DIRECT AND INVERSE APPROXIMATION THEOREMS OF FUNCTIONS IN THE MUSIELAK–ORLICZ TYPE SPACES

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Abstract. In Musilak-Orlicz type spaces $\mathscr{S}_{\mathbf{M}}$, direct and inverse approximation theorems are obtained in terms of the best approximations of functions and generalized moduli of smoothness. The question of the exact constants in Jackson-type inequalities is studied.

1. Introduction

In Musilak-Orlicz type spaces $\mathscr{S}_{\mathbf{M}}$, we prove direct and inverse approximation theorems in terms of the best approximations of functions and generalized moduli of smoothness. Such theorems establish a connection between the smoothness properties of functions and the behavior of the error of their approximation by various methods. In particular, direct theorems show that good smoothness properties of a function (the existence of derivatives of a given order, the specific behavior of the modulus of smoothness, etc.) imply a good estimate of the error of its approximation. In the case of best approximation by polynomials, these results are also known as Jackson-type theorems or Jackson-type inequalities [18]. Inverse theorems characterize smoothness properties of functions depending on the rapidity with which the errors of best, or any other, approximations tend to zero. The problem of obtaining inverse theorems in the approximation of functions was first stated, and in some cases solved, by Bernstein [7]. In ideal cases, the direct and inverse theorems complement each other, and this allows us to fully characterize a functional class having certain smoothness properties, using, for example, sequences of best approximations. The results concerning direct and inverse connection between the smoothness properties of functions and the errors of their approximations in classical functional spaces (such as Lebesgue and Hilbert spaces, the spaces of continues functions, etc) are described quite fully in the monographs [31], [10], [14], [15], [32] and others.

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In 2001, Stepanets [27] considered the spaces $\mathcal{S}^p=\mathcal{S}^p(\mathbb{T})$ of 2π-periodic Lebesgue summable functions f $(f\in L)$ with the finite norm

$$\|f\|_p:=\|f\|_{\mathcal{S}^p}=\|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_p(\mathbb{Z})}:=\Big(\sum_{k\in\mathbb{Z}}|\widehat{f}(k)|^p\Big)^{1/p}, \tag{1}$$

where $\widehat{f}(k):=[f]^{\widehat{}}(k)=(2\pi)^{-1}\int_0^{2\pi}f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,\ k\in\mathbb{Z}$, are the Fourier coefficients of the function f, and investigated some approximation characteristics of these spaces. Stepanets and Serdyuk [29] introduced the notion of kth modulus of smoothness in $\mathcal{S}^p$ and proved direct and inverse theorems on approximation in terms of these moduli of smoothness and the best approximations of functions. Also this topic was investigated actively in [30], [33], [34], [28, Ch. 11], [32, Ch. 3], etc.

In [11] and [1], some results for the spaces $\mathcal{S}^p$ were extended to the Orlicz type spaces $\mathcal{S}_M$ and $\mathcal{S}_{\mathbf{p},\mu}$. In particular, in [11] and [1], direct and inverse approximation theorems were proved in terms of best approximations of functions and moduli of smoothness of fractional order and a connection was established between K-functional and such moduli of smoothness. In other Banach spaces, in particular, in Banach spaces of Orlicz type, topics related to direct and inverse approximation theorems, were investigated in [16], [4], [19], [20], [26], [3] and others.

Here, we continue such studies and consider the Musilak-Orlicz type spaces $\mathcal{S}_{\mathbf{M}}$, which are natural generalizations of the spaces $\mathcal{S}_M$ and $\mathcal{S}_{\mathbf{p},\mu}$. In these spaces, we give direct and inverse approximation theorems in terms of best approximations of functions and generalized moduli of smoothness. Particular attention is paid to the study of the accuracy of constants in Jackson-type inequalities.

2. Preliminaries

Let $\mathbf{M}=\{M_k(u)\}_{k\in\mathbb{Z}}$, $u\ge 0$, be a sequence of Orlicz functions. In other words, for every $k\in\mathbb{Z}$, the function $M_k(u)$ is a nondecreasing convex function for which $M_k(0)=0$ and $M_k(u)\to\infty$ as $u\to\infty$. The modular space (or Musilak-Orlicz space) $\mathcal{S}_{\mathbf{M}}$ is the space of all functions $f\in L$ such that the following quantity (which is also called the Luxemburg norm of f) is finite:

$$\|f\|_{\mathbf{M}}:=\|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_{\mathbf{M}}(\mathbb{Z})}:=\inf\Big\{a>0:\ \sum_{k\in\mathbb{Z}}M_k(|\widehat{f}(k)|/a)\le 1\Big\}. \tag{2}$$

By definition, we say that the functions $f\in L$ and $g\in L$ are assumed to be equivalent in the space $\mathcal{S}_{\mathbf{M}}$, when $\|f-g\|_{\mathbf{M}}=0$.

The spaces $\mathcal{S}_{\mathbf{M}}$ defined in this way are Banach spaces. Sequence spaces of this type have been studied by mathematicians since the 1940s (see, for example, the monographs [23], [24]). If all functions M_k are identical (namely, $M_k(u)\equiv M(u)$, $k\in\mathbb{Z}$), the spaces $\mathcal{S}_{\mathbf{M}}$ coincide with the ordinary Orlicz type spaces $\mathcal{S}_M$ [11]. If $M_k(u)=\mu_k u^{p_k}$, $p_k\ge 1$, $\mu_k\ge 0$, then $\mathcal{S}_{\mathbf{M}}$ coincide with the weighted spaces $\mathcal{S}_{\mathbf{p},\mu}$ with variable exponents [1]. If all $M_k(u)=u^p$, $p\ge 1$, then the spaces $\mathcal{S}_{\mathbf{M}}$ are the above-defined spaces $\mathcal{S}^p$.

