\leq \frac{1}{2\lambda} \int_M \Psi''(f+\varepsilon) \prec A\nabla(f+\varepsilon), \nabla(f+\varepsilon) \succ d\mu = \tag{3.10 }$$

$$= \frac{1}{2\lambda} \int_M \Psi''(f+\varepsilon) \prec A\nabla f, \nabla f \succ d\mu$$

for $f \in C_0^2(M)$, we obtain the weighted inequality (3.2) as an application of Lebesgue domination theorem to (3.10).

4. The representation (3.8) enables us to rewrite inequality (3.2) as

$$\int_0^\infty \int_M \Psi''(P_t f) \prec A\nabla P_t f, \nabla P_t f \succ d\mu dt \leq$$

$$\leq \frac{1}{\lambda} \int_M \Psi''(f) \prec A\nabla f, \nabla f \succ d\mu$$

Therefore it is sufficient to prove that

$$0 \leq \int_M \Psi''(P_t f) \prec A\nabla P_t f, \nabla P_t f \succ d\mu \leq$$

$$\leq e^{-\lambda t} \int_M \Psi''(f) \prec A\nabla f, \nabla f \succ d\mu$$

or that

$$K'(t) \leq -\lambda K(t) \tag{3.11 }$$

for function

$$K(t) = \int_M \Psi''(P_t f) \prec A\nabla P_t f, \nabla P_t f \succ d\mu$$

Introduce notations $\psi = \Psi''$, $P_t f(x) = h(u(x))$ with some function h on line $\mathcal{R}^1$ which we will choose later. Then inequality (3.11) adopts the form

$$\lambda \int\limits_M \psi(h(u))[h'(u)]^2 \prec A\nabla u, \nabla u \succ d\mu \leq$$
$$\leq 2 \int\limits_M \psi(h(u)) \prec A\nabla h(u), \nabla H^A_\mu h(u) \succ d\mu +$$
$$+ \int\limits_M \psi'(h(u))[h'(u)]^2 H^A_\mu h(u) \prec A\nabla u, \nabla u \succ d\mu$$

Using identities

$$H^A_\mu h(u) = h' H^A_\mu u - \frac{h''}{2} \prec A\nabla u, \nabla u \succ$$

and

$$[\nabla, \nabla^*_\mu]v = R_\mu v, \quad v \in C^1(M, T^{1,0}M)$$

we have

$$\lambda \int\limits_M \psi(h)[h']^2 \prec A\nabla u, \nabla u \succ d\mu \leq$$
$$\leq 2 \int\limits_M \psi(h)[h']^2 \prec AR_\mu A\nabla u, \nabla u \succ d\mu +$$
$$+ \int\limits_M \prec \nabla_2(\psi h'(A\nabla u)_1), \nabla_1(h'(A\nabla u)_2) \succ d\mu + \tag{3.12 }$$
$$+ \int\limits_M \psi'[h']^3 H^A_\mu u \prec A\nabla u, \nabla u \succ d\mu -$$
$$- \int\limits_M \frac{\psi'[h']^2 h''}{2} | \prec A\nabla u, \nabla u \succ |^2 d\mu$$

5.The second term on the l.h.s. of (3.12) could be transformed to

$$\int\limits_M \prec \nabla_2(\psi h'(A\nabla u)_1), \nabla_1(h'(A\nabla u)_2) \succ d\mu =$$
$$= \int\limits_M \psi[h']^2 \prec \nabla_2(A\nabla u)_1, \nabla_1(A\nabla u)_2 \succ d\mu +$$
$$+ \int\limits_M [\psi h']' h'' | \prec A\nabla u, \nabla u \succ |^2 d\mu +$$
$$+ \int\limits_M \{[\psi h']' h' + [\psi h'] h''\} \prec \nabla_2(A\nabla u)_1, \nabla_1 u \otimes (A\nabla u)_2 \succ d\mu =$$
$$= \frac{1}{2} \int\limits_M \psi[h']^2 \prec \nabla_2(A\nabla u)_1, \nabla_1(A\nabla u)_2 \succ d\mu +$$
$$+ \frac{1}{2} \int\limits_M \psi[h']^2 \prec (A_{13} \otimes A_{24}) \nabla_1 \nabla_2 u, \nabla_3 \nabla_4 u \succ d\mu +$$
$$+ \frac{1}{2} \int\limits_M \psi[h']^2 \prec (\nabla_2 A_{13}) \nabla_3 u, (\nabla_1 A_{24}) \nabla_4 u \succ d\mu +$$
$$+ \int\limits_M [\psi h']' h'' | \prec A\nabla u, \nabla u \succ |^2 d\mu + B_1 + B_2 \tag{3.13 }$$

In (3.13) terms B_1 and B_2 are equal to

$$B_1 = \int\limits_M \psi[h']^2 \prec (\nabla_2 A_{13})\nabla_3 u, A_{24}\nabla_1\nabla_4 u \succ d\mu =$$

$$= \frac{1}{2}\int\limits_M \psi[h']^2 \prec A_{24}(\nabla_2 A_{13}), \nabla_4(\nabla_3 u \otimes \nabla_1 u) \succ d\mu =$$

$$= \int\limits_M \psi[h']^2 \prec (H^A_\mu A)\nabla u, \nabla u \succ d\mu -$$

$$-\frac{1}{2}\int\limits_M (\psi[h']^2)' \prec \nabla_2 A_{13}, A_{24}\nabla_4 u \otimes \nabla_3 u \otimes \nabla_1 u \succ d\mu$$

$$B_2 = \int\limits_M (\psi[h']^2)' \prec \nabla_2(A\nabla u)_1, \nabla_1 u \otimes (A\nabla u)_2 \succ d\mu =$$

$$= \int\limits_M (\psi[h']^2)' \prec \nabla_2 A_{13} \otimes (A\nabla u)_2, \nabla_1 u \otimes \nabla_3 u \succ d\mu +$$

$$+\frac{1}{2}\int\limits_M (\psi[h']^2)' \prec A_{13}, \prec A\nabla u, \nabla u \succ (\nabla_1 u \otimes \nabla_3 u) \succ d\mu =$$

$$= \frac{1}{2}\int\limits_M (\psi[h']^2)''| \prec A\nabla u, \nabla u \succ |^2 d\mu +$$

$$+ \int\limits_M (\psi[h']^2)' H^A_\mu u \prec A\nabla u, \nabla u \succ d\mu +$$

$$+\frac{1}{2}\int\limits_M (\psi[h']^2)' \prec \nabla_2 A_{13} \otimes (A\nabla u)_2, \nabla_1 u \otimes \nabla_3 u \succ d\mu$$

Arranging all terms we transform estimate (3.12) to

$$\lambda \int\limits_M \psi(h)[h']^2 \prec A\nabla u, \nabla u \succ d\mu \le$$

$$\le \int\limits_M \psi[h']^2 \prec \{AR_\mu A + H^A_\mu A\}\nabla u, \nabla u \succ d\mu +$$

$$+\frac{1}{2}\int\limits_M \psi[h']^2 \prec \nabla_2(A\nabla u)_1, \nabla_1(A\nabla u)_2 \succ d\mu +$$

$$+\frac{1}{2}\int\limits_M \psi[h']^2 \prec (A\otimes A)\nabla\nabla u, \nabla\nabla u \succ d\mu +$$

$$+\frac{1}{2}\int\limits_M \psi[h']^2 \prec (\nabla_2 A_{13})\nabla_3 u, (\nabla_1 A_{24})\nabla_4 u \succ d\mu +$$

$$+ \int\limits_M \{\psi'[h']^3 + (\psi[h']^2)'\} H^A_\mu u \prec A\nabla u, \nabla u \succ d\mu +$$

$$+ \int\limits_M \{-\tfrac{1}{2}\psi'[h']^2 h'' + (\psi h')' h'' - \tfrac{1}{2}(\psi[h']^2)''\}| \prec A\nabla u, \nabla u \succ |^2 d\mu$$

6. We ask the next restriction on the dependence between functions ψ and h

$$\psi'[h']^3 + (\psi[h']^2)' = 2h'(\psi h')' = 0$$

i.e. $\psi h' = C$ for some constant $C \in \mathcal{R}^1\backslash\{0\}$.

Then the derivatives of h could be expressed in the terms of ψ

$$h'' = (C/\psi(h))' = -C\psi'_h h'/\psi^2 = -C^2\psi'_h/\psi^3$$

$$h''' = -C^2\psi''_h h'/\psi^3 + 3C^2(\psi'_h)^2 h'/\psi^4 =$$

$$= C^2\{-\psi''_h/\psi^4 + 3(\psi'_h)^2/\psi^5\}$$

Therefore we have the next estimate

$$-\frac{1}{2}\psi'[h']^2 h'' + (\psi h')' h'' - \frac{1}{2}(\psi[h']^2)'' =$$

$$= \frac{1}{2}C^4\{+(\psi'_h)^2/\psi^5 + \psi''_h/\psi^4 - 3(\psi'_h)^2/\psi^5\} =$$

$$= \frac{1}{2\psi^2}C^4\{(\psi\psi''_h - 2(\psi'_h)^2)/\psi^3\} = \frac{1}{2\psi^2}C^4\{-1/\psi\}''_h \geq 0$$

due to the assumptions of Theorem 3.1.

7. Finally the estimate (3.11) adopts the form

$$\lambda \int_M \psi[h']^2 \prec A\nabla u, \nabla u \succ d\mu \leq$$

$$\leq \int_M \psi[h']^2 \prec \{AR_\mu A + H^A_\mu A\}\nabla u, \nabla u \succ d\mu +$$

$$+\frac{1}{2}\int_M \psi[h']^2 \prec \nabla_2(A\nabla u)_1, \nabla_1(A\nabla u)_2 \succ d\mu +$$

$$+\frac{1}{2}\int_M \psi[h']^2 \prec (A\otimes A)\nabla\nabla u, \nabla\nabla u \succ d\mu +$$

$$+\frac{1}{2}\int_M \psi[h']^2 tr_{12}\{\prec (\nabla_2 A_{13})\nabla_3 u, [(\nabla_1 A_{24})\nabla_4 u]^* \succ\} d\mu +$$

$$+\frac{1}{2}C^4\int_M \{\,-(1/\psi)''/\psi^2\}| \prec A\nabla u, \nabla u \succ |^2 d\mu$$

As four last terms above are nonnegative condition (3.1) gives the statement of Theorem 3.1. ■

Remark 3.2. *Condition* (3.1) *of Theorem* 3.1 *applied to* $\Psi(x) = x^2$ *improves the spectral gap inequality* (2.7) *to*

$$\int_M |f - \int_M f\, d\mu|^2 d\mu \leq \frac{1}{\lambda}\int_M \prec A\nabla f, \nabla f \succ d\mu \qquad (3.14\ )$$

Note that in the case of autoparallel field $A(x)$ (i.e. $\nabla A(x) = 0$ for all $x \in M$) condition (3.1) transforms to

$$A^{1/2}R_\mu A^{1/2} \geq \lambda > 0$$

In particular case when $A = Id_x$ on T_xM, $x \in M$ we have Bakry-Emery condition $R_\mu \geq \lambda > 0$ for the spectral gap [7]

$$H_\mu = H^{Id}_\mu = \frac{1}{2}\nabla^*_\mu\nabla \geq \lambda/2$$

on the orthocomplement to constant in $L_2(M, \mu)$.

Remark 3.3. *Theorem* 3.1 *applied with* $\Psi(x) = x\ln x$ *and* $f = u^2$ *gives the weighted Logarithmic Sobolev inequality*

$$\int\limits_M u^2 \ln u^2 d\mu - \|u\|_{L_2}^2 \ln \|u\|_{L_2}^2 \leq$$
$$\leq \tfrac{2}{\lambda} \int\limits_M \prec A\nabla u, \nabla u \succ d\mu \tag{3.15 }$$

For the sake of convenience we give self-contained and more simple proof of this fact in Appendix A for $A(x) = S(x)Id_x$, $S \in C^3 \cap C_b(M)$.

