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Direct and inverse approximation theorems of 2π-periodic functions by Taylor–Abel–Poisson means

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We obtain direct and inverse approximation theorems of 2π-periodic functions by Taylor–Abel–Poisson operators in the integral metric.

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It is well-known that any function $f\in L_p$, $f\not\equiv$ const, can be approximated by its Abel-Poisson means $f(\varrho,\cdot)$ with a precision not better than $1-\varrho$. It relates to the so-called saturation property of this approximation method. From this property, it follows that for any $f\in L_p$, the relation $\|f-f(\varrho,\cdot)\|_p=o(1-\varrho)$, $\varrho\to 1-$, holds only in the trivial case where $f\equiv$ const. Therefore, any additional restrictions on the smoothness of functions don't give us the order of approximation better than $1-\varrho$. In this connection, a natural question is to find a linear operator, constructed similarly to the Poisson operator, which takes into account the smoothness properties of functions and at the same time, for a given functional class, is the best in a certain sense. In [17], for classes of convolutions, whose kernels were generated by some moment sequences, the authors proposed the general method of construction of similar operators that take into account properties of such kernels and hence, the smoothness of functions from corresponding classes. One example of such operators are the operators $A_{\varrho,r}$, which are the main subject of study in this paper.

The operators $A_{\varrho,r}$ were first studied in [14], where in the terms of these operators, the author gave the structural characteristic of Hardy-Lipschitz classes $H^r_p\,\mathrm{Lip}\,\alpha$ of functions of one variable, holomorphic on the unit circle of the complex plane. In [15], in terms of approximation estimates of such operators in some spaces S^p of Sobolev type, the authors give a constructive description of classes of functions of several variables, whose generalized derivatives belong to the classes S^pH_ω.

Similar operators of polynomial type were studied in [7], [9], [11], [6] etc. In particular, in [7], the authors found the degree of convergence of the well-known Euler and Taylor means to the functions f from some subclasses of the Lipschitz classes Lipα in the uniform norm. In [11], the analogical results for Taylor means were obtained in the L_p–norm.

In this paper, we continue the study of approximation properties of the operators $A_{\varrho,r}$. In particular, we find the relation of these operators with the operators $L_{\varrho,r}$ and $B_{\varrho,r}$, considered in [10] and [3]. Also we give direct and inverse approximation theorems by the operators $A_{\varrho,r}$ in the terms of K–functionals of functions, generated by their radial derivatives.

Let $L_p = L_p(\mathbb{T})$, $1 \le p \le \infty$, be the space of all functions f, given on the torus $\mathbb{T}$, with the usual norm

$$\|f\|_p := \|f\|_{L_p(\mathbb{T})} := \begin{cases} \Big(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(x)|dx\Big)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}\limits_{x\in[0,2\pi]} |f(x)|, & p = \infty. \end{cases}$$

Further, let $f \in L_1$, the Fourier coefficients of f are given by

$$\widehat{f}_k := \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}kt}dx, \quad k \in \mathbb{Z}.$$

We denote by $f\,(\varrho, x)$, $0 \le \varrho < 1$, the Poisson integral (the Poisson operator) of f, i.e.,

$$f\,(\varrho, x) := \frac{1}{2\pi}\int_0^{2\pi} f(t)P(\varrho, x-t)dt, \tag{1}$$

where $P(\varrho, t) = \frac{1-\varrho^2}{|1-\varrho\mathrm{e}^{\mathrm{i}t}|^2}$ is the Poisson kernel.

Leis [10] considered the transformation

$$L_{\varrho,r}(f)(x) := \sum_{k=0}^{r-1} \frac{\mathrm{d}^k f(x)}{\mathrm{d}n^k}\cdot\frac{(1-\varrho)^k}{k!}, \quad r \in \mathbb{N},$$

where

$$\frac{\mathrm{d}f(x)}{\mathrm{d}n} = -\frac{\partial f(\varrho, x)}{\partial\varrho}\bigg|_{\varrho=1}$$

is the normal derivative of the function f. He showed that if $1 < p < \infty$ and

$$\|f(\varrho,\cdot) - L_{\varrho,r}(f)(\cdot)\|_p = O\Big(\frac{(1-\varrho)^r}{r!}\Big), \quad \varrho \to 1-,$$

then $\mathrm{d}^r/\mathrm{d}n^r f \in L_p$.

Butzer and Sunouchi [3] considered the transformation

$$B_{\varrho,r}(f)(x) := \sum_{k=0}^{r-1}(-1)^{\frac{k+1}{2}} f^{\{k\}}(x)\frac{(-\ln\varrho)^k}{k!},$$

where

$$f^{\{k\}}(x) = \begin{cases} f^{(k)}, & k \in 2\mathbb{Z}_+, \\ \widetilde{f}^{(k)}, & k-1 \in 2\mathbb{Z}_+. \end{cases}$$

They proved the following theorem:

Theorem A [3]. *Assume that* $f \in L_p$, $1 \le p < \infty$.

i) If the derivatives $f^{\{j\}}$, $j = 0, 1, \ldots, r-1$, *are absolutely continuous and* $f^{\{r\}} \in L_p$, *then*

$$\|f(\varrho, \cdot) - B_{\varrho,r}(f)(\cdot)\|_p = O\Big(\frac{(-\ln \varrho)^r}{r!}\Big), \quad \varrho \to 1- \quad . \tag{2}$$

ii) If the derivatives $f^{\{j\}}$, $j = 0, 1, \ldots, r-2$, $r \ge 2$, *are absolutely continuous,* $f^{\{r-1\}} \in L_p$, $1 < p < \infty$, *and relation (2) holds, then* $\widetilde{f}^{\{r-1\}}$ *is absolutely continuous and* $\widetilde{f}^{\{r\}} \in L_p$.

