

*Widths of functional classes defined by majorants of generalized moduli of smoothness in the spaces \mathcal{S}^p

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Abstract.

Exact Jackson-type inequalities are obtained in terms of best approximations and averaged values of generalized moduli of smoothness in the spaces \mathcal{S}^p . The values of Kolmogorov, Bernstein, linear, and projective widths in the spaces \mathcal{S}^p are found for classes of periodic functions defined by certain conditions on the averaged values of the generalized moduli of smoothness.

Keywords: Kolmogorov width Bernstein width best approximation module of smoothness Jackson-type inequality

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1 Introduction

Let \mathcal{S}^p , $1 \leq p < \infty$, (see, for example, [16], [17, Ch. 11]) be the space of 2π -periodic complex-valued Lebesgue summable functions f , defined on the real axis ($f \in L$), with finite norm

$$\|f\|_{\mathcal{S}^p} := \left(\sum_{k \in \mathbb{Z}} |\widehat{f}(k)|^p \right)^{1/p}, \quad (1.1)$$

where $\widehat{f}(k) = \int_0^{2\pi} f(x) e^{-ikx} \frac{dx}{2\pi}$ are the Fourier coefficients of the function f .

In the case $p = 2$, the spaces \mathcal{S}^2 are ordinary Lebesgue spaces L_2 of functions $f \in L$ with finite norm

$$\|f\|_{L_2} = \|f\|_{\mathcal{S}^2} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt \right)^{1/2}.$$

For arbitrary $1 \leq p < \infty$, these spaces possess some important properties of Hilbert spaces, in particular, the minimal property of Fourier sums, which will be formulated below in the relation (2.8).

An active study of the approximative characteristics of the spaces \mathcal{S}^p originates from the papers of Stepanets [16], [17, Ch. 11], [18], etc. Stepanets and Serdyuk [19] introduced

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the notion of k th modulus of smoothness in \mathcal{S}^p and proved direct and inverse theorems on approximation in terms of these moduli of smoothness and the best approximations of functions. Also this topic was investigated actively in [20], [29], [13], [25], [17, Ch. 11], [26], [24, Ch. 3], [12], [2], etc. In the paper, this research continues. In particular, exact Jackson-type inequalities are obtained in terms of best approximations and averaged values of generalized moduli of smoothness in the spaces \mathcal{S}^p . The values of Kolmogorov, Bernstein, linear, and projection widths in the spaces \mathcal{S}^p are found for classes of periodic functions defined by certain conditions on the averaged values of generalized moduli of smoothness.

2 Preliminaries

2.1 Generalized moduli of smoothness and their averaged values

Let Φ be the set of all continuous bounded non-negative pair functions $\varphi(t)$ such that $\varphi(0) = 0$ and the Lebesgue measure of the set $\{t \in \mathbb{R} : \varphi(t) = 0\}$ is equal to zero.

Developing ideas of the papers [15], [5], [6], for a fixed $\varphi \in \Phi$ define the generalized modulus of smoothness of the function $f \in \mathcal{S}^p$ by the equality

$$\omega_\varphi(f, t)_{\mathcal{S}^p} := \sup_{|h| \leq t} \left(\sum_{k \in \mathbb{Z}} \varphi^p(kh) |\widehat{f}(k)|^p \right)^{1/p}, \quad t \geq 0. \quad (2.1)$$

Let $\omega_\alpha(f, t)_{\mathcal{S}^p}$ be the ordinary modulus of smoothness of $f \in \mathcal{S}^p$ of order $\alpha > 0$, that is,

$$\omega_\alpha(f, t)_{\mathcal{S}^p} := \sup_{|h| \leq t} \|\Delta_h^\alpha f\|_{\mathcal{S}^p} = \sup_{|h| \leq t} \left\| \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(\cdot - jh) \right\|_{\mathcal{S}^p}, \quad (2.2)$$

where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!}$ for $j \in \mathbb{N}$ and $\binom{\alpha}{j} = 1$ for $j = 0$.