In the situation when $A = Id_x$ we have condition (3.1) in form [8]

$$R_\mu \geq \lambda$$

and the classic Gross' Log-Sobolev inequality [29]

$$\int\limits_M u^2 \ln u^2 d\mu - \|u\|_{L_2}^2 \ln \|u\|_{L_2}^2 \leq \frac{2}{\lambda} \int\limits_M |\nabla u|^2 d\mu \tag{3.16 }$$

4 Examples and applications.

Below we investigate situations when we can find at given R_μ-object the coefficient operator A in (3.15) such that it behaves like R_μ^{-1} and satisfies condition (3.1) of Theorem 3.1.

In the case when R_μ - object grows at the infinity of manifold M, such choice of coefficient operator A enables us to improve inequality (3.16) to the weighted Log-Sobolev inequality (3.15). The similar problem has already been solved in Theorem 2.2, inequality (2.5).

Example 4.1. We consider situation when $A = R_\mu^{-1}$.

Let $M = \mathcal{R}^1$ and density $\Phi(x) = ch(x)$ in (2.1). Then $R_\mu(x) = ch(x)$ and if we put $A(x) = 1/ch(x)$ the condition (3.1) would have the form

$$\frac{1}{ch(x)} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{ch(x)}\frac{\partial}{\partial x}\left(\frac{1}{ch(x)}\right)\right) +$$
$$+\frac{1}{2}\frac{sh(x)}{ch(x)}\frac{\partial}{\partial x}\left(\frac{1}{ch(x)}\right) \geq \lambda/ch(x)$$

with constant $\lambda = \frac{1}{2} - (38\sqrt{19} - 56)/2 \cdot 3^5 \approx 0.2744$.

Therefore for measure

$$d\nu(x) = \exp(-ch(x)\,)dx/(\int_{-\infty}^{\infty} \exp(-ch(x)\,)dx)$$

we have obtained inequality

$$\int_{-\infty}^{\infty} |f|^2 \ln |f|^2 d\nu - \|f\|^2 \ln \|f\|^2 \leq C \int_{-\infty}^{\infty} \frac{|\nabla f(x)|^2}{ch(x)} d\nu(x)$$

with constant $C \approx 7.2884$.

Example 4.2 Here we show that condition (3.1) enables to obtain classic Log-Sobolev inequalities (3.16) even in the case when R_μ-object is negative at some points (4.2), (4.3).

Consider $M = \mathcal{R}^1$ and a family of measures μ_a (2.1) with densities

$$\Phi(x) = ax^2 + x^4, \ a > 0$$

Then $R_\mu = \Phi'' = 2a + 12x^2$ and choosing

$$A = 1/(1 + x^2)$$

we have condition (3.1) in the form

$$\begin{aligned} AR_\mu A + H_\mu^A A &= \frac{2a + 12x^2}{(x^2+1)^2} - \frac{1}{2}\frac{\partial}{\partial x}(\frac{1}{x^2+1}\frac{\partial}{\partial x}(\frac{1}{x^2+1}))+ \\ &+\frac{1}{2}\frac{2ax + 4x^3}{x^2+1}\frac{\partial}{\partial x}(\frac{1}{x^2+1}) = \frac{2a}{(x^2+1)^2} + \frac{12}{x^2+1} - \\ &-\frac{12}{(x^2+1)^2} + C_1 + C_2 \geq \lambda/(x^2+1) \end{aligned} \tag{4.1 }$$

where

$$C_1 = \frac{\partial}{\partial x}(\frac{x}{(x^2+1)^3}) = -\frac{5}{(x^2+1)^3} + \frac{6}{(x^2+1)^4}$$

and

$$\begin{aligned} C_2 = -\frac{2ax^2 + 4x^4}{(x^2+1)^3} &= -\frac{2a}{(x^2+1)^2} + \frac{2a}{(x^2+1)^3} - \\ &-\frac{4}{(x^2+1)} + \frac{8}{(x^2+1)^2} - \frac{4}{(x^2+1)^3} \end{aligned}$$

Condition (4.1) adopts the next form for $y = 1/(x^2+1) \in [0,1]$ and $x \in \mathcal{R}^1$

$$f(1,y) = \{6y^3 - 9y^2 - 4y + 8\} + 2ay^2 - \lambda \geq 0$$

But the polynomial in brakets is semibounded from below on $y \in [0,1]$

$$\inf_{y\in[0,1]}(6y^3 - 9y^2 - 4y + 8) = 1$$

Therefore (4.1) follows from the next condition

$$1 + 2ay^2 - \lambda \geq 0, \quad y \in [0,1]$$

If $a \geq 0$ then we can choose $\lambda \equiv 1$, for $a \in (-1/2, 0]$ constant λ equals to $(1+2a)$.

So for all $a \in (-1/2, \infty)$ we have obtained the next family of weighted Log-Sobolev inequalities

$$\begin{aligned} \int_{-\infty}^{\infty} |f|^2 \ln |f|^2 d\mu_a - \|f\|^2_{L_2(\mu_a)} \ln \|f\|^2_{L_2(\mu_a)} \leq \\ \leq \frac{2}{\min(1, 1+2a)} \int_{-\infty}^{\infty} \frac{|\nabla f|^2}{1+x^2} d\mu_a(x) \end{aligned} \tag{4.2 }$$

Note that inequality (4.2) is proved even in the case when R_μ-object is negative in region $\{x \, : \, |x| < \sqrt{-a/6}\}$ for $a \in (-1/2, 0)$.

As $1/(1+x^2) \le 1$, $x \in \mathcal{R}^1$ weighted inequality (4.2) implies the classic Log-Sobolev inequality

$$\int_{-\infty}^{\infty} |f|^2 \ln |f|^2 d\mu_a - \|f\|^2_{L_2(\mu_a)} \ln \|f\|^2_{L_2(\mu_a)} \le$$

$$\le \frac{2}{\min(1, 1+2a)} \int_{-\infty}^{\infty} |\nabla f|^2 d\mu_a \tag{4.3 }$$

for measures μ_a, $a \in (-1/2, 0]$ when Bakry-Emery criterion doesn't work directly.

The next theorem gives weighted Log-Sobolev inequalities for logarithmically concave polynomial measures on $\mathcal{R}^1$.

Remark that tending $b \to \infty$ in (4.4) we obtain classic Log-Sobolev inequality (3.16) as a limit of weighted inequalities (4.4), (3.15).

Theorem 4.3. *Consider probability measure* μ (2.1) *on manifold* $M = \mathcal{R}^1$.

$$d\mu = \frac{1}{Z} \exp(-P(x)\,)\; dx$$

where

$$P(x) = a_0 x^{2n+2} + ... + a_{2n+2}$$

is polynomial with $n \ge 1$ *and* $a_0 > 0$.

Suppose that

$$\exists \varepsilon > 0 \ \forall x \in \mathcal{R}^1 \quad P''(x) \ge \varepsilon$$

Then $\exists b_0 = b_0(P)$ *such that* $\forall b \ge b_0$

$$\int_{\mathcal{R}} |f|^2 \ln |f|^2 d\mu - \int_{\mathcal{R}} |f|^2 d\mu \ \ln(\int_{\mathcal{R}} |f|^2 d\mu) \le$$

$$\le \frac{2}{\varepsilon - \delta(b)} \int_{\mathcal{R}} \frac{\varepsilon + b}{P'' + b} |\nabla f|^2 d\mu \tag{4.4 }$$

for all $f \in C_0^2(\mathcal{R}^1)$ *where*

$$\forall b \ge b_0 \ |\delta(b)| < \varepsilon \quad \& \quad \lim_{b \to \infty} \delta(b) = 0$$

Proof. Note that for $A(x) = \dfrac{1}{P'' + b}$, $b > 0$

$$AR_\mu A + H^A_\mu A = \frac{P''}{(P''+b)^2} + \frac{1}{2} \nabla^*_\mu (\frac{1}{P''+b} \nabla (\frac{1}{P''+b})\,)$$

Therefore condition (3.1) has form

$$\frac{P''}{P''+b} - \frac{P''+b}{2} \nabla^*_\mu (\frac{P'''}{(P''+b)^3}) = \frac{P''}{P''+b} +$$

$$+ \frac{1}{2} \frac{P^{IV}}{(P''+b)^2} - \frac{3}{2} \frac{(P''')^2}{(P''+b)^3} - \frac{1}{2} \frac{P' P'''}{(P''+b)^2} \ge \lambda \tag{4.5 }$$

As $P'P''' - \dfrac{2n}{2n+1}(P'')^2$ is a polynomial of power $(4n-1)$ then there are constants $D, C > 0$ and $\beta = \dfrac{4n-1}{2n} = 2 - \dfrac{1}{2n}$ such that

$$\forall x \in \mathcal{R}^1 \quad |P'P''' - \frac{2n}{2n+1}(P'')^2|(x) \leq D(P''(x) + C)^\beta \tag{4.6 }$$

Now we add and substract term $\dfrac{2n}{2n+1}(P'')^2$ in the nominator of last fraction in (4.5)

$$\frac{P''}{P''+b} - \frac{n}{2n+1}\frac{(P'')^2}{(P''+b)^2} - \frac{1}{2}\frac{P'P''' - \frac{2n}{2n+1}(P'')^2}{(P''+b)^2} + \tag{4.7 }$$

$$+\frac{1}{2}\frac{P^{IV}}{(P''+b)^2} - \frac{3}{2}\frac{(P''')^2}{(P''+b)^3} - \lambda \geq 0 \tag{4.8 }$$

Using (4.6) and setting $y = \dfrac{b}{P''+b} \in [0, \dfrac{1}{(1+\varepsilon/b)}]$ we estimate terms in (4.7) from below by

$$(4.7) \geq 1 - y - \frac{n}{2n+1}\{1 - 2y + y^2\} - \tag{4.9 }$$
$$-\frac{D}{2b^{2-\beta}}y^{2-\beta}(1 - \frac{b-C}{b}y)^\beta = f(y)$$

Now we find the condition on b when minimum on interval $[0, \dfrac{1}{1+\varepsilon/b}]$ of function $f(y)$ is attained at point $y = \dfrac{1}{1+\varepsilon/b}$

$$f'(y) = -\frac{1}{2n+1} - \frac{2ny}{2n+1} - \frac{D(2-\beta)}{2b^{2-\beta}}y^{1-\beta}(1 - \frac{b-C}{b}y)^\beta +$$
$$+\frac{D\beta}{2b^{2-\beta}}\frac{b-C}{b}y^{2-\beta}(1 - \frac{b-C}{b}y)^{\beta-1} \tag{4.10 }$$

Due to $1 < \beta = 2 - 1/2n < 2$ we have that the last term in (4.10) uniformly on $y \in [0,1]$ tends to zero when $b \to \infty$.

Therefore using that first three terms in (4.10) are less than $-1/(2n+1)$ we have that

$$\exists b_0' \ \forall b \geq b_0' \ \forall y \in [0, \frac{1}{1+\varepsilon/b}] \subset [0,1] \quad f'(y) < 0$$

This leads to

$$(4.7) \geq \min_{y \in [0, \frac{1}{1+\varepsilon/b}]} f(y) = f(\frac{1}{1+\varepsilon/b}) =$$
$$= \frac{\varepsilon}{b+\varepsilon} - \frac{n\varepsilon^2}{(2n+1)(b+\varepsilon)^2} - \frac{D}{2}\frac{(\varepsilon+C)^\beta}{(b+\varepsilon)^2} \tag{4.11 }$$

for all $b \geq b_0'$.

Terms in (4.8) are estimated from above with usage of next lemma.

