# DIRECT AND INVERSE APPROXIMATION THEOREMS OF FUNCTIONS IN THE ORLICZ TYPE SPACES $\mathcal{S}_{M}$ 

STANISLAV CHAICHENKO*, ANDRII SHIDLICH** AND FAHREDDIN ABDULLAYEV***


#### Abstract

In the Orlicz type spaces $\mathcal{S}_{M}$, we prove direct and inverse approximation theorems in terms of the best approximations of functions and moduli of smoothness of fractional order. We also show the equivalence between moduli of smoothness and Peetre $K$-functionals in the spaces $\mathcal{S}_{M}$


## 1. Introduction

Direct approximation theorems are statements asserting that smoothness of the function $f$ implies a quick decrease to zero of its error of approximation by polynomials or other approximating aggregates. On classes of continuously differentiable functions, such theorems were first proved in terms of the first-order modulus of continuity by Jackson [14] in 1911. Later, Zygmund [32] and Akhiezer [3] generalized Jackson's results to the second-order modulus of continuity, and Stechkin [21] extended these results to the moduli of continuity of an arbitrary integer order $k, k \geq 3$.

Inverse approximation theorems are the converse statements that characterize the smoothness properties of a function depending on the speed of convergence to zero of its approximation by some approximating aggregates. These theorems were first obtained by Bernstein 5 in 1912. And already in 1919, direct and inverse approximation theorems, due to Jackson and Bernstein, were given in the book on approximation theory by de la Vallée Poussin [31.

Investigations of the connection (direct and inverse) between the smoothness properties of functions and the possible orders of their approximations were carried out by many authors on various classes of functions and for various approximating aggregates. Such results constitute the classics of modern approximation theory and they are also described quite fully in the monographs [6], 9], [10], [29].

For the last decades, in addition to the classical direction of theory of direct and inverse approximation theorems, a number of "non-classical" directions have also been developed fruitfully. It should be mention the studies on direct and inverse approximation theorems in the Orlicz function spaces, which results are contained, in particular, in the papers of Ramazanov [17, Garidi 11, Runovski [18, Israfilov and Guven [13, [12], Akgün and Izrafilov [2], Akgün [1], Chaichenko [8] and others.

In 2001, Stepanets [22] considered the spaces $\mathcal{S}^{p}$ of $2 \pi$-periodic Lebesgue summable functions $f$ $(f \in L)$ with the finite norm

$$
\|f\|_{\mathcal{S}^{p}}=\left\|\{\widehat{f}(k)\}_{k \in \mathbb{Z}}\right\|_{l_{p}(\mathbb{Z})}=\left(\sum_{k \in \mathbb{Z}}|\widehat{f}(k)|^{p}\right)^{1 / p},
$$

[^0]where $\widehat{f}(k):=[f]^{\wedge}(k)=\frac{1}{2 \pi} \int_{0}^{2 \pi} f(t) \mathrm{e}^{-\mathrm{i} k t} \mathrm{~d} t, k \in \mathbb{Z}$, are the Fourier coefficients of the function $f$, and investigated some approximation characteristics of these spaces, including in the context of direct and inverse theorems. Stepanets and Serdyuk [25] introduced the notion of $k$ th modulus of smoothness in $\mathcal{S}^{p}$ and established direct and inverse theorems on approximation in terms of these moduli of smoothness and the best approximations of functions. Also this topic was investigated actively in [26, 30, [24, [29, Ch. 3] and others.

In the papers [19, 20] some results for the spaces $\mathcal{S}^{p}$ were extended to the Orlicz sequence spaces. In particular, in 20 the authors found the exact values of the best $n$-term approximations and Kolmogorov widths of certain sets of images of the diagonal operators in the Orlicz spaces. The purpose of this paper is to combine the above mentioned studies and prove direct and inverse theorems in the Orlicz type spaces $\mathcal{S}_{M}$ in terms of best approximations of functions and moduli of smoothness of fractional order.

## 2. Preliminaries

An Orlicz function $M(t)$ is a non-decreasing convex function defined for $t \geq 0$ such that $M(0)=0$ and $M(t) \rightarrow \infty$ as $t \rightarrow \infty$. Let $\mathcal{S}_{M}$ be the space of all functions $f \in L$ such that the following quantity (which is also called the Luxemburg norm of $f$ ) is finite:

$$
\begin{equation*}
\|f\|_{M}:=\left\|\{\widehat{f}(k)\}_{k \in \mathbb{Z}}\right\|_{l_{M}(\mathbb{Z})}=\inf \left\{a>0: \sum_{k \in \mathbb{Z}} M(|\widehat{f}(k)| / a) \leq 1\right\} . \tag{2.1}
\end{equation*}
$$

Functions $f \in L$ and $g \in L$ are equivalent in the space $\mathcal{S}_{M}$, when $\|f-g\|_{M}=0$.
The spaces $\mathcal{S}_{M}$ defined in this way are Banach spaces. In case $M(t)=t^{p}, p \geq 1$, they coincide with the above-defined spaces $\mathcal{S}^{p}$.

Let $\mathcal{T}_{n}, n=0,1, \ldots$, be the set of all trigonometric polynomials $\tau_{n}(x):=\sum_{|k| \leq n} c_{k} \mathrm{e}^{i k x}$ of the order $n$, where $c_{k}$ are arbitrary complex numbers. For any function $f \in \mathcal{S}_{M}$, we denote by

$$
\begin{equation*}
E_{n}(f)_{M}:=\inf _{\tau_{n-1} \in \mathcal{T}_{n-1}}\left\|f-\tau_{n-1}\right\|_{M} \tag{2.2}
\end{equation*}
$$

the best approximation of $f$ by trigonometric polynomials $\tau_{n-1} \in \mathcal{T}_{n-1}$ in the space $\mathcal{S}_{M}$.
Similarly to [7], we define the (right) difference of $f \in L$ of fractional order $\alpha>0$ with respect to the increment $h \in \mathbb{R}$ by

$$
\begin{equation*}
\Delta_{h}^{\alpha} f(x):=\sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha}{j} f(x-j h) \tag{2.3}
\end{equation*}
$$

where $\binom{\alpha}{j}=\frac{\alpha(\alpha-1) \cdot \ldots \cdot(\alpha-j+1)}{j!}, j \in \mathbb{N},\binom{\alpha}{0}:=1$, and assemble some basic properties of the fractional differences.

Lemma 2.1. Assume that $f \in \mathcal{S}_{M}, \alpha, \beta>0, x, h \in \mathbb{R}$. Then
(i) $\left\|\Delta_{h}^{\alpha} f\right\|_{M} \leq K(\alpha)\|f\|_{M}$, where $K(\alpha):=\sum_{j=0}^{\infty}\left|\binom{\alpha}{j}\right| \leq 2^{\{\alpha\}},\{\alpha\}=\inf \{k \in \mathbb{N}: k \geq \alpha\}$.
(ii) $\left[\Delta_{h}^{\alpha} f\right]^{\wedge}(k)=\left(1-\mathrm{e}^{-i k h}\right)^{\alpha} \widehat{f}(k), k \in \mathbb{Z}$.
(iii) $\left(\Delta_{h}^{\alpha}\left(\Delta_{h}^{\beta} f\right)\right)(x)=\Delta_{h}^{\alpha+\beta} f(x)$ (a.e.).
(iv) $\left\|\Delta_{h}^{\alpha+\beta} f\right\|_{M} \leq 2^{\{\beta\}}\left\|\Delta_{h}^{\alpha} f\right\|_{M}$.
(v) $\lim _{|h| \rightarrow 0}\left\|\Delta_{h}^{\alpha} f\right\|_{M}=0$.

The proof of Lemma 2.1 and other auxiliary statements of the paper will be given in Section 7. Based on definition (2.3), the modulus of smoothness of $f \in \mathcal{S}_{M}$ of the index $\alpha>0$ is defined by

$$
\omega_{\alpha}(f, \delta)_{M}:=\omega_{\alpha}(f, \delta)_{\mathcal{S}_{M}}=\sup _{|h| \leq \delta}\left\|\Delta_{h}^{\alpha} f\right\|_{M}
$$

For convenience, we also assume that $\Delta_{h}^{0} f:=f$ and $\omega_{0}(f, \delta)_{M}:=\|f\|_{M}$. Using the standard arguments, it can be shown that the functions $\omega_{\alpha}(f, \delta)_{M}$ possess all the basic properties of ordinary moduli of smoothness. Before formulating them, we give the definition of the $\psi$-derivative of a function.