Since for any $k \in \mathbb{Z}$, the Fourier coefficients

$$|\widehat{\Delta_h^\alpha f}(k)| = |1 - e^{-ikh}|^\alpha |\widehat{f}(k)| = 2^{\frac{\alpha}{2}} (1 - \cos kh)^{\frac{\alpha}{2}} |\widehat{f}(k)|,$$

then in view of (1.1) and (2.1), we have

$$\omega_\alpha(f, t)_{\mathcal{S}^p} = \sup_{|h| \leq t} \left(\sum_{k \in \mathbb{Z}} 2^{\frac{\alpha p}{2}} (1 - \cos kh)^{\frac{\alpha p}{2}} |\widehat{f}(k)|^p \right)^{1/p} = \omega_{\varphi_\alpha}(f, \delta)_{\mathcal{S}^p},$$

where $\varphi_\alpha(t) = 2^{\frac{\alpha}{2}} (1 - \cos t)^{\frac{\alpha}{2}}$. In the general case, such modules were considered, in particular, in [28], [9], [27], [4], etc.

Further, let $M(\tau)$, $\tau > 0$, be the set of all functions μ , bounded non-decreasing and non-constant on the segment $[0, \tau]$. By $\Omega_\varphi(f, \tau, \mu, u)_{\mathcal{S}^p}$, $u > 0$, denote the average value of the generalized modulus of smoothness ω_φ of the function f with the weight $\mu \in M(\tau)$, that is,

$$\Omega_\varphi(f, \tau, \mu, u)_{\mathcal{S}^p} := \left(\frac{1}{\mu(\tau) - \mu(0)} \int_0^u \omega_\varphi^p(f, t)_{\mathcal{S}^p} d\mu\left(\frac{\tau t}{u}\right) \right)^{1/p}. \quad (2.3)$$

In particular, $\Omega_\alpha(f, \tau, \mu, u)_{\mathcal{S}^p}$ denotes the average value of the modulus of smoothness of the order α of the function f with the weight $\mu \in M(\tau)$, that is, $\Omega_\alpha(f, \tau, \mu, u)_{\mathcal{S}^p} := \Omega_\varphi(f, \tau, \mu, u)_{\mathcal{S}^p}$ when $\varphi(t) = \varphi_\alpha(t) = 2^{\frac{\alpha}{2}}(1 - \cos t)^{\frac{\alpha}{2}}$.

Note that for arbitrary $f \in \mathcal{S}^p$, $\tau > 0$, $\mu \in M(\tau)$, $u > 0$ the functionals $\Omega_\varphi(f, \tau, \mu, u)_{\mathcal{S}^p}$ do not exceed the value $\omega_\varphi(f, u)_{\mathcal{S}^p}$, and therefore in a number of questions they can be more effective for characterizing the structural and approximative properties of the function f .

2.2 Definition of ψ -derivatives derivatives and functional classes

Let $\psi = \{\psi(k)\}_{k=-\infty}^\infty$ be an arbitrary sequence of complex numbers. If for a given function $f \in L$ with the Fourier series $\sum_{k \in \mathbb{Z}} \widehat{f}(k)e^{ikx}$ the series $\sum_{k \in \mathbb{Z}} \psi(k)\widehat{f}(k)e^{ikx}$ is the Fourier series of a certain function $F \in L$, then F is called (see, for example, [17, Ch. 11]) ψ -integral of the function f and is denoted as $F = \mathcal{J}^\psi(f, \cdot)$. In turn, the function f is called the ψ -derivative of the function F and is denoted as $f = F^\psi$. In this case, the Fourier coefficients of functions f and F^ψ are related by the equalities

$$\widehat{f}(k) = \psi(k)\widehat{F^\psi}(k), \quad k \in \mathbb{Z}. \quad (2.4)$$

The set of ψ -integrals of functions f of L is denoted as L^ψ . If $\mathfrak{N} \subset L$, then $L^\psi\mathfrak{N}$ denotes the set of ψ -integrals of functions $f \in \mathfrak{N}$. In particular, $L^\psi\mathcal{S}^p$ is the set of ψ -integrals of functions $f \in \mathcal{S}^p$.

In the case when $\psi(k) = (ik)^{-r}$, $r = 0, 1, \dots$, we denote $L^\psi =: L^r$ and $L^\psi\mathfrak{N} =: L^r\mathfrak{N}$.