These results summarize the approximation behaviour of the operators $L_{\varrho,r}$ and $B_{\varrho,r}$ in the space L_p. In particular, Leis's result and the statement *ii*) of Theorem A represent the so-called inverse theorems and the statement *i*) is the so-called direct theorem. Direct and inverse theorems are one of the central theorems of approximation theory. They were studied by many authors. Here, we mention only the books [2, 8, 16], which contain fundamental results in this subject. The given results are based on the investigations in the papers [4, 5], where the authors find the direct and inverse approximation theorems for the one-parameter semi-groups of bounded linear transformations $\{T(t)\}$ of some Banach space X into itself by the "Taylor polynomial" $\sum_{k=0}^{r-1}(t^k/k!)A^k f$, where Af is the infinitesimal operator of a semi–group $\{T(t)\}$.

The transformations $A_{\varrho,r}$, considered in this paper, are similar to the transformations $L_{\varrho,r}$ and $B_{\varrho,r}$ as they are also based on the "Taylor polynomials". The transformation $A_{\varrho,r}$ are defined in the following way:

For $\varrho \in [0, 1)$, $r \in \mathbb{N}$ and $f \in L_1$, we set

$$A_{\varrho,r}(f)(t) := \sum_{k\in\mathbb{Z}} \lambda_{|k|,r}(\varrho)\widehat{f}_k \mathrm{e}^{\mathrm{i}kt}, \tag{3}$$

where for $k = 0, 1, \ldots, r-1$, the numbers $\lambda_{k,r}(\varrho) \equiv 1$ and

$$\lambda_{k,r}(\varrho) := \sum_{j=0}^{r-1} \binom{k}{j} (1-\varrho)^j \varrho^{k-j}, k = r, r+1, \ldots, \qquad \varrho \in [0, 1]. \tag{4}$$

The transformation $A_{\varrho,r}$ can be considered as a linear operator on L_1 into itself. Indeed, $\lambda_{k,r}(0)=0$ and for all $k = r, r+1, \ldots$ and $\varrho \in (0, 1)$, we have

$$\sum_{j=0}^{r-1} \binom{k}{j} (1-\varrho)^j \varrho^{k-j} \le r q^k k^{r-1}, \text{ where } 0 < q := \max\{1-\varrho, \varrho\} < 1.$$

Therefore, for any function $f \in L_1$ and for any $0 < \varrho < 1$, the series on the right-hand side of (3) is majorized by the convergent series $2r\|f\|_1 \sum_{k=r}^{\infty} q^k k^{r-1}$.

Note that if the function $f \in L_1$ and it has the Fourier series of power type, i.e., $f(x) \sim \sum_{k=0}^{\infty} \widehat{f}_k \mathrm{e}^{\mathrm{i}kx}$, then $f(\varrho, x) = f(z) := \sum_{k=0}^{\infty} \widehat{f}_k z^k, \quad z = \varrho \mathrm{e}^{\mathrm{i}x}$.

The relation between the operators $A_{\varrho,r}$ and the "Taylor polynomials" is shown in the following statement.

Lemma 1. *Assume that $f \in L_1$. Then for any numbers $r \in \mathbb{N}$, $\varrho \in [0,1)$ and $x \in \mathbb{T}$,*

$$A_{\varrho,r}(f)(x) = \sum_{k=0}^{r-1} \frac{\partial^k f(\varrho,x)}{\partial \varrho^k} \cdot \frac{(1-\varrho)^k}{k!}. \tag{5}$$

Proof. Let us associate the function f with the functions

$$f_1(z) := \widehat{f}_0/2 + \sum_{k=1}^{\infty} \widehat{f}_k z^k \quad \text{and} \quad f_2(z) := \widehat{f}_0/2 + \sum_{k=1}^{\infty} \widehat{f}_{-k} z^k, \tag{6}$$

which are holomorphic in the disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

From Lemma 4 in [14], it follows that for any $z \in \overline{\mathbb{D}}$,

$$\frac{\widehat{f}_0}{2} + \sum_{k=1}^{r-1} \widehat{f}_k z^k + \sum_{k=r}^{\infty} \lambda_{k,r}(\varrho) \widehat{f}_k z^k = \frac{\widehat{f}_0}{2} + \sum_{k=1}^{r-1} z^k f_1^{(k)}(\varrho z) \frac{(1-\varrho)^k}{k!} \tag{7}$$

and

$$\frac{\widehat{f}_0}{2} + \sum_{k=1}^{r-1} \widehat{f}_{-k} \overline{z}^k + \sum_{k=r}^{\infty} \lambda_{k,r}(\varrho) \widehat{f}_{-k} \overline{z}^k = \frac{\widehat{f}_0}{2} + \sum_{k=1}^{r-1} \overline{z}^k f_2^{(k)}(\varrho \overline{z}) \frac{(1-\varrho)^k}{k!}, \tag{8}$$

where for $r = 1$, the sums $\sum_{k=1}^{0}$ are assumed to be zero.

Actually, in [14], the relations of the kind as in (7) and (8) were proved for $z \in \mathbb{D}$, but such restrictions are not important.

Adding these two equalities at $z = \mathrm{e}^{\mathrm{i}x}$ and taking into account the relation

$$\mathrm{e}^{\mathrm{i}kx} f_1^{(k)}(\varrho \mathrm{e}^{\mathrm{i}x}) + \mathrm{e}^{-ikx} f_2^{(k)}(\varrho \mathrm{e}^{-ix}) = \frac{\partial^k f(\varrho,x)}{\partial \varrho^k}, \tag{9}$$

we get (5), which proves the Lemma.

Now let us formulate direct and inverse approximation theorems by the operators $A_{\varrho,r}$ in the terms of K–functionals of functions, generated by their radial derivatives.