Let $\psi=\left\{\psi_{k}\right\}_{k=-\infty}^{\infty}$ be an arbitrary sequence of complex numbers, $\psi_{k} \neq 0, k \in \mathbb{Z}$. If for a given function $f \in L$ with the Fourier series of the form $S[f](x)=\sum_{k \in \mathbb{Z}} \widehat{f}(k) \mathrm{e}^{\mathrm{i} k x}$, the series $\sum_{k \in \mathbb{Z} \backslash\{0\}} \widehat{f}(k) \mathrm{e}^{\mathrm{i} k x} / \psi_{k}$ is the Fourier series of a certain function $g \in L$, then $g$ is called (see, for example, [23, Ch. 9]) $\psi$-derivative of the function $f$ and is denoted as $g:=f^{\psi}$. It is clear that the Fourier coefficients of functions $f$ and $f^{\psi}$ are related by equality

$$
\begin{equation*}
\widehat{f}(k)=\psi_{k} \widehat{f}^{\psi}(k), \quad k \in \mathbb{Z} \backslash\{0\} \tag{2.4}
\end{equation*}
$$

and $\widehat{f}^{\psi}(0)=0$. In case $\psi_{k}=|k|^{-r}, r>0, k \in \mathbb{Z} \backslash\{0\}$, we use the notation $f^{\psi}=: f^{(r)}$.
Lemma 2.2. Assume that $f, g \in \mathcal{S}_{M}, \alpha \geq \beta>0$ and $\delta, \delta_{1}, \delta_{2}>0$. Then
(i) $\omega_{\alpha}(f, \delta)_{M}$ is a non-negative increasing continuous function of $\delta$ on $(0, \infty)$ such that $\lim _{\delta \rightarrow 0+} \omega_{\alpha}(f, \delta)_{M}=0$.
(ii) $\omega_{\alpha}(f, \delta)_{M} \leq 2^{\{\alpha-\beta\}} \omega_{\beta}(f, \delta)_{M}$.
(iii) $\omega_{\alpha}(f+g, \delta)_{M} \leq \omega_{\alpha}(f, \delta)_{M}+\omega_{\alpha}(g, \delta)_{M}$.
(iv) $\omega_{1}\left(f, \delta_{1}+\delta_{2}\right)_{M} \leq \omega_{1}\left(f, \delta_{1}\right)_{M}+\omega_{1}\left(f, \delta_{2}\right)_{M}$.
(v) $\omega_{\alpha}(f, \delta)_{M} \leq 2^{\{\alpha\}}\|f\|_{M}$.
(vi) if there exists a derivative $f^{(\beta)} \in \mathcal{S}_{M}$, then $\omega_{\alpha}(f, \delta)_{M} \leq \delta^{\beta} \omega_{\alpha-\beta}\left(f^{(\beta)}, \delta\right)_{M}$.
(vii) $\omega_{\alpha}(f, p \delta)_{M} \leq p^{\alpha} \omega_{\alpha}(f, \delta)_{M} \quad(\alpha \in \mathbb{N}, \quad p \in \mathbb{N})$.
(viii) $\omega_{\alpha}(f, \eta)_{M} \leq \delta^{-\alpha}(\delta+\eta)^{\alpha} \omega_{\alpha}(f, \delta)_{M} \quad(\alpha \in \mathbb{N})$.

## 3. Direct approximation theorems

Proposition 3.1. Let $\psi=\left\{\psi_{k}\right\}_{k=-\infty}^{\infty}$ be an arbitrary sequence of complex numbers such that $\psi_{k} \neq 0$ and $\lim _{|k| \rightarrow \infty}\left|\psi_{k}\right|=0$. If for the function $f \in \mathcal{S}_{M}$ there exists a derivative $f^{\psi} \in \mathcal{S}_{M}$, then

$$
E_{n}(f)_{M} \leq \varepsilon_{n} E_{n}\left(f^{\psi}\right)_{M}
$$

where $\varepsilon_{n}=\max _{|k| \geq n}\left|\psi_{k}\right|$.
Proof. For a fixed $a>0$ and arbitrary numbers $c_{k} \in \mathbb{C}$,

$$
\sum_{|k| \leq n-1} M\left(\left|\widehat{f}(k)-c_{k}\right| / a\right)+\sum_{|k| \geq n} M(|\widehat{f}(k)| / a) \geq \sum_{|k| \geq n} M(|\widehat{f}(k)| / a)
$$

therefore, for any function $f \in \mathcal{S}_{M}$ we have

$$
\begin{equation*}
E_{n}(f)_{M}=\left\|f-S_{n-1}(f)\right\|_{M}=\inf \left\{a>0: \sum_{|k| \geq n} M(|\widehat{f}(k)| / a) \leq 1\right\} \tag{3.1}
\end{equation*}
$$

where $S_{n-1}(f, x)=\sum_{|k| \leq n-1} \widehat{f}(k) \mathrm{e}^{\mathrm{i} k x}$ is the Fourier sum of the function $f$.
According to (3.1) and (2.4), we have

$$
\begin{aligned}
E_{n}(f)_{M} & =\inf \left\{a>0: \sum_{|k| \geq n} M\left(\left|\psi_{k} \widehat{f}^{\psi}(k)\right| / a\right) \leq 1\right\} \\
& \leq \inf \left\{a>0: \sum_{|k| \geq n} M\left(\varepsilon_{n}\left|\widehat{f}^{\psi}(k)\right| / a\right) \leq 1\right\} \leq \varepsilon_{n} E_{n}\left(f^{\psi}\right)_{M}
\end{aligned}
$$

In this case, if $\varepsilon_{n}=\max _{|k| \geq n}\left|\psi_{k}\right|=\left|\psi_{k_{0}}\right|$, where $k_{0}$ is an integer, $\left|k_{0}\right| \geq n$, then for an arbitrary polynomial of the form $\tilde{\tau}_{k_{0}}(x):=c \mathrm{e}^{\mathrm{i} k_{0} x}, c \neq 0$, obviously, the equality holds

$$
E_{n}\left(\tilde{\tau}_{k_{0}}\right)_{M}=\varepsilon_{n} E_{n}\left(\tilde{\tau}_{k_{0}}^{\psi}\right)_{M} .
$$

Theorem 3.2. If $f \in \mathcal{S}_{M}$, then for any numbers $\alpha>0$ and $n \in \mathbb{N}$ the following inequality holds:

$$
\begin{equation*}
E_{n}(f)_{M} \leq C(\alpha) \omega_{\alpha}\left(f, n^{-1}\right)_{M} \tag{3.2}
\end{equation*}
$$

where $C=C(\alpha)$ is a constant that does not depend on $f$ and $n$.
Before proving Theorem 3.2, we formulate the auxiliary Lemma 3.1. This assertion establishes the equivalence of the Luxemburg norm (2.1) and the Orlicz norm, where the latter is defined as follows. Consider the function

$$
\begin{equation*}
\tilde{M}(v):=\sup \{u v-M(u): u \geq 0\} \tag{3.3}
\end{equation*}
$$

and the set $\Lambda=\Lambda(\tilde{M})$ of all sequences of positive numbers $\lambda=\left\{\lambda_{k}\right\}_{k \in \mathbb{Z}}$ such that $\sum_{k \in \mathbb{Z}} \tilde{M}\left(\lambda_{k}\right) \leq 1$. For any function $f \in \mathcal{S}_{M}$, define its Orlicz norm by the equality

$$
\begin{equation*}
\|f\|_{M}^{*}:=\sup \left\{\sum_{k \in \mathbb{Z}} \lambda_{k}|\widehat{f}(k)|: \quad \lambda \in \Lambda\right\} \tag{3.4}
\end{equation*}
$$

Lemma 3.1. For any function $f \in \mathcal{S}_{M}$, the following relation holds:

$$
\begin{equation*}
\|f\|_{M} \leq\|f\|_{M}^{*} \leq 2\|f\|_{M} \tag{3.5}
\end{equation*}
$$

Proof of Theorem 3.2. Let us use the proof scheme from [21]. Let $\left\{K_{n}(t)\right\}_{n=1}^{\infty}$ be a sequence of kernels (where $K_{n}(t)$ is a trigonometric polynomial of order not greater than $n$ ) such that for all $n=1,2, \ldots$ the following conditions are fulfilled:

$$
\begin{gather*}
\int_{-\pi}^{\pi} K_{n}(t) \mathrm{d} t=1  \tag{3.6}\\
\int_{-\pi}^{\pi}|t|^{r}\left|K_{n}(t)\right| \mathrm{d} t \leq C(r)(n+1)^{-r}, \quad r=0,1,2, \ldots \tag{3.7}
\end{gather*}
$$

In the role of such kernels, in particular, we can take the well-known Jackson kernels of sufficiently great order, that is,

$$
K_{n}(t)=b_{p}\left(\frac{\sin p t / 2}{\sin t / 2}\right)^{2 k_{0}}
$$

where $k_{0}$ is an integer that does not depend on $n, 2 k_{0} \geq r+2$, the positive integer $p$ is determined from the inequality $n /\left(2 k_{0}\right)<p \leq n /\left(2 k_{0}\right)+1$, and the constant $b_{p}$ is chosen due to the normalization condition (3.6).