Lemma. Let Q be polynomial of power m and $0 < \gamma = \dfrac{m}{2n} < k$. Then under conditions of Theorem 4.3 $\exists b_0'' = b_0''(Q, P'', k)$ such that $\forall b \geq b_0''$

$$\sup_{x \in \mathcal{R}} \left| \frac{Q(x)}{(P''(x)+b)^k} \right| \leq D_1 \frac{\gamma^\gamma (k-\gamma)^{k-\gamma}}{k^k (b - C_1)^{k-\gamma}}$$

where constants C_1 and D_1 are such that

$$|Q(x)| \leq D_1 (P''(x) + C_1)^\gamma, \quad x \in \mathcal{R}^1$$

Proof. We have that

$$\sup_{x \in \mathcal{R}} \left| \frac{Q(x)}{(P''(x)+b)^k} \right| \leq \sup_{P'' \geq \varepsilon} F(P'')$$

with function

$$F(t) = D_1 \frac{(t + C_1)^\gamma}{(t+b)^k}$$

The maximum of function $F(t)$ on interval $t \in [\varepsilon, \infty)$ is attained at point $t_0 = \dfrac{\gamma b - k C_1}{k - \gamma}$

$$\max_{t \geq \varepsilon} F(t) = F(t_0) = D_1 \frac{\gamma^\gamma (k-\gamma)^{k-\gamma}}{k^k (b - C_1)^{k-\gamma}}$$

where t_0 should satisfy $t_0 \geq \varepsilon$ or

$$b \geq b_0'' = \frac{1}{\gamma}\{k C_1 + \frac{\varepsilon}{k-\gamma}\} \qquad \blacksquare$$

Note that in notations adopted in lemma above we have $k - \gamma = 2 - \dfrac{2n-2}{2n} = 1 + \dfrac{1}{n}$ for the first term in (4.8) and $k - \gamma = 3 - \dfrac{2(2n-1)}{2n} = 1 + \dfrac{1}{n}$ for the second term in (4.8). Application of this lemma imply $\exists b_0''' \; \exists D_2$ such that $\forall b \geq b_0'''$

$$\sup_{x \in \mathcal{R}} \left| \frac{1}{2} \frac{P^{IV}}{(P''+b)^2} - \frac{3}{2} \frac{(P''')^2}{(P''+b)^3} \right| \leq \frac{D_2}{(b+\varepsilon)^{1+1/n}} \tag{4.12 }$$

Therefore due to (4.11) and (4.12) condition (4.7 - 4.8) finally appears in the next form

$$\frac{\varepsilon}{b+\varepsilon} - \lambda - \frac{1}{b+\varepsilon} \delta(b) \geq 0$$

with

$$\delta(b) = \frac{D_2}{(b+\varepsilon)^{1/n}} + \frac{n\varepsilon^2}{(2n+1)(b+\varepsilon)} + \frac{D}{2} \frac{(\varepsilon + C)^\beta}{b+\varepsilon}$$

So we can choose $b_0 > \max(b_0', b_0''')$ such that $|\delta(b)| < \varepsilon$ for all $b \geq b_0$. Then choice

$$\lambda = \frac{\varepsilon - \delta(b)}{b + \varepsilon}$$

ensures condition (3.1) for $A = \dfrac{1}{P'' + b}$, $b \geq b_0$. $\blacksquare$

The next theorem gives the conditions on integrability of geodetic distance

$$\tilde{\rho}_A(x,x_0)=\inf\int_0^T\{(A^{-1})^{jk}(z)\frac{\partial z_j}{\partial\tau}\frac{\partial z_k}{\partial\tau}\}^{1/2}d\tau$$

where infimum is taken along smooth paths $\{z(\tau),\ \tau\in[0,T]\}$ such that $z(0)=x_0$ and $z(T)=x$. In the case $M=\mathcal{R}^n$, $\rho(x,y)=|x-y|$ the similar result for renormalized Schrödinger operators $-\Delta+V$ was obtained in [20].

For uniformly logarithmically concave measures on manifold M (i.e. $\exists\lambda>0$ $R_\mu(x)\geq\lambda Id_x$ pointwisely on $x\in M$) and for coefficient operator $A(x)=Id_x$ such result was obtained in [7] for Riemannian distance $\rho(x,y)$ (see also [6] for the rigged Hilbert space case).

Theorem 4.4. *Let probability measure* μ (2.1) *have density* $\Phi\in C^3(M)$. *Consider coefficient operator* $A\in C^3\cap C_b(M,T^{1,1}M)$, $A>0$ *pointwisely.*

1. If $AR_\mu A\geq\lambda A$, $\lambda>0$ *then*

$$\forall x_0\in M\quad\forall p\geq 1\quad\tilde{\rho}_A(\cdot,x_0)\in L_p(M,\mu)$$

2. If $AR_\mu A+H_\mu^A A\geq\lambda A$, $\lambda>0$ *then* $\forall x_0\in M$ $\tilde{\rho}_A(\cdot,x_0)\in L_p(M,\mu)$, $p\geq 1$ *and the next estimate holds for* $a\in(0,\lambda/2)$

$$\int\limits_M\exp(a\tilde{\rho}_A^2(x,x_0)\,)d\mu(x)\leq$$

$$\leq\exp\{\frac{a}{1-\frac{2a}{\lambda}}\int\limits_M\tilde{\rho}_A^2(x,x_0)d\mu(x)\}$$

Proof. 1. Inequality $AR_\mu A\geq\lambda A$ implies the spectral gap inequality (2.7)

$$\frac{\lambda}{2}\int\limits_M|f-\int\limits_M f\,d\mu|^2d\mu\leq\int\limits_M\prec A\nabla f,\nabla f\succ d\mu \tag{4.13 }$$

Consider increasing on $n\geq 1$ sequence of functions

$$f_n=\begin{cases}\tilde{\rho}_A(x,x_0),\ \tilde{\rho}_A(x,x_0)\leq n\\ n,\quad\tilde{\rho}_A(x,x_0)>n\end{cases} \tag{4.14 }$$

We have that $0<\int\limits_M f_n^2d\mu<\infty$ and

$$\lim_{n\to\infty}\|f_n\|_{L_2}^2=\int\limits_M\tilde{\rho}_A^2(x,x_0)d\mu(x)$$

Suppose for contradiction that $\lim\limits_{n\to\infty}\|f_n\|_{L_2}^2=\infty$. Due to the next estimate for $k=1$ (we use that $\prec A\nabla\tilde{\rho}_A,\nabla\tilde{\rho}_A\succ==1(mod\ \sigma)$)

$$\int\limits_M\prec A\nabla f_n^k,\nabla f_n^k\succ d\mu(x)=$$

$$=k^2\int\limits_{\tilde{\rho}(x,x_0)\leq n}\tilde{\rho}_A^{2k-2}(x,x_0)d\mu(x)\leq k^2\int\limits_M\tilde{\rho}_A^{2k-2}(x,x_0)d\mu(x)$$

we have that $\int\limits_M\prec A\nabla f_n,\nabla f_n\succ d\mu\leq 1$.

Put $g_n=f_n/\|f_n\|_{L_2}$ then

$$\int\limits_M\prec A\nabla g_n,\nabla g_n\succ d\mu\to 0,\quad n\to\infty$$

and spectral gap (4.13) implies that

$$\int_M |g_n - \int_M g_n d\mu|^2 d\mu = 1 - (\int_M g_n d\mu)^2 \to 0 \qquad (4.15\ )$$

when $n \to \infty$.

But $\|g_n\|_{L_2} = 1$ therefore $\exists g^*$ and $\{g_{n(i)}\}_{i\geq 1}$ such that $g_{n(i)} \to g^*$ weakly in $L_2(M, \mu)$.

Due to the uniform convergence of $g_{n(i)}$ to zero on sets $\{y : \tilde\rho_A(x,y) < C\}_{x\in M,\, C>0}$ we have $g^* \equiv 0$, which contradicts to (4.15). So we have obtained that

$$\int_M \tilde\rho_A^2(x, x_0) d\mu(x) < \infty$$

Repeating inductively on $k \geq 1$ the proof above for functions f_n^k we have statement 1.

2. Under condition $AR_\mu A + H_\mu^A A \geq \lambda A$ by Remark 3.2 we have spectral gap inequality (3.14) which gives

$$\tilde\rho_A(x, x_0) \in L_p(M, \mu),\ p \geq 1$$

like in the proof of statement 1 of this theorem. Moreover this condition gives the weighted Log-Sobolev inequality (3.15).

Consider the sequence of functions on half-line $a \in [0, \infty)$

$$h_n(a) = \int_M \exp(a f_n^2) d\mu \geq 1$$

increasing on both a and n with all derivatives $h_n^{(k)}(a) > 0,\ a > 0$. Here functions f_n are defined in (4.14). Then for functions $g_n = \exp(a f_n^2/2)$ we apply weighted Log-Sobolev inequality (3.15)

$$a h_n'(a) = \int_M a f_n^2 \exp(a f_n^2) d\mu = \int_M g_n^2 \ln g_n^2\, d\mu \leq$$

$$\leq \frac{2}{\lambda} \int_M \prec A\nabla g_n, \nabla g_n \succ d\mu + h_n(a) \ln h_n(a) \leq$$

$$\leq \frac{2}{\lambda} a^2 \int_M f_n^2 \exp(a f_n^2) d\mu + h_n(a) \ln h_n(a)$$

So we have family of functions $h_n(a)$ increasing on both n and $a \geq 0$, $h_n(0) = 1$ satisfying inequality

$$a(1 - \frac{2a}{\lambda}) h_n'(a) \leq h_n(a) \ln h_n(a)$$

For to find the major function we must set $h(0) = 1$ and take the highest growth of its derivative, so

$$a(1 - \frac{2a}{\lambda}) h'(a) = h(a) \ln h(a)$$

It its obvious that $h(a) = \exp(\frac{aD}{1 - \frac{2a}{\lambda}})$ for some D. The restriction on D we obtain from the highest growth of h_n at zero

$$h_n'(0) = \int_M f_n^2 d\mu \leq \int_M \tilde\rho_A^2(x, x_0) d\mu(x) = D < \infty$$

So we have achieved estimate

$$h_n(a) \le \exp\{\frac{a}{1-\frac{2a}{\lambda}} \int\limits_M \tilde{\rho}_A^2(x,x_0)d\mu(x)\}$$

and tending $n \to \infty$ we obtain theorem proved. ■

5 Gross - Föllmer scheme for decay of correlations and Weighted Spectral Gap.

Below we investigate the connection between the weighted spectral gap inequality (2.5) and the decay of correlations for Gibbs measure with pair interaction on infinite product of *n o n c o m p a c t* manifolds. We consider the class of lattice Gibbs measures with convex pair interaction which is dominated by one-point potentials.

We follow partially the approach of L.Gross and H.Föllmer [30], [26]. But the weighted spectral gap inequality (5.5) enables us to omit supremum type estimates on densities of one-point conditional measures.

Let $\mathcal{Z}^d$ be d - dimensional lattice to each point $k = = (k_1,\dots,k_d) \in \mathcal{Z}^d$ corresponds smooth complete connected Riemannian manifold M_k with probability measure P_k

$$dP_k(x_k) = \frac{\exp(-\Phi_k(x_k)\,)d\sigma_k(x_k)}{\int\limits_{M_k} \exp(-\Phi_k(x_k)\,)d\sigma_k(x_k)}$$

where $\Phi_k \in C^3(M_k)$.

For $\Lambda \in \mathcal{Z}^d$, $|\Lambda| < \infty$ denote $M^\Lambda = \underset{k\in\Lambda}{\times} M_k$, $x_\Lambda = \{x_k\}_{k\in\Lambda}$, $x_k \in M_k$ and let $\mathcal{F}_\Lambda$ be the Borel σ - algebra on product of manifolds M^Λ, $\Lambda^c = \mathcal{Z}^d \backslash \Lambda$.