Let us give all necessary definitions. If for a function $f \in L_1$ and for a positive integer n, there exists the function $g \in L_1$ such that

$$\widehat{g}_k = \begin{cases} 0, & \text{if} \quad |k| < n, \\ \dfrac{|k|!}{(|k|-n)!} \widehat{f}_k, & \text{if} \quad |k| \geq n, \end{cases}, \quad k \in \mathbb{Z}$$

then we say that for the function f, there exists the radial derivative g of order n, for which we use the notation $f^{[n]}$. Here, we use the term "radial derivative" in view of the following fact.

If the function $f^{[r]} \in L_1$, then its Poisson integral can be presented as

$$f^{[r]}(\varrho,x) = (f(\varrho,\cdot))^{[r]}(x) = \varrho^r \frac{\partial^r f(\varrho,x)}{\partial \varrho^r} \quad \varrho \in [0,1), \ \forall\, x \in \mathbb{T}. \tag{10}$$

Hence, by virtue of the theorem of limit values of Poisson integral (see, for example, [13, p. 27]), for almost all $x \in \mathbb{T}$, we have $f^{[r]}(x) = \lim_{\varrho\to 1-} f^{[r]}(\varrho, x)$.

Relation (10) can be easily proved by term by term differentiation with respect to the variable ϱ of the decomposition of Poisson integral into the uniformly convergent series

$$f\,(\varrho, x) = \sum_{k\in\mathbb{Z}} \varrho^{|k|} \widehat{f}_k \mathrm{e}^{\mathrm{i}kx} \quad \forall\ \varrho \in [0,1),\ x \in \mathbb{T}. \tag{11}$$

From the definition of radial derivative, in particular, it follows the differentiation rule: *If* $f(x) = \sum_{|k|\le m} \widehat{f}_k \mathrm{e}^{\mathrm{i}kx}$, $m \in \mathbb{Z}_+$, *then*

$$f^{[n]}(x) = \begin{cases} 0, & \text{if} \quad m < n, \\ \displaystyle\sum_{n\le|k|\le m} \frac{|k|!}{(|k|-n)!} \widehat{f}_k \mathrm{e}^{\mathrm{i}kx}, & \text{if} \quad m \ge n. \end{cases} \tag{12}$$

In the space L_p, the K–functional of a function f (see, for example, [8, Chap. 6]) generated by the radial derivative of order n, is the following quantity:

$$K_n(\delta, f)_p := \inf\left\{ \|f-h\|_p + \delta^n \left\|h^{[n]}\right\|_p : h^{[n]} \in L_p \right\}, \quad \delta > 0.$$

Further, we consider the functions $\omega(t)$, $t \in [0,1]$, satisfying the following conditions:

1) $\omega(t)$ is continuous on $[0,1]$;

2) $\omega(t)\uparrow$;

3) $\omega(t) \ne 0$ for any $t \in (0,1]$;

4) $\omega(t) \to 0$ as $t \to 0$;

and the well-known Zygmund–Bari–Stechkin conditions (see, for example, [1]):

$$(\mathscr{Z}) \qquad \int_0^\delta \frac{\omega(t)}{t} dt = O(\omega(\delta)), \qquad \delta > 0,$$

$$(\mathscr{Z}_n) \qquad \int_\delta^1 \frac{\omega(t)}{t^{n+1}} dt = O\Big(\frac{\omega(\delta)}{\delta^n}\Big), \quad \delta > 0,\ n \in \mathbb{N}.$$

The main results of this paper are contained in the following two statements:

Theorem 1. *Assume that* $f \in L_p$, $1 \le p \le \infty$, $n, r \in \mathbb{N}$, $n \le r$ *and the function* $\omega(t)$, $t \in [0,1]$, *satisfies conditions 1)–4) and* $(\mathscr{Z})$. *If*

$$K_n\left(\delta, f^{[r-n]}\right)_p = O(\omega(\delta)), \quad \delta \to 0+, \tag{13}$$

then

$$\|f - A_{\varrho,r}(f)\|_p = O\left((1-\varrho)^{r-n}\omega(1-\varrho)\right), \quad \varrho \to 1-. \tag{14}$$

Theorem 2. *Assume that $f \in L_p$, $1 \le p \le \infty$, $n, r \in \mathbb{N}$, $n \le r$ and the function $\omega(t)$, $t \in [0,1]$, satisfies conditions 1)–4), $(\mathscr{Z})$ and $(\mathscr{Z}_n)$. If relation (14) holds, then $f^{[r-n]} \in L_p$ and relation (13) also holds.*

We note that in the case where $\omega(t)$ is a power function: $\omega(t) = t^\alpha$, $\alpha > 0$, the results of the Theorems 1 and 2 were announced in [12].

***Remark* 1.** For a given $n \in \mathbb{N}$, from condition $(\mathscr{Z}_n)$ it follows that $\liminf\limits_{\delta \to 0+}(\delta^{-n}\omega(\delta)) > 0$ or, equivalently, that $(1-\varrho)^{r-n}\omega(1-\varrho) \ll (1-\varrho)^r$ as $\varrho \to 1-$. Therefore, if condition $(\mathscr{Z}_n)$ is satisfied, then the quantity on the right-hand side of (14) decreases to zero as $\varrho \to 1-$ not faster, than the function $(1-\varrho)^r$. Also note that the relation $\|f - A_{\varrho,r}(f)\|_p = o\left((1-\varrho)^r\right)$, $\varrho \to 1-$, holds only in the trivial case when $f(x) = \sum_{|k| \le r-1} \widehat{f}_k \mathrm{e}^{\mathrm{i}kx}$, and in such case, the theorems are easily true. This fact is related to the so-called saturation property of the approximation method, generated by the operator $A_{\varrho,r}$. In particular, in [14], it was shown that the operator $A_{\varrho,r}$ generates the linear approximation method of holomorphic functions, which is saturated in the space H_p with the saturation order $(1-\varrho)^r$ and the saturation class $H_p^{r-1}\,\mathrm{Lip}\,1$.

