Towards upper bounds on general second order operators on metric function

Alexander Val. Antoniouk

Department of Nonlinear Analysis, Institute of Mathematics NAS Ukraine,
Tereschenkivska 3, 01601 MSP Kiev-4, Ukraine
antoniouk@imath.kiev.ua

Abstract. We discuss how one can get the upper bounds on the general second order operator acting on metric function. The suggested approach does not use traditional formulas for deviations of geodesics and Jacobi fields construction and leads to the manifolds version of coercitivity and dissipativity conditions. These conditions generalize known Krylov-Rosovskii-Pardoux conditions from linear space to the manifold setting.

The research was supported by A.von Humboldt Foundation (Germany).

In this paper we turn to the geometric problem of optimal upper estimates on the general second order operators, acting on metric function. We consider operators of form

$$\mathcal{L} = \frac{1}{2}\sum_{\sigma=1}^{d} \nabla^2_{A_\sigma} + \nabla_{A_0}, \tag{1}$$

where A_0, A_α represent smooth globally defined vector fields on the oriented smooth connected Riemannian manifold M without boundary. Such operators are important in applications, because they give generators of the following Stratonovich diffusions on manifold

$$y_t^x = x + \int_0^t A_0(y_s^x)ds + \sum_\sigma \int_0^t A_\sigma(y_t^x)\delta W_t^\sigma \tag{2}$$

where δW^σ means Stratonovich differential with respect to a family of one dimensional independent Wiener processes W_t^σ, range of index σ corresponds to the dimension of manifold.

The known procedures of differential geometry were mostly invented for the case of Δ or similar operators [5, 6], because it was hard to find the implicit representations for arbitrary differential operators on metric function, defined as a minimum of length functional

$$\rho^2(x,y) = \inf\{\int_0^1 |\dot\gamma(\ell)|^2 d\ell,\ \gamma(0) = x,\ \gamma(1) = y\} \tag{3}$$

Corresponding techniques were related with geodesic deviations and arising Jacobi fields, harmonic tensors and Betti numbers, etc, e.g. [5, 6].

However, for upper bounds we do not need the precise representations for differential operators on metric (!). Below we develop such estimates and demonstrate, that the traditional approach of geodesic deviations is a little advanced for such simple problem.

These upper estimates are also important in applications, because, for example, to have nonexplosion and continuity on initial data estimates on process y^x_t

$$\mathbf{E}\rho^2(y^x_t, y^y_t) \le e^{Kt}\rho^2(x, y), \quad \mathbf{E}\rho^2(y^x_t, o) \le e^{Kt}(1 + \rho^2(x, o))$$

one should write Itô formula for geodesic distance

$$\rho^2(y^x_t, y^y_t) = \rho^2(x, y) + \sum_{\sigma=1}^{d} \int_0^t [(A^1_\sigma + A^2_\sigma)\rho^2](y^x_s, y^y_s)dW^\sigma_s +$$

$$+ \int_0^t \{A^1_0 + A^2_0 + \frac{1}{2}\sum_{\sigma=1}^{d}(A^1_\sigma + A^2_\sigma)^2\}\rho^2(y^x_s, y^y_s)ds \tag{4}$$

or similar for $\rho^2(y^x_t, o)$ ($o \in M$ is some point of manifold), and ask for upper estimates by $K\rho^2$. Notation A^1, A^2 means vector fields, acting on corresponding first and second variables.

The found conditions on coefficients of operator $\mathcal{L}$ generalize the classical Krylov - Rosovskii - Pardoux conditions [7, 8] from the linear base space to manifold. They relate the coefficients of operator with the geometric properties of manifold, without traditional separation of geometry:

- **dissipativity**: $\exists\, o \in M$ such that $\forall C \in I\!R^1\ \exists\, K_C \in I\!R^1$ such that $\forall x \in M$

$$< \widetilde{A_0}(x), \nabla^x\rho^2_M(x, o) > +C\sum_{\sigma=1}^{d}\|A_\sigma(x)\|^2 \le K_C(1 + \rho^2_M(x, o)) \tag{5}$$