It was shown in [21] that for any sequence of kernels $\left\{K_{n}(t)\right\}$ satisfying conditions (3.6)-(3.7), the following estimate holds:

$$
\begin{equation*}
\int_{-\pi}^{\pi}\left(|t|+n^{-1}\right)^{r}\left|K_{n}(t)\right| \mathrm{d} t \leq C(r) n^{-r}, \quad(r, n=1,2, \ldots) \tag{3.8}
\end{equation*}
$$

Let us first consider the case of $\alpha \in \mathbb{N}$. Set

$$
\sigma_{n-1}(x)=(-1)^{\alpha+1} \int_{-\pi}^{\pi} K_{n-1}(t) \sum_{j=1}^{\alpha}(-1)^{j}\binom{\alpha}{j} f(x-j t) \mathrm{d} t
$$

It is clear that $\sigma_{n-1}(x)$ is a trigonometric polynomial which order does not exceed $n$. Further, in view of (3.6), we have

$$
f(x)-\sigma_{n-1}(x)=(-1)^{\alpha} \int_{-\pi}^{\pi} K_{n-1}(t) \sum_{j=0}^{\alpha}(-1)^{j}\binom{\alpha}{j} f(x-j t) \mathrm{d} t=(-1)^{\alpha} \int_{-\pi}^{\pi} K_{n-1}(t) \Delta_{t}^{\alpha} f(x) \mathrm{d} t
$$

Hence, taking into account relations (3.4)-(3.5) and the definition of the set $\Lambda$, we obtain

$$
\begin{aligned}
& E_{n}(f)_{M} \leq\left\|f-\sigma_{n-1}\right\|_{M} \leq\left\|f-\sigma_{n-1}\right\|_{M}^{*}=\left\|(-1)^{\alpha} \int_{-\pi}^{\pi} K_{n-1}(t) \Delta_{t}^{\alpha} f \mathrm{~d} t\right\|_{M}^{*} \\
& \quad=\sup \left\{\sum_{k \in \mathbb{Z}} \lambda_{k}\left|\frac{1}{2 \pi} \int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} K_{n-1}(t) \Delta_{t}^{\alpha} f(x) \mathrm{d} t\right) \mathrm{e}^{-\mathrm{i} k x} \mathrm{~d} x\right|: \quad \lambda \in \Lambda\right\} .
\end{aligned}
$$

Applying now the Fubini theorem and again using estimate (3.5), we find

$$
\begin{align*}
E_{n}(f)_{M} & \leq \int_{-\pi}^{\pi}\left|K_{n-1}(t)\right| \sup \left\{\sum_{k \in \mathbb{Z}} \lambda_{k}\left|\frac{1}{2 \pi} \int_{-\pi}^{\pi} \Delta_{t}^{\alpha} f(x) \mathrm{e}^{-\mathrm{i} k x} \mathrm{~d} x\right|: \lambda \in \Lambda\right\} \mathrm{d} t \\
& \leq 2 \int_{-\pi}^{\pi}\left|K_{n-1}(t)\right|\left\|\Delta_{t}^{\alpha} f(x)\right\|_{M}^{*} \mathrm{~d} t \leq 2 \int_{-\pi}^{\pi}\left|K_{n-1}(t)\right|\left\|\Delta_{t}^{\alpha} f(x)\right\|_{M} \mathrm{~d} t \\
& \leq 2 \int_{-\pi}^{\pi}\left|K_{n-1}(t)\right| \omega_{\alpha}(f,|t|)_{M} \mathrm{~d} t \tag{3.9}
\end{align*}
$$

To estimate the integral on the right-hand side of (3.9), we use the property (viii) of Lemma 2.2. Setting $\eta=|t|, \delta=n^{-1}$, we see that

$$
\omega_{\alpha}(f ;|t|)_{M} \leq n^{\alpha}\left(|t|+n^{-1}\right)^{\alpha} \omega_{\alpha}\left(f, n^{-1}\right)_{M}
$$

This inequality together with (3.8) yields

$$
\int_{-\pi}^{\pi}\left|K_{n-1}(t)\right| \omega_{\alpha}(f,|t|)_{M} \mathrm{~d} t \leq n^{\alpha} \omega_{\alpha}\left(f, n^{-1}\right)_{M} \int_{-\pi}^{\pi}\left(|t|+n^{-1}\right)^{\alpha}\left|K_{n-1}(t)\right| \mathrm{d} t \leq C(\alpha) \omega_{\alpha}\left(f, n^{-1}\right)_{M}
$$

Thus, in the case of $\alpha \in \mathbb{N}$, the theorem is proved.
If $\alpha>0, \alpha \notin \mathbb{N}$, then we denote by $\beta$ an arbitrary positive integer satisfying the condition $\beta-1<\alpha<\beta$. Due to property (ii) of Lemma 2.2, we obtain

$$
E_{n}(f)_{M} \leq C(\beta) \omega_{\beta}\left(f, n^{-1}\right)_{M} \leq C(\beta) \omega_{\alpha}\left(f, n^{-1}\right)_{M}
$$

## 4. InVERSE APPROXIMATION THEOREMS

The key role in proving of the inverse approximation theorems is played by the known Bernstein inequality in which the norm of the derivative of a trigonometric polynomial is estimated in terms of the norm of this polynomial (see, e.g. [27, Ch. 4], [29, Ch. 4]).

Proposition 4.1. Let $\psi=\left\{\psi_{k}\right\}_{k=-\infty}^{\infty}$ be an arbitrary sequence of complex numbers, $\psi_{k} \neq 0$. Then for any $\tau_{n} \in \mathcal{T}_{n}, n \in \mathbb{N}$, the following inequality holds:

$$
\begin{equation*}
\left\|\tau_{n}^{\psi}\right\|_{M} \leq \frac{1}{\epsilon_{n}}\left\|\tau_{n}\right\|_{M}, \quad \epsilon_{n}:=\min _{0<|k| \leq n}\left|\psi_{k}\right| \tag{4.1}
\end{equation*}
$$

Proof. Let $\tau_{n}(x)=\sum_{|k| \leq n} c_{k} \mathrm{e}^{\mathrm{i} k x}, c_{k} \in \mathbb{C}$. By the definition of the $\psi$-derivative and equalities (2.4), we get

$$
\begin{aligned}
& \left\|\tau_{n}^{\psi}\right\|_{M}=\inf \left\{a>0: \sum_{0<|k| \leq n} M\left(\left|c_{k}\right| /\left|a \psi_{k}\right|\right) \leq 1\right\} \\
& \quad \leq \max _{0<|k| \leq n}\left|\psi_{k}\right|^{-1} \inf \left\{a>0: \sum_{0<|k| \leq n} M\left(\left|c_{k}\right| / a\right) \leq 1\right\}=\frac{1}{\epsilon_{n}}\left\|\tau_{n}\right\|_{M}
\end{aligned}
$$

In this case, if $\epsilon_{n}=\min _{0<|k| \leq n}\left|\psi_{k}\right|=\left|\psi_{k_{0}}\right|$, then for an arbitrary polynomial of the form $\tilde{\tau}_{k_{0}}(x):=$ $c \mathrm{e}^{\mathrm{i} k_{0} x}, c \neq 0$, we have

$$
\left\|\tilde{\tau}_{k_{0}}^{\psi}\right\|_{M}=\inf \left\{a>0: M\left(|c| /\left|a \psi_{k_{0}}\right|\right) \leq 1\right\}=\frac{1}{\epsilon_{n}}\left\|\tau_{k_{0}}\right\|_{M}
$$

Corollary 4.1.1. Let $\psi=\left\{\psi_{k}\right\}_{k=-\infty}^{\infty}$ be an arbitrary sequence of complex numbers such that $\left|\psi_{-k}\right|=\left|\psi_{k}\right| \geq\left|\psi_{k+1}\right|>0$. Then for any $\tau_{n} \in \mathcal{T}_{n}, n \in \mathbb{N}$,

$$
\begin{equation*}
\left\|\tau_{n}^{\psi}\right\|_{M} \leq \frac{1}{\left|\psi_{n}\right|}\left\|\tau_{n}\right\|_{M} \tag{4.2}
\end{equation*}
$$

In particular, if $\psi_{k}=|k|^{-r}, r>0, k \in \mathbb{Z} \backslash\{0\}$, then

$$
\left\|\tau_{n}^{\psi}\right\|_{M}=\left\|\tau_{n}^{(r)}\right\|_{M} \leq n^{r}\left\|\tau_{n}\right\|_{M}
$$

Theorem 4.2. If $f \in \mathcal{S}_{M}$, then for any $\alpha>0$ and $n \in \mathbb{N}$, the following inequality is true:

$$
\begin{equation*}
\omega_{\alpha}\left(f, n^{-1}\right)_{M} \leq \frac{C(\alpha)}{n^{\alpha}} \sum_{\nu=1}^{n} \nu^{\alpha-1} E_{\nu}(f)_{M} \tag{4.3}
\end{equation*}
$$

where $C=C(\alpha)$ is a constant that does not depend on $f$ and $n$.
For the spaces $L_{p}$ of $2 \pi$-periodic functions integrable to the $p$ th power with the usual norm, inequalities of the type (4.3) were proved in [28] (see also [27, Ch. 6], [29, Ch. 2]). In the spaces $\mathcal{S}^{p}$, similar results were obtained in [25, [26].