Before proving the Theorems 1 and 2, let us give some auxiliary results. For any $f \in L_1$, $1 \le p \le \infty$, $0 \le \varrho < 1$ and $r = 0, 1, 2, \ldots$, we set

$$M_p(\varrho, f, r) := \varrho^r \left\| \frac{\partial^r f(\varrho, \cdot)}{\partial \varrho^r} \right\|_p = \left\| (f(\varrho,\cdot))^{[r]}(\cdot) \right\|_p . \tag{15}$$

Lemma 2. *Assume that $f \in L_p$, $1 \le p \le \infty$. Then for any numbers $n \in \mathbb{N}$ and $\varrho \in [1/2, 1)$,*

$$\begin{aligned} \frac{1}{2n!}(1-\varrho)^n M_p(\varrho, f, n) &\le K_n(1-\varrho, f)_p \\ &\le \|f - A_{\varrho,n}(f)\|_p + \frac{4^n - 1}{3}(1-\varrho)^n M_p(\sqrt{\varrho}, f, n). \end{aligned}$$

Proof. First, let us note that the statement of Lemma 2 is trivial in the case, if f is a trigonometric polynomial of order not exceeding $n-1$, i.e., if $f(x) = \sum_{|k| \le n-1} \widehat{f}_k \mathrm{e}^{\mathrm{i}kx}$, as well as in the case, if $\varrho = 0$. Therefore, further in the proof, we exclude these two cases.

Let g be a function such that $g^{[n]} \in L_p$.

Since

$$\frac{1-\varrho^2}{|1 - \mathrm{e}^{\mathrm{i}(x-t)}\varrho|^2} = \frac{1}{1 - \mathrm{e}^{\mathrm{i}(x-t)}\varrho} + \frac{1}{1 - e^{-i(x-t)}\varrho} - 1,$$

then by virtue of (1), for any numbers $\varrho \in [0,1)$ and $x \in \mathbb{T}$, we have

$$\frac{\partial^n f(\varrho, x)}{\partial \varrho^n} = \frac{1}{2\pi}\int_0^{2\pi} (f(t) - g(t)) \frac{\partial^n}{\partial \varrho^n}\left(\frac{1-\varrho^2}{|1 - \mathrm{e}^{\mathrm{i}(x-t)}\varrho|^2}\right) dt + \frac{\partial^n g(\varrho, x)}{\partial \varrho^n}$$

$$= \frac{n!}{2\pi}\int_0^{2\pi} (f(t) - g(t)) \left(\frac{\mathrm{e}^{\mathrm{i}r(x-t)}}{(1 - \mathrm{e}^{\mathrm{i}(x-t)}\varrho)^{n+1}} + \frac{\mathrm{e}^{-\mathrm{i}r(x-t)}}{(1 - \mathrm{e}^{-\mathrm{i}(x-t)}\varrho)^{n+1}}\right) dt + \frac{\partial^n g(\varrho, x)}{\partial \varrho^n}$$

$$= \frac{n!}{\pi}\int_0^{2\pi}(f(t)-g(t))\,\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}r(x-t)}}{(1-\mathrm{e}^{\mathrm{i}(x-t)}\varrho)^{n+1}}dt+\frac{\partial^n g(\varrho,x)}{\partial\varrho^n}.$$

Hence, changing the variables of integration and using the integral Minkowski inequality, we obtain

$$\begin{aligned}\left\|\frac{\partial^n f(\varrho,\cdot)}{\partial\varrho^n}\right\|_p &\le \frac{n!}{\pi}\int_0^{2\pi}\frac{dt}{|1-\varrho\mathrm{e}^{\mathrm{i}t}|^{n+1}}\|f-g\|_p+\left\|\frac{\partial^n g(\varrho,\cdot)}{\partial\varrho^n}\right\|_p\\ &\le \frac{2n!}{(1-\varrho)^n}\|f-g\|_p+\left\|\frac{\partial^n g(\varrho,\cdot)}{\partial\varrho^n}\right\|_p.\end{aligned}$$

Taking into account (10), (15) and the inequality $\|g^{[n]}(\varrho,\cdot)\|_p \le \|g^{[n]}\|_p$, we see that for any $\varrho\in(0,1)$,

$$\frac{1}{2n!}(1-\varrho)^n M_p(\varrho,f,n)\le\|f-g\|_p+(1-\varrho)^n\left\|g^{[n]}\right\|_p.$$

Considering the infimum over all functions g such that $g^{[n]}\in L_p$, we conclude that

$$\frac{1}{2n!}(1-\varrho)^n M_p(\varrho,f,n)\le K_n(1-\varrho,f)_p.$$

On the other hand, from the definition of the K–functional, it follows that

$$K_n(1-\varrho,f)_p\le\|f-A_{\varrho,n}(f)\|_p+(1-\varrho)^n\left\|(A_{\varrho,n}(f))^{[n]}\right\|_p. \tag{16}$$

According to (5) and (10), we have

$$\begin{aligned}(A_{\varrho,n}(f))^{[n]}(x) &= \Big(\sum_{k=0}^{n-1}\frac{(f(\varrho,\cdot))^{[k]}(\cdot)}{\varrho^k k!}(1-\varrho)^k\Big)^{[n]}(x)\\ &= \sum_{k=0}^{n-1}\frac{((f(\varrho,\cdot))^{[k]}(\cdot))^{[n]}(x)}{\varrho^k k!}(1-\varrho)^k.\end{aligned}$$

Since for any nonnegative integers k and n

$$((f(\varrho,\cdot))^{[n]}(\cdot))^{[k]}(x)=((f(\varrho,\cdot))^{[k]}(\cdot))^{[n]}(x), \tag{17}$$

we obtain

$$(A_{\varrho,n}(f))^{[n]}(x)=\sum_{k=0}^{n-1}\frac{((f(\varrho,\cdot))^{[n]}(\cdot))^{[k]}(x)}{\varrho^k k!}(1-\varrho)^k.$$