In addition to the Luxembourg norm (2), in the space $\mathscr{S}_{\mathbf{M}}$, consider the Orlicz norm that is defined as follows. Let $\tilde{\mathbf{M}} = \{\tilde{M}_k(v)\}_{k\in\mathbb{Z}}$ be the sequence of functions defined by the relations

$$\tilde{M}_k(v) := \sup\{uv - M_k(u) :\ u \ge 0\}, \quad k \in \mathbb{Z}.$$

Consider the set $\Lambda{=}\Lambda(\tilde{\mathbf{M}})$ of sequences of positive numbers $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}} \tilde{M}_k(\lambda_k){\le}1$. For any function $f \in \mathscr{S}_{\mathbf{M}}$, define its Orlicz norm by the equality

$$\|f\|^*_{\mathbf{M}} := \|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l^*_{\mathbf{M}}(\mathbb{Z})} := \sup\Big\{ \sum_{k\in\mathbb{Z}} \lambda_k |\widehat{f}(k)| : \quad \lambda \in \Lambda \Big\}. \tag{3}$$

The following auxiliary Lemma 1 establishes the equivalence of the Luxembourg norm (2) and the Orlicz norm (3).

LEMMA 1. *For any function $f \in \mathscr{S}_{\mathbf{M}}$, the following relation holds:*

$$\|f\|_{\mathbf{M}} \le \|f\|^*_{\mathbf{M}} \le 2\,\|f\|_{\mathbf{M}}. \tag{4}$$

Relation (4) follows from the similarly relation for corresponding norms in the modular Orlicz sequence spaces (see, for example [23, Ch. 4]).

Further, denote by $\|\cdot\|$ one of the norms $\|\cdot\|_{\mathbf{M}}$ or $\|\cdot\|^*_{\mathbf{M}}$.

Let $\mathscr{T}_n$, $n = 0, 1, \ldots$, be the set of trigonometric polynomials $t_n(x) = \sum_{|k|\le n} c_k \mathrm{e}^{\mathrm{i}kx}$ of the order n, where c_k are arbitrary complex numbers. For any $f \in \mathscr{S}_{\mathbf{M}}$, denote by $E_n(f)_{\mathbf{M}}$ and $E_n(f)^*_{\mathbf{M}}$ the best approximations of f by trigonometric polynomials $t_{n-1} \in \mathscr{T}_{n-1}$ in the space $\mathscr{S}_{\mathbf{M}}$ with respect to the norms $\|\cdot\|_{\mathbf{M}}$ and $\|\cdot\|^*_{\mathbf{M}}$ respectively, i.e.,

$$E_n(f)_{\mathbf{M}} := \inf_{t_{n-1}\in\mathscr{T}_{n-1}} \|f - t_{n-1}\|_{\mathbf{M}} \quad \text{and} \quad E_n(f)^*_{\mathbf{M}} := \inf_{t_{n-1}\in\mathscr{T}_{n-1}} \|f - t_{n-1}\|^*_{\mathbf{M}}. \tag{5}$$

The following auxiliary Lemma 2 characterizes the polynomial of the best approximation in $\mathscr{S}_{\mathbf{M}}$.

LEMMA 2. *Assume that $f \in \mathscr{S}_{\mathbf{M}}$. Then*

$$E_n(f) := \inf_{t_{n-1}\in\mathscr{T}_{n-1}} \|f - t_{n-1}\| = \|f - S_{n-1}(f)\|, \tag{6}$$

where $S_{n-1}(f) = S_{n-1}(f,\cdot) = \sum_{|k|\le n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}k\cdot}$ is the Fourier sum of the function f.

Proof. Indeed, for any polynomial $t_{n-1} = \sum_{|k|\le n-1} c_k \mathrm{e}^{\mathrm{i}k\cdot} \in \mathscr{T}_{n-1}$, the quantities $|(f - t_{n-1})\widehat{\ }(k)| = |\widehat{f}(k) - c_k|$ when $|k| \le n-1$ and $|(f - t_{n-1})\widehat{\ }(k)| = |\widehat{f}(k)|$ when $|k| \ge n$. Therefore, in view of (2) and (3), the infimum in (6) is reached in the case when all $c_k = \widehat{f}(k)$, i.e., when $t_{n-1} = S_{n-1}(f)$. $\square$

Let $\omega_\alpha(f,\delta)$ be the modulus of smoothness of a function $f \in \mathscr{S}_{\mathbf{M}}$ of order $\alpha > 0$, i.e.,

$$\omega_\alpha(f,\delta) := \sup_{|h|\le\delta} \|\Delta^\alpha_h f\| = \sup_{|h|\le\delta} \Big\| \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(\cdot - jh) \Big\|, \tag{7}$$

where $\binom{\alpha}{j}=\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-j+1)}{j!}$ for $j\in\mathbb{N}$ and $\binom{\alpha}{j}=1$ for $j=0$. By the definition, for any $k\in\mathbb{Z}$, we have

$$|[\Delta_h^\alpha f]^\frown(k)|=|1-\mathrm{e}^{-\mathrm{i}kh}|^\alpha|\widehat{f}(k)|=2^\alpha\left|\sin\frac{kh}{2}\right|^\alpha|\widehat{f}(k)|. \tag{8}$$

Now consider the set Φ of all continuous bounded nonnegative pair functions φ such that $\varphi(0)=0$ and the Lebesgue measure of the set $\{t\in\mathbb{R}:\ \varphi(t)=0\}$ is equal to zero. For a fixed function $\varphi\in\Phi$, $h\in\mathbb{R}$ and for any $f\in\mathscr{S}_{\mathbf{M}}$, we denote by $\{[\Delta_h^\varphi f]^\frown(k)\}_{k\in\mathbb{Z}}$ the sequence of numbers such that for any $k\in\mathbb{Z}$,