Proof. Let us use the proof scheme from [27, Ch. 6]. Let $f \in \mathcal{S}_{M}$ and $S_{n}:=S_{n}(f)$ be the Fourier sum of $f$. Then, due to Lemma 2.2 (v) and relation (3.1) for an arbitrary $m \in \mathbb{N}$, we have

$$
\begin{align*}
\omega_{\alpha}\left(f, n^{-1}\right)_{M} & \leq \omega_{\alpha}\left(f-S_{2^{m+1}}, n^{-1}\right)_{M}+\omega_{\alpha}\left(S_{2^{m+1}}, n^{-1}\right)_{M} \\
& \leq 2^{\{\alpha\}} E_{2^{m+1}+1}(f)_{M}+\omega_{\alpha}\left(S_{2^{m+1}}, n^{-1}\right)_{M} \tag{4.4}
\end{align*}
$$

Further, using property (vi) of Lemma 2.2 and the properties of the norm, we obtain

$$
\omega_{\alpha}\left(S_{2^{m+1}}, n^{-1}\right)_{M} \leq n^{-\alpha}\left\|S_{2^{m+1}}^{(\alpha)}\right\|_{M} \leq n^{-\alpha}\left(\left\|S_{1}^{(\alpha)}\right\|_{M}+\sum_{k=0}^{m}\left\|S_{2^{k+1}}^{(\alpha)}-S_{2^{k}}^{(\alpha)}\right\|_{M}\right)
$$

Moreover, on the basis of Corollary 4.1.1,

$$
\left\|S_{2^{k+1}}^{(\alpha)}-S_{2^{k}}^{(\alpha)}\right\|_{M} \leq 2^{(k+1) \alpha}\left\|S_{2^{k+1}}-S_{2^{k}}\right\|_{M} \leq 2^{(k+1) \alpha+1} E_{2^{k}+1}(f)_{M}
$$

and $\left\|S_{1}^{(\alpha)}\right\|_{M}=\left\|S_{1}^{(\alpha)}-S_{0}^{(\alpha)}\right\|_{M} \leq 2 E_{1}(f)_{M}$. Therefore,

$$
\omega_{\alpha}\left(S_{2^{m+1}}, n^{-1}\right)_{M} \leq n^{-\alpha}\left\|S_{2^{m+1}}^{(\alpha)}\right\|_{M} \leq 2 n^{-\alpha}\left(E_{1}(f)_{M}+\sum_{k=0}^{m} 2^{(k+1) \alpha} E_{2^{k}+1}(f)_{M}\right)
$$

Taking into account the relation

$$
2^{(k+1) \alpha} E_{2^{k}+1}(f)_{M} \leq 2^{2 \alpha} \sum_{\nu=2^{k-1}+2}^{2^{k}+1} \nu^{\alpha-1} E_{\nu}(f)_{M}, \quad k=1,2, \ldots
$$

we get

$$
\begin{aligned}
\omega_{\alpha}\left(S_{2^{m+1}}, n^{-1}\right)_{M} & \leq 2^{2 \alpha+1} n^{-\alpha}\left(E_{1}(f)_{M}+E_{2}(f)_{M}+\sum_{k=1}^{m} \sum_{\nu=2^{k-1}+2}^{2^{k}+1} \nu^{\alpha-1} E_{\nu}(f)_{M}\right) \\
& \leq \frac{c^{*}(\alpha)}{n^{\alpha}} \sum_{k=1}^{2^{m}+1} k^{\alpha-1} E_{k}(f)_{M}
\end{aligned}
$$

Choosing now an integer $m$ so that $2^{m}+1 \leq n \leq 2^{m+1}$ and substituting this estimate into (4.4), we get (4.3).
Corollary 4.2.1. Assume that the sequence of the best approximations $E_{n}(f)_{M}$ of a function $f \in \mathcal{S}_{M}$ satisfies the following relation for some $\beta>0$ :

$$
E_{n}(f)_{M}=\mathcal{O}\left(n^{-\beta}\right)
$$

Then, for any $\alpha>0$, one has

$$
\omega_{\alpha}(f, t)_{M}=\left\{\begin{array}{cl}
\mathcal{O}\left(t^{\beta}\right) & \text { for } \beta<\alpha \\
\mathcal{O}\left(t^{\alpha}|\ln t|\right) & \text { for } \beta=\alpha \\
\mathcal{O}\left(t^{\alpha}\right) & \text { for } \beta>\alpha
\end{array}\right.
$$

## 5. Constructive characteristics of the classes of functions defined by the $\alpha$ TH MODULI OF SMOOTHNESS

In the following two sections some applications of the obtained results are considered. In particular, in this section we give the constructive characteristics of the classes $\mathcal{S}_{M} H_{\alpha}^{\omega}$ of functions for which the $\alpha$ th moduli of smoothness do not exceed some majorant.

Let $\omega$ be a function defined on interval $[0,1]$. For a fixed $\alpha>0$, we set

$$
\begin{equation*}
\mathcal{S}_{M} H_{\alpha}^{\omega}=\left\{f \in \mathcal{S}_{M}: \quad \omega_{\alpha}(f, \delta)_{M}=\mathcal{O}(\omega(\delta)), \quad \delta \rightarrow 0+\right\} \tag{5.1}
\end{equation*}
$$

Further, we consider the functions $\omega(\delta), \delta \in[0,1]$, satisfying the following conditions 1)-4): 1) $\omega(\delta)$ is continuous on $[0,1]$; 2) $\omega(\delta) \uparrow$; 3) $\omega(\delta) \neq 0$ for any $\delta \in(0,1]$; 4) $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0+$; as well-known condition $\left(\mathcal{B}_{\alpha}\right), \alpha>0: \sum_{v=1}^{n} v^{\alpha-1} \omega\left(v^{-1}\right)=\mathcal{O}\left(n^{\alpha} \omega\left(n^{-1}\right)\right)$ (see, e.g. [4]).
Theorem 5.1. Assume that $\alpha>0$ and $\omega$ is a function, satisfying conditions 1)-4) and ( $\mathcal{B}_{\alpha}$ ). Then, in order a function $f \in \mathcal{S}_{M}$ to belong to the class $\mathcal{S}_{M} H_{\alpha}^{\omega}$, it is necessary and sufficient that

$$
\begin{equation*}
E_{n}(f)_{M}=\mathcal{O}\left(\omega\left(n^{-1}\right)\right) \tag{5.2}
\end{equation*}
$$

Proof. Let $f \in \mathcal{S}_{M} H_{\alpha}^{\omega}$, by virtue of Theorem 3.2, we have

$$
\begin{equation*}
E_{n}(f)_{M} \leq C(\alpha) \omega_{\alpha}\left(f ; n^{-1}\right)_{M} \tag{5.3}
\end{equation*}
$$

Therefore, relation (5.1) yields (5.2). On the other hand, if relation (5.2) holds, then by virtue of Theorem 4.2, taking into account the condition $\left(\mathcal{B}_{\alpha}\right)$, we obtain

$$
\begin{equation*}
\omega_{\alpha}\left(f, n^{-1}\right)_{M} \leq \frac{C(\alpha)}{n^{\alpha}} \sum_{\nu=1}^{n} \nu^{\alpha-1} E_{\nu}(f)_{M} \leq \frac{C_{1}}{n^{\alpha}} \sum_{\nu=1}^{n} \nu^{\alpha-1} \omega\left(v^{-1}\right)=\mathcal{O}\left(\omega\left(n^{-1}\right)\right) \tag{5.4}
\end{equation*}
$$

Thus, the function $f$ belongs to the set $\mathcal{S}_{M} H_{\alpha}^{\omega}$.
The function $\varphi(t)=t^{r}, r \leq \alpha$, satisfies the condition $\left(\mathcal{B}_{\alpha}\right)$. Hence, denoting by $\mathcal{S}_{M} H_{\alpha}^{r}$ the class $\mathcal{S}_{M} H_{\alpha}^{\omega}$ for $\omega(t)=t^{r}, 0<r \leq \alpha$, we establish the following statement:
Corollary 5.1.1. Let $\alpha>0,0<r \leq \alpha$. In order a function $f \in S_{M}$ to belong to $\mathcal{S}_{M} H_{\alpha}^{r}$, it is necessary and sufficient that

$$
E_{n}(f)_{M}=\mathcal{O}\left(n^{-r}\right)
$$

## 6. The equivalence between $\alpha$ Th moduli of smoothness and $K$-functionals

$K$-functionals were introduced by Lions and Peetre in 1961, and defined in their usual form by Peetre in the monograph [16] as a basis for his theory of operator interpolation. Unlike the moduli of continuity expressing the smooth properties of functions, $K$-functionals express some of their approximative properties. In this section we prove the equivalence between our moduli of smoothness and certain Peetre $K$-functionals. This connection is important for studying the properties of the modulus of smoothness and the $K$-functional, and also for their further application to the problems of approximation theory.