For arbitrary fixed $\varphi \in \Phi$, $\tau > 0$ and $\mu \in M(\tau)$, define the following functional classes:

$$L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p} := \left\{ f \in L^\psi\mathcal{S}^p : \Omega_\varphi(f^\psi, \tau, \mu, \frac{\tau}{n})_{\mathcal{S}^p} \leq 1, \quad n \in \mathbb{N} \right\}, \quad (2.5)$$

$$L^\psi(\varphi, \tau, \mu, \Omega)_{\mathcal{S}^p} := \left\{ f \in L^\psi\mathcal{S}^p : \Omega_\varphi(f^\psi, \tau, \mu, u)_{\mathcal{S}^p} \leq \Omega(u), \quad 0 \leq u \leq \tau \right\}, \quad (2.6)$$

where $\Omega(u)$ is a fixed continuous monotonically increasing function of the variable $u \geq 0$ such that $\Omega(0) = 0$. Also we set $L^\psi(\alpha, \tau, \mu, n)_{\mathcal{S}^p} := L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p}$ and $L^\psi(\alpha, \tau, \mu, \Omega)_{\mathcal{S}^p} := L^\psi(\varphi, \tau, \mu, \Omega)_{\mathcal{S}^p}$ for $\varphi(t) = \varphi_\alpha(t) = 2^{\frac{\alpha}{2}}(1 - \cos kt)^{\frac{\alpha}{2}}$.

Note that for $p = 2$, $\psi(k) = k^{-r}$, $r \in \mathbb{N}$, and the weight function $\mu(t) = t$, Taikov [21], [22] first considered the functional classes similar to the classes $L^\psi(\alpha, \tau, \mu, n)_{\mathcal{S}^p}$ and $L^\psi(\alpha, \tau, \mu, \Omega)_{\mathcal{S}^p}$. He found the exact values of the widths of such classes in the spaces L_2 in the case when the majorants Ω of the averaged values of the moduli of smoothness satisfied some constraints. Later, the problem of finding the exact values of the widths in the spaces L_2 and \mathcal{S}^p of functional classes of this kind generated by some specific weighting functions μ , was studied in [3], [11, Ch. 4], [30], [31], [14], [8], [13], [25], [27], etc.

2.3 Best approximations and widths of functional classes

Let \mathcal{T}_{2n+1} , $n = 0, 1, \dots$, be the set of trigonometric polynomials $T_n(x) = \sum_{|k| \leq n} c_k e^{ikx}$ of the order n , where c_k are arbitrary complex numbers.

For any function $f \in \mathcal{S}^p$ denote by $E_n(f)_{\mathcal{S}^p}$ its best approximation by the trigonometric polynomials $T_{n-1} \in \mathcal{T}_{2n-1}$ in the space \mathcal{S}^p , that is,

$$E_n(f)_{\mathcal{S}^p} := \inf_{T_{n-1} \in \mathcal{T}_{2n-1}} \|f - T_{n-1}\|_{\mathcal{S}^p}. \quad (2.7)$$

From relation (1.1), it follows (see, for example, [17, Ch. 11, relation (11.4)]) that for any function $f \in \mathcal{S}^p$ and all $n = 0, 1, \dots$,

$$E_n^p(f)_{\mathcal{S}^p} = \|f - S_{n-1}(f)\|_{\mathcal{S}^p}^p = \sum_{|k| \geq n} |\widehat{f}(k)|^p, \quad (2.8)$$

where $S_{n-1}(f) = S_{n-1}(f, \cdot) = \sum_{|k| \leq n-1} \widehat{f}(k) e^{ik \cdot}$ is the partial Fourier sum of the order $n - 1$ of the function f .

Further, let K be a convex centrally symmetric subset of \mathcal{S}^p and let B be a unit ball of the space \mathcal{S}^p . Let also F_N be an arbitrary N -dimensional subspace of space \mathcal{S}^p , $N \in \mathbb{N}$, and $\mathcal{L}(\mathcal{S}^p, F_N)$ be a set of linear operators from \mathcal{S}^p to F_N . By $\mathcal{P}(\mathcal{S}^p, F_N)$ denote the subset of projection operators of the set $\mathcal{L}(\mathcal{S}^p, F_N)$, that is, the set of the operators A of linear projection onto the set F_N such that $Af = f$ when $f \in F_N$. The quantities

$$b_N(K, \mathcal{S}^p) = \sup_{F_{N+1}} \sup\{\varepsilon > 0 : \varepsilon B \cap F_{N+1} \subset K\},$$