$$[\Delta_h^\varphi f]^\frown(k)=\varphi(kh)\widehat{f}(k). \tag{9}$$

If there exists a function $\Delta_h^\varphi f\in L$ whose Fourier coefficients coincide with the numbers $[\Delta_h^\varphi f]^\frown(k)$, $k\in\mathbb{Z}$, then, as above, the expressions $\|\Delta_h^\varphi f\|_{\mathbf{M}}$ and $\|\Delta_h^\varphi f\|_{\mathbf{M}}^*$ denote Luxemburg and Orlicz norms of the function $\Delta_h^\varphi f$. If such a function does not exist, then we also keep the notation $\|\Delta_h^\varphi f\|_{\mathbf{M}}$ and $\|\Delta_h^\varphi f\|_{\mathbf{M}}^*$. But in this case, by these notations we mean the corresponding norm $\|\cdot\|_{l_{\mathbf{M}}(\mathbb{Z})}$ or $\|\cdot\|_{l_{\mathbf{M}}^*(\mathbb{Z})}$ of the sequence $\{[\Delta_h^\varphi f]^\frown(k)\}_{k\in\mathbb{Z}}$. Also we denote by $\|\Delta_h^\varphi f\|$ any of the expressions $\|\Delta_h^\varphi f\|_{\mathbf{M}}$ and $\|\Delta_h^\varphi f\|_{\mathbf{M}}^*$.

Similarly to [25], [8], [9], [22], define the generalized modulus of smoothness ω_φ of a function $f\in\mathscr{S}_{\mathbf{M}}$ by the equality:

$$\omega_\varphi(f,\delta)=\sup_{|h|\le\delta}\|\Delta_h^\varphi f\|. \tag{10}$$

In particular, we set

$$\omega_\varphi(f,\delta)_{\mathbf{M}}:=\sup_{|h|\le\delta}\|\Delta_h^\varphi f\|_{\mathbf{M}}\quad\text{and}\quad\omega_\varphi(f,\delta)_{\mathbf{M}}^*:=\sup_{|h|\le\delta}\|\Delta_h^\varphi f\|_{\mathbf{M}}^*.$$

It follows from (8) that $\omega_\alpha(f,\delta)=\omega_\varphi(f,\delta)$ when $\varphi(t)=2^\alpha|\sin(t/2)|^\alpha$.

3. Direct approximation theorems

In this section, we prove direct approximation theorems in the space $\mathscr{S}_{\mathbf{M}}$ in terms of the best approximations and generalized moduli of smoothness, and also establish Jackson type inequalities with the constants that are the best possible in some important cases.

Let $V(\tau)$, $\tau>0$, be a set of bounded nondecreasing functions v that differ from a constant on $[0,\tau]$.

THEOREM 1. *Assume that $f\in\mathscr{S}_{\mathbf{M}}$. Then for any $\tau>0$, $n\in\mathbb{N}$ and $\varphi\in\Phi$, the following inequality holds:*

$$E_n(f)_{\mathbf{M}}^*\le C_{n,\varphi}(\tau)\,\omega_\varphi\Big(f,\frac{\tau}{n}\Big)_{\mathbf{M}}^*, \tag{11}$$

where

$$C_{n,\varphi}(\tau) := \inf_{v\in V(\tau)} \frac{v(\tau)-v(0)}{I_{n,\varphi}(\tau,v)}, \tag{12}$$

and

$$I_{n,\varphi}(\tau,v) := \inf_{k\in\mathbb{N}:k\geq n} \int_0^\tau \varphi\Big(\frac{ku}{n}\Big) dv(u). \tag{13}$$

*In this case, there exists a function $v^*\in V(\tau)$ that realizes the greatest lower bound in (*13*).*

Proof. Let $f\in\mathscr{S}_{\mathbf{M}}$, $n\in\mathbb{N}$ and $h\in\mathbb{R}$. According to (6) and (3), we have

$$E_n(f)^*_{\mathbf{M}} = \|f-S_{n-1}(f)\|^*_{\mathbf{M}} = \sup\Big\{\sum_{|k|\geq n}\lambda_k|\widehat{f}(k)| : \ \lambda\in\Lambda\Big\}, \tag{14}$$

and by the definition of supremum, for arbitrary $\varepsilon>0$ there exists a sequence $\tilde{\lambda}\in\Lambda$, $\tilde{\lambda}=\tilde{\lambda}(\varepsilon)$, such that the following relations holds:

$$\sum_{|k|\geq n}\tilde{\lambda}_k|\widehat{f}(k)|+\varepsilon \geq \sup\Big\{\sum_{|k|\geq n}\lambda_k|\widehat{f}(k)| : \ \lambda\in\Lambda\Big\}.$$

In view of (3) and (9), we have

$$\|\Delta_h^\varphi f\|^*_{\mathbf{M}} \geq \sup\Big\{\sum_{|k|\geq n}\lambda_k\varphi(kh)|\widehat{f}(k)| : \ \lambda\in\Lambda\Big\} \geq \sum_{|k|\geq n}\tilde{\lambda}_k\varphi(kh)|\widehat{f}(k)| =$$

$$= \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\sum_{|k|\geq n}\tilde{\lambda}_k|\widehat{f}(k)| + \sum_{|k|\geq n}\tilde{\lambda}_k|\widehat{f}(k)|\Big(\varphi(kh)-\frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\Big).$$

For any $u\in[0,\tau]$, we get

$$\|\Delta^\varphi_{\frac{u}{n}} f\|^*_{\mathbf{M}} \geq \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\sum_{|k|\geq n}\tilde{\lambda}_k|\widehat{f}(k)| + \sum_{|k|\geq n}\tilde{\lambda}_k|\widehat{f}(k)|\Big(\varphi\Big(\frac{ku}{n}\Big)-\frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\Big). \tag{15}$$

The both sides of inequality (15) are nonnegative and, in view of the boundedness of the function φ, the series on its right-hand side is majorized on the entire real axis by the absolutely convergent series $C(\varphi)\sum_{|k|\geq n}\tilde{\lambda}_k|\widehat{f}(k)|$, where $C(\varphi):=\max_{u\in\mathbb{R}}\varphi(u)$. Then integrating this inequality with respect to $dv(u)$ from 0 to τ, we get

$$\int_0^\tau \|\Delta^\varphi_{\frac{u}{n}} f\|^*_{\mathbf{M}} dv \geq I_{n,\varphi}(\tau,v)\sum_{|k|\geq n}\tilde{\lambda}_k|\widehat{f}(k)| + \sum_{|k|\geq n}\tilde{\lambda}_k|\widehat{f}(k)|\Big(\int_0^\tau\varphi\Big(\frac{ku}{n}\Big)dv - I_{n,\varphi}(\tau,v)\Big).$$