In the space $\mathcal{S}_{M}$, the Petree $K$-functional of a function $f$ (see, e.g. [9, Ch. 6]), which generated by its derivative of order $\alpha>0$, is the following quantity:

$$
K_{\alpha}(\delta, f)_{M}=\inf \left\{\|f-h\|_{M}+\delta^{\alpha}\left\|h^{(\alpha)}\right\|_{M}: h^{(\alpha)} \in \mathcal{S}_{M}\right\}, \quad \delta>0
$$

Theorem 6.1. For each $f \in \mathcal{S}_{M}, \alpha>0$, there exist constants $C_{1}(\alpha), C_{2}(\alpha)>0$, such that for $\delta>0$

$$
\begin{equation*}
C_{1}(\alpha) \omega_{\alpha}(f, \delta)_{M} \leq K_{\alpha}(\delta, f)_{M} \leq C_{2}(\alpha) \omega_{\alpha}(f, \delta)_{M} \tag{6.1}
\end{equation*}
$$

Proof. Consider an arbitrary function $h \in \mathcal{S}_{M}$ such that $h^{(\alpha)} \in \mathcal{S}_{M}$. Then we have by Lemman 2.2 (iii), (v) and (vi)

$$
\omega_{\alpha}(f, \delta)_{M} \leq \omega_{\alpha}(f-h, \delta)_{M}+\omega_{\alpha}(h, \delta)_{M} \leq 2^{\{\alpha\}}\|f-h\|_{M}+\delta^{\alpha}\left\|h^{(\alpha)}\right\|_{M}
$$

Taking the infimum over all $h \in \mathcal{S}_{M}$ such that $h^{(\alpha)} \in \mathcal{S}_{M}$, we get the left-hand side of (6.1).
To prove the right-hand side of (6.1), let us formulate the following auxiliary lemma.
Lemma 6.1. Assume that $\alpha>0, n \in \mathbb{N}$ and $0 \leq h \leq 2 \pi / n$. Then for any $\tau_{n} \in \mathcal{T}_{n}$

$$
\begin{equation*}
\left(\frac{\sin (n h / 2)}{n / 2}\right)^{\alpha}\left\|\tau_{n}^{(\alpha)}\right\|_{M} \leq\left\|\Delta_{h}^{\alpha} \tau_{n}\right\|_{M} \leq h^{\alpha}\left\|\tau_{n}^{(\alpha)}\right\|_{M} \tag{6.2}
\end{equation*}
$$

Now let $\delta \in(0,2 \pi)$ and $n \in \mathbb{N}$ such that $\pi / n<\delta<2 \pi / n$. Let also $S_{n}:=S_{n}(f)$ be the Fourier sum of $f$. Using Lemma 6.1 with $h=\pi / n$ and property (i) of Lemma 2.1 we obtain

$$
\begin{gather*}
\left\|S_{n}^{(\alpha)}\right\|_{M} \leq 2^{-\alpha+1} n^{\alpha}\left\|\Delta_{\pi / n}^{\alpha} S_{n}\right\|_{M} \leq 2(\pi / \delta)^{\alpha}\left(\left\|\Delta_{\pi / n}^{\alpha}\left(S_{n}-f\right)\right\|_{M}+\left\|\Delta_{\pi / n}^{\alpha} f\right\|_{M}\right) \\
\leq 2(\pi / \delta)^{\alpha}\left(2^{\{\alpha\}}\left\|f-S_{n}\right\|_{M}+\left\|\Delta_{\pi / n}^{\alpha} f\right\|_{M}\right) \tag{6.3}
\end{gather*}
$$

By virtue of (3.1) and Theorem 3.2, we have

$$
\begin{equation*}
\left\|f-S_{n}\right\|_{M}=E_{n}(f)_{M} \leq C(\alpha) \omega_{\alpha}(f, \delta)_{M} \tag{6.4}
\end{equation*}
$$

Combining (6.3), (6.4) and the definition of modulus of smoothness, we obtain the relation

$$
\left\|S_{n}^{(\alpha)}\right\|_{M} \leq C_{2}(\alpha) \delta^{-\alpha} \omega_{\alpha}(f, \delta)_{M}
$$

where $C_{2}(\alpha):=2 \pi^{\alpha}\left(2^{\{\alpha\}} C(\alpha)+1\right)$, which yields the right-hand side of (6.1):

$$
K_{\alpha}(\delta, f)_{M} \leq\left\|f-S_{n}\right\|_{M}+\delta^{\alpha}\left\|S_{n}^{(\alpha)}\right\|_{M} \leq C_{2}(\alpha) \omega_{\alpha}(f, \delta)_{M} .
$$

## 7. Proof of the auxiliary statements

Proof of Lemma 2.1. Let us set $f(x-j h)=: f_{j h}(x)$. For any $k \in \mathbb{Z}$ and $j=0,1, \ldots$, we have $\widehat{f}_{j h}(k)=\widehat{f}(k) \mathrm{e}^{-\mathrm{i} k j h}$. Therefore,

$$
\begin{aligned}
& \left\|\Delta_{h}^{\alpha} f\right\|_{M}=\inf \left\{a>0: \sum_{k \in \mathbb{Z}} M\left(\left|\left[\Delta_{h}^{\alpha} f\right]^{\wedge}(k)\right| / a\right) \leq 1\right\} \\
& =\quad \inf \left\{a>0: \sum_{k \in \mathbb{Z}} M\left(\left|\left[\sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha}{j} f_{j h}\right] \wedge(k)\right| / a\right) \leq 1\right\} \\
& =\quad \inf \left\{a>0: \sum_{k \in \mathbb{Z}} M\left(\left|\widehat{f}(k) \sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha}{j} \mathrm{e}^{-\mathrm{i} k j h}\right| / a\right) \leq 1\right\} .
\end{aligned}
$$

For a fixed $a>0$

$$
\begin{align*}
\sum_{k \in \mathbb{Z}} M\left(\left|\widehat{f}(k) \sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha}{j} \mathrm{e}^{-\mathrm{i} k j h}\right| / a\right) & \leq \sum_{k \in \mathbb{Z}} M\left(\sum_{j=0}^{\infty}\left|\binom{\alpha}{j}\right||\widehat{f}(k)| / a\right) \\
& \leq \sum_{k \in \mathbb{Z}} M\left(2^{\{\alpha\}}|\widehat{f}(k)| / a\right) . \tag{7.1}
\end{align*}
$$

and hence (i) holds. Property (ii) is obvious:

$$
\begin{equation*}
\left[\Delta_{h}^{\alpha} f\right]^{\wedge}(k)=\left[\sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha}{j} f_{j h}\right] \wedge(k)=\widehat{f}(k) \sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha}{j} \mathrm{e}^{-\mathrm{i} k j h}=\left(1-\mathrm{e}^{-\mathrm{i} k h}\right)^{\alpha} \widehat{f}(k), \tag{7.2}
\end{equation*}
$$

and property (iii) is its consequence. Part (iv) follows by (i)-(iii).
To prove ( v ) we first show that the following relation holds:

$$
\begin{equation*}
\lim _{|h| \rightarrow 0}\left\|\Delta_{h}^{\alpha} \tau_{n}\right\|_{M}=0 \tag{7.3}
\end{equation*}
$$

where $\tau_{n}$ is an arbitrary polynomial of the form $\tau_{n}(x)=\sum_{|k| \leq n} c_{k} \mathrm{e}^{\mathrm{i} k x}, n \in \mathbb{N}, c_{k} \in \mathbb{C}$.
Since $\left\|\tau_{n}\right\|_{M}=\inf \left\{a>0: \sum_{|k| \leq n} M\left(\left|c_{k}\right| / a\right) \leq 1\right\}$, then taking into the account (ii), for $a_{0}=|n h|^{\alpha}\left\|\tau_{n}\right\|_{M}$, we obtain

$$
\begin{align*}
& \sum_{|k| \leq n} M\left(\left|\left[\Delta_{h}^{\alpha} \tau_{n}\right]^{\wedge}(k)\right| / a_{0}\right)=\sum_{|k| \leq n} M\left(\left|1-\mathrm{e}^{-\mathrm{i} k h}\right|^{\alpha}\left|c_{k}\right| / a_{0}\right)=\sum_{|k| \leq n} M\left(2^{\alpha}\left|\sin \frac{k h}{2}\right|^{\alpha}\left|c_{k}\right| / a_{0}\right) \\
& \leq \sum_{|k| \leq n} M\left(|k h|^{\alpha}\left|c_{k}\right| / a_{0}\right) \leq \sum_{|k| \leq n} M\left(|n h|^{\alpha}\left|c_{k}\right| / a_{0}\right)=\sum_{|k| \leq n} M\left(\left|c_{k}\right| /\left\|\tau_{n}\right\|_{M}\right) \leq 1 . \tag{7.4}
\end{align*}
$$