- **differential coercitivity**: $\forall C, C' \in I\!R^1\ \exists\, K_C \in I\!R^1$ such that $\forall x \in M$, $\forall h \in T_xM$

$$< \nabla\widetilde{A_0}(x)[h], h > +C\sum_{\sigma=1}^{d}\|\nabla A_\sigma(x)[h]\|^2 + C'\sum_{\sigma=1}^{d} < R_x(A_\sigma(x), h)A_\sigma(x), h > \le K_C\|h\|^2 \tag{6}$$

  where $\widetilde{A_0} = A_0 + \frac{1}{2}\sum_{\sigma=1}^{d}\nabla_{A_\sigma}A_\sigma$ and $[R(A, h)A]^m = R_p{}^m{}_{\ell q}A^pA^\ell h^q$ denotes the curvature operator, related with (1,3) curvature tensor with components

$$R_1{}^2{}_{34} = \frac{\partial\Gamma_1{}^2{}_3}{\partial x^4} - \frac{\partial\Gamma_1{}^2{}_4}{\partial x^3} + \Gamma_1{}^j{}_3\Gamma_j{}^2{}_4 - \Gamma_1{}^j{}_4\Gamma_j{}^2{}_3 \tag{7}$$

  For simplicity of further calculations we only point the positions of corresponding indexes.

  Notation $\nabla H[h]$ means the directional covariant derivative, defined by

$$(\nabla H(x)[h])^i = \nabla_j H^i(x) \cdot h^j \tag{8}$$

Main result of article provides

Theorem 1 *Suppose that conditions (5) - (6) on smooth vector fields A_α, A_0 hold.*
Then there is constant K such that

$$\{A^1_0 + A^2_0 + \frac{1}{2}\sum_{\sigma=1}^{d}(A^1_\sigma + A^y_\sigma)^2\}\rho^2(x, y) \le K\rho^2(x, y) \tag{9}$$

Similarly $\forall C\ \exists\, K_C$ such that

$$\mathcal{L}^1\rho^2(x, y) + C\sum_{\sigma=1}^{d}\frac{(A^1_\sigma\rho^2(x, y))^2}{\rho^2(x, y)} \le K(1 + \rho^2(x, y)) \tag{10}$$

Proof. First note, that for any vector field H in a vicinity of some point z of manifold there are following representations

$$Hf(z) = \lim_{\varepsilon\to 0} \frac{f(z^\varepsilon) - f(z)}{\varepsilon}, \quad H(Hf)(z) = \lim_{\varepsilon\to 0} \frac{f(z^\varepsilon) + f(z^{-\varepsilon}) - 2f(z)}{\varepsilon^2} \tag{11}$$

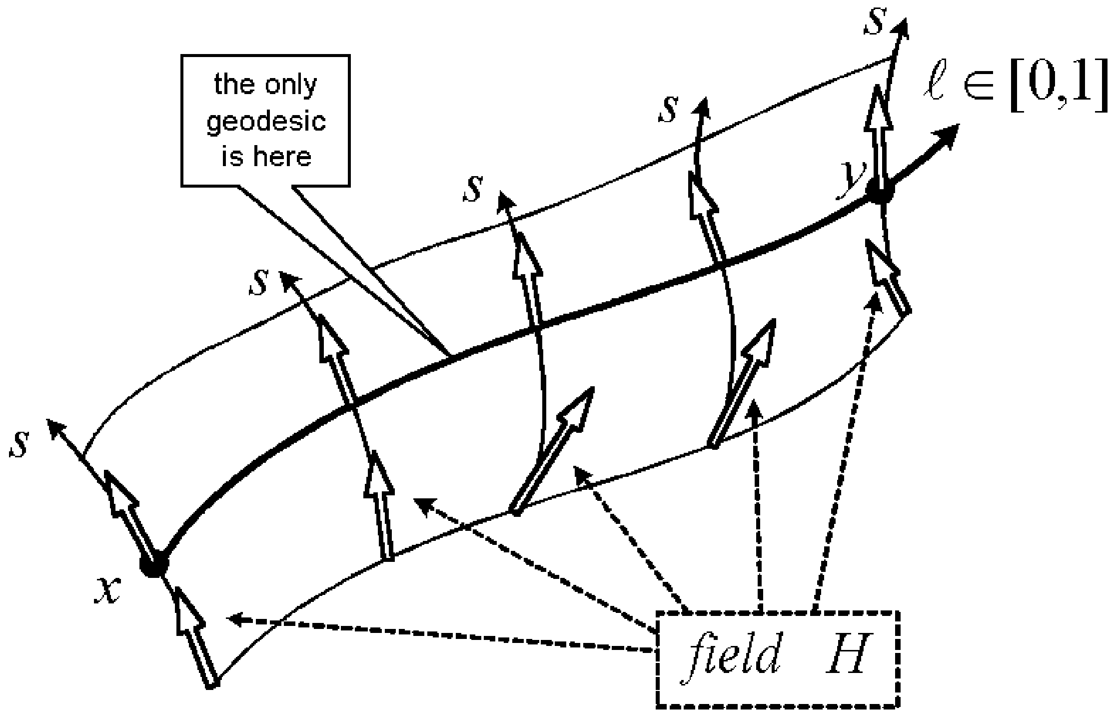
that can be verified in local coordinates. Here z^ε is a shift along field H: $z^\varepsilon = z + \int_0^\varepsilon H(z^s)ds$.