Denote by $C^\infty_{c,cyl}(M^{\mathcal{Z}^d})$ the set of functions f such that $\exists\ const_f \in \mathcal{R}^1$ and $u_f \in C^\infty_{0,cyl}(M^{\mathcal{Z}^d})$:

$$f = const_f + u_f \qquad (5.1\ )$$

Here $C^\infty_{0,cyl}(M^{\mathcal{Z}^d})$ denote the space of smooth cylinder functions on $M^{\mathcal{Z}^d}$ with compact support, i.e

$$\forall f \in C^\infty_{0,cyl}(M^{\mathcal{Z}^d})\ \exists \Lambda \in \mathcal{Z}^d\ |\Lambda| < \infty\ \exists F_f \in C_0^\infty(M^\Lambda)$$

$$such\ that\ \forall x \in M^{\mathcal{Z}^d} f(x) = F(\{x_j,\ j \in \Lambda\})$$

Consider the family of interactive potentials

$$\{W_{kj}\ : k \neq j,\ k,j \in \mathcal{Z}^d\}$$

which satisfy the next assumptions:

$$\begin{aligned} &1.\ Function\ W_{kj} \in C^3(M_k \times M_j),\quad k \neq j \\ &2.\ \exists\, r_0 > 0 :\ \forall k,j \in \mathcal{Z}^d :\ |k-j| > r_0\ W_{kj} \equiv 0 \\ &3.\ \exists \alpha_{kj} \in \mathcal{R}^1\ such\ that\ W_{kj} \ge \alpha_{kj} \end{aligned} \qquad (5.2\ )$$

For finite subset $\Lambda \subset \mathcal{Z}^d$, $|\Lambda| < \infty$ we introduce the Gibbs measure in volume Λ with fixed boundary condition $y \in M^{\mathcal{Z}^d}$ by next expression:

$$d\mu_\Lambda(x_\Lambda|y) = \frac{1}{Z_\Lambda(y)} \exp(-\lambda V_\Lambda(x_\Lambda|y)\,) \underset{k\in\Lambda}{\times} dP_k(x_k) \qquad (5.3\ )$$

with interactive constant $\lambda > 0$. Here

$$V_\Lambda(x_\Lambda|y) = \sum_{\{k,j\}\cap\Lambda\neq\emptyset} W_{kj}(z),\ \ z = (x_\Lambda, y_{\Lambda^c})$$

and

$$Z_\Lambda = \int_{M^\Lambda} e^{-\lambda V_\Lambda(x_\Lambda|y)} \underset{k\in\Lambda}{\times} dP_k(x_k)$$

These measures are correctly defined because $0 < Z_\Lambda(y) < \infty$ due to condition 3 in (5.2).

Let μ^y_Λ denote the expectation with respect to measure $\mu_\Lambda(\cdot|y)$ and $\mu(f) = \int f\, d\mu$. Then the next consistency condition is satisfied:

$$\mu^y_{\Lambda_1}\mu^\bullet_{\Lambda_2} = \mu^y_{\Lambda_1},\ \ \Lambda_1 \supset \Lambda_2$$

Definition 5.1 *The probability measure μ on $M^{\mathcal{Z}^d}$ is a Gibbs one with local specifications $\{\mu_\Lambda,\ \Lambda \subset \mathcal{Z}^d\}$ iff for each $\Lambda \subset \mathcal{Z}^d$: $\mu(\mu^\bullet_\Lambda) = \mu$ (Notation $\mu \in \mathcal{G}\{\mu_\Lambda\}$).*

Remark. For Gibbs measure condition $\mu(\mu_\Lambda) = \mu$ is equivalent to assumption that family $\{\mu_\Lambda\}$ form a set of conditional measures for measure μ with respect to $\mathcal{F}_{\Lambda^c}$ [22], [40], [42].

Below we investigate conditions when the weighted spectral gap inequality (2.5) implies uniqueness and decay of correlations for measures from $\mathcal{G}\{\mu_\Lambda\}$. In this paper we do not investigate the general conditions when $\mathcal{G}\{\mu_\Lambda\} \neq \emptyset$, in each model situation nonemptiness of $\mathcal{G}\{\mu_\Lambda\}$ is obtained independently (Section 6).

Now we restrict our considerations to the case when one point conditional measures $\{\mu_k\}_{k\in\mathcal{Z}^d}$ (5.3) satisfy

$$\forall k \in \mathcal{Z}^d\ \ \exists\varepsilon_k > 0\ \ \exists B_k \in C^\infty(M_k, T^{1,1}M_k)\ \textit{such that}$$

$$R_{\mu_k}(x_k|y) \geq B_k(x_k) \geq \varepsilon_k \qquad (5.4\ )$$

pointwisely on $x_k \in M_k$, $y \in M^{\mathcal{Z}^d}$. Here

$$R_{\mu_k}(x_k|y) = Ric_k(x_k) + \nabla_k\nabla_k\{\Phi_k + \lambda V_{\{k\}}\}$$

like in (2.4).

Then due to the Theorem 2.2 we have the next spectral gap inequality, like in (2.5)

$$\frac{1}{2}cov_{\mu_k}(f,f) = \frac{1}{2}\int_{M_k} |f(x_k|y) - \int_{M_k} f(\cdot|y)d\mu_k(\cdot|y)\,|^2 d\mu_k(x_k|y) \leq$$

$$\leq \int\limits_{M_k} \prec R^{-1}_{\mu_k}(x_k|y)\nabla_k f, \nabla_k f \succ d\mu_k(x_k|y) \leq \tag{5.5 }$$

$$\leq \int\limits_{M_k} |B_k^{-1/2}(x_k)\nabla_k f(x_k|y)|^2_{TM_k} d\mu_k(x_k|y)$$

for $f \in C^\infty_{c,cyl}(M^{\mathcal{Z}^d})$.

Denote by $\delta_k(f)$ the next expression

$$\delta_k(f) = \sup_{x \in M^{\mathcal{Z}^d}} |B_k^{-1/2}(x_k)\nabla_k f(x)|_{TM_k} \tag{5.6 }$$

for $f \in C^\infty_{c,cyl}(M^{\mathcal{Z}^d})$.

Remark 5.2. The expression for $\delta_k(f)$ in (5.6) could be transformed to the more convenient form like in [26]

Introduce the next metric on manifold M_k

$$\tilde{\rho}_k(x,y) = \inf_{z(0)=x,\, z(T)=y} \int_0^T |B_k^{1/2}(z(\tau))\frac{\partial z(\tau)}{\partial \tau}|_{TM_k} d\tau \tag{5.7 }$$

Then it simply follows from the Lagrange formula that

$$\delta_k(f) = \sup_{x,y \in M^{\mathcal{Z}^d}} \{\frac{|f(x) - f(y)|}{\tilde{\rho}_k(x_k, y_k)},\ x_j = y_j,\ j \neq k\}$$

on $f \in C^\infty_{c,cyl}(M^{\mathcal{Z}^d})$.

Lemma 5.3. *Let potentials $\{W, \Phi\}$ satisfy conditions (5.2) and (5.4).*

Suppose that $\forall k \neq j \in \mathcal{Z}^d$

$$\sup_{x \in M^{\mathcal{Z}^d}} |B_k^{-1/2}B_j^{-1/2}\nabla_k\nabla_j W_{kj}|_{TM_k \otimes TM_j} < \infty \tag{5.8 }$$

Then for all $f \in C^1_{cyl}(M^{\mathcal{Z}^d})$ such that $\sum_{j \in \mathcal{Z}^d} \delta_j(f) < \infty$ we have

$$\delta_j(\mu_k(f)\ ) \leq \delta_j(f) + C_{jk}\delta_k(f)$$

for $j \neq k \in \mathcal{Z}^d$. Here

$$C_{jk} = 2\lambda \sup_{x \in M^{\mathcal{Z}^d}} |B_k^{-1/2}B_j^{-1/2}\nabla_k\nabla_j W_{kj}| \tag{5.9 }$$

Proof. The next identity for $j \neq k,\ j,k \in \mathcal{Z}^d$

$$\nabla_j \mu_k(f) = \mu_k(\nabla_j f) - \lambda cov_{\mu_k}(f, \nabla_j W_{kj})$$

leads to

$$\delta_j(\mu_k(f)\ ) = \sup |B_j^{-1/2}\nabla_j(\mu_k(f)\ )|_j =$$

$$= \sup |\mu_k(B_j^{-1/2}\nabla_j f) - \lambda cov_{\mu_k}(f, B_j^{-1/2}\nabla_j W_{kj})|_j \leq$$

$$\leq \delta_j(f) + \lambda \sup |cov_{\mu_k}(f, B_j^{-1/2}\nabla_j W_{kj})|_j$$

Here $|u|_j = (\prec u, u \succ_{TM_j})^{1/2}$, $j \in \mathcal{Z}^d$.

Using the inequality

$$|cov_\mu(f,g)| \le cov_\mu^{1/2}(f,f)\cdot cov_\mu^{1/2}(g,g)$$

with function $g(x) = B_j^{-1/2}\nabla_j W_{kj}(x)$, $\mu = \mu_k(x_k)$ and the following modification of the weigted spectral gap inequality (5.5)

$$cov_{\mu_k}(f,f) \le 2\delta_k^2(f)$$

we have

$$\sup_{x\in M^{\mathcal{Z}^d}} |cov_{\mu_k}(f, B_j^{-1/2}\nabla_j W_{kj})|_j \le$$

$$\le \sup\ cov_{\mu_k}^{1/2}(f,f)(2\int_{M_k}|B_k^{-1/2}B_j^{-1/2}\nabla_k\nabla_j W_{kj}|^2_{TM_k\times TM_j}d\mu_k)^{1/2} \le$$

$$\le 2\delta_k(f)\sup_{x\in M^{\mathcal{Z}^d}}|B_k^{-1/2}B_j^{-1/2}\nabla_k\nabla_j W_{kj}| =$$

$$= 2C_{kj}\delta_k(f) \qquad \blacksquare$$

Below we investigate conditions on the uniqueness and the fast decay of corelations for the Gibbs measures μ with conditional measures (5.3) from the Dobrushin's uniqueness region i.e. for which the following holds

$$\sup_{k\in\mathcal{Z}^d}\int_{M^{\mathcal{Z}^d}}\tilde\rho_k(x_k,z_k)d\mu(x) < \infty \qquad (5.10\ )$$

for some point $z\in M^{\mathcal{Z}^d}$ and $\tilde\rho$ defined in (5.7). In Section 6 we control this condition for systems with non-Gaussian interaction.

Definition 5.4. *Let μ,ν be arbitrary probability measures on Tichonov σ-algebra on $M^{\mathcal{Z}^d}$. The vector $a_f = \{a_j\}_{j\in\mathcal{Z}^d}$ is called an estimate for μ and ν if for $f\in C^1_{cyl}(M^{\mathcal{Z}^d})$ such that $\sum_{j\in\mathcal{Z}^d}\delta_j(f)<\infty$ we have*

$$|\int_{M^{\mathcal{Z}^d}} f\,d\mu - \int_{M^{\mathcal{Z}^d}} f\,d\nu| \le \sum_{j\in\mathcal{Z}^d} a_j\delta_j(f) \qquad (5.11\ )$$

where $\delta_j(f)$ are defined in (5.6).

Lemma 5.5. *Let μ,ν be arbitrary probability measures on Tichonov σ-algebra on $M^{\mathcal{Z}^d}$ which satisfy condition (5.10) with some $z\in M^{\mathcal{Z}^d}$*

Then $\forall\ f\in C^1_{cyl}(M^{\mathcal{Z}^d})$, $\sum_{j\in\mathcal{Z}^d}\delta_j(f)<\infty$ there is the next estimate

$$|\int_{M^{\mathcal{Z}^d}} f\,d\mu - \int_{M^{\mathcal{Z}^d}} f\,d\nu| \le \alpha\sum_{j\in\mathcal{Z}^d}\delta_j(f)$$

with constant

$$\alpha = \sup_{k\in\mathcal{Z}^d}\int_{M^{\mathcal{Z}^d}}\tilde\rho_k(x_k,z_k)\{d\mu(x)+d\nu(x)\} < \infty$$

Proof. The remark 5.2 implies that for any function $f\in C^1_{cyl}(M^{\mathcal{Z}^d})$ such that $\sum_{j\in\mathcal{Z}^d}\delta_j(f)<\infty$ we have inequality

$$\forall x,z\in M^{\mathcal{Z}^d}\quad |f(x)-f(z)| \le \sum_{k\in\mathcal{Z}^d}\delta_j(f)\tilde\rho_k(x_k,z_k)$$

Therefore

$$|\int_{M^{\mathcal{Z}^d}} f(x)d\mu(x) - \int_{M^{\mathcal{Z}^d}} f(x)d\nu(x)| \leq$$

$$\leq |\int_{M^{\mathcal{Z}^d}}[f(x) - f(z)]d\mu(x)| +$$

$$+|\int_{M^{\mathcal{Z}^d}}[f(x) - f(z)]d\nu(x)| \leq$$

$$\leq \sum_{j\in\mathcal{Z}^d} \delta_j(f) \sup_{k\in\mathcal{Z}^d} \int_{M^{\mathcal{Z}^d}} \tilde{\rho}_k(x_k, z_k)\{d\mu(x) + d\nu(x)\} \quad \blacksquare$$

Theorem 5.6. *Let potentials $\{W, \Phi\}$ satisfy conditions (5.2), (5.4) and (5.8).*

1. Let probability measures $\mu, \nu \in \mathcal{G}\{\mu_\Lambda\}$ (5.3). If the vector $\{a_j\}_{j\in\mathcal{Z}^d}$ is an estimate (5.11) then vector

$$aC = \{\sum_{j\in\mathcal{Z}^d} a_j C_{jk}\}_{k\in\mathcal{Z}^d}$$

is an estimate too. Here matrix $C = \{C_{kj}\}$ is defined in (5.9).