Therefore, $\left\|\Delta_{h}^{\alpha} \tau_{n}\right\|_{M} \leq|n h|^{\alpha}\left\|\tau_{n}\right\|_{M}$. For an arbitrary $\varepsilon>0$, we set $\delta:=\delta(\varepsilon)=\left(\varepsilon / n^{\alpha}\left\|\tau_{n}\right\|_{M}\right)^{1 / \alpha}$. Then for all $|h|<\delta$, we have $\left\|\Delta_{h}^{\alpha} \tau_{n}\right\|_{M}<\varepsilon$, i.e., relation (7.3) is indeed fulfilled.

Now let $f$ is an arbitrary function from $\mathcal{S}_{M}$ and $S_{n}(f, x)=\sum_{|k| \leq n} \widehat{f}(k) \mathrm{e}^{\mathrm{i} k x}$ is its Fourier sum. Since the value $\left\|f-S_{n}(f)\right\|_{M}$ tends to zero as $n \rightarrow \infty$, then for any $\varepsilon>0$ there exist a positive integer $n_{0}=n_{0}(\varepsilon)$ such that for any $n>n_{0}$, we have

$$
\left\|f-S_{n}(f)\right\|_{M}<\varepsilon / 2^{\{\alpha\}+1}
$$

Furthermore, by virtue of (7.3), there exist a number $\delta:=\delta(\varepsilon, n)$ such that $\left\|\Delta_{h}^{\alpha} S_{n}(f)\right\|_{M}<\frac{\varepsilon}{2}$ when $|h|<\delta$. Then using properties of norm and (i), for $n>n_{0}$ we get

$$
\left\|\Delta_{h}^{\alpha} f\right\|_{M} \leq\left\|\Delta_{h}^{\alpha}\left(f-S_{n}(f)\right)\right\|_{M}+\left\|\Delta_{h}^{\alpha} S_{n}(f)\right\|_{M} \leq 2^{\{\alpha\}}\left\|f-S_{n}(f)\right\|_{M}+\left\|\Delta_{h}^{\alpha} S_{n}(f)\right\|_{M}<\varepsilon
$$

which yields (v).
Proof of Lemma 2.2. In (i), the convergence to zero for $\delta \rightarrow 0+$ follows by Lemma 2.1 (v). Part (v) is the consequence of Lemma 2.1(i). Property (iii), non-negativity and increasing of the function $\omega_{\alpha}(f, t)_{M}$ follow from the definition of modulus of smoothness. According to Lemma 2.1 (i) and (iii), for arbitrary numbers $0<\beta \leq \alpha$, we have

$$
\left\|\Delta_{h}^{\alpha} f\right\|_{M}=\left\|\Delta_{h}^{\alpha-\beta}\left(\Delta_{h}^{\beta} f\right)\right\|_{M} \leq 2^{\{\alpha-\beta\}}\left\|\Delta_{h}^{\beta} f\right\|_{M}
$$

whence passing to the exact upper bound over all $|h| \leq \delta$, we obtain (ii). Part (iv) is proved by the following standard arguments:

$$
\begin{aligned}
\omega_{1}\left(f, \delta_{1}+\delta_{2}\right)_{M}= & \sup _{\left|h_{1}\right| \leq \delta_{1},\left|h_{2}\right| \leq \delta_{2}}\left\|f\left(x+h_{1}+h_{2}\right)-f(x)\right\|_{M} \leq \sup _{\left|h_{2}\right| \leq \delta_{2}}\left\|f\left(x+h_{1}+h_{2}\right)-f\left(x+h_{1}\right)\right\|_{M} \\
& +\sup _{\left|h_{1}\right| \leq \delta_{1}}\left\|f\left(x+h_{1}\right)-f(x)\right\|_{M} \leq \omega_{1}\left(f, \delta_{2}\right)_{M}+\omega_{1}\left(f, \delta_{1}\right)_{M}
\end{aligned}
$$

In particular, this yields the continuity of the function $\omega_{1}(f, \delta)_{M}$, since for arbitrary $\delta_{1}>\delta_{2}>0$, $\omega_{1}\left(f, \delta_{1}\right)_{M}-\omega_{1}\left(f, \delta_{2}\right)_{M} \leq \omega_{1}\left(\delta_{1}-\delta_{2}\right)_{M} \rightarrow 0$ as $\delta_{1}-\delta_{2} \rightarrow 0$.

Let us prove the continuity of $\omega_{\alpha}(f, \delta)_{M}$ for arbitrary $\alpha>0$. Let $0<\delta_{1}<\delta_{2}$ and $h=h_{1}+h_{2}$, where $0<h_{1} \leq \delta_{1}, 0<h_{2} \leq \delta_{2}-\delta_{1}$. Since $\Delta_{h}^{\alpha} f(\delta)=\Delta_{h_{1}}^{\alpha} f(\delta)+\sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^{j} \Delta_{j h_{2}}^{1} f\left(\delta-j h_{1}\right)$ and

$$
\begin{gathered}
\left\|\sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^{j} \Delta_{j h_{2}}^{1} f_{j h_{1}}\right\|_{M}=\inf \left\{a>0: \sum_{k \in \mathbb{Z}} M\left(\left|\left[\sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^{j} \Delta_{j h_{2}}^{1} f_{j h_{1}}\right] \wedge(k)\right| / a\right) \leq 1\right\} \\
\leq \inf \left\{a>0: \sum_{k \in \mathbb{Z}} M\left(\left|2^{\{\alpha\}} \alpha\left[\Delta_{h_{2}}^{1} f\right] \wedge(k)\right| / a\right) \leq 1\right\} \leq 2^{\{\alpha\}} \alpha\left\|\Delta_{h_{2}}^{1} f\right\|_{M}
\end{gathered}
$$

then $\left\|\Delta_{h}^{\alpha} f\right\|_{M} \leq\left\|\Delta_{h_{1}}^{\alpha} f\right\|_{M}+2^{\{\alpha\}} \alpha\left\|\Delta_{h_{2}}^{1} f\right\|_{M}$ and $\omega_{\alpha}\left(f, \delta_{2}\right)_{M} \leq \omega_{\alpha}\left(f, \delta_{1}\right)_{M}+2^{\{\alpha\}} \alpha \omega_{1}\left(f, \delta_{2}-\delta_{1}\right)_{M}$. Hence, we obtain the necessary relation:

$$
\omega_{\alpha}\left(f, \delta_{2}\right)_{M}-\omega_{\alpha}\left(f, \delta_{1}\right)_{M} \leq 2^{\{\alpha\}} \alpha \omega_{1}\left(f, \delta_{2}-\delta_{1}\right)_{M} \rightarrow 0, \quad \delta_{2}-\delta_{1} \rightarrow 0
$$