In the vicinity of geodesic γ from x to y that minimizes (3) consider a vector field H. Introduce a family of paths

$$[0,1] \times (-\delta, \delta) \ni (\ell, s) \to \gamma(\ell, s) \in M$$

such that at $s = 0$ path $\gamma(\ell, s)\Big|_{s=0} = \gamma(\ell)$ gives the minimal geodesic above and parameter s appears as a result of evolution along H:

$$\frac{\partial}{\partial s}\gamma(\ell, s) = H(\gamma(\ell, s)) \tag{12}$$



Picture 1. How to avoid complications with geodesic deviations.

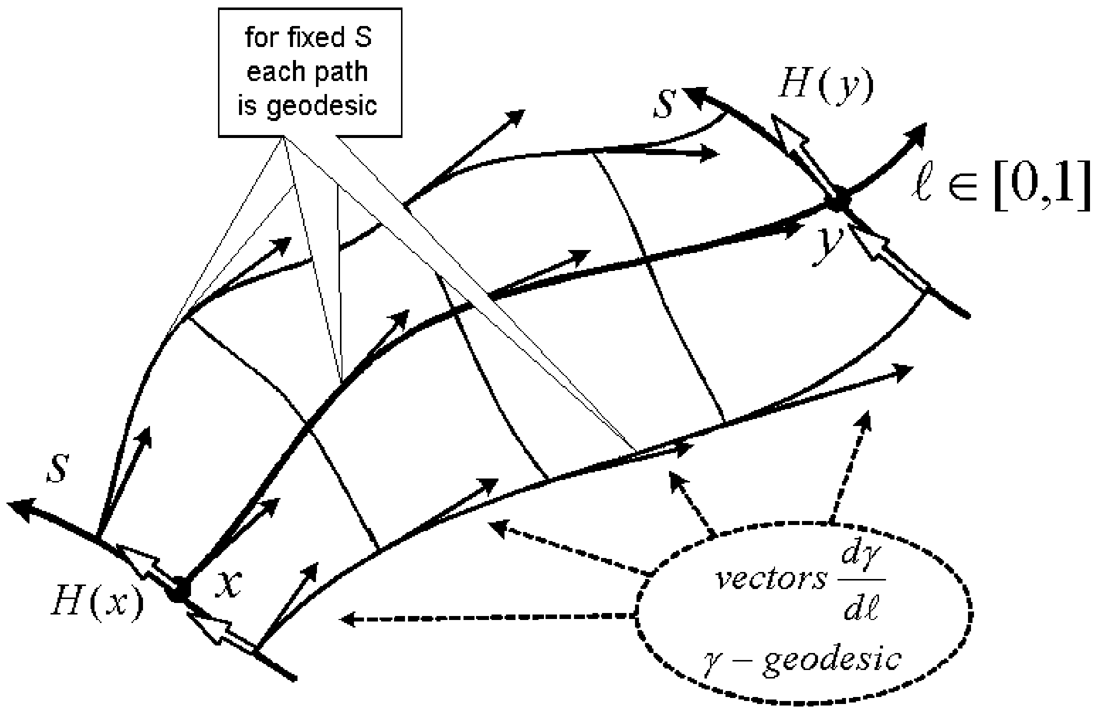
Field H (white vectors) in a vicinity of geodesic from x to y determines a set of paths, parameterized by s. The resulting smooth 2-dimensional surface is parameterized by $(\ell, s) \in [0,1] \times (-\delta, \delta)$. Note that for $s \neq 0$ each path $\gamma(\ell, s)_{s-\text{fixed}}$ must not be geodesic, unlike in formulas for geodesic deviations.