By virtue of the definition of $I_{n,\varphi}(\tau,v)$, we see that the second term on the right-hand side of the last relation is nonnegative. Therefore, for any function $v\in V(\tau)$, we have

$$\int_0^\tau \|\Delta^\varphi_{\frac{u}{n}} f\|^*_{\mathbf{M}} dv \geq I_{n,\varphi}(\tau,v)\sum_{|k|\geq n}\tilde{\lambda}_k|\widehat{f}(k)| \geq I_{n,\varphi}(\tau,v)\Big(\sup\Big\{\sum_{|k|\geq n}\lambda_k|\widehat{f}(k)| : \lambda\in\Lambda\Big\}-\varepsilon\Big),$$

wherefrom due to an arbitrariness of choice of the number ε, we conclude that the inequality

$$\int_0^\tau \|\Delta_{\frac{u}{n}}^{\varphi} f\|_{\mathbf{M}}^* dv \geq I_{n,\varphi}(\tau,v) E_n(f)_{\mathbf{M}}^*$$

is true. Hence,

$$E_n(f)_{\mathbf{M}}^* \leq \frac{1}{I_{n,\varphi}(\tau,v)} \int_0^\tau \|\Delta_{\frac{u}{n}}^{\varphi} f\|_{\mathbf{M}}^* dv \leq \frac{1}{I_{n,\varphi}(\tau,v)} \int_0^\tau \omega_\varphi \Big(f, \frac{u}{n}\Big)_{\mathbf{M}}^* dv,$$

whence taking into account nondecreasing of the function ω_φ, we immediately obtain relation (11). The existence of the function $v^* \in V(\tau)$ realizing the greatest lower bound in (13) will be given below in the proof of Theorem 2. $\square$

COROLLARY 1. *Assume that $f \in \mathscr{S}_{\mathbf{M}}$. Then for any $\tau > 0$, $n \in \mathbb{N}$ and $\varphi \in \Phi$, the following inequality holds:*

$$E_n(f)_{\mathbf{M}} \leq 2C_{n,\varphi}(\tau)\, \omega_\varphi \Big(f, \frac{\tau}{n}\Big)_{\mathbf{M}}, \tag{16}$$

where the quantity $C_{n,\varphi}(\tau)$ is defined by (19).

COROLLARY 2. *Assume that $f \in \mathscr{S}_{\mathbf{M}}$. Then for any $\tau > 0$, $n \in \mathbb{N}$ and $\alpha > 0$ the following inequality holds:*

$$E_n(f) \leq 2C_{n,\alpha}(\tau) \omega_\alpha \Big(f, \frac{\tau}{n}\Big),$$

where the quantity $C_{n,\alpha}(\tau)$ is defined by (19) with $\varphi(t) = 2^\alpha |\sin(t/2)|^\alpha$.

For moduli of smoothness $\omega_\alpha(f,\delta)_{\mathbf{M}}$, in the mentioned above spaces $\mathscr{S}_M$ and $\mathscr{S}_{\mathbf{p},\mu}$, the inequalities of the type (16) were proved in [11] and [1] correspondingly. Unlike to [11] and [1], here we find the constant $C_{n,\varphi}(\tau)$ in Jackson-type inequality (11). Let us see how accurate this constant is. For this, consider the case where all functions $M_k(u) = u^p \Big(p^{-1/p} q^{-1/q}\Big)^p$, $p > 1$, $1/p + 1/q = 1$. In this case, all functions $\tilde{M}_k(v) = v^q$, the set Λ is a set of all sequences of positive numbers $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\|\lambda\|_{l_q(\mathbb{Z})} \leq 1$. Then the spaces $\mathscr{S}_{\mathbf{M}}$ coincide with the spaces $\mathscr{S}^p$, $p > 1$, and by Hölder inequality for any $f \in \mathscr{S}^p$, the following relation holds:

$$\|f\|_{\mathbf{M}}^* = \sup_{\lambda\in\Lambda} \sum_{k\in\mathbb{Z}} \lambda_k |\widehat{f}(k)| \leq \sup_{\lambda\in\Lambda} \|\lambda\|_{l_p(\mathbb{Z})} \cdot \|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_p(\mathbb{Z})} \leq \|f\|_p.$$

Furthermore, if $f \not\equiv 0$, then for the sequence $\lambda_k^* = |\widehat{f}(k)|^{p/q} \Big(\sum_{j\in\mathbb{Z}} |\widehat{f}(k)|^p\Big)^{-1/q}$, $k \in \mathbb{Z}$, we have $\sum_{k\in\mathbb{Z}} \lambda_k^* |\widehat{f}(k)| = \|f\|_p$ and $\|\lambda^*\|_{l_q(\mathbb{Z})} = 1$. Therefore, in this case $\|f\|_{\mathbf{M}}^* = \|f\|_p$, $p > 1$.

In the case $p=1$, the similar equality for norms

$$\|f\|^*_{\mathbf{M}} = \|f\|_1 \tag{17}$$

obviously can be obtained if we consider all $M_k(u)=u$, $k\in\mathbb{Z}$, and the set Λ is a set of all sequences of positive numbers $\lambda=\{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\|\lambda\|_{l_\infty(\mathbb{Z})}=\sup_{k\in\mathbb{Z}}\lambda_k\le 1$.