This yields

$$\|(A_{\varrho,n}(f))^{[n]}\|_p\le\sum_{k=0}^{r-1}\frac{\|((f(\varrho,\cdot))^{[n]}(\cdot))^{[k]}\|_p}{\varrho^k k!}(1-\varrho)^k. \tag{18}$$

By virtue of the definition of the Poisson integral, for any $k=0,1,\ldots,r-1$, we have

$$((f(\varrho,\cdot))^{[n]}(\cdot))^{[k]}(x) = \Big(\sum_{|j|\ge n}\frac{|j|!}{(|j|-n)!}\widehat{f}_j\varrho^{\frac{|j|}{2}}\mathrm{e}^{\mathrm{i}jx}\varrho^{\frac{|j|}{2}}\Big)^{[k]}(x)$$

$$= \left(\frac{1}{2\pi}\int_0^{2\pi}(f(\sqrt{\varrho},\cdot))^{[n]}(t)P(\sqrt{\varrho},t-\cdot)dt\right)^{[k]}(x)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(f(\sqrt{\varrho},\cdot))^{[n]}(t)\sum_{|\nu|\geq k}\frac{|\nu|!}{(|\nu|-k)!}\varrho^{\frac{|\nu|}{2}}\mathrm{e}^{\mathrm{i}\nu(t-x)}dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(f(\sqrt{\varrho},\cdot))^{[n]}(t+x)\sum_{|\nu|\geq k}\frac{|\nu|!}{(|\nu|-k)!}\varrho^{\frac{|\nu|}{2}}\mathrm{e}^{\mathrm{i}\nu t}dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(f(\sqrt{\varrho},\cdot))^{[n]}(t+x)\Big(\tau^k\frac{\partial^k}{\partial\tau^k}P(\tau,t)\Big)\Big|_{\tau=\sqrt{\varrho}}dt.$$

Using the integral Minkowski inequality, for $k=0$, we obtain

$$\|((f(\varrho,\cdot))^{[n]}(\cdot))^{[k]}\|_p = \|(f(\varrho,\cdot))^{[n]}\|_p$$

$$\leq M_p(\sqrt{\varrho},f,n)\frac{1}{2\pi}\int_0^{2\pi}|P(\sqrt{\varrho},t)|dt = M_p(\sqrt{\varrho},f,n). \tag{19}$$

If $k=1,2,\ldots$, then

$$\frac{\partial^k}{\partial\tau^k}P(\tau,t) = \frac{\partial^k}{\partial\tau^k}\Big(\frac{1}{1-\tau\mathrm{e}^{\mathrm{i}t}}+\frac{\tau\mathrm{e}^{-\mathrm{i}t}}{1-\tau\mathrm{e}^{-\mathrm{i}t}}\Big) = \frac{k!\,\mathrm{e}^{\mathrm{i}kt}}{(1-\tau\mathrm{e}^{\mathrm{i}t})^{k+1}}+\frac{k!\,\mathrm{e}^{-\mathrm{i}kt}}{(1-\tau\mathrm{e}^{-\mathrm{i}t})^{k+1}}.$$

This similarly yields

$$\|(f^{[n]}(\varrho,\cdot))^{[k]}\|_p \leq M_p(\sqrt{\varrho},f,n)\frac{1}{2\pi}\int_0^{2\pi}\Big|\Big(\tau^k\frac{\partial^k}{\partial\tau^k}P(\tau,t)\Big)\Big|_{\tau=\sqrt{\varrho}}\Big|dt$$

$$\leq 2k!\,M_p(\sqrt{\varrho},f,n)\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|1-\sqrt{\varrho}\mathrm{e}^{\mathrm{i}t}|^{k+1}} \leq M_p(\sqrt{\varrho},f,n)\frac{2^k k!}{(1-\varrho)^k}. \tag{20}$$

Combining relations (18)–(20), we see that for any $\varrho\in[1/2,1)$,

$$\int_\varrho^1\|(A_{\varrho,n}(f))^{[n]}\|_p \leq M_p(\sqrt{\varrho},f,n)+M_p(\sqrt{\varrho},f,n)\sum_{k=1}^{n-1}4^k$$

$$= M_p(\sqrt{\varrho},f,n)\frac{4^n-1}{3}. \tag{21}$$

By virtue of (21) and (16), we conclude that

$$K_n\,(1-\varrho,f)_p \leq \|f-A_{\varrho,n}(f)\|_p+\frac{4^n-1}{3}(1-\varrho)^n M_p(\sqrt{\varrho},f,n)$$

which proves the Lemma.

Lemma 3. *Assume that $r \in \mathbb{N}$, $1 \le p \le \infty$ and $\varrho \in [1/2, 1)$. Then for any function $f \in L_p$*

$$\|(A_{\varrho,r}(f))^{[r]}\|_p \le C_r \frac{\|f\|_p}{(1-\varrho)^r}, \tag{22}$$

where the constant C_r depends only on r.

Proof. By virtue of (10), for any function $f \in L_p$ and all $x \in \mathbb{T}$, we have

$$\begin{aligned} (f(\varrho,\cdot))^{[r]}(x) &= \frac{\varrho^r}{2\pi}\int_0^{2\pi} f(t)\frac{\partial^r}{\partial\varrho^r}\left(\frac{1-\varrho^2}{|1-\mathrm{e}^{\mathrm{i}(x-t)}\varrho|^2}\right)dt \\ &= \frac{r!\varrho^r}{2\pi}\int_0^{2\pi} f(t)\left(\frac{\mathrm{e}^{\mathrm{i}r(x-t)}}{(1-\mathrm{e}^{\mathrm{i}(x-t)}\varrho)^{r+1}} + \frac{\mathrm{e}^{-\mathrm{i}r(x-t)}}{(1-\mathrm{e}^{-\mathrm{i}(x-t)}\varrho)^{r+1}}\right)dt \\ &= \frac{r!\varrho^r}{\pi}\int_0^{2\pi} f(t)\,\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}r(x-t)}}{(1-\mathrm{e}^{\mathrm{i}(x-t)}\varrho)^{r+1}}dt. \end{aligned}$$