Using the minimal property of geodesic, i.e. that the path path $\gamma(\ell, s)$ is longer than geodesic from $\gamma(0, s)$ to $\gamma(1, s)$, we can estimate terms with $\pm\varepsilon$ in (11) from above and obtain (point at which we get rid of implicit representations)

$$(H^1 + H^2)\rho^2(x, y) = \lim_{\varepsilon\to 0} \frac{\rho^2(\gamma(1, \varepsilon), \gamma(0, \varepsilon)) - \rho^2(x, y)}{\varepsilon} \leq$$

$$\leq \lim_{\varepsilon\to 0} \frac{\int_0^1 |\frac{\partial}{\partial \ell}\gamma(\ell,s)|^2 d\ell - \int_0^1 |\frac{\partial}{\partial \ell}\gamma(\ell,0)|^2 d\ell}{\varepsilon} = \int_0^1 \frac{\partial}{\partial s}\Big|_{s=0} |\frac{\partial}{\partial \ell}\gamma(\ell,s)|^2 d\ell \tag{13}$$

Let us now make a comparison of approach (13) with the classical deviations of geodesic formulas. If we had not used the above estimate $\leq$ in (13), the path $\gamma(\ell, s)$, that realizes the distance $\rho^2(\gamma(0,s),\gamma(1,s))$, would **have to be geodesic** and we would obtain a standard picture



Picture 2. Classical deviations of geodesics.
Field H constructs only paths $\gamma(0,s)$ and $\gamma(1,s)$ that go through points x and y correspondingly. After that for any $s \in (-\delta,\delta)$ the points $\gamma(0,s)$ and $\gamma(1,s)$ must be connected by geodesic $\gamma(\ell,s)$, parameter $\ell \in [0,1]$ being proportional to the geodesic length parameter.

Before turning back to the proof, let us briefly remark how Jacobi fields appear in the approach of picture 2. One calculates a formal line

$$\nabla_\ell(\nabla_\ell \frac{\partial\gamma}{\partial\ell}) \underset{(1)}{=} \nabla_\ell(\nabla_s \frac{\partial\gamma}{\partial\ell}) \underset{(2)}{=} \nabla_s(\nabla_\ell \frac{\partial\gamma}{\partial\ell}) - R(\frac{\partial\gamma}{\partial\ell}, \frac{\partial\gamma}{\partial s})\frac{\partial\gamma}{\partial\ell} \tag{14}$$

where at step (1) one uses that fields $\dfrac{\partial\gamma}{\partial s}$ and $\dfrac{\partial\gamma}{\partial\ell}$ commute as forming 2-dimensional surface, so $\nabla_\ell \dfrac{\partial\gamma}{\partial s} = \nabla_s \dfrac{\partial\gamma}{\partial\ell}$. At step (2) one should commute $[\nabla_\ell, \nabla_s]$ to get the curvature.

Taking into account that $\nabla_\ell \dfrac{\partial\gamma}{\partial\ell} = 0$ (equation of geodesic $\gamma(\ell,s)_{s-\text{fixed}}$) one gets

$$\nabla_\ell^2 \frac{\partial\gamma}{\partial s} + R(\frac{\partial\gamma}{\partial\ell}, \frac{\partial\gamma}{\partial s})\frac{\partial\gamma}{\partial\ell} = 0 \quad \text{or} \quad \frac{d^2}{d\ell^2}(\frac{\partial\gamma}{\partial s}) + R(\frac{\partial\gamma}{\partial\ell}, \frac{\partial\gamma}{\partial s})\frac{\partial\gamma}{\partial\ell} = 0 \tag{15}$$

i.e. the second order Sturm-Liouville equation for Jacobi field $\dfrac{\partial\gamma}{\partial s}$. Jacobi fields describe small increments of geodesics, that again form geodesics, the study of their different properties is seriously complicated by the behaviour of geometry in the whole [5, 6].