$$d_N(K, \mathcal{S}^p) = \inf_{F_N} \sup_{f \in K} \inf_{u \in F_N} \|f - u\|_{\mathcal{S}^p},$$

$$\lambda_N(K, \mathcal{S}^p) = \inf_{F_N} \inf_{A \in \mathcal{L}(\mathcal{S}^p, F_N)} \sup_{f \in K} \|f - Af\|_{\mathcal{S}^p},$$

$$\pi_N(K, \mathcal{S}^p) = \inf_{F_N} \inf_{A \in \mathcal{P}(\mathcal{S}^p, F_N)} \sup_{f \in K} \|f - Af\|_{\mathcal{S}^p},$$

are called Bernstein, Kolmogorov, linear, and projection N -widths of the set K in the space \mathcal{S}^p , respectively.

3 Main results

3.1 Jackson-type inequalities

In this subsection, Jackson-type inequalities are obtained in terms of best approximations and averaged values of generalized moduli of smoothness in the spaces \mathcal{S}^p .

Theorem 3.1. *Assume that $f \in L^\psi \mathcal{S}^p$, $1 \leq p < \infty$, $\varphi \in \Phi$, $\tau > 0$, $\mu \in M(\tau)$ and $\{\psi(k)\}_{k \in \mathbb{Z}}$ is a sequence of complex numbers such that $|\psi(k)| \leq K < \infty$. Then for any $n \in \mathbb{N}$ the following inequality is true:*

$$E_n(f)_{\mathcal{S}^p} \leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)} \right)^{1/p} \nu(n) \Omega_\varphi \left(f^\psi, \tau, \mu, \frac{\tau}{n} \right)_{\mathcal{S}^p}, \quad (3.1)$$

where $\nu(n) := \nu(n, \psi) = \sup_{|k| \geq n} |\psi(n)|$,

$$I_{n,\varphi,p}(\tau, \mu) := \inf_{\substack{k \geq n \\ k \in \mathbb{N}}} \int_0^\tau \varphi^p\left(\frac{kt}{n}\right) d\mu(t). \quad (3.2)$$

If, in addition, the function φ is non-decreasing on the interval $[0, \tau]$, the quantity $\nu(n) = \max\{|\psi(n)|, |\psi(-n)|\}$, and the condition

$$I_{n,\varphi,p}(\tau, \mu) = \int_0^\tau \varphi^p(t) d\mu(t), \quad (3.3)$$

holds, then inequality (3.1) can not be improved and therefore,

$$\sup_{\substack{f \in L^\psi \mathcal{S}^p \\ f \neq \text{const}}} \frac{E_n(f)_{\mathcal{S}^p}}{\Omega_\varphi(f^\psi, \tau, \mu, \frac{\tau}{n})_{\mathcal{S}^p}} = \left(\frac{\mu(\tau) - \mu(0)}{\int_0^\tau \varphi^p(t) d\mu(t)} \right)^{1/p} \nu(n). \quad (3.4)$$

Proof. Let $f \in L^\psi \mathcal{S}^p$, $1 \leq p < \infty$. By virtue of (2.4) and (2.8), we have

$$\begin{aligned} E_n^p(f)_{\mathcal{S}^p} &= \sum_{|k| \geq n} |\widehat{f}(k)|^p \leq \sum_{|k| \geq n} \left| \frac{\nu(n)}{\psi(k)} \right|^p |\widehat{f}(k)|^p \\ &= \nu^p(n) \sum_{|k| \geq n} \left| \frac{\widehat{f}(k)}{\psi(k)} \right|^p = \nu^p(n) E_n^p(f^\psi)_{\mathcal{S}^p}. \end{aligned} \quad (3.5)$$

As shown in [1, Proof of Theorem 2], for any $g \in \mathcal{S}^p$, $1 \leq p < \infty$, $\tau > 0$, $\varphi \in \Phi$, $\mu \in M(\tau)$ and $n \in \mathbb{N}$

$$E_n^p(g)_{\mathcal{S}^p} \leq \frac{1}{I_{n,\varphi,p}(\tau, \mu)} \int_0^\tau \omega_\varphi^p\left(g, \frac{t}{n}\right)_{\mathcal{S}^p} d\mu(t). \quad (3.6)$$