2. Consider probability measure $\mu \in \mathcal{G}\{\mu_\Lambda\}$ (5.3) which satisfies (5.10) with some $z \in M^{\mathcal{Z}^d}$ and probability measure ν such that

$$d\nu = g\ d\mu, \quad g \in C^\infty_{c,cyl}(M^{\mathcal{Z}^d}), \quad \int_{M^{\mathcal{Z}^d}} g\ d\mu = 1$$

If the vector $\{a_j\}_{j\in\mathcal{Z}^d}$ is an estimate for measures μ and ν (5.11) then vector

$$aC + b = \{\sum_{j\in\mathcal{Z}^d} a_j C_{jk} + b_k\}_{k\in\mathcal{Z}^d}$$

is an estimate too. Matrix C is defined in (5.9) and vector b has coordinates

$$b_k = 2\delta_k(g) \tag{5.12 }$$

Proof. This theorem is proved by induction on set $J \subset \mathcal{Z}^d$, $|J| < \infty$, like in [26].

Put

$$a_k^J = \begin{cases} a_k, & k \notin J \\ min(a_k, (aC + b)_k), & k \in J \end{cases}$$

Here $b_k \equiv 0,\ k \in \mathcal{Z}^d$ in the first situation for $\mu, \nu \in \mathcal{G}\{\mu_\Lambda\}$ and b_k are defined in (5.12) in the second situation $d\nu = g\,d\mu$, $g \in C^\infty_{c,cyl}(M^{\mathcal{Z}^d})$, $\mu(g) = 1$.

The inductive base $J = \emptyset$ is trivial because $\{a\}$ is an estimate.

First of all we remark that for the function $f \in C^1_{cyl}(M^{\mathcal{Z}^d})$, $\sum_{j\in\mathcal{Z}^d} \delta_j(f) < \infty$ from Lemma 5.3 it follows that

$$\sum_{j\in\mathcal{Z}^d} \delta_j(\mu_k(f)) < \infty, \quad k \in \mathcal{Z}^d$$

Moreover the assumptions on the potentials $\{\Phi, W\}$ give that

$$\mu_k(f) \in C^1_{cyl}(M^{\mathcal{Z}^d})$$

Let now $K = J \cup \{k\}$ and suppose that vector $\{a_k^J\}_{k\in\mathcal{Z}^d}$ is an estimate for measures μ and ν. Then

$$|\mu(f) - \nu(f)| \le |(\mu-\nu)_y\{ \int\limits_{M_k} f(\cdot|y) d\mu_k(\cdot|y) \}| +$$

$$+|\nu_y\{ \int\limits_{M_k} f(\cdot|y) d\mu_k(\cdot|y) - \int\limits_{M_k} f(\cdot|y) d\nu_k(\cdot|y)\}| \le$$

$$\le \sum_{j\in\mathcal{Z}^d} a_j^J \delta_j(\mu_k(f)) + \{b - term\}$$

where $\nu_k(\cdot|y)$ and $\mu_k(\cdot|y)$ are one point conditional measures for ν and μ.

The $\{b - term\}$ equals to zero in the first situation for $\mu, \nu \in \mathcal{G}\{\mu_\Lambda\}$ because $\mu_k \equiv \nu_k$ under definition of $\mathcal{G}\{\mu_\Lambda\}$.

In the second situation we estimate $\{b - term\}$ from above using the spectral gap inequality (5.5). We use below that the conditional measures

$$d\nu_k = \frac{g}{\mu_k(g)} d\mu_k$$

and that $d\nu = g\ d\mu$:

$$\{b - term\} \equiv |\nu_y\{\int f\ d\mu_k(\cdot|y) - \int f\ d\nu_k(\cdot|y)\}| =$$

$$= |\nu_y\{ \int\limits_{M_k} (f - \mu_k(f))[d\mu_k - \frac{g}{\mu_k(g)} d\mu_k]\}| =$$

$$= |\mu_y\{\frac{g}{\mu_k(g)} \int\limits_{M_k} (f - \mu_k(f))(g - \mu_k(g))d\mu_k\}|$$

The expression under the integral on M_k doesn't depend on variable $x_k \in M_k$ therefore

$$\{b - term\} = |\mu\{\mu_k(\frac{g}{\mu_k(g)}) \cdot \int\limits_{M_k} (f - \mu_k(f))(g - \mu_k(g))d\mu_k\}| =$$

$$= |\mu\{ \int\limits_{M_k} (f - \mu_k(f))(g - \mu_k(g))d\mu_k\}| \le$$

$$\le \sup_{y\in M^{\mathcal{Z}^d}} (\int\limits_{M_k} [f(\cdot|y) - \mu_k^y(f)]^2 d\mu_k(\cdot|y))^{1/2} \cdot$$

$$\cdot \sup_{y\in M^{\mathcal{Z}^d}} (\int\limits_{M_k} [g(\cdot|y) - \mu_k^y(g)]^2 d\mu_k(\cdot|y))^{1/2} \le$$

$$\le 2 \sup_{y\in M^{\mathcal{Z}^d}} (\int\limits_{M_k} |B_k^{-1/2}(x_k)\nabla_k f|^2 d\mu_k)^{1/2} \cdot$$

$$\cdot \sup_{y\in M^{\mathcal{Z}^d}} (\int\limits_{M_k} |B_k^{-1/2}(x_k)\nabla_k g|^2 d\mu_k)^{1/2} \le 2\delta_k(f)\delta_k(g)$$

Here we have used the spectral gap inequality (5.5).

Now we apply Lemma 5.3 for to obtain the final estimate

$$|\int f\ d\mu - \int f\ d\nu| \le \sum_{j\in\mathcal{Z}^d} a_j^J \delta_j(\mu_k(f)) + \delta_k(f) b_k \le$$

$$\le \sum_{j\ne k} a_j^J \{\delta_j(f) + C_{jk}\delta_k(f)\} + \delta_k(f) b_k + a_k^J \cdot 0 =$$

$$= \sum_{j\neq k} a_j^J \delta_j(f) + (aC+b)_k\delta_k(f) \leq \sum_{j\in\mathcal{Z}^d} a_j^K \delta_j(f)$$

Recall that $b_k \equiv 0$ in the first situation and $b_k = 2\delta_k(g)$ in the second one. ■

Theorem 5.7. *Let potentials $\{W,\Phi\}$ satisfy conditions* (5.2), (5.4) *and* (5.8).

Suppose that

$$\vartheta = \sup_{k\in\mathcal{Z}^d} \sum_{j\in\mathcal{Z}^d} C_{kj} < 1 \tag{5.13 }$$

where constants C_{kj} are defined in (5.9).

Then

1. Set of $\mu \in \mathcal{G}\{\mu_\Lambda\}$, such that for some $z \in M^{\mathcal{Z}^d}$ the condition (5.10) is satisfied, consists of only one point $\tilde{\mu} \in \mathcal{G}\{\mu_\Lambda\}$.

2. For $\tilde{\mu}$, $\forall f, g \in C^\infty_{c,cyl}(M^{\mathcal{Z}^d})$ we have

$$|cov_{\tilde{\mu}}(f,g)| \leq 2 \sum_{k,j\in\mathcal{Z}^d} D_{kj}\delta_k(f)\delta_j(g) \tag{5.14 }$$

where $D = \sum_{m=0}^{\infty} C^m$ with matrix C defined in (5.9).

Proof. Theorem 5.6 implies that

$$aC^n + \sum_{m=0}^{n-1} bC^m$$

is an estimate if a is an estimate. Therefore

$$aC^n + \sum_{m=0}^{\infty} bC^m$$

is an estimate too. As measure μ satisfies condition (5.10) with some $z \in M^{\mathcal{Z}^d}$, then measure

$$d\nu = g\,d\mu, \quad g \in C^\infty_{c,cyl}(M^{\mathcal{Z}^d}), \quad \int g\,d\mu = 1$$

also satisfies (5.10) with the same $z \in M^{\mathcal{Z}^d}$.

Lemma 5.5 implies that there is an estimate $a = \{a_k\}$ for the measures μ, ν.

Then

$$\|aC^n\|_{\ell_\infty(\mathcal{Z}^d)} = \alpha \sup_{k\in\mathcal{Z}^d} \sum_{j\in\mathcal{Z}^d} |C^n_{kj}| =$$

$$= \alpha \sup_{k\in\mathcal{Z}^d} \sum_{j(1)\in\mathcal{Z}^d} \cdot\cdot \sum_{j(n-1)\in\mathcal{Z}^d} \sum_{j\in\mathcal{Z}^d} C_{kj(1)} \cdot\cdot C_{j(n-1)j} \leq \tag{5.15 }$$

$$\leq \alpha (\sup_{k\in\mathcal{Z}^d} \sum_{j\in\mathcal{Z}^d} C_{kj})^n \leq \alpha\vartheta^n$$

1. For measures $\mu, \nu \in \mathcal{G}\{\mu_\Lambda\}$ we have from Theorem 5.6 (statement 1) and (5.15) that for $f \in C^\infty_{c,cyl}(M^{\mathcal{Z}^d})$

$$|\int_{M^{\mathcal{Z}^d}} f\,d\mu - \int_{M^{\mathcal{Z}^d}} f\,d\nu| \leq \sum_{i\in\mathcal{Z}^d} (aC^n)_i \delta_i(f) \leq$$

$$\leq \sum_{i\in\mathcal{Z}^d} \delta_i(f) \|aC^n\|_{\ell_\infty(\mathcal{Z}^d)} \leq \alpha\vartheta^n \sum_{i\in\mathcal{Z}^d} \delta_i(f)$$

Tending $n \to \infty$ we obtain that

$$\forall f \in C^{\infty}_{c,cyl}(M^{\mathcal{Z}^d}) \quad \int\limits_{M^{\mathcal{Z}^d}} f\, d\mu = \int\limits_{M^{\mathcal{Z}^d}} f\, d\nu \tag{5.16 }$$

for every $\mu, \nu \in \mathcal{G}\{\mu_\Lambda\}$.

Relation (5.16) implies that projections of measures from $\mathcal{G}\{\mu_\Lambda\}$ coincide on Borel σ-algebras

$$\{\mathcal{B}(\underset{k\in\Lambda}{\times} M_k),\ \Lambda \subset \mathcal{Z}^d,\ |\Lambda| < \infty\}$$

Therefore due to the Kolmogorov theorem $\mu \equiv \nu$ and the set of all Gibbs measures consists of only one point.

2. In (5.14) we can always make by linearity $\int\limits_{M^{\mathcal{Z}^d}} g\, d\mu = 1$ and $g > 0$ to be strictly positive because

$$cov_\mu(f, g) = cov_\mu(f + c_1, g + c_2)$$

for every $c_1, c_2 \in \mathcal{R}^1$.

Then Theorem 5.6, part 2, implies that for measures μ and ν such that $d\nu = g\, d\mu$ the vector $aC^n + D$ is an estimate, so

$$|cov_\mu(f, g)| = |\int\limits_{M^{\mathcal{Z}^d}} f\, d\mu - \int\limits_{M^{\mathcal{Z}^d}} f\, d\nu| \le$$

$$\le 2 \sum_{k,j\in\mathcal{Z}^d} D_{kj}\delta_k(f)\delta_j(g) + \alpha \sum_{j\in\mathcal{Z}^d} \delta_j(f)\, \vartheta^n$$

Tending $n \to \infty$ we obtain Theorem 5.7 proved. ■

Corollary 5.8. Under conditions of Theorem 5.7 we have statements (5.16) and (5.14) for functions $f \in E$ where E is a class of functions which are limits of $f_n \in C^{\infty}_{c,cyl}(M^{\mathcal{Z}^d})$ in seminorm $\sum\limits_{j\in\mathcal{Z}^d} \delta_j(f)$, i.e.

$$\sum_{j\in\mathcal{Z}^d} \delta_j(f - f_n) \to 0, \quad n \to \infty$$

The next theorem provides conditions on decay of correlations for Gibbs measures [26].