For fixed $n\in\mathbb{N}$, $\tau>0$ and for a given $\varphi\in\Phi$, consider the quantity

$$K_{n,\varphi}(\tau)_p := \sup_{\substack{f\in\mathscr{S}^p\\ f\not\equiv\text{const}}} \frac{E_n(f)_p}{\omega_\varphi(f,\tau/n)_p} = \sup_{\substack{f\in\mathscr{S}^p\\ f\not\equiv\text{const}}} \frac{\inf\limits_{t_{n-1}\in\mathscr{T}_{n-1}}\|f-t_{n-1}\|_p}{\sup\limits_{|h|\le\delta}\|\Delta_h^\varphi f\|_p}.$$

THEOREM 2. *Assume that* $f\in\mathscr{S}^p$, $1\le p<\infty$. *Then for any* $\tau>0$, $n\in\mathbb{N}$ *and* $\varphi\in\Phi$, *the following inequality holds:*

$$E_n(f)_p \le C_{n,\varphi,p}(\tau)\,\omega_\varphi\Big(f,\frac{\tau}{n}\Big)_p, \tag{18}$$

where

$$C_{n,\varphi,p}(\tau) := \bigg(\inf_{v\in V(\tau)} \frac{v(\tau)-v(0)}{I_{n,\varphi,p}(\tau,v)}\bigg)^{1/p}, \tag{19}$$

and

$$I_{n,\varphi,p}(\tau,v) := \inf_{k\in\mathbb{N}:k\ge n}\int_0^\tau \varphi^p\Big(\frac{ku}{n}\Big)dv(u). \tag{20}$$

In this case, there exists a function $v^*\in V(\tau)$ *that realizes the greatest lower bound in* (20). *Inequality* (18) *is unimprovable on the set of all functions* $f\in\mathscr{S}^p$, $f\not\equiv\text{const}$, *in the sense that the following equality is true:*

$$K_{n,\varphi}(\tau)_p = C_{n,\varphi,p}(\tau). \tag{21}$$

Proof. Here, we basically use the arguments given in [5], [12], [13] and [29]. Let $f\in\mathscr{S}^p$, $n\in\mathbb{N}$ and $h\in\mathbb{R}$. By virtue of (9) and (1), we have

$$\|\Delta_h^\varphi f\|_p^p \ge \sum_{|k|\ge n}\varphi^p(kh)|\widehat{f}(k)|^p$$

$$= \frac{I_{n,\varphi,p}(\tau,v)}{v(\tau)-v(0)}\sum_{|k|\ge n}|\widehat{f}(k)|^p + \sum_{|k|\ge n}|\widehat{f}(k)|^p\Big(\varphi^p(kh) - \frac{I_{n,\varphi,p}(\tau,v)}{v(\tau)-v(0)}\Big).$$

For any $u\in[0,\tau]$, we get

$$\|\Delta^\varphi_{\frac{u}{n}} f\|_p^p \ge \frac{I_{n,\varphi,p}(\tau,v)}{v(\tau)-v(0)}\sum_{|k|\ge n}|\widehat{f}(k)|^p + \sum_{|k|\ge n}|\widehat{f}(k)|^p\bigg(\varphi^p\Big(\frac{ku}{n}\Big) - \frac{I_{n,\varphi,p}(\tau,v)}{v(\tau)-v(0)}\bigg). \tag{22}$$

The both sides of inequality (22) are nonnegative and, in view of the boundedness of the function φ, the series on its right-hand side is majorized on the entire real axis by the absolutely convergent series $C^p(\varphi)\sum_{|k|\geq n}|\widehat{f}(k)|^p$, where $C(\varphi):=\max_{u\in\mathbb{R}}\varphi(u)$. Then integrating this inequality with respect to $dv(u)$ from 0 to τ, we get

$$\int\limits_0^\tau \|\Delta_{\frac{u}{n}}^{\varphi} f\|_p^p dv \geq I_{n,\varphi,p}(\tau,v)\sum_{|k|\geq n}|\widehat{f}(k)|^p$$

$$+\sum_{|k|\geq n}|\widehat{f}(k)|^p\bigg(\int\limits_0^\tau \varphi^p\Big(\frac{ku}{n}\Big)dv - I_{n,\varphi,p}(\tau,v)\bigg). \tag{23}$$

By virtue of the definition of $I_{n,\varphi,p}(\tau,v)$, we see that the second term on the right-hand side of (3) is nonnegative. Therefore, for any function $v\in V(\tau)$, we have

$$\int\limits_0^\tau \|\Delta_{\frac{u}{n}}^{\varphi} f\|_p^p dv \geq I_{n,\varphi,p}(\tau,v)\sum_{|k|\geq n}|\widehat{f}(k)|^p \geq I_{n,\varphi,p}(\tau,v)E_n^p(f)_p.$$

Hence,

$$E_n^p(f)_p \leq \frac{1}{I_{n,\varphi,p}(\tau,v)}\int\limits_0^\tau \|\Delta_{\frac{u}{n}}^{\varphi} f\|_p^p dv \leq \frac{1}{I_{n,\varphi,p}(\tau,v)}\int\limits_0^\tau \omega_\varphi^p\Big(f,\frac{u}{n}\Big)_p dv. \tag{24}$$

whence taking into account nondecreasing of the function ω_φ, we immediately obtain relation (18) and the estimate

$$K_{n,\varphi}(\tau)_p \leq C_{n,\varphi,p}(\tau). \tag{25}$$

Let us show that relation (25) is the equality. By virtue of Lemma 2, we have

$$K_{n,\varphi}(\tau)_p = \sup_{\substack{f\in\mathscr{S}^p\\ f\not\equiv\text{const}}}\frac{\sum_{|k|\geq n}|\widehat{f}(k)|^p}{\sup_{|h|\leq\tau}\sum_{|k|\geq n}\varphi^p(kh/n)|\widehat{f}(k)|^p}. \tag{26}$$

and in (26), it is sufficient to consider supremum over all functions $f\in\mathscr{S}^p$ such that $\sum_{|k|\geq n}|\widehat{f}(k)|^p\leq 1$. Therefore, taking into account the parity of the function φ, we get

$$K_{n,\varphi}^{-p}(\tau)_p \leq J_{n,\varphi,p}(\tau) := \inf_{w\in W_{n,\varphi,p}}\|w\|_{C_{[0,\tau]}}, \tag{27}$$

where the set

$$W_{n,\varphi,p} := \Big\{\omega(u)=\sum_{j=n}^\infty \rho_j\varphi^p(ju/n):\rho_j\geq 0,\ \sum_{j=n}^\infty\rho_j=1\Big\}. \tag{28}$$

For what follows, we need a duality relation in the space $C_{[a,b]}$, (see, e.g., [21, Ch. 1.4]).