Setting $g = f^\psi$ in (3.6), we get

$$\begin{aligned} E_n^p(f^\psi)_{\mathcal{S}^p} &\leq \frac{\mu(\tau) - \mu(0)}{I_{n,\varphi,p}(\tau, \mu)} \frac{\int_0^\tau \omega_\varphi^p(f^\psi, \frac{t}{n})_{\mathcal{S}^p} d\mu(t)}{\mu(\tau) - \mu(0)} \\ &\leq \frac{\mu(\tau) - \mu(0)}{I_{n,\varphi,p}(\tau, \mu)} \Omega_\varphi^p\left(f^\psi, \tau, \mu, \frac{\tau}{n}\right)_{\mathcal{S}^p}. \end{aligned} \quad (3.7)$$

Combining inequalities (3.5) and (3.7), we obtain (3.1).

Now let the function φ be non-decreasing on $[0, \tau]$, the condition (3.3) holds and $\nu(n) = \max\{|\psi(n)|, |\psi(-n)|\}$. Then by virtue of (3.1), we have

$$\sup_{\substack{f \in L^\psi \mathcal{S}^p \\ f \neq \text{const}}} \frac{E_n(f)_{\mathcal{S}^p}}{\Omega_\varphi(f^\psi, \tau, \mu, \frac{\tau}{n})_{\mathcal{S}^p}} \leq \left(\frac{\mu(\tau) - \mu(0)}{\int_0^\tau \varphi^p(u) d\mu(u)} \right)^{1/p} \nu(n). \quad (3.8)$$

To prove the unimprovability of inequality (3.8), consider the function

$$f_n(x) = \gamma + \varepsilon_{-n}\delta e^{-inx} + \varepsilon_n\delta e^{inx},$$

where γ and δ are arbitrary complex numbers, and the quantity ε_k , $k \in \{-n, n\}$, is equal to 1 when $\nu(n) = |\psi(k)|$ and $\varepsilon_k = 0$ when $\nu(n) > |\psi(k)|$.

Since the function $\varphi(nt)$ is non-decreasing on the interval $[0, \frac{\tau}{n}]$, then by virtue of (2.1) and (2.4) we have

$$\omega_\varphi(f_n^\psi, t) = |\delta|(\varepsilon_{-n} + \varepsilon_n)^{1/p} \frac{\varphi(nt)}{\nu(n)}. \quad (3.9)$$

Taking into account (2.3), (3.9) and the equality $E_n(f_n)_{\mathcal{S}^p} = |\delta|(\varepsilon_{-n} + \varepsilon_n)^{1/p}$, we see that

$$\begin{aligned} \sup_{\substack{f \in L^\psi \mathcal{S}^p \\ f \neq \text{const}}} \frac{E_n(f)_{\mathcal{S}^p}}{\Omega_\varphi(f^\psi, \tau, \mu, \frac{\tau}{n})_{\mathcal{S}^p}} &\geq \frac{E_n(f_n)_{\mathcal{S}^p}}{\Omega_\varphi(f_n^\psi, \tau, \mu, \frac{\tau}{n})_{\mathcal{S}^p}} \\ &= \frac{|\delta|(\varepsilon_{-n} + \varepsilon_n)^{1/p}(\mu(\tau) - \mu(0))^{1/p}\nu(n)}{\left(\int_0^{\tau/n} |\delta|^p(\varepsilon_{-n} + \varepsilon_n)\varphi^p(nt)d\mu(nt)\right)^{1/p}} = \left(\frac{\mu(\tau) - \mu(0)}{\int_0^\tau \varphi^p(u)d\mu(u)}\right)^{1/p} \nu(n). \end{aligned} \quad (3.10)$$

Relations (3.8) and (3.10) yield (3.4). □

Combining relations (3.5) and (3.6) with $g = f^\psi$, given that the modulus $\omega_\varphi(f, t)_{\mathcal{S}^p}$ is non-decreasing for $t \geq 0$, we conclude that the following statement holds:

Corollary 3.1. *Assume that $f \in L^\psi \mathcal{S}^p$, $1 \leq p < \infty$, $\varphi \in \Phi$, $\tau > 0$, $\mu \in M(\tau)$ and $\{\psi(k)\}_{k \in \mathbb{Z}}$ is a sequence of complex numbers such that $|\psi(k)| \leq K < \infty$. Then for any $n \in \mathbb{N}$*

$$E_n(f)_{\mathcal{S}^p} \leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)}\right)^{1/p} \nu(n) \omega_\varphi\left(f, \frac{\tau}{n}\right)_{\mathcal{S}^p}, \quad (3.11)$$

where $\nu(n) = \sup_{|k| \geq n} |\psi(k)|$ and the quantity $I_{n, \varphi, p}(\tau, \mu)$ is defened by (3.2).