Making a change of variables of integration and using the integral Minkowski inequality, we obtain

$$M_p(\varrho, f, r) \le \frac{r!}{\pi}\int_0^{2\pi}\frac{dt}{|1-\varrho\mathrm{e}^{\mathrm{i}t}|^{r+1}}\|f\|_p \le \frac{2r!}{(1-\varrho)^r}\|f\|_p. \tag{23}$$

Combining this relation and relation (21) with $n = r$, we conclude that

$$\begin{aligned} \|(A_{\varrho,r}(f))^{[r]}\|_p &\le M_p(\sqrt{\varrho}, f, r)\frac{4^r-1}{3} \le \frac{2r!(4^r-1)}{3(1-\sqrt{\varrho})^r}\|f\|_p \\ &\le \frac{r!(2^{3r+1}-2^{r+1})}{3}\cdot\frac{\|f\|_p}{(1-\varrho)^r}. \end{aligned}$$

Lemma 4. *Assume that $r \in \mathbb{N}$ and $0 \le \varrho < 1$. Then for any function $f \in L_p$, $1 \le p \le \infty$, such that*

$$\int_\varrho^1 \left\|\frac{\partial^r f(\zeta,\cdot)}{\partial\zeta^r}\right\|_p (1-\zeta)^{r-1}d\zeta < \infty \tag{24}$$

and for almost all $x \in \mathbb{T}$,

$$f(x) - A_{\varrho,r}(f)(x) = \frac{1}{(r-1)!}\int_\varrho^1 \frac{\partial^r f(\zeta,x)}{\partial\zeta^r}(1-\zeta)^{r-1}d\zeta. \tag{25}$$

Proof. For fixed $r \in \mathbb{N}$ and $0 \le \varrho < 1$, the integral on the right-hand side of (25), defines a certain function $F(x)$. By virtue of (24) and the integral Minkowski inequality, we conclude that the function F belongs to the space L_p. Let us find the Fourier coefficients of F and compare them with the Fourier coefficients of the function $G := f - A_{\varrho,r}(f)$. Since for $r \in \mathbb{N}$,

$$\frac{\partial^r f(\zeta,x)}{\partial\zeta^r} = \sum_{|k|\ge r}\frac{|k|!}{(|k|-r)!}\widehat{f}_k\zeta^{|k|-r}\mathrm{e}^{\mathrm{i}kx},$$

then $\widehat{F}_k = 0$, when $|k| < r$. If $|k| \ge r$, then integrating by parts, we see that

$$\widehat{F}_k = \frac{1}{2\pi}\int_0^{2\pi} F(t)\mathrm{e}^{-\mathrm{i}kt}dt = \widehat{f}_k\sum_{j=r}^{|k|}\binom{|k|}{j}(1-\varrho)^j\varrho^{|k|-j}. \tag{26}$$

On the other hand, if $|k|<r$ the Fourier coefficients $\widehat{G}_k$ of the function G are equal to zero. If $|k|\ge r$, then in view of the equality

$$\sum_{j=0}^{k}\binom{k}{j}(1-\varrho)^j\varrho^{k-j}=((1-\varrho)+\varrho)^k=1,\qquad k=0,1,\ldots,$$

we see that

$$\widehat{G}_k=(1-\lambda_{|k|,r}(\varrho))\widehat{f}_k=\widehat{f}_k\sum_{j=r}^{|k|}\binom{|k|}{j}(1-\varrho)^j\varrho^{|k|-j}.$$

Therefore, for all $k\in\mathbb{Z}$, we have $\widehat{F}_k=\widehat{G}_k$. Hence, for almost all $x\in\mathbb{T}$, relation (25) holds.

Proof of Theorem 1. Assume that the function f is such that $f^{[r-n]}\in L_p$ and relation (13) is satisfied. Let us apply the first inequality of Lemma 2 to the function $f^{[r-n]}$. In view of (10) and (15), we obtain

$$\frac{1}{2n!}(1-\varrho)^nM_p\left(\varrho,f,r\right)\le K_n\left(1-\varrho,f^{[r-n]}\right)_p.$$

This yields

$$M_p\left(\varrho,f,r\right)\le C\frac{\omega(1-\varrho)}{(1-\varrho)^n},\quad \varrho\to1-. \tag{27}$$

Using relations (15), (27) and $(\mathscr{Z})$ and the integral Minkowski inequality, we obtain

$$\begin{aligned}\int_\varrho^1\left\|\frac{\partial^rf(\zeta,\cdot)}{\partial\zeta^r}\right\|_p(1-\zeta)^{r-1}d\zeta &\le \int_\varrho^1 M_p\left(\zeta,f,r\right)\frac{(1-\zeta)^{r-1}}{\zeta^r}d\zeta\\ &\le 2^rC(1-\varrho)^{r-n}\int_\varrho^1\frac{\omega(1-\zeta)}{1-\zeta}d\zeta\\ &= O\left((1-\varrho)^{r-n}\omega(1-\varrho)\right),\ \varrho\to1-. \end{aligned}\tag{28}$$

Therefore, for almost all $x\in\mathbb{T}$, relation (25) holds. Hence, by virtue of (25), using the integral Minkowski inequality and (28), we finally get (14):

$$\begin{aligned}\|f-A_{\varrho,r}(f)\|_p &\le \frac{1}{(r-1)!}\int_\varrho^1 M_p\left(\zeta,f,r\right)\frac{(1-\zeta)^{r-1}}{\zeta^r}d\zeta\\ &= O\left((1-\varrho)^{r-n}\omega(1-\varrho)\right),\quad \varrho\to1-.\end{aligned}$$

Proof of Theorem 2. First, let us note that for any function $f\in L_p$, $1\le p\le\infty$, and all fixed numbers $s,r\in\mathbb{N}$ and $\varrho\in(0,1)$

$$\begin{aligned}\|A_{\varrho,r}^{[s]}(f)\|_p &= \Big\|\sum_{|k|\ge s}\frac{|k|!}{(|k|-s)!}\omega_{|k|}(\varrho)\widehat{f}_k\mathrm{e}^{\mathrm{i}kt}\Big\|_p\\ &\le 2r\|f\|_p\Big(C+\sum_{k\ge\max\{s,r\}}q^kk^{s+r-1}\Big)<\infty,\end{aligned}$$

where $0 < q = \max\{1-\varrho, \varrho\} < 1$.