PROPOSITION A. [21, Ch. 1.4] *If F is a convex set in the space $C_{[a,b]}$, then for any function $x \in C_{[a,b]}$,*

$$\inf_{u\in F} \|x-u\|_{C_{[a,b]}} = \sup_{\substack{b\\ \bigvee\limits_a (g)\le 1}} \Big(\int_a^b x(t)dg(t) - \sup_{u\in F} \int_a^b u(t)dg(t) \Big). \tag{29}$$

For $x \in C_{[a,b]} \setminus \bar{F}$, where $\bar{F}$ is the closure of a set F, there exists a function g_* with variation equal to 1 on $[a,b]$ that realizes the least upper bound in (29).

It is easy to show that the set $W_{n,\varphi,p}$ is a convex subset of the space $C_{[0,\tau]}$. Therefore, setting $a=0,\ b=\tau,\ x(t)\equiv 0,\ u(t)=w(t)\in W_{n,\varphi,p},\ F=W_{n,\varphi,p},$ from relation (29) we get

$$J_{n,\varphi,p}(\tau) = \inf_{w\in W_{n,\varphi,p}} \|0-w\|_{C_{[0,\tau]}}$$

$$= \sup_{\bigvee\limits_0^\tau (g)\le 1} \Big(0 - \sup_{w\in W_{n,\varphi,p}} \int_0^\tau w(t)dg(t)\Big) = \sup_{\bigvee\limits_0^\tau (g)\le 1} \inf_{w\in W_{n,\varphi,p}} \int_0^\tau w(t)dg(t). \tag{30}$$

Furthermore, according to the Proposition A, there exists a function $g_*(t)$, that realizes the least upper bound in (3) and such that $\bigvee\limits_0^\tau (g_*) = 1$. Since every function $w \in W_{n,\varphi,p}$ is nonnegative, it suffices to take the supremum on the right-hand side of (3) over the set of nondecreasing functions $v(t)$ for which $v(\tau)-v(0)\le 1$. For such functions, by virtue of (13) and (28), the following equality is true:

$$\inf_{w\in W_{n,\varphi,p}} \int_0^\tau w(t)dv(t) = I_{n,\varphi,p}(\tau,v). \tag{31}$$

Hence, there exists a function $v_* \in V(\tau)$ such that $v_*(\tau)-v_*(0)=1$ and

$$I_{n,\varphi,p}(\tau,v_*) = \sup_{v\in V(\tau):\bigvee\limits_0^\tau (v)\le 1} I_{n,\varphi,p}(\tau,v) = J_{n,\varphi,p}(\tau). \tag{32}$$

From relations (27) and (32), we get the necessary estimate:

$$K^p_{n,\varphi}(\tau)_p \ge \frac{1}{J_{n,\varphi,p}(\tau)} = \frac{1}{I_{n,\varphi,p}(\tau,v_*)} = \frac{v_*(\tau)-v_*(0)}{I_{n,\varphi,p}(\tau,v_*)} = C^p_{n,\varphi,p}(\tau).$$

□

From Theorem 2, in particular, follows that the constant $C_{n,\varphi}(\tau) = C_{n,\varphi,1}(\tau)$ is exact in the Jackson-type inequality (11) in the case when $\mathscr{S}_{\mathbf{M}} = \mathscr{S}^1$. In this case, estimate (25) in the proof obviously follows directly from estimate (11) and relation (17). For $p>1$, estimate (25) is more accurate than the estimate that can be obtained using similar arguments from Theorem 1.

In the Lebesgue space $L_2(\mathbb{T})$, such result was proved for ordinary moduli of smoothness $\omega_\alpha(f,\delta)_p$ with $\alpha = 1$ by Babenko [5]. In the spaces $\mathscr{S}^p$, for moduli $\omega_\alpha(f,\delta)_p$, this theorem was proved by Stepanets and Serdyuk [29]. In the spaces $\mathscr{S}^p(\mathbb{T}^d)$ of functions of several variables, for moduli $\omega_\alpha(f,\delta)_p$, such result was obtained in [2]. For generalized moduli of smoothness, the similar result was proved by Vasil'ev [36] in $L_2(\mathbb{T})$. We also mention the paper of Vakarchuk [35] which, in particular, contains a survey of the main results on Jackson-Type inequalities with generalized moduli of smoothness in the spaces $L_2(\mathbb{T})$.

4. Inverse approximation theorem.

THEOREM 3. *Let $f \in \mathscr{S}_{\mathbf{M}}$, the function $\varphi \in \Phi$ is nondecreasing on an interval $[0,\tau]$ and $\varphi(\tau) = \max\{\varphi(t) : t \in \mathbb{R}\}$. Then for any $n \in \mathbb{N}$, the following inequality holds:*

$$\omega_\varphi\Big(f, \frac{\tau}{n}\Big) \le \sum_{\nu=1}^{n} \Big(\varphi\Big(\frac{\tau\nu}{n}\Big) - \varphi\Big(\frac{\tau(\nu-1)}{n}\Big)\Big) E_\nu(f). \tag{33}$$

Proof. Let us use the proof scheme from [29], modifying it taking into account the peculiarities of the spaces $\mathscr{S}_{\mathbf{M}}$ and the definition of the modulus of smoothness ω_φ.