In comparison to the approach of picture 2 the one, suggested in picture 1 and inequality (13) avoids these complications. Moreover, it is in some sense dual to the approach of picture 2, because further calculations at lines (19)-(25) are similar to the lines (14)-(15), but we calculate below $\dfrac{d^2}{ds^2}\dfrac{d\gamma}{d\ell}$ instead of $\dfrac{d^2}{d\ell^2}\dfrac{d\gamma}{ds}$ and, therefore, do not need to introduce Jacobi fields (recall that $\gamma(\ell,s)_{s-\text{fixed},s\neq 0}$ should not be geodesic in our case).

In such a sense the approach of picture 1 changes the implicit approach of geodesic deviations (picture 2) to the explicit level of direct calculations. We conclude that the approach of geodesic deviations is too advanced for such a simple problem, like estimates (9), (10).

Let us now return to the proof of theorem. To proceed further, we need a preparatory calculation from Riemannian geometry: for any vector fields U, V in vicinity of path $\gamma(\ell,s)$ one has

$$\frac{\partial}{\partial s} < U, V >_{T_{\gamma(\ell,s)}} = \frac{\partial}{\partial s}[\, g_{ij}(\gamma)U^i(\gamma)V^j(\gamma)\,] =$$

$$= g_{ij}[\, V^j\partial_k U^i \frac{\partial\gamma^k}{\partial s} + U^i\partial_k V^j\frac{\partial\gamma^k}{\partial s}] + \partial_k g_{ij}\frac{\partial\gamma^k}{\partial s}U^iV^j =$$

$$=< \nabla U[\frac{\partial\gamma}{\partial s}], V > + < U, \nabla V[\frac{\partial\gamma}{\partial s}] > \tag{16}$$

where we applied autoparallel property of Riemannian connection

$$\partial_k g_{mn}(x) = g_{hn}\Gamma_{k\ m}^{\ h} + g_{mh}\Gamma_{k\ n}^{\ h} \tag{17}$$

and substituted the definition of covariant derivative of vector field. We also used notation (8).

To find from (16) the representation of $\dfrac{\partial}{\partial s}|\dot\gamma(\ell,s)|^2$ we first calculate this expression in local coordinates, that have (ℓ,s) as the coordinate functions and return, after that, to the invariant notations

$$\{\nabla\frac{\partial\gamma}{\partial\ell}\}[\frac{\partial\gamma}{\partial s}]^i = \nabla_k\frac{\partial\gamma^i}{\partial\ell}\cdot\frac{\partial\gamma^k}{\partial s} = [\partial_k\frac{\partial\gamma^i}{\partial\ell} - \Gamma_{k\ h}^{\ i}\frac{\partial\gamma^h}{\partial\ell}]\frac{\partial\gamma^k}{\partial s} =$$

$$= \frac{\partial}{\partial s}(\frac{\partial\gamma^i}{\partial\ell}) - \Gamma_{k\ h}^{\ i}\frac{\partial\gamma^h}{\partial\ell}\frac{\partial\gamma^k}{\partial s} = \frac{\partial}{\partial\ell}(\frac{\partial\gamma^i}{\partial s}) - \Gamma_{k\ h}^{\ i}\frac{\partial\gamma^h}{\partial\ell}\frac{\partial\gamma^k}{\partial s} =$$

$$= \frac{\partial}{\partial\ell}H^i(\gamma) - \Gamma_{k\ h}^{\ i}\frac{\partial\gamma^h}{\partial\ell}\frac{\partial\gamma^k}{\partial s} = \{\nabla H\}[\frac{\partial\gamma}{\partial\ell}]^i \tag{18}$$

Above we commute $\dfrac{\partial}{\partial s}$ and $\dfrac{\partial}{\partial\ell}$, because by construction and C^∞ differentiability of field H variables (ℓ,s) form a part of local coordinate system in the vicinity of geodesic for a.e. ℓ.