Put $\varrho_k := 1 - 2^{-k}$, $k \in \mathbb{N}$, and $A_k := A_k(f) := A_{\varrho_k, r}(f)$. For any $x \in \mathbb{T}$ and $s \in \mathbb{N}$, consider the series

$$A_0^{[s]}(f)(x) + \sum_{k=1}^{\infty}(A_k^{[s]}(f)(x) - A_{k-1}^{[s]}(f)(x)). \tag{29}$$

According to the definition of the operator $A_{\varrho,r}$, we see that for any $\varrho_1, \varrho_2 \in [0,1)$ and $r \in \mathbb{N}$,

$$A_{\varrho_1,r}\left(A_{\varrho_2,r}(f)\right) = A_{\varrho_2,r}\left(A_{\varrho_1,r}(f)\right).$$

By virtue of Lemma 3 and relation (14), for any $k \in \mathbb{N}$ and $s \in \mathbb{N}$, we have

$$\begin{aligned}\left\|A_k^{[s]} - A_{k-1}^{[s]}\right\|_p &= \left\|A_k^{[s]}(f - A_{k-1}(f)) - A_{k-1}^{[s]}(f - A_k(f))\right\|_p \\ &\le \left\|A_k^{[s]}(f - A_{k-1}(f))\right\|_p + \left\|A_{k-1}^{[s]}(f - A_k(f))\right\|_p \\ &\le C_s \frac{\|f - A_{k-1}(f)\|_p}{(1-\varrho_k)^s} + C_s \frac{\|f - A_k(f)\|_p}{(1-\varrho_{k-1})^s} \\ &= O\left(\frac{\omega(1-\varrho_{k-1})}{(1-\varrho_k)^{s-r+n}}\right) + O\left(\frac{\omega(1-\varrho_k)}{(1-\varrho_{k-1})^{s-r+n}}\right), \quad k \to +\infty.\end{aligned} \tag{30}$$

Therefore, for any $s \le r - n$,

$$\left\|A_k^{[s]} - A_{k-1}^{[s]}\right\|_p = O\left(\omega(1-\varrho_{k-1})\right) = O\left(\omega(2^{-(k-1)})\right), \quad k \to +\infty. \tag{31}$$

Consider the sum $\sum_{k=1}^N \omega(2^{-(k-1)})$, $N \in \mathbb{N}$. Taking into account the monotonicity of the function ω and $(\mathscr{Z})$, we see that for all $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=1}^N \omega(2^{-(k-1)}) &\le \omega(1) + \int_1^N \omega(2^{-(t-1)})dt \\ &= \omega(1) + \frac{1}{\ln 2}\int_{2^{-N+1}}^1 \frac{\omega(\tau)}{\tau}d\tau \le C\omega(1) < \infty.\end{aligned} \tag{32}$$

Combining relations (31) and (32), we conclude that for all $1 \le p \le \infty$, the series in (29) converges in the norm of the space L_p. Hence, by virtue of the Banach–Alaoglu theorem, for any $s = 0, 1, \ldots, r-n$, there exists the subsequence

$$S_{N_j}^{[s]}(x) = A_0^{[s]}(f)(x) + \sum_{k=1}^{N_j}(A_k^{[s]}(f)(x) - A_{k-1}^{[s]}(f)(x)), \quad j = 1, 2, \ldots \tag{33}$$

of partial sums of this series, converging to a certain function $g \in L_p$ almost everywhere on $\mathbb{T}$ as $j \to \infty$.

Let us show that $g = f^{[s]}$. For this, let us find the Fourier coefficients of the function g. For any fixed $k \in \mathbb{Z}$ and all $j = 1, 2, \ldots$, we have

$$\widehat{g}_k := \frac{1}{2\pi}\int_0^{2\pi} S^{[s]}_{N_j}(t)\mathrm{e}^{-\mathrm{i}kt}dt + \frac{1}{2\pi}\int_0^{2\pi} (g(t) - S^{[s]}_{N_j}(t))\mathrm{e}^{-\mathrm{i}kt}dt.$$

Since the sequence $\{S^{[s]}_{N_j}\}_{j=1}^{\infty}$ converges almost everywhere on $\mathbb{T}$ to the function g, then the second integral on the right-hand side of the last equality tends to zero as $j \to \infty$. By virtue of (33) and the definition of the radial derivative, for $|k| < s$ the first integral is equal to zero, and for all $|k| \geq s$,

$$\frac{1}{2\pi}\int_0^{2\pi} S^{[s]}_{N_j}(t)\mathrm{e}^{-\mathrm{i}kt}dt = \lambda_{|k|,r}(1 - 2^{-N_j})\frac{|k|!}{(|k|-s)!}\widehat{f}_k \underset{j\to\infty}{\longrightarrow} \frac{|k|!}{(|k|-s)!}\widehat{f}_k.$$

Therefore, the equality $g = f^{[s]}$ is true. Hence, for the function f and all $s = 0, 1, \ldots, r-n$, there exists the derivative $f^{[s]}$ and $f^{[s]} \in L_p$.