Let $f \in \mathscr{S}_{\mathbf{M}}$. For any $\varepsilon > 0$ there exist a number $N_0 = N_0(\varepsilon) \in \mathbb{N}$, $N_0 > n$, such that for any $N > N_0$, we have

$$E_N(f) = \|f - S_{N-1}(f)\| < \varepsilon/\varphi(\tau).$$

Let us set $f_0 := S_{N_0}(f)$. Then in view of (9), we see that

$$\|\Delta_h^\varphi f\| \le \|\Delta_h^\varphi f_0\| + \|\Delta_h^\varphi (f - f_0)\| \le \|\Delta_h^\varphi f_0\| + \varphi(\tau) E_{N_0+1}(f) < \|\Delta_h^\varphi f_0\| + \varepsilon. \tag{34}$$

Further, let $S_{n-1} := S_{n-1}(f_0)$ be the Fourier sum of f_0. Then by virtue of (8), for $|h| \le \tau/n$, we have

$$\|\Delta_h^\varphi f_0\| = \|\Delta_h^\varphi (f_0 - S_{n-1}) + \Delta_h^\varphi S_{n-1}\| \le \Big\|\varphi(\tau)(f_0 - S_{n-1}) + \sum_{|k|\le n-1} \varphi(kh)|\widehat{f}(k)|\mathrm{e}^{\mathrm{i}k\cdot}\Big\|$$

$$\le \Big\|\varphi(\tau)\sum_{\nu=n}^{N_0} H_\nu + \sum_{\nu=1}^{n-1} \varphi\Big(\frac{\tau\nu}{n}\Big) H_\nu\Big\|, \tag{35}$$

where $H_\nu(x) := H_\nu(f,x) = |\widehat{f}(\nu)|\mathrm{e}^{\mathrm{i}\nu x} + |\widehat{f}(-\nu)|\mathrm{e}^{-\mathrm{i}\nu x}$, $\nu = 1,2,\ldots$

Now we use the following assertion which is proved directly.

LEMMA 3. *Let $\{c_\nu\}_{\nu=1}^\infty$ and $\{a_\nu\}_{\nu=1}^\infty$ be arbitrary numerical sequences. Then the following equality holds for all natural m, M and N $m \le M < N$:*

$$\sum_{\nu=m}^{M} a_\nu c_\nu = a_m \sum_{\nu=m}^{N} c_\nu + \sum_{\nu=m+1}^{M} (a_\nu - a_{\nu-1}) \sum_{i=\nu}^{N} c_i - a_M \sum_{\nu=M+1}^{N} c_\nu. \tag{36}$$

Setting $a_\nu=\varphi\left(\frac{\tau\nu}{n}\right),\ c_\nu=H_\nu(x),\ m=1,\ M=n-1$ and $N=N_0$ in (36), we get

$$\sum_{\nu=1}^{n-1}\varphi\Big(\frac{\tau\nu}{n}\Big)H_\nu(x)=\sum_{\nu=1}^{N_0}H_\nu(x)$$

$$+\sum_{\nu=2}^{n-1}\left(\varphi\Big(\frac{\tau\nu}{n}\Big)-\varphi\Big(\frac{\tau(\nu-1)}{n}\Big)\right)\sum_{i=\nu}^{N_0}H_i(x)-\varphi\Big(\frac{\tau(n-1)}{n}\Big)\sum_{\nu=n}^{N_0}H_\nu(x).$$

Therefore,

$$\left\|\varphi(\tau)\sum_{\nu=n}^{N_0}H_\nu+\sum_{\nu=1}^{n-1}\varphi\Big(\frac{\tau\nu}{n}\Big)H_\nu\right\|$$

$$\le\left\|\varphi(\tau)\sum_{\nu=n}^{N_0}H_\nu+\sum_{\nu=1}^{n-1}\left(\varphi\Big(\frac{\tau\nu}{n}\Big)-\varphi\Big(\frac{\tau(\nu-1)}{n}\Big)\right)\sum_{i=\nu}^{N_0}H_i-\varphi\Big(\frac{\tau(n-1)}{n}\Big)\sum_{\nu=n}^{N_0}H_\nu\right\|$$

$$\le\left\|\sum_{\nu=1}^{n}\left(\varphi\Big(\frac{\tau\nu}{n}\Big)-\varphi\Big(\frac{\tau(\nu-1)}{n}\Big)\right)\sum_{i=\nu}^{N_0}H_i\right\|$$

$$\le\sum_{\nu=1}^{n}\left(\varphi\Big(\frac{\tau\nu}{n}\Big)-\varphi\Big(\frac{\tau(\nu-1)}{n}\Big)\right)E_\nu(f_0). \tag{37}$$

Combining relations (34), (35) and (4) and taking into account the definition of the function f_0, we see that for $|h|\le\tau/n$, the following inequality holds:

$$\|\Delta_h^\varphi f\|\le\sum_{\nu=1}^{n}\left(\varphi\Big(\frac{\tau\nu}{n}\Big)-\varphi\Big(\frac{\tau(\nu-1)}{n}\Big)\right)E_\nu(f)+\varepsilon$$

which, in view of arbitrariness of ε, gives us (33). □

As noted above, for $\varphi(t)=2^\alpha|\sin(t/2)|^\alpha$, $\alpha>0$, we have $\omega_\varphi(f,\delta)=\omega_\alpha(f,\delta)$. In this case, the number $\tau=\pi$. If $\alpha\ge1$, then using the inequality $x^\alpha-y^\alpha\le\alpha x^{\alpha-1}(x-y),\ x>0,y>0$ (see, for example, [17, Ch. 1]), and the usual trigonometric formulas, for $\nu=1,2,\ldots,n,$ we get

$$\varphi\Big(\frac{\tau\nu}{n}\Big)-\varphi\Big(\frac{\tau(\nu-1)}{n}\Big)=2^\alpha\left(\Big|\sin\Big(\frac{\pi\nu}{n}\Big)\Big|^\alpha-\Big|\sin\Big(\frac{\pi(\nu-1)}{n}\Big)\Big|^\alpha\right)\le$$

$$\le2^\alpha\alpha\Big|\sin\Big(\frac{\pi\nu}{n}\Big)\Big|^{\alpha-1}\Big|\sin\Big(\frac{\pi\nu}{n}\Big)-\sin\Big(\frac{\pi(\nu-1)}{n}\Big)\Big|\le\alpha\Big(\frac{2\pi}{n}\Big)^\alpha\nu^{\alpha-1}.$$