In the case when $p = 2$, $\mu_1(t) = 1 - \cos t$, $\tau = \pi$ and the function $\varphi_1(t) = 2^{\frac{1}{2}}(1 - \cos t)^{\frac{1}{2}}$, that is, when ω_φ is the ordinary modulus of smoothness of the order 1, the inequality of the form (3.11) was obtained by Stepanets [17, Ch. 8]. As follows from formula (3.16) below, in this case $\left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)}\right)^{1/p} = 2^{-1/2}$.

The function $\varphi_\alpha(t) = 2^{\frac{\alpha}{2}}(1 - \cos t)^{\frac{\alpha}{2}}$, $\alpha > 0$, is non-decreasing on the interval $[0, \pi]$. Therefore, in this case the following statement holds:

Corollary 3.2. *Assume that $f \in L^\psi \mathcal{S}^p$, $1 \leq p < \infty$, $\tau > 0$, $\mu \in M(\tau)$ and $\{\psi(k)\}_{k \in \mathbb{Z}}$ is a sequence of complex numbers such that $|\psi(k)| \leq K < \infty$. Then for any numbers $\alpha > 0$ and $n \in \mathbb{N}$*

$$E_n(f)_{\mathcal{S}^p} \leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \alpha, p}(\tau, \mu)}\right)^{1/p} \nu(n) \Omega_\varphi\left(f^\psi, \tau, \mu, \frac{\tau}{n}\right)_{\mathcal{S}^p}, \quad (3.1')$$

where $\nu(n) = \sup_{|k| \geq n} |\psi(n)|$, the quantity $I_{n,\alpha,p}(\tau, \mu)$ is defined by (3.2) with $\varphi(t) = \varphi_\alpha(t) = 2^{\frac{\alpha}{2}}(1 - \cos t)^{\frac{\alpha}{2}}$.

If, in addition, $\nu(n) = \max\{|\psi(n)|, |\psi(-n)|\}$ and

$$I_{n,\alpha,p}(\tau, \mu) = 2^{\frac{\alpha p}{2}} \int_0^\tau (1 - \cos t)^{\frac{\alpha p}{2}} d\mu(t), \quad (3.3')$$

then for $\tau \in (0, \pi]$ inequality (3.1') can not be improved and thus,

$$\begin{aligned} \sup_{\substack{f \in L^\psi \mathcal{S}^p \\ f \neq \text{const}}} \frac{E_n(f)_{\mathcal{S}^p}}{\Omega_\alpha(f^\psi, \tau, \mu, \frac{\tau}{n})_{\mathcal{S}^p}} &= \left(\frac{\mu(\tau) - \mu(0)}{2^{\frac{\alpha p}{2}} \int_0^\tau (1 - \cos t)^{\frac{\alpha p}{2}} d\mu(t)} \right)^{1/p} \nu(n) \\ &= \left(\frac{\mu(\tau) - \mu(0)}{2^{\alpha p} \int_0^\tau \sin^{\alpha p} \frac{t}{2} d\mu(t)} \right)^{1/p} \nu(n). \end{aligned} \quad (3.4')$$

Consider some consequences of this statement for specific weight functions $\mu_1(t) = 1 - \cos t$ and $\mu_2(t) = t$.