If there exists a derivative $f^{(\beta)} \in \mathcal{S}_{M}, 0<\beta \leq \alpha$, then by virtue of (7.2) and (2.4), for arbitrary numbers $k \in \mathbb{Z} \backslash\{0\}$ and $h \in[0, \delta]$, we have

$$
\left|\left[\Delta_{h}^{\alpha} f\right]^{\wedge}(k)\right|=2^{\beta}\left|\sin \frac{k h}{2}\right|^{\beta}\left|1-\mathrm{e}^{-\mathrm{i} k h}\right|^{\alpha-\beta}|\widehat{f}(k)| \leq \delta^{\beta}|k|^{\beta}\left|1-\mathrm{e}^{-\mathrm{i} k h}\right|^{\alpha-\beta}|\widehat{f}(k)| \leq \delta^{\beta}\left|\left[\Delta_{h}^{\alpha-\beta} f^{(\beta)}\right]^{\wedge}(k)\right|
$$

and therefore property (vi) holds.
If $\alpha \in \mathbb{N}$ and $p \in \mathbb{N}$, then using the representation

$$
\Delta_{p h}^{\alpha} f(x)=\sum_{k_{1}=0}^{p-1} \ldots \sum_{k_{\alpha}=0}^{p-1} \Delta_{h}^{\alpha} f\left(x-\left(k_{1}+k_{2}+\ldots+k_{\alpha}\right) h\right)
$$

and the relation

$$
\begin{aligned}
\left|\left[\Delta_{h}^{\alpha} f\left(x-\left(k_{1}+\ldots+k_{\alpha}\right) h\right)\right]^{\wedge}(k)\right| & =\left|\frac{1}{2 \pi} \int_{-\pi}^{\pi} \sum_{j=0}^{\alpha}(-1)^{j}\binom{\alpha}{j} f_{j h}\left(x-\left(k_{1}+\ldots+k_{\alpha}\right) h\right) \mathrm{e}^{-\mathrm{i} k x} \mathrm{~d} x\right| \\
& \leq\left|\frac{1}{2 \pi} \int_{-\pi}^{\pi} \sum_{j=0}^{\alpha}(-1)^{j}\binom{\alpha}{j} f_{j h}(x) \mathrm{e}^{-\mathrm{i} k x} \mathrm{~d} x\right|=\left|\left[\Delta_{h}^{\alpha} f(x)\right]^{\wedge}(k)\right|
\end{aligned}
$$

we get

$$
\begin{aligned}
\left\|\Delta_{p h}^{\alpha} f(x)\right\|_{M} & =\inf \left\{a>0: \sum_{k \in \mathbb{Z}} M\left(\left|\sum_{k_{1}=0}^{p-1} \cdots \sum_{k_{\alpha}=0}^{p-1}\left[\Delta_{h}^{\alpha} f\left(x-\left(k_{1}+\ldots+k_{\alpha}\right) h\right)\right]^{\wedge}(k)\right| / a\right) \leq 1\right\} \\
& \leq \inf \left\{a>0: \sum_{k \in \mathbb{Z}} M\left(p^{\alpha}\left|\left[\Delta_{h}^{\alpha} f(x)\right]^{\wedge}(k)\right| / a\right) \leq 1\right\} \leq p^{\alpha}\left\|\Delta_{h}^{\alpha} f(x)\right\|_{M},
\end{aligned}
$$

and property (vii) is proved. To prove (viii) it is sufficient to consider the case $\delta<\eta$ (for $\delta \geq \eta$, property (viii) is obvious). Choosing the number $p$ such that $\frac{\eta}{\delta} \leq p<\frac{\eta}{\delta}+1$, by virtue (i) and (vii), we obtain

$$
\omega_{\alpha}(f ; \eta) \leq \omega_{\alpha}(f ; p \delta)_{M} \leq p^{\alpha} \omega_{\alpha}(f ; \delta)_{M} \leq\left(\frac{\eta}{\delta}+1\right)^{\alpha} \omega_{\alpha}(f, \delta)_{M}
$$

Proof of Lemma [3.1] The right-hand side of inequality (3.5) is obtained from the Young inequality $u v \leq M(u)+\tilde{M}(v)$, where $u, v \geq 0$, (see, e.g. [15, Ch. 1, $\S 2$ ) as follows

$$
\begin{aligned}
\|f\|_{M}^{*} /\|f\|_{M} & =\|f /\| f\left\|_{M}\right\|_{M}^{*}=\sup \left\{\sum_{k \in \mathbb{Z}} \lambda_{k}|\widehat{f}(k)| /\|f\|_{M}: \lambda \in \Lambda\right\} \\
& \leq \sup \left\{\sum_{k \in \mathbb{Z}}\left(M\left(|\widehat{f}(k)| /\|f\|_{M}\right)+\tilde{M}\left(\lambda_{k}\right)\right): \lambda \in \Lambda\right\} \leq 2 .
\end{aligned}
$$

To prove the left-hand side of the inequality (3.5), we choose an arbitrary function $f \in \mathcal{S}_{M}$ such that $\|f\|_{M}^{*}=1$, and show that for this function the inequality $\|f\|_{M} \leq 1$ holds.

Using the relation $M(u)=\int_{0}^{u} p(t) \mathrm{d} t, u \geq 0$, we define the function $p=p(t), t \geq 0$, and consider the sequence $\lambda^{*}=\left\{\lambda_{k}^{*}\right\}_{k \in \mathbb{Z}}$, where $\lambda_{k}^{*}:=p(|\widehat{f}(k)|)$. Then for any $k \in \mathbb{Z}$, the inequality

$$
\begin{equation*}
\lambda_{k}^{*}|\widehat{f}(k)|=M(|\widehat{f}(k)|)+\tilde{M}\left(\lambda_{k}^{*}\right) . \tag{7.5}
\end{equation*}
$$

holds (see [15]). Also note that for any Orlicz function $M$ the function $\tilde{M}$, defined by (3.3), is also convex (see [15]) and satisfies the inequality

$$
\begin{equation*}
\tilde{M}(\mu v) \leq \mu \tilde{M}(v), \quad 0 \leq \mu \leq 1 . \tag{7.6}
\end{equation*}
$$

If we assume that $\sum_{k \in \mathbb{Z}} \tilde{M}\left(\lambda_{k}^{*}\right)>1$, then by (7.6) we see that

$$
\begin{equation*}
\sum_{k \in \mathbb{Z}} \tilde{M}\left(\frac{\lambda_{k}^{*}}{\sum_{j \in \mathbb{Z}} \tilde{M}\left(\lambda_{j}^{*}\right)}\right) \leq \sum_{k \in \mathbb{Z}} \frac{\tilde{M}\left(\lambda_{k}^{*}\right)}{\sum_{j \in \mathbb{Z}} \tilde{M}\left(\lambda_{j}^{*}\right)} \leq 1 . \tag{7.7}
\end{equation*}
$$

Taking into account (7.5), (7.7), the definition of the set $\Lambda$ and the equality $\|f\|_{M}^{*}=1$, we get the contradiction

$$
\begin{aligned}
\sum_{k \in \mathbb{Z}} M(|\widehat{f}(k)|)+\sum_{k \in \mathbb{Z}} \tilde{M}\left(\lambda_{k}^{*}\right) & =\sum_{k \in \mathbb{Z}} \lambda_{k}^{*}|\widehat{f}(k)|=\sum_{i \in \mathbb{Z}} \tilde{M}\left(\lambda_{i}^{*}\right) \sum_{k \in \mathbb{Z}}|\widehat{f}(k)| \frac{\lambda_{k}^{*}}{\sum_{i \in \mathbb{Z}} \tilde{M}\left(\lambda_{i}^{*}\right)} \\
& \leq \sum_{i \in \mathbb{Z}} \tilde{M}\left(\lambda_{i}^{*}\right) \sup \left\{\sum_{k \in \mathbb{Z}} \lambda_{k}|\widehat{f}(k)|: \quad \lambda \in \Lambda\right\}=\sum_{k \in \mathbb{Z}} \tilde{M}\left(\lambda_{k}^{*}\right) .
\end{aligned}
$$

Consequently, $\sum_{k \in \mathbb{Z}} \tilde{M}\left(\lambda_{k}^{*}\right) \leq 1$ and therefore $\lambda_{k}^{*} \in \Lambda$. Then, taking into account (7.5), we obtain

$$
\sum_{k \in \mathbb{Z}} M(\widehat{f}(k)) \leq \sum_{k \in \mathbb{Z}}|\widehat{f}(k)| \lambda_{k}^{*} \leq\|f\|_{M}^{*}=1,
$$

hence, $\|f\|_{M} \leq 1$.