Theorem 5.9. *Under conditions of Theorem* 5.7 *and condition*

$$\vartheta_d = \sup_{k\in\mathcal{Z}^d} \sum_{j\in\mathcal{Z}^d} e^{d(k,j)} C_{kj} < 1$$

for some semimetric $d(k, j)$ on lattice $\mathcal{Z}^d$ (for example $d(k, j) = |k - j|$ or $d(k, j) = \ln(1 + |k - j|)$) for any $\mu \in \mathcal{G}\{\mu_\Lambda\}$ which satisfies (5.10) we have estimate

$$\sum_{k\in\mathcal{Z}^d} |cov_\mu(f, \tau_k g)| e^{d(k,0)} \le \frac{2}{1 - \vartheta_d} \|f\|_d \|g\|_d \tag{5.17 }$$

where τ_k denotes the shift on vector $k \in \mathcal{Z}^d$ and

$$\|f\|_d = \sum_{k\in\mathcal{Z}^d} e^{d(k,0)} \delta_k(f)$$

Note that always $\vartheta_d \geq \vartheta$.

Proof. Using the triangle inequality for semimetric $d(\cdot,\cdot)$ and inequality (5.14) we have

$$|cov_\mu(f,\tau_i g)|e^{d(i,0)} \leq$$

$$\leq 2\sum_{k,j\in\mathcal{Z}^d} e^{d(j,k)}D_{jk}e^{d(k,0)}\delta_k(f)e^{d(i,j)}\delta_{j-i}(g)$$

Summing up on $i \in \mathcal{Z}^d$ we have the stated decay of correlations proved. ■

6 Gibbs measures with convex pair interaction.

In this section we consider $M_k = \mathcal{R}^1$, $k \in \mathcal{Z}^d$ and potentials $\{W,\Phi\}$ such that

1. For function $F(t) = F(-t) \in C^\infty(\mathcal{R})$ such that

$$\exists\varepsilon > 0 \quad F''(t) \geq \varepsilon \quad t \in \mathcal{R}^1 \tag{6.1 }$$

we have

$$\Phi_k(x_k) = F(x_k), \quad k \in \mathcal{Z}^d$$

2. For functions $G_j(t) = G_j(-t) \in C^\infty(\mathcal{R}^1)$, $j \in \mathcal{Z}^d$ such that

$$\forall j \in \mathcal{Z}^d \quad G_j''(t) \geq 0 \quad t \in \mathcal{R} \tag{6.2 }$$

we have

$$W_{kj}(x_k,x_j) = G_{k-j}(x_k - x_j), \quad k,j \in \mathcal{Z}^d$$

and

$$\exists r_0 > 0\ \forall j \in \mathcal{Z}^d\ |j| > r_0 \quad G_j \equiv 0 \tag{6.3 }$$

Lemma 6.1. *Under conditions* (6.1 - 6.3) *on potentials* $\{W,\Phi\}$ *for all* $\lambda \geq 0$ *the set of Gibbs measures* $\mu \in \mathcal{G}\{\mu_\Lambda\}$ *which satisfy condition*

$$\sup_{k\in\mathcal{Z}^d}\int_{M^{\mathcal{Z}^d}} q^2(x_k)d\mu(x) < \infty \tag{6.4 }$$

is non-empty. Here

$$q(x_k) = \int_0^{x_k}\sqrt{F''(\tau)}\,d\tau$$

and $|q(x_k)| = \tilde{\rho}_k(x_k,0)$ *(see Remark 5.2).*

Proof. Choose boundary condition $y = \{y_j\} \in \mathcal{R}^{\mathcal{Z}^d}$, $y_j \equiv 0$. Conditional measures (5.3) have the next representation

$$d\mu_\Lambda(x_\Lambda \mid 0) = \frac{1}{Z_\Lambda(0)}\exp(-\mathcal{U}_\Lambda(x_\Lambda|0)\,)\mathop{\times}_{k\in\Lambda} dx_k$$

where

$$\mathcal{U}_\Lambda(x_\Lambda|0) = \sum_{k\in\Lambda}F(x_k) + \lambda\sum_{\{k,j\}\cap\Lambda\neq\emptyset}G_{k-j}(\xi_k - \xi_j)$$

and $\xi = (x_\Lambda, 0_{\Lambda^c})$. Due to assumptions (6.1 - 6.3) we have for all $\lambda \geq 0$

$$R_{\mu_\Lambda(\cdot|0)} = \{\nabla_k\nabla_j\mathcal{U}_\Lambda(x_\Lambda|0)\}_{k,j\in\Lambda} \tag{6.5 }$$

$$\geq \{\delta_{kj}F''(x_k)\}_{k,j\in\Lambda} \geq \varepsilon\{\delta_{kj}\}_{k,j\in\Lambda} > 0$$

as the functions G_j are convex.

Inequality (6.5) and the weighted spectral gap (Thm.2.2) give the next inequality

$$\frac{1}{2}\int_{\mathcal{R}^\Lambda}(f(\cdot)-\int_{\mathcal{R}^\Lambda}f(x_\Lambda)d\mu_\Lambda(x|0))^2 d\mu_\Lambda(\cdot|0)\leq$$

$$\leq\int_{\mathcal{R}^\Lambda}\sum_{k\in\Lambda}\frac{|\nabla_k f|^2}{F''(x_k)}d\mu_\Lambda(\cdot|0)$$

Applying this estimate to the function $q(x_k)=\int_0^{x_k}\sqrt{F''(\tau)}d\tau$ and using the symmetry of measure $\mu_\Lambda(\cdot|0)$:

$$\int_{\mathcal{R}^\Lambda}q(x_k)d\mu_\Lambda(x_\Lambda|0)=0$$

we obtain estimate

$$\forall k\in\Lambda\subset\mathcal{Z}^d\ \ \frac{1}{2}\int_{\mathcal{R}^\Lambda}q^2(x_k)d\mu_\Lambda(\cdot|0)\leq 1 \tag{6.6 }$$

As function q is a compact function on $\mathcal{R}^1=M_k:\ |q(x_k)|\geq\varepsilon|x_k|$ then by the Prochorov's Theorem [49,Ch.1,§5] it follows the existence of the weak limit $\tilde{\mu}:\ \tilde{\mu}(\mathcal{R}^{\mathcal{Z}^d})=1$ of measures $\{\mu_{\Lambda_n}(\cdot|0)\}_{n\geq 1}$ for some subsuquence $\{\Lambda_n\}$ which exhaust the lattice $\mathcal{Z}^d$.

The assumption on the finiteness of interactive radius give the Gibbs property of the limit measure $\tilde{\mu}$.

Tending $n\to\infty$ we also have the property (6.6) for $\tilde{\mu}$. ■

Due to the assumptions (6.2) on functions G_j we can estimate constants C_{kj} (5.9) from above by

$$C_{kj}\leq 2\lambda\sup_{x\in\mathcal{R}^{\mathcal{Z}^d}}|\frac{G''_{k-j}(x_k-x_j)}{\sqrt{F''(x_k)}\sqrt{F''(x_j)}}|$$

Theorem 6.2. *Let potentials* $\{W,\Phi\}$ *satisfy conditions* (6.1 - 6.3) *and*

$$\gamma_d=\sum_{k\in\mathcal{Z}^d}e^{d(k,0)}\sup_{x\in\mathcal{R}^{\mathcal{Z}^d}}|\frac{G''_k(x_k-x_0)}{\sqrt{F''(x_k)}\sqrt{F''(x_0)}}|<\infty$$

with some semimetric $d(\cdot,\cdot)$ *on* $\mathcal{Z}^d$.

Then for

$$\lambda\in[0,1/2\gamma_d\ )$$

we have that

1. *Set* $\mathcal{G}\{\mu_\Lambda\}\neq\emptyset$ *and consists of at most one point* $\tilde{\mu}$.

2. *For measure* $\tilde{\mu}$ *the fast decay of correlations holds*

$$\sum_{k\in\mathcal{Z}^d}e^{d(k,0)}|\,cov_{\tilde{\mu}}(f,\tau_j g)|\leq$$

$$\leq\frac{1}{1-2\lambda\gamma_d}(\sum_{k\in\mathcal{Z}^d}\delta_k(f))(\sum_{j\in\mathcal{Z}^d}\delta_j(g)) \tag{6.7 }$$

for $f,g\in C^\infty_{c,cyl}(\mathcal{R}^{\mathcal{Z}^d})$.

3. The support of measure $\tilde{\mu}$ belongs to

$$supp\ \tilde{\mu} \subset \{y \in \mathcal{R}^{\mathcal{Z}^d} : \sum_{k\in\mathcal{Z}^d} |a_k| q^2(y_k) < \infty\} \quad \forall a \in \ell_1(\mathcal{Z}^d)$$

for function $q(t) = \int_0^t \sqrt{F''(\tau)} d\tau$

Proof. Set $\mathcal{G}\{\mu_\Lambda\} \neq \emptyset$ due to Lemma 6.1. Statement 1) and 2) follow from Theorems 5.7 and 5.9.

From inequality (6.6) for the measure $\tilde{\mu}$ it follows that

$$\frac{1}{2} \int_{\mathcal{R}^{\mathcal{Z}^d}} \sum_{j\in\mathcal{Z}^d} a_j q^2(x_j)\, d\tilde{\mu}(x) \leq \sum_{j\in\mathcal{Z}^d} a_j < \infty$$

Therefore the set of points where the following function

$$x = \{x_j\}_{j\in\mathcal{Z}^d} \to \sum_{j\in\mathcal{Z}^d} a_j q^2(x_j)$$

is bounded has the full $\tilde{\mu}$ - measure. ■

Model 1. (Anharmonic oscilators)

Potentials of finite volumes $\Lambda \subset \mathcal{Z}^d$ are defined by the next expressions

$$\mathcal{U}_\Lambda = \sum_{k\in\Lambda} F(x_k) + \lambda \sum_{\{k,j\}\cap\Lambda\neq\emptyset} a(k-j)\ (x_k - x_j)^2$$

Here function F satisfies requirement (6.1) and

$$\forall j \in \mathcal{Z}^d\ a(j) \geq 0\ \ \&\ \ \exists r_0\ \forall |j| > r_0 : a(j) = 0$$

Then for

$$0 \leq \lambda < \frac{\varepsilon}{8\|a\|_d}$$

we have statements of Theorem 6.2. Here

$$\|a\|_d = \sum_{j\in\mathcal{Z}^d} a(j) e^{d(j,0)} < \infty$$

for some semimetric $d(\cdot,\cdot)$ on $\mathcal{Z}^d$.

Model 2. Here potentials are defined by

$$\mathcal{U}_\Lambda = \sum_{k\in\Lambda} (1 + x_k^2)^{2n+1} + \lambda \sum_{\{k,j\}\cap\Lambda\neq\emptyset} a(k-j)(x_k - x_j)^{2n+2}$$

and assumptions on coefficients $\{a(j)\}$ are as in Model 1.

Then for

$$0 \leq \lambda < \frac{1}{(n+1)2^{2n+1}\|a\|_d}$$

the statement of Theorem 6.2 is valid.

Below we show that for interactive potentials (6.1 - 6.3) the corresponding Gibbs measure has Log-Sobolev inequality.

The discussions of various applications and conditions on Log-Sobolev inequalities for Gibbs measures could be found in papers cited in the introduction to the paper.

Remark 6.3. Under conditions of Theorem 6.2 we have Log-Sobolev inequality for measure $\mu \in \mathcal{G}\{\mu_\Lambda\}$

$$\int\limits_{\mathcal{R}^{\mathcal{Z}^d}} |f|^2 \ln |f|^2 d\mu - \|f\|^2 \ln \|f\|^2 \leq \\ \leq \tfrac{2}{\varepsilon} \int\limits_{\mathcal{R}^{\mathcal{Z}^d}} \sum_{k \in \mathcal{Z}^d} |\nabla_k f|^2 d\mu \qquad (6.8\ )$$

on $f \in C^\infty_{c,cyl}(\mathcal{R}^{\mathcal{Z}^d})$.