We continue (13) and find from (16) and (18) that

$$(H^1 + H^2)\rho^2(x,y) \leq 2\int_0^1 < \nabla H[\dot\gamma], \dot\gamma > d\ell$$

In case when $\dfrac{\partial}{\partial \ell}$ and $\dfrac{\partial}{\partial s}$ do not commute, i.e. there are points along γ where vector field H is parallel to $\dfrac{\partial\gamma}{\partial\ell}$, one should first consider a small deformation $H_\delta = H + \delta V$ by C^∞ field V, such that $V = 0$ at points where $H = 0$ or is not parallel to $\dfrac{\partial\gamma}{\partial\ell}$. From another side we demand for all small δ that field H_δ should be not parallel to $\dfrac{\partial\gamma}{\partial\ell}$ in parts of geodesic, where H was parallel. Then the above commutation becomes correct for these small $\delta > 0$.

Tending the resulting inequality to limit $\delta \to 0$, one comes to the statements above. We remark that in such a way we also avoid the usual problems of zeros and behaviour on the infinity of Jacobi fields, traditional for the deviations of geodesics approach.

In a similar way,

$$(H^1 + H^2)(H^1 + H^2)\rho^2(x, y)) = \lim_{\varepsilon\to 0} \frac{\rho^2(\gamma(1, \varepsilon)\gamma(0, \varepsilon)) + \rho^2(\gamma(1, -\varepsilon), \gamma(0, \varepsilon)) - 2\rho^2(x, y)}{\varepsilon^2} \leq$$

$$\leq \lim_{\varepsilon\to 0} \frac{\int_0^1 |\frac{\partial}{\partial\ell}\gamma(\ell, \varepsilon)|^2 d\ell + \int_0^1 |\frac{\partial}{\partial\ell}\gamma(\ell, -\varepsilon)|^2 d\ell - 2\int_0^1 |\frac{\partial}{\partial\ell}\gamma(\ell, 0)|^2 d\ell}{\varepsilon^2} = \int_0^1 \frac{\partial^2}{\partial s^2}\Big|_{s=0} |\frac{\partial}{\partial\ell}\gamma(\ell, \varepsilon)|^2 d\ell$$

By (16) and (18) we find, like before, that

$$\frac{\partial^2}{\partial s^2}|\frac{\partial\gamma(\ell, s)}{\partial\ell}|^2 = 2\frac{\partial}{\partial s} < \nabla H[\frac{\partial\gamma}{\partial\ell}], \frac{\partial\gamma}{\partial\ell} >=$$

$$= 2|\nabla H[\frac{\partial\gamma}{\partial\ell}]\,|^2 + < \nabla\{\nabla H[\frac{\partial\gamma}{\partial\ell}]\}[\frac{\partial\gamma}{\partial s}], \frac{\partial\gamma}{\partial\ell} > \tag{19}$$

and have to calculate the last expression.

Turning to the coordinate representations we find

$$(\nabla\{\nabla H[\frac{\partial\gamma}{\partial\ell}]\}[\frac{\partial\gamma}{\partial s}])^n = \frac{\partial\gamma^k}{\partial s}\nabla_k[\nabla_j H^n \cdot \frac{\partial\gamma^j}{\partial\ell}] = \tag{20}$$

$$= \nabla_j H^n(\frac{\partial\gamma^k}{\partial s}\nabla_k\frac{\partial\gamma^j}{\partial\ell}) + \nabla_k\nabla_j H^n \frac{\partial\gamma^k}{\partial s}\frac{\partial\gamma^j}{\partial\ell} = \tag{21}$$

$$= \nabla_j H^n \frac{\partial\gamma^k}{\partial\ell}\nabla_k H^j + H^k\frac{\partial\gamma^j}{\partial\ell}\nabla_k\nabla_j H^n = \tag{22}$$

$$= (\nabla(\nabla_H H)[\frac{\partial\gamma}{\partial\ell}])^n - H^j\frac{\partial\gamma^k}{\partial\ell}\nabla_k\nabla_j H^n + H^k\frac{\partial\gamma^j}{\partial\ell}\nabla_k\nabla_j H^n = \tag{23}$$

$$= (\nabla(\nabla_H H)[\frac{\partial\gamma}{\partial\ell}])^n + H^k\frac{\partial\gamma^j}{\partial\ell}R_{\ell\ kj}^{\ n}H^\ell = \tag{24}$$

$$= (\nabla(\nabla_H H)[\frac{\partial\gamma}{\partial\ell}])^n + R(H, \frac{\partial\gamma}{\partial\ell})H \tag{25}$$

where we have made the following transformations

– come from invariant to coordinate representations in (20)