Proof of Lemma 6.1. Since for any polynomial of the form $\tau_{n}(x)=\sum_{|k| \leq n} c_{k} \mathrm{e}^{\mathrm{i} k x}$ we have $\left\|\tau_{n}^{(\alpha)}\right\|_{M}=\inf \left\{a>0: \sum_{|k| \leq n} M\left(|k|^{\alpha}\left|c_{k}\right| / a\right) \leq 1\right\}$, then similarly to (7.4), we obtain

$$
\sum_{|k| \leq n} M\left(\left|\left[\Delta_{h}^{\alpha} \tau_{n}\right]^{\wedge}(k)\right| / a_{1}\right) \leq \sum_{|k| \leq n} M\left(|k h|^{\alpha}\left|c_{k}\right| / a_{1}\right) \leq \sum_{|k| \leq n} M\left(|k|^{\alpha}\left|c_{k}\right| /\left\|\tau_{n}^{(\alpha)}\right\|_{M}\right) \leq 1
$$

when $a_{1}:=|h|^{\alpha}\left\|\tau_{n}^{(\alpha)}\right\|_{M}$. Therefore, $\left\|\Delta_{h}^{\alpha} \tau_{n}\right\|_{M} \leq|h|^{\alpha}\left\|\tau_{n}^{(\alpha)}\right\|_{M}$.
In (6.2), the first inequality is trivial in the cases where $h=0$ or $|h|=2 \pi / n$. So, now let $0<|h|<2 \pi / n$. Since

$$
\left\|\Delta_{h}^{\alpha} \tau_{n}\right\|_{M}=\inf \left\{a>0: \sum_{|k| \leq n} M\left(2^{\alpha}\left|\sin \frac{k h}{2}\right|^{\alpha}\left|c_{k}\right| / a\right) \leq 1\right\}
$$

and the function $t / \sin t$ increase on $(0, \pi)$, then for $a_{2}:=\left|\frac{n / 2}{\sin (n h / 2)}\right|^{\alpha}\left\|\Delta_{h}^{\alpha} \tau_{n}\right\|_{M}$ we get

$$
\begin{gathered}
\sum_{|k| \leq n} M\left(|k|^{\alpha}\left|c_{k}\right| / a_{2}\right)=\sum_{|k| \leq n} M\left(\left|\frac{k h / 2}{\sin (k h / 2)}\right|^{\alpha}\left|\frac{\sin (k h / 2)}{h / 2}\right|^{\alpha}\left|c_{k}\right| / a_{2}\right) \\
\leq \sum_{|k| \leq n} M\left(\left|\frac{n h / 2}{\sin (n h / 2)}\right|^{\alpha}\left|\frac{\sin (k h / 2)}{h / 2}\right|^{\alpha}\left|c_{k}\right| / a_{2}\right)=\sum_{|k| \leq n} M\left(2^{\alpha}\left|\sin \frac{k h}{2}\right|^{\alpha}\left|c_{k}\right| /\left\|\Delta_{h}^{\alpha} \tau_{n}\right\|_{M}\right) \leq 1
\end{gathered}
$$

Thus, the first inequality in (6.2) also holds.

## References

[1] AKGÜN R.: Approximating polynomials for functions of Weighted Smirnov-Orlicz spaces, Journ. of Funct. Spaces and Applic. 2012 (2012), 1-41.
[2] AKGÜN R.-IZRAFILOV D.: Approximations in wieghted Orlicz spaces, Math. Slovaca, 61 (4) (2011), 601618.
[3] AKHIEZER N. I.: Lectures on Approximation Theory, 2nd ed. Nauka, Moscow, 1965 [in Russian]; English translation of the 1st ed. (1947): Theory of approximation, Ungar, New York, 1956.
[4] BARI N. K.-STECHKIN S. B.: Best approximations and differential properties of two conjugate functions, Trudy Moskov. Mat. Obsch., 5 (1956), 483-522 [in Russian].
[5] BERNSTEIN S. N.: On the best approximation of continuous functions by polynomials of given degree (1912), in: Collected Works, 1, Acad. Nauk SSSR, Moscow, 11-104 (1952) [in Russian].
[6] BUTZER P.-NESSEL R.: Fourier Analysis and Approximation. One-Dimensional Theory, Birkhäuser, Basel, 1971.
[7] BUTZER P. L.-WESTPHAL U.: An access to fractional differentiation via fractional difference quotients, in Fractional Calculus and its Applications (edited by B. Ross), Lecture Notes in Math. 457, Springer, Berlin, 116-145 (1975).
[8] CHAICHENKO S. O.: Inverse approximation theorems in the weighted Orlicz spaces, Bukovyn. Mat. Zh. 1 (3-4) (2013), 148-157 [in Ukrainian].
[9] DEVORE R. A.-LORENTZ G. G.: Constructive Approximation, Springer, Berlin, 1993.
[10] DZYADYK V. K.-SHEVCHUK I. A.: Theory of uniform approximation of functions by polynomials, Walter de Gruyter GmbH \& Co. KG, Berlin, 2008.
[11] GARIDI W.: On approximation by polynomials in Orlicz spaces, Approx. Theory Appl. (N.S.), 7 (1991), 97-110.
[12] GUVEN A.-ISRAFILOV D. M.: On approximation in weigted Orlicz spaces, Math. Slovaca, 62 (1) (2012), 77-86.
[13] ISRAFILOV D. M.-GUVEN A.: Approximation by trigonometric polynomials in weighted Orlicz spaces, Studia Math. 174 (2006), 147-168.
[14] JACKSON D.: Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung, Göttingen, 1911, Thesis.
[15] KRASNOSEL'SKII M. A.-RUTICKII YA.B.: Convex Functions and Orlicz Spaces, P.Noordhoff Ltd, Groningen 1961.
[16] PEETRE J.: A theory of interpolation of normed spaces. Notes, Brasilia, 1963.
[17] RAMAZANOV A. R-K.: On approximation by polynomials and rational functions in Orlicz spaces, Anal. Math. 10 (1984), 117-132.
[18] RUNOVSKI K.: On Jackson type inequality in Orlicz classes, Rev. Mat. Complut. 14 (2001), 395-404.
[19] SHIDLICH A. L.-CHAICHENKO S. O.: Some extremal problems in the Orlicz spaces, Mat. Stud. 42 (1) (2014), 21-32 [in Ukrainian].
[20] SHIDLICH A.L.-CHAICHENKO S. O.: Approximative properties of diagonal operators in Orlicz spaces, Numer. Funct. Anal. Optim. 36 (10) (2015), 1339-1352.
[21] STECHKIN S. B.: On the order of the best approximations of continuous functions, Izv. Akad. Nauk SSSR, Ser. Mat. 15 (3) (1951), 219-242 [in Russian].
[22] STEPANETS A. I.: Approximation characteristics of the spaces $\mathcal{S}_{\varphi}^{p}$, Ukrainian Math. J. 53 (3) (2001), 446-475.
[23] STEPANETS A. I.: Methods of Approximation Theory, VSP, Leiden-Boston, 2005.
[24] STEPANETS A. I.: Problems of approximation theory in linear spaces, Ukrainian Math. J. 58 (1) (2006), 54-102.
[25] STEPANETS A.I.-SERDYUK A.S.: Direct and inverse theorems in the theory of the approximation of functions in the space $\mathcal{S}^{p}$, Ukrainian Math. J. 54 (1) (2002), 126-148.
[26] STERLIN M. D.: Exact constants in inverse theorems of approximation theory, Dokl. Akad. Nauk SSSR 202 (1972), 545-547 [in Russian].
[27] TIMAN A. F.: Theory of approximation of functions of a real variable, Fizmatgiz, Moscow, 1960 [in Russian]; English translation by J. Berry, International Series of Monographs on Pure and Applied Mathematics 34, Pergamon Press and MacMillan, Oxford, 1963.
[28] TIMAN M. F., Inverse theorems of the constructive theory of functions in $L_{p}$ spaces $(1 \leq p \leq \infty)$, Mat. Sb . 46 (1) (1958), 125-132.
[29] TIMAN M. F.: Approximation and properties of periodic functions. Nauk. dumka, Kiev, 2009 [in Russian].
[30] VAKARCHUK S. B.: Jackson-type inequalities and exact values of widths of classes of functions in the spaces $S^{p}, 1 \leq p<\infty$, Ukrainian Math. J. 56 (5) (2004), 718-729.
[31] VALLÉE-POUSSIN C.-J.: Leşons sur l'Approximation des Fonctions d'une Variable Réelle, Gauthier-Villars, Paris, 1919.
[32] ZYGMUND A.: On the continuity module of the sum of the series conjugate to a Fourier series, Prace Mat.Fiz. 33 (1924), 25-132 [in Polish].

* Donbas State Pedagogical University, 19, G. Batyuka st., Slaviansk, Donetsk region, UKRAINE, 84116
E-mail address: s.chaichenko@gmail.com
** Department of Theory of Functions,
Institute of Mathematics of NAS of Ukraine,
3, Tereshchenkivska str., Kyiv, UKRAInE, 01601
E-mail address: shidlich@gmail.com
*** Faculty of Sciences, Kygyz-Turkish Manas University,
56, Chyngyz Aitmatov avenue, Bishkek, KYRGYZ REPUBLIC, 720044;
Faculty of Science and Letters, Mersin University,
Çiftlikköy Kampüsü, Yenişehir, Mersin, TURKEY, 33342
E-mail address: fahreddinabdullayev@gmail.com


[^0]:    2010 Mathematics Subject Classification. Primary 41A27; Secondary 42B05.
    Key words and phrases. best approximation, modulus of smoothness, direct theorem, inverse theorem.
    This work was supported in part by the Ministry of Education and Science of Ukraine within the framework of the fundamental research No. 0118U003390 and the Kyrgyz-Turkish Manas University (Bishkek / Kyrgyz Republic), project No. KTMÜ-BAP-2018.FBE. 05 .