Indeed, log-concavity of measures $\{\mu_\Lambda(\cdot|0)\}$ imply the set of Log-Sobolev inequalities (see Remark 3.3)

$$\int\limits_{\mathcal{R}^\Lambda} |f|^2 \ln |f|^2 d\mu^0_\Lambda - \|f\|^2_{L_2(\mu^0_\Lambda)} \ln \|f\|^2_{L_2(\mu^0_\Lambda)} \leq \\ \leq \frac{2}{\varepsilon} \int\limits_{\mathcal{R}^\Lambda} \sum_{k \in supp_{cyl} f} |\nabla_k f|^2 d\mu^0_\Lambda$$

for all $f \in C^\infty_{c,cyl}(\mathcal{R}^{\mathcal{Z}^d})$, $supp_{cyl} f \subset \Lambda$.

By [49, Ch.1,§5] and (6.6) we have local weak convergence for sequence of measures

$$\mu^0_\Lambda \otimes \big(\bigotimes_{k \in \mathcal{Z}^d \setminus \Lambda} \delta_0(x_k) \big)$$

Then from finite radius of interaction assumption we see that the weak local limit should be Gibbs measure, so tending $\Lambda \nearrow \mathcal{Z}^d$ we have inequality (6.8).

See also [3, 2] where different conditions for Gibbs measures with Gaussian interaction were obtained.

7 Appendix A.

Theorem A.1. *Let probability measure μ (2.1) have density $\Phi \in C^3(M)$. Suppose that there is a function $S \in C^3 \cap C_b(M)$, $S > 0$ pointwisely, and*

$$SR_\mu S + H^S_\mu S \geq \lambda S(x) \qquad (7.1\ )$$

with constant $\lambda > 0$.

Then we have weighted Logarithmic Sobolev inequality

$$\int\limits_M |f|^2 \ln |f|^2 d\mu - \|f\|_{L_2} \ln \|f\|_{L_2} \leq \\ \leq \tfrac{2}{\lambda} \int\limits_M S(x) |\nabla f|^2 d\mu(x) \qquad (7.2\ )$$

for $f \in C^2_0(M)$.

Proof.

Inequality (7.2) it is sufficient to prove on non-negative functions $f \geq 0$, $f \in C^2_0(M)$ due to the following estimate

$$\int\limits_M S |\nabla |u|\,|^2 d\mu \leq \int\limits_M S |\nabla u|^2 d\mu$$

on functions $u \in C_0^2(M)$. This is a common fact from the theory of Dirichlet forms. The simplest proof could be found in [43, Ch.X, Th.X.27].

Moreover we only have to prove inequality (7.2) for strictly positive functions

$$f_\varepsilon = f+\varepsilon,\ f \in C_0^2(M),\ \varepsilon > 0 \tag{7.3 }$$

Indeed, tending $\varepsilon \to 0$ in inequality

$$\begin{aligned} &\int_M |f+\varepsilon|^2 \ln |f+\varepsilon|^2 d\mu - \|f+\varepsilon\|^2 \ln \|f+\varepsilon\|^2 \leq \\ &\leq \tfrac{2}{\lambda} \int_M S|\nabla(f+\varepsilon)|^2 d\mu = \tfrac{2}{\lambda} \int_M S|\nabla f|^2 d\mu \end{aligned} \tag{7.4 }$$

for $f \in C_0^2(M)$, we obtain the weighted inequality (7.2) as an application of Lebesgue domination theorem to (7.4).

In this place the steps 1.-2. of Theorem 3.1 proof should be repeated literally for $\Psi(x) = x \ln x$ and $A(x) = S(x)Id_x$.

In particular we have representation

$$\begin{aligned} &\int_M v \ln v\, d\mu - (\int_M v\, d\mu) \ln(\int_M v\, d\mu) = \\ &= \tfrac{1}{2} \int_0^\infty \int_M \frac{1}{P_t v} S|\nabla P_t v|^2\, d\mu\, dt \end{aligned} \tag{7.5 }$$

for $P_t = \exp(-tH_\mu^S)$ and $v \in C^2(M)$, to be strictly positive. Then inequality (7.2) transforms to

$$\int_0^\infty \int_M \frac{1}{P_t v} S|\nabla P_t v|^2 d\mu dt \leq \frac{1}{\lambda} \int_M \frac{1}{v} S|\nabla v|^2 d\mu$$

Here we used representation (7.5). Therefore it is sufficient to prove that

$$0 \leq \int_M \frac{1}{P_t v} S|\nabla P_t v|^2 d\mu \leq e^{-\lambda t} \int_M \frac{1}{v} S|\nabla v|^2 d\mu$$

or that

$$K'(t) \leq -\lambda K(t) \tag{7.6 }$$

for function

$$K(t) = \int_M \frac{1}{P_t v} S|\nabla P_t v|^2 d\mu$$

Introduce notations $P_t v = \exp(u(x))$ Then using

$$\nabla e^u = e^u \nabla u \ \ \&\ \ H_\mu^S e^u = e^u H_\mu^S u - \frac{1}{2} e^u S|\nabla u|^2$$

inequality (7.6) adopts the form

$$\begin{aligned} &\lambda \int_M e^u S|\nabla u|^2 d\mu \leq \\ &\leq 2 \int_M S \prec \nabla u, \nabla H_\mu^S e^u \succ d\mu - \int_M H_\mu^S e^u \cdot S|\nabla u|^2 d\mu \end{aligned} \tag{7.7 }$$

The first term in (7.7) could be transformed to the next terms

$$2\int\limits_M S \prec \nabla u, \nabla H_\mu^S e^u \succ d\mu = 4\int\limits_M H_\mu^S u \cdot H_\mu^S e^u d\mu =$$

$$= \int\limits_M e^u \prec SR_\mu S\nabla u, \nabla u \succ d\mu +$$

$$+ \int\limits_M tr(\nabla(S\nabla u)[\nabla(S\nabla e^u)]^*) d\mu =$$

$$= \int\limits_M e^u \prec SR_\mu S\nabla u, \nabla u \succ d\mu +$$

$$+\frac{1}{2}\int\limits_M e^u tr(\nabla(S\nabla u)[\nabla(S\nabla u)]^*) d\mu +$$

$$+\frac{1}{2}\int\limits_M e^u | \prec \nabla S, \nabla u \succ |^2 d\mu +$$

$$+\frac{1}{2}\int\limits_M e^u S^2 tr(\nabla\nabla u[\nabla\nabla u]^*) d\mu +$$

$$+\frac{1}{2}\int\limits_M e^u S \prec \nabla S, \nabla \succ |\nabla u|^2 d\mu +$$

$$+ \int\limits_M e^u tr(\nabla(S\nabla u) \cdot S[\nabla u \otimes \nabla u]^*) d\mu$$

The second term in (7.7) is transformed to

$$-\int\limits_M H_\mu^S e^u \cdot S|\nabla u|^2 d\mu = +\frac{1}{2}\int\limits_M e^u S^2 |\nabla u|^4 d\mu -$$

$$-\int\limits_M e^u H_\mu^S u \cdot S|\nabla u|^2 d\mu = \frac{1}{2}\int\limits_M e^u S^2 |\nabla u|^2 d\mu -$$

$$-\frac{1}{2}\int\limits_M e^u S^2 |\nabla u|^2 d\mu - \frac{1}{2}\int\limits_M e^u \prec S\nabla u, \nabla(S|\nabla u|^2) \succ d\mu =$$

$$= -\int\limits_M e^u S \cdot tr((\nabla u \otimes \nabla u) \cdot [\nabla(S\nabla u)]^*) d\mu +$$

$$+\frac{1}{2}\int\limits_M e^u \prec S\nabla S, \nabla u \succ |\nabla u|^2 d\mu =$$

$$= -\int\limits_M e^u S \cdot tr(\nabla(S\nabla u) \cdot [\nabla u \otimes \nabla u]^*) d\mu -$$

$$-\frac{1}{2}\int\limits_M e^u \prec S\nabla S, \nabla \succ |\nabla u|^2 d\mu + \int\limits_M e^u (H_\mu^S S)|\nabla u|^2 d\mu$$

Finally estimate (7.7) transforms to

$$\lambda\int\limits_M e^u S|\nabla u|^2 d\mu \le \int\limits_M \prec (SR_\mu S + H_\mu^S S)\nabla u, \nabla u \succ d\mu +$$

$$+\frac{1}{2}\int\limits_M e^u tr(\nabla(S\nabla u) \cdot [\nabla(S\nabla u)]^*) d\mu +$$

$$+\frac{1}{2}\int\limits_M e^u \{S^2 tr(\nabla\nabla u \cdot [\nabla\nabla u]^*) + | \prec \nabla S, \nabla u \succ |^2\} d\mu$$

therefore the condition (7.1) is sufficient. ■

8 Appendix B.

Proof of Lemma 2.1. Consider function $\xi \in C_0^\infty(M)$, $0 \le \xi \le 1$ such that $\xi(x) = 1$ on some ball $x \in B_R(x_0)= \{y \in M : \rho(x_0, y) < R\}$. The operator $H_\mu^{\xi A}$ is essentially self-adjoint in $L_2(M, \mu)$ because the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = -H_\mu^{\xi A} u(x,t) \\ u(x,0) = u_0(x) \end{cases}$$

could be smoothly solved in $C_0^2(M)$ due to the assumptions on the coefficients of operator $H_\mu^{\xi A}$ [27, Ch.9, §6, Th.8].

We have to show that from condition

$$\forall \varphi \in C_0^\infty(M) : \int\limits_M \prec (1 + H_\mu^A)\varphi, u \succ d\mu = 0 \qquad (8.1\ )$$

follows that $u \equiv 0$ which gives the required essential self-adjointness for operator H_μ^A [43, Ch.X, Th.X.1].

Consider function $\eta \in C_0^\infty(M)$ such that $0 \le \xi \le \eta \le 1$ and $\eta \upharpoonright_{supp\,\xi}= 1$. Then restrictions $H_\mu^{\eta A} \upharpoonright_{supp\,\xi}= H_\mu^A \upharpoonright_{supp\,\xi}$ coincide.

Therefore

$$\begin{aligned} &\int\limits_M \prec (1 + H_\mu^A)\varphi, u\xi \succ d\mu = \int\limits_M \prec (1 + H_\mu^{\eta A})\varphi, u\xi \succ d\mu = \\ &= \int\limits_M \prec (1 + H_\mu^{\eta A})\varphi\xi, u \succ d\mu + \int\limits_M \prec [\xi, H_\mu^{\eta A}]\varphi, u \succ d\mu = \\ &= \int\limits_M \prec \{\prec A\nabla\xi, \nabla\cdot \succ \varphi - (H_\mu^A \xi)\varphi\}, u \succ d\mu \end{aligned} \qquad (8.2\ )$$

due to $(\varphi\xi) \in C_0^\infty(M)$ and (8.1). Put $\psi = (1+H_\mu^{\eta A})^{1/2}\varphi$ then the above identity transforms to

$$\begin{aligned} &\int\limits_M \prec (1 + H_\mu^{\eta A})^{1/2}\psi, u\xi \succ d\mu = \\ &= \int\limits_M \prec \prec A\nabla\xi, \nabla\cdot \succ (1 + H_\mu^{\eta A})^{-1/2}\psi, u \succ d\mu - \\ &- \int\limits_M \prec (H_\mu^A\xi)(1 + H_\mu^{\eta A})^{-1/2}\psi, u \succ d\mu = B_1 + B_2 \end{aligned}$$