If $0<\alpha<1$, then the similar estimate can be obtained using the inequality $x^\alpha-y^\alpha\le\alpha y^{\alpha-1}(x-y)$, which holds for any $x>0,y>0,$ [17, Ch. 1]. Hence, we get the following statement:

COROLLARY 3. *Let $f\in\mathscr{S}_{\mathbf{M}}$ and $\alpha>0$. Then for any $n\in\mathbb{N}$, the following inequality holds:*

$$\omega_\alpha\Big(f,\frac{\pi}{n}\Big)\le\alpha\Big(\frac{2\pi}{n}\Big)^\alpha\sum_{\nu=1}^{n}\nu^{\alpha-1}E_\nu(f). \tag{38}$$

Note that in the above-mentioned spaces $\mathscr{S}_M$ and $\mathscr{S}_{\mathbf{p},\mu}$, the similar estimates were obtained for moduli of smoothness and best approximations determined with respect to the corresponding Luxemburg norms in [1] and [11]. In $\mathscr{S}^p$, such estimates were obtained in [30] and [29]. For the Lebesgue spaces L_p, inequalities of the type (38) were proved by M. Timan (see, for example, [32, Ch. 2], [31, Ch. 6]).

COROLLARY 4. *Assume that the sequence of the best approximations $E_n(f)$ of a function $f \in \mathscr{S}_{\mathbf{M}}$ satisfies the following relation for some $\beta > 0$:*

$$E_n(f) = \mathscr{O}(n^{-\beta}).$$

Then, for any $\alpha > 0$, one has

$$\omega_\alpha(f,t) = \begin{cases} \mathscr{O}(t^\beta) & \text{for } \beta < \alpha, \\ \mathscr{O}(t^\alpha |\ln t|) & \text{for } \beta = \alpha, \\ \mathscr{O}(t^\alpha) & \text{for } \beta > \alpha. \end{cases}$$

5. Constructive characteristics of the classes of functions defined by the αth moduli of smoothness

In this section we give the constructive characteristics of the classes $\mathscr{S}_{\mathbf{M}} H^\omega_\alpha$ of functions for which the αth moduli of smoothness $\omega_\alpha(f,\delta)$ do not exceed some majorant.

Let ω be a function defined on interval $[0,1]$. For a fixed $\alpha > 0$, we set

$$\mathscr{S}_{\mathbf{M}} H^\omega_\alpha = \Big\{ f \in \mathscr{S}_{\mathbf{M}} : \quad \omega_\alpha(f,\delta) = \mathscr{O}(\omega(\delta)), \quad \delta \to 0+ \Big\}. \tag{39}$$

Further, we consider the functions $\omega(\delta)$, $\delta \in [0,1]$, satisfying the following conditions 1)–4): **1)** $\omega(\delta)$ is continuous on $[0,1]$; **2)** $\omega(\delta)\uparrow$; **3)** $\omega(\delta) \neq 0$ for any $\delta \in (0,1]$; **4)** $\omega(\delta) \to 0$ as $\delta \to 0+$; and the well-known condition $(\mathscr{B}_\alpha)$, $\alpha > 0$ (see, e.g. [6]):

$$(\mathscr{B}_\alpha): \qquad \sum_{v=1}^n v^{\alpha-1}\omega(v^{-1}) = \mathscr{O}(n^\alpha \omega(n^{-1})), \quad n \to \infty.$$

THEOREM 4. *Assume that $\alpha > 0$ and the function ω satisfies conditions $1)-4)$ and $(\mathscr{B}_\alpha)$. Then, in order a function $f \in \mathscr{S}_{\mathbf{M}}$ to belong to the class $\mathscr{S}_{\mathbf{M}} H^\omega_\alpha$, it is necessary and sufficient that*

$$E_n(f) = \mathscr{O}(\omega(n^{-1})). \tag{40}$$

Proof. Let $f \in \mathscr{S}_{\mathbf{M}} H^\omega_\alpha$, by virtue of Corollary 2, we have

$$E_n(f) \le 2C_{n,\alpha}(1)\omega_\alpha(f;n^{-1}), \tag{41}$$

Therefore, relation (39) yields (40). On the other hand, if relation (40) holds, then by virtue of (38), taking into account the condition $(\mathscr{B}_\alpha)$, we obtain

$$\omega_\alpha(f,n^{-1}) \le \alpha\Big(\frac{2\pi}{n}\Big)^\alpha \sum_{\nu=1}^n \nu^{\alpha-1}E_\nu(f) \le \frac{C}{n^\alpha}\sum_{\nu=1}^n \nu^{\alpha-1}\omega(\nu^{-1}) = \mathscr{O}(\omega(n^{-1})). \tag{42}$$

Thus, the function f belongs to the set $\mathscr{S}_{\mathbf{M}}H_{\alpha}^{\omega}$. $\square$

The function $h(t)=t^r$, $r\le\alpha$, satisfies the condition $(\mathscr{B}_{\alpha})$. Hence, denoting by $\mathscr{S}_{\mathbf{M}}H_{\alpha}^{r}$ the class $\mathscr{S}_{\mathbf{M}}H_{\alpha}^{\omega}$ for $\omega(t)=t^r$, $0<r\le\alpha$, we establish the following statement:

COROLLARY 5. *Let $\alpha>0$, $0<r\le\alpha$. In order a function $f\in S_{\mathbf{M}}$ to belong to $\mathscr{S}_{\mathbf{M}}H_{\alpha}^{r}$, it is necessary and sufficient that*

$$E_n(f)=\mathscr{O}(n^{-r}).$$

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