– splitted $\nabla(AB) = (\nabla A)B + A(\nabla B)$ to transfer from (20) to (21)

– substituted representation (18) for the first term in (21) to get the first term in (22)

– rewrote term $(22)_1$ to the first two terms in (23)

– extracted curvature from commutator $H^k\dfrac{\partial\gamma^j}{\partial\ell}[\nabla_k, \nabla_j]H^n$ in terms $(23)_{2+3}$.

Returning to (19) we finally get estimate

$$(H^1+H^2)^2\rho^2(x,y)) \le \int_0^1 \{|\nabla H[\dot\gamma]\,|^2 + <\nabla(\nabla_H H)[\dot\gamma],\dot\gamma> + <R(H,\dot\gamma)H,\dot\gamma>\}d\ell$$

Taking now $H=A_0$ for the first order estimate and $H=A_\alpha$ for the second order estimate we find

$$\{A_0^1+A_0^2+\frac{1}{2}\sum_{\alpha=1}^{d}(A_\alpha^1+A_\alpha^y)^2\}\rho^2(x,y) \le \int_0^1 (<\nabla\widetilde{A_0}[\dot\gamma],\dot\gamma> +$$

$$+\frac{1}{2}\sum_{\alpha=1}^{d}\{|\nabla A_\alpha[\dot\gamma]\,|^2 + <R(A_\alpha,\dot\gamma)A_\alpha,\dot\gamma>\})d\ell$$

Coercitivity assumption (6), in view of (3) leads to the statement (9).

To get estimate (10), one proceeds like above with choice

$$\frac{\partial}{\partial s}\gamma(\ell,s)=c(\ell)H(\gamma(\ell,s))$$

instead of (12), taking $c=a$ and $c=b$ for the first A_0 and second $(A_\alpha)^2$ order operators correspondingly. One comes to the estimate

$$\{A_0^1+\frac{1}{2}\sum_{\alpha=1}^{d}(A_\alpha^1)\}\rho^2(x,y) \le \int_0^1 (a<\nabla A_0[\dot\gamma],\dot\gamma> + \frac{\partial a}{\partial \ell}<A_0,\dot\gamma> +$$

$$+\sum_{\alpha=1}^{d}\{b^2|\nabla A_\alpha[\dot\gamma]\,|^2 + \frac{\partial b^2}{\partial \ell}<A_\alpha,\nabla A_\alpha[\dot\gamma]> + (\frac{\partial b}{\partial \ell})^2|A_\alpha(\gamma)|^2 +$$

$$+b^2<\nabla(\nabla_{A_\alpha}A_\alpha)[\dot\gamma],\dot\gamma> + b^2<R(A_\alpha,\dot\gamma)A_\alpha,\dot\gamma> + \frac{\partial b^2}{\partial \ell}<\nabla_{A_\alpha}A_\alpha,\dot\gamma>\})d\ell \tag{26}$$

Taking further $a(\ell)=b^2(\ell),\ b(\ell)=1-\ell$ and using estimate

$$|<\nabla A_\alpha[\dot\gamma],A_\alpha>| \le \frac{(1-\ell)}{2}|\nabla A_\alpha[\dot\gamma]\,|^2 + \frac{1}{2(1-\ell)}|A_\alpha|^2$$

we find

$$\mathcal{L}^1\rho^2(x,y) \le 2\int_0^1 \{(1-\ell)^2(<\nabla\widetilde{A_0}[\dot\gamma],\dot\gamma> + \sum_{\alpha=1}^{d} 3/2|\nabla A_\alpha[\dot\gamma]\,|^2 + R(A_\alpha,\dot\gamma)A_\alpha,\dot\gamma>) + \tag{27}$$

$$+2(1-\ell)<\widetilde{A_0}(\gamma),\dot\gamma> + 2\sum_{\alpha=1}^{d}|A_\alpha|^2\}d\ell \tag{28}$$