We obtain estimates

$$|B_1| \le \|u\|_{L_2(\mu)} (\int\limits_M | \prec A\nabla\xi, \nabla \succ (1 + H_\mu^{\eta A})^{-1/2}\psi|^2 d\mu)^{1/2} \le$$

$$\le \|u\|_{L_2} \sup_M | \prec A\nabla\xi, \nabla\xi \succ |^{1/2} \cdot$$

$$\cdot (\int\limits_{supp\,\xi} \prec A\nabla(1 + H_\mu^{\eta A})^{-1/2}\psi, \nabla(1 + H_\mu^{\eta A})^{-1/2}\psi \succ d\mu)^{1/2} \le$$

$$\le C'_{\xi,u} (\int\limits_M \prec \eta A\nabla(1 + H_\mu^{\eta A})^{-1/2}\psi, \nabla(1 + H_\mu^{\eta A})^{-1/2}\psi \succ d\mu)^{1/2} =$$

$$= C'_{\xi,u} \|(H_\mu^{\eta A})^{1/2}(1 + H_\mu^{\eta A})^{-1/2}\psi\|_{L_2(\mu)} \le C'_{\xi,u}\|\psi\|_{L_2(\mu)}$$

$$|B_2| \le \max |H_\mu^A\xi| \cdot \|u\|_{L_2} \cdot \|(1 + H_\mu^{\eta A})^{-1/2}\psi\|_{L_2} \le C''_{\xi,u}\|\psi\|_{L_2}$$

The previous estimates and identity (8.2) imply that

$$(u\xi) \in \mathcal{D}(\,(H_\mu^{\eta A})^{1/2})$$

Therefore the weak derivation D on function $(u\xi)$ is defined. Put $\varphi = u\xi$ in (8.2). Then

$$\|u\xi\|_{L_2}^2 + \|(H_\mu^{\eta A})^{1/2} u\xi\|_{L_2}^2 =$$

$$= \int_M \{-|u|^2 \xi H_\mu^{\eta A}\xi + \prec u, \prec A\nabla\xi, D\cdot \succ u\xi \succ\} d\mu =$$

$$= \int_{supp\,\xi} \{-|u|^2 \xi H_\mu^{\eta A}\xi + \eta \prec D\frac{|u|^2}{2}, A\nabla\frac{\xi^2}{2} \succ\} d\mu +$$

$$+ \int_{supp\,\xi} |u|^2 \prec A\nabla\xi, \nabla\xi \succ d\mu = - \int_{supp\,\xi} |u|^2 \xi H_\mu^{\eta A}\xi d\mu +$$

$$+ \int_{supp\,\xi} \{\frac{1}{2}|u|^2 H_\mu^{\eta A}\xi^2 + |u|^2 \prec A\nabla\xi, \nabla\xi \succ\} d\mu =$$

$$= \frac{1}{2}\int_M |u|^2 \prec A\nabla\xi, \nabla\xi \succ d\mu$$

So we have proved the following estimate

$$\|u\xi\|_{L_2}^2 \le C \int_M |u|^2 |\nabla\xi|^2 d\mu$$

Here we use that $A \in C_b(M, T^{1,1}M)$.

Let F be smooth function on half-line $\mathcal{R}_+$ such that

$$F(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \in [2,\infty) \end{cases}$$

and $|F'| \le \alpha,\ x \in \mathcal{R}_+$. Then choosing sequence $\xi_n = F(\rho(x, x_0)/n)$ we have $\xi_n \nearrow 1$ and

$$|\nabla\xi_n| \le \max_{t\in[1,2]} \frac{F'(t)}{n} \to 0, \quad n \to \infty$$

Therefore $\|u\|_{L_2} = 0$ and so $u \equiv 0$. ■

References

[1] S. Albeverio and R. Hoegh-Krohn. Dirichlet forms and diffusion processes on Rigged Hilbert space. *Z. Wahrsch. und verw. Geb.*, 40:1–57, 1977.

[2] S. Albeverio, Yu.G. Kondratiev, and M. Röckner. Dirichlet Operators and Gibbs Measures. *BiBoS preprints, FRG*, (541), 1992.

[3] S. Albeverio, Yu.G. Kondratiev, and M. Röckner. Dirichlet Operators via Stochastic Analysis. *BiBoS preprints, FRG*, 1993.

[4] S. Albeverio and M. Röckner. Classical Dirichlet Forms on Topological Vector Spaces - the Construction of the Associated Diffusion Process. *BiBoS preprints, FRG*, (342), 1988.

[5] S. Albeverio and M. Röckner. New developments in the theory and applications of Dirichlet forms. *Ruhr Universität preprints, FRG*, 65, 1989.

[6] A.Val. Antoniouk and Yu.G. Kondratiev. Log-concave smooth measures on Rigged Hilbert space and some properties of corresponding Dirichlet operators. *Bielefeld-Bochum Stochastics preprint*, 478:1–16, 1991.

[7] A.Val. Antoniouk and Yu.G. Kondratiev. Log-Sobolev inequality for Dirichlet operators on Riemannian manifold and its applications. *Bielefeld-Bochum Stochastics preprint*, 460:1–29, 1991.

[8] D. Bakry and M. Emery. Hypercontractivite de semigroups des diffusion. *Compt.Rend.Acad.Sci., Paris, Ser.1*, 299:775–777, 1984.

[9] Yu.M. Berezansky. *Eigenfunctions Expansions of Self-Adjoint Operators.* Kiev: Naukova dumka, 1965. in Russian.

[10] Yu.M. Berezansky and Yu.G. Kondratiev. *Spectral Methods in Infinite Dimensional Analysis.* Naukova dumka, Kiev, 1988. to appear in Kluwer Academic Publishers.

[11] A.L. Besse. *Einstein Manifolds.* Springer-Verlag, 1987.

[12] S. Bochner. Vector fields and Ricci curvature. *Bull.Amer. Math.Soc.*, 52:776–797, 1946.

[13] E.A. Carlen and D.W. Stroock. An application of the Bakry-Emery criterion to infinite dimensional diffusions. *Strasbourg Sem. de Probabilite (Azima and Yor, eds.)*, XX, 1986.

[14] M. Cassandro, E. Olivieri, A. Pelegrinotti, and E. Presutti. Existence and uniqueness of DRL measures for unbounded spin systems. *Z.Wahrs.verw.Geb.*, 41:313–334, 1978.

[15] I. Chavel. *Eigenvalues in Riemannian Geometry.* Academic Press, 1984.

[16] Yu.L. Daletsky and Ya.I. Belopolskaya. *Stochastic equations and differential geometry.* Kiev: Vyscha Shkola, 1989. in Russian.

[17] Yu.L. Daletsky and S.V. Fomin. *Measures and Differential Equations on Infinite Dimensional Spaces.* Moscow: Nauka, 1984. (Kluwer Academic Publishers (1992)).

[18] Yu.L. Daletsky and B.D. Marianin. Smooth measures on infinite-dimensional manifolds. *Soviet Math.Dokl.*, 285(6):863–866, 1985.

[19] Yu.L. Daletsky and S.N. Paramonova. Integration by parts for measures on functional spaces and some applications. *Theor.Probab.and Math.Statist.*, I. 17:51-60, 1977; II. 18:37-45, 1978.

[20] E.B. Davies and B. Simon. Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *Journ.Funct.Anal.*, 59:335–395, 1984.

[21] J.-D. Deushel and D.W. Stroock. Hypercontractivity and spectral gap of symmetric diffusions with applications to the stochastic Ising models. *Journ.Funct.Anal.*, 92:30–48, 1990.

[22] R.L. Dobrushin. The description of a random field by means of conditional probabilities and the conditions governing its regularity. *Theor.Probab.Appl.*, 13:197–224, 1968.

[23] R.L. Dobrushin. The problem of uniqueness of a Gibbs field and the problem of phase transition. *Funct.Anal.Appl.*, 2:302–312, 1968.

[24] R.L. Dobrushin. Prescribing a system of random variables by conditional distributions. *Theor.Probab.Appl.*, 15:458–486, 1970.

[25] R.L. Dobrushin and S.B. Shlosman. Constrictive criterion for the uniqueness of Gibbs field. *Statistical Physics and Dynamical Systems, Rigorous results (Fritz, Jaffe and Szasz, eds.), Birkhäuser*, pages 347–370, 1985.

[26] H. Föllmer. A covariance estimate for Gibbs measures. *Journ.Funct.Anal.*, 46:387–395, 1982.

[27] A. Friedman. *Partial Differential Equations of Parabolic Type.* Prentice - Hall Inc., 1964.

[28] M. Fukushima. *Dirichlet Forms and Markov Processes.* North - Holland, 1980.

[29] L. Gross. Logarithmic Sobolev Inequalities. *Amer.Journ.of Math.*, 97:1061–1083, 1975.

[30] L. Gross. Decay of correlations in classical lattice models at high temperatures. *Comm.Math.Phys.*, 68:9–27, 1979.

[31] R. Holley and D.W. Stroock. Diffusions on an infinite dimensional torus. *Journ.Funct.Anal.*, 42:29–63, 1981.

[32] R. Holley and D.W. Stroock. Logarithmic Sobolev inequalities and stochastic Ising models. *Journ.of Stat.Phys.*, 46:1159–1194, 1987.

[33] R. Holley and D.W. Stroock. Uniform and L_2 - convergence in one dimensional stochastic Ising models. *Comm.Math.Phys.*, 123:85–93, 1989.

[34] D. Klein. Dobrushin uniqueness techniques and the decay of correlations in continuum ststistical mechanics. *Comm.Math.Phys.*, 86:227–246, 1982.

[35] Yu.G. Kondratiev. Dirichlet operators and smoothness of solutions of elliptic equations. *Soviet Math.Dokl.*, 282:269–273, 1985.

[36] Yu.G. Kondratiev and T.V. Tsikalenko. Dirichlet operators and differential equations. *Selecta Math.Sov.*, 10:345–397, 1991.

[37] D. Künsch. Decay of correlations under Dobrushin's uniqueness condition and its applications. *Comm.Math.Phys.*, 84:207–222, 1979.

[38] S. Kusuoka and D. Stroock. Some boundedness properties of certain stationary diffusion semigroups. *Journ.Funct.Anal.*, 60:243–264, 1985.

[39] O.E. Lanford. *Entrophy and equilibrium states in classical Statistical Mechanics*, volume 20. Lecture Notes in Physics, 1973.

[40] O.E. Lanford and D. Ruelle. Observables at infinity and states with short range correlations in Statistical Mechanics. *Comm. Math. Phys.*, 13:194–215, 1969.

[41] V.A. Malyshev and R.A. Minlos. *Gibbs random fields. Method of cluster expansions.* Moscow: Nauka, 1985.

[42] C. Preston. Random fields. In *Lecture Notes in Mathematics*, volume 534. Springer - Verlag, 1976.

[43] M. Reed and B. Simon. *Methods of Modern Mathematical Physics: Fourier Analysis. Self - Adjointness*, volume 2. Academic Press, 1972.

[44] J. Rosen. Sobolev inequalities for weight spaces and supercontractivity. *Trans.Amer.Math.Soc.*, 222:367–376, 1976.

[45] O.E. Rothaus. Analytic inequalities, isoperimetric inequalities and logarithmic Sobolev inequalities. *Journ.Funct.Anal.*, 64:296–313, 1985.

[46] H. Rund. *The Differential Geometry of Finsler Spaces.* Springer-Verlag, 1959.

[47] C.G. Simader. Essential self-adjointness of Schrödinger operators bounded from below. *Math.Z.*, 159:47–50, 1978.

[48] B. Simon. A remark on Dobrushin's uniqueness theorem. *Comm.Math.Phys.*, 68:183–186, 1979.

[49] Ya.G. Sinai. *Theory of Phase Transitions. Rigorous Results.* Moskow: Nauka, 1980. in Russian.

[50] D.W. Stroock and B. Zegarlinski. The equivalence of the logarithmic Sobolev inequality and Dobrushin - Schlosman mixing condition. *Comm. Math. Phys*, 144:303–323, 1992.

[51] D.W. Stroock and B. Zegarlinski. The logarithmic Sobolev inequality for continuous spins on a lattice. *Journ.Funct.Anal.*, 104:299–326, 1992.

[52] L.N. Vasserstein. Markov processes over denumerable products of spaces, describing large systems of automata. *Probl.Trans.Inf.*, 5:64–72, 1969.

[53] P. Wienholtz. Halbbeschrankte partielle differentialoperatoren zweiter ordnung vom elliptischen typus. *Math.Ann.*, 135:50–80, 1958.

[54] B. Zegarlinski. Dobrushin uniqueness theorem and logarithmic Sobolev inequalities. *Journ.Funct.Anal.*, 105:77–111, 1992.

Contents