Now, let us prove the estimate (27). By virtue of (15), (30), for any $k \in \mathbb{N}$ and $\varrho \in (0,1)$, we have

$$\begin{aligned} M_p\left(\varrho, A_k - A_{k-1}, r\right) &\leq \left\| A_k^{[r]} - A_{k-1}^{[r]} \right\|_p = O\left(\frac{\omega(1-\varrho_{k-1})}{(1-\varrho_k)^n}\right) \\ &+ O\left(\frac{\omega(1-\varrho_k)}{(1-\varrho_{k-1})^n}\right) = O\left(2^{kn}\omega(2^{-k+1}) + 2^{(k-1)n}\omega(2^{-k})\right) \\ &= O\left(2^{(k-1)n}\omega(2^{-(k-1)})\right), \quad k \to +\infty. \end{aligned} \tag{34}$$

According to (23) and (14), for any $r \in \mathbb{N}$, $\varrho \in (0,1)$ and $x \in \mathbb{T}$, we obtain

$$M_p\left(\varrho, f - A_{\varrho,r}(f), r\right) \leq 2r!\frac{\|f - A_{\varrho,r}(f)\|_p}{(1-\varrho)^r} = O\left(\frac{\omega(1-\varrho)}{(1-\varrho)^n}\right), \quad \varrho \to 1-.$$

Therefore, for any positive integer N,

$$\begin{aligned} M_p\left(\varrho_N, f - A_N(f), r\right) &= O\left(\frac{\omega(1-\varrho_N)}{(1-\varrho_N)^n}\right) \\ &= O\left(2^{Nn}\omega(2^{-N})\right), \ N \to +\infty. \end{aligned} \tag{35}$$

Consider the sum $\sum_{k=1}^N 2^{(k-1)n}\omega(2^{-(k-1)})$, $N \in \mathbb{N}$. Since the function ω satisfies the condition $(\mathscr{Z}_n)$, the function $\omega(t)/t^n$ almost decreases on $[0,1]$ (see, for example [1]). Therefore,

$$\begin{aligned} C_1\sum_{k=1}^N 2^{(k-1)n}\omega(2^{-(k-1)}) &\leq 2^{(N-1)n}\omega(2^{-(N-1)}) + \int_1^N 2^{(t-1)n}\omega(2^{-(t-1)})dt \\ &\leq 2^{(N-1)n}\omega(2^{-(N-1)}) + \frac{1}{\ln 2}\int_{2^{-N+1}}^{1} \omega(\tau)/\tau^{n+1}d\tau \leq C_2 2^{(N-1)n}\omega(2^{-(N-1)}). \end{aligned} \tag{36}$$

Putting $\varrho = \varrho_N$ and taking into account relations (34), (35), (36) and

$$A_0(x) = S_{r-1}(f)(x) = \sum_{|k|\le r-1} \widehat{f}_k \mathrm{e}^{\mathrm{i}kx},$$

we get

$$\begin{aligned} M_p(\varrho_N, f, r) &= M_p(\varrho_N, f - S_{r-1}(f), r) \\ = M_p\left(\varrho_N, f - A_{\varrho_N} + \sum_{k=1}^{N}(A_k - A_{k-1}), r\right) &= O\left(\sum_{k=1}^{N} 2^{(k-1)n}\omega(2^{-(k-1)})\right) \\ = O\left(2^{Nn}\omega(2^{-N})\right) &= O\left(\frac{\omega(1-\varrho_N)}{(1-\varrho_N)^n}\right), \quad N \to +\infty. \end{aligned} \tag{37}$$

If the function ω satisfies the condition $(\mathscr{Z}_n)$, then for all $t \in [0,1]$ $\omega(2t) \le C\omega(t)$ (see, for example [1]). Furthermore, for all $\varrho \in [\varrho_{N-1}, \varrho_N]$, we have $1 - \varrho_N \le 1 - \varrho \le 2(1 - \varrho_N)$. Hence, relation (37) yields the estimate (27).

Now, applying the second inequality in Lemma 2 to the function $f^{[r-n]}$, we get

$$\begin{aligned} K_n\left(1-\varrho, f^{[r-n]}\right)_p &\le \|f^{[r-n]} - A_{\varrho,n}(f^{[r-n]})\|_p \\ &+ \frac{4^n - 1}{3}(1-\varrho)^n M_p(\sqrt{\varrho}, f, r). \end{aligned} \tag{38}$$

By virtue of (15) and (27), we see that for $\varrho \in [1/2, 1)$,

$$\begin{aligned} \int_\varrho^1 \left\|\frac{\partial^n f^{[r-n]}(\zeta,\cdot)}{\partial\zeta^n}\right\|_p (1-\zeta)^{n-1} d\zeta &= \int_\varrho^1 \left\|(f(\zeta,\cdot))^{[r]}(x)\right\|_p \frac{(1-\zeta)^{n-1}}{\zeta^n} d\zeta \\ = \int_\varrho^1 M_p(\zeta, f, r) \frac{(1-\zeta)^{n-1}}{\zeta^n} d\zeta &\le 2^n C \int_\varrho^1 \frac{\omega(1-\zeta)}{1-\zeta} d\zeta \\ &= O(\omega(1-\varrho)), \quad \varrho \to 1-. \end{aligned} \tag{39}$$

Therefore, we can apply Lemma 4 to the function $f^{[r-n]}$. Taking into account (15), we obtain

$$f^{[r-n]}(x) - A_{\varrho,n}(f^{[r-n]})(x) = \frac{1}{(n-1)!}\int_\varrho^1 (f(\zeta,\cdot))^{[r]}(x) \frac{(1-\zeta)^{n-1}}{\zeta^n} d\zeta.$$

Using the integral Minkowski inequality and (39), we conclude

$$\begin{aligned} \|f^{[r-n]} - A_{\varrho,n}(f^{[r-n]})\|_p &\le \frac{1}{(n-1)!}\int_\varrho^1 M_p(\zeta, f, r) \frac{(1-\zeta)^{n-1}}{\zeta^n} d\zeta \\ &= O(\omega(1-\varrho)), \quad \varrho \to 1-. \end{aligned} \tag{40}$$

Combining relations (38), (27) and (40), we get (13).

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