Now by proportionality arguments

$$\nabla\rho^2(\gamma(\ell),y) = 2\rho(\gamma(\ell),y)\nabla\rho(\gamma(\ell),y) = 2(1-\ell)\rho(x,y)\frac{\dot\gamma(\ell)}{\rho(x,y)} = 2(1-\ell)\dot\gamma$$

so the first term in the last line gives

$$2(1-\ell)<\widetilde{A_0}(\gamma),\dot\gamma> = <\widetilde{A_0}(\gamma),\nabla^{\gamma(\ell)}\rho^2(\gamma(\ell),y)>$$

Finally, using the coercitivity and dissipativity assumption (5-6) for lines (27) and (28) correspondingly, we conclude

$$\mathcal{L}^1\rho^2(x,y) \le 2\int_0^1 \{K_C(1-\ell)^2|\dot\gamma|^2K_{C'}(1+\rho^2(\gamma(\ell),y))\}d\ell \le K(1+\rho^2(x,y))$$

Term $\frac{(A^1_\alpha \rho^2(x,y))^2}{\rho^2(x,y)}$ in (10) is treated like the first order term in (13) with choice of coefficient $c(\ell) = b(\ell)$. We get

$$A^1_\alpha \rho^2(x,y) \leq \int_0^1 \{ \frac{\partial b}{\partial \ell} < A_\alpha(\gamma), \dot\gamma > + b < \nabla A_\alpha[\dot\gamma], \dot\gamma > \} d\ell$$

therefore

$$(A^1_\alpha \rho^2(x,y))^2 \leq 2 \int_0^1 \{ (\frac{\partial b}{\partial \ell})^2 \|A_\alpha\|^2 + b^2 \|\nabla A_\alpha[\dot\gamma]\,\|^2 \} d\ell \cdot \int_0^1 |\dot\gamma|^2 d\ell$$

The last integral gives $\rho^2(x,y)$ by (3), therefore we can add these expressions to (26) and, applying again the dissipativity and coercitivity conditions (5)-(6), finish estimate (10).

Let us again recall that for $x, y \in M$ such that there appear points along geodesic γ at which field H is parallel to $\frac{\partial \gamma}{\partial \ell}$, one should first consider its perturbations H_δ, like discussed for the first order estimate above. One can also go into extra dimensions by embedding theorems if prefers. ■

The applications of upper bounds (9)-(10) to the smooth properties of parabolic equations on manifolds are discussed in [1]-[4].

References

[1] A.Val.Antoniouk, Nonlinear Symmetries of Variational Calculus and Regularity Properties of Differential Flows on Non-Compact Manifolds, to appear in Proceedings of 5^{th} Intern. Conf. "Symmetry in nonlinear mathematical physics", 2003, 9 pp.

[2] A.Val.Antoniouk, Nonlinear Estimates on Regularity of Differential Flows on Manifolds, to appear in *Mathematical methods and physical-mechanical fields*, 8 pp.

[3] Antoniouk A.Val., Antoniouk A.Vict., Regularity of Nonlinear Flows on Noncompact Riemannian Manifolds: Differential vs. Stochastic Geometry or What kind of Variations are Natural?, submitted to press.

[4] A.Val.Antoniouk, A.Vict.Antoniouk, Nonlinear Calculus of variations for differential flows on manifolds: geomentrically correct generalization of covariant and stochastic variations, to appear in *Ukrainean Math. Bulletin*, 25 pp.

[5] Besse A.L., *Manifolds all of whose geodesics are closed*, Springer-Verlag, 1978.

[6] Cheeger J., Ebin D.G., *Comparison theorems in Riemannian geometry*, North-Holland Publ.Co., 1975

[7] Krylov N.V., Rozovskii B.L., On the evolutionary stochastic equations, *Ser. "Contemporary problems of Mathematics"*, VINITI, Moscow, **14**, 71-146 (1979).

[8] Pardoux E., Stochastic partial differential equations and filtering of diffusion processes, *Stochastics*, **3**, 127-167 (1979).