Corollary 3.3. Assume that $f \in L^\psi \mathcal{S}^p$, $1 \leq p < \infty$, and $\{\psi(k)\}_{k \in \mathbb{Z}}$ is a sequence of complex numbers such that $|\psi(k)| \leq K < \infty$. Then for any numbers $\alpha > 0$ and $n \in \mathbb{N}$

$$E_n(f)_{\mathcal{S}^p} \leq \left(\frac{2}{I_{n,\alpha,p}(\pi, \mu_1)} \right)^{1/p} \Omega_\alpha\left(f^\psi, \pi, \mu_1, \frac{\pi}{n}\right)_{\mathcal{S}^p} \nu(n), \quad (3.12)$$

where $\nu(n) = \sup_{|k| \geq n} |\psi(n)|$,

$$I_{n,\alpha,p}(\pi, \mu_1) = 2^{\frac{\alpha p}{2}} \inf_{\substack{k \geq n \\ k \in \mathbb{N}}} \int_0^\pi \left(1 - \cos \frac{kt}{n}\right)^{\frac{\alpha p}{2}} \sin t dt \quad (3.13)$$

If, in addition, $\nu(n) = \max\{|\psi(n)|, |\psi(-n)|\}$ and the number $\frac{\alpha p}{2} \in \mathbb{N}$, then inequality (3.12) on the set $L^\psi \mathcal{S}^p$ can not be improved and

$$\sup_{\substack{f \in L^\psi \mathcal{S}^p \\ f \neq \text{const}}} \frac{E_n(f)_{\mathcal{S}^p}}{\Omega_\alpha(f^\psi, \pi, \mu_1, \frac{\pi}{n})_{\mathcal{S}^p}} = \frac{(\frac{\alpha p}{2} + 1)^{1/p}}{2^\alpha} \nu(n). \quad (3.14)$$

Proof. Indeed, if $\tau = \pi$ and $\mu(t) = 1 - \cos t$ in Corollary 3.2, then relation (3.12) follows from inequality (3.1'). If the number $\frac{\alpha p}{2}$ is positive integer, then use the following formula (see [19, relation (52)]):

$$\inf_{\theta \geq 1} \int_0^\pi (1 - \cos \theta t)^\lambda \sin t dt = \frac{2^{\lambda+1}}{\lambda + 1}, \quad \lambda \in \mathbb{N}. \quad (3.15)$$

Setting $\lambda = \frac{\alpha p}{2}$ and $\theta = \frac{k}{n}$, $k = n, n+1, n+2, \dots$, we see that $\theta \geq 1$. Therefore,

$$\inf_{\substack{k \geq n \\ k \in \mathbb{N}}} \int_0^\tau \left(1 - \cos \frac{kt}{n}\right)^{\frac{\alpha p}{2}} \sin t \, dt = \int_0^\tau \left(1 - \cos t\right)^{\frac{\alpha p}{2}} \sin t \, dt = \frac{2^{\frac{\alpha p}{2}+1}}{\frac{\alpha p}{2}+1}, \quad (3.16)$$

and equality (3.14) follows from relation (3.4') of Corollary 3.2 with $\tau = \pi$ and $\mu(t) = 1 - \cos t$. \square

Remark 3.1. In case $p = 2$ and $\psi(k) = (ik)^{-r}$, $r = 0, 1, \dots$, equality (3.14) can be given in the form

$$\sup_{\substack{f \in L^r \mathcal{S}^2 \\ f \neq \text{const}}} \frac{E_n(f)_{\mathcal{S}^2}}{\Omega_\alpha(f^{(r)}, \pi, \mu_1, \frac{\pi}{n})_{\mathcal{S}^2}} = \frac{\sqrt{\alpha+1}}{2^\alpha} n^{-r}, \quad \alpha > 0, \, n \in \mathbb{N}. \quad (3.14')$$

For $\alpha = 1$ this relation follows from the result of Chernykh [7]. For arbitrary $\alpha = k \in \mathbb{N}$ and $n \in \mathbb{N}$, the exact values of the quantities on the left-hand side of (3.14') were obtained by Yussef [30] in a slightly different form.

Corollary 3.4. Let $0 < \tau \leq \frac{3\pi}{4}$, $\mu_2(t) = t$, the numbers $1 \leq p < \infty$ and $\alpha > 0$ be such that $\alpha p \geq 1$. Let also $n \in \mathbb{N}$ and $\psi \in \Psi$ be the sequence such that $\nu(n) = \sup_{|k| \geq n} |\psi(n)| = \max\{|\psi(n)|, |\psi(-n)|\}$. Then

$$\sup_{\substack{f \in L^\psi \mathcal{S}^p \\ f \neq \text{const}}} \frac{E_n(f)_{\mathcal{S}^p}}{\Omega_\alpha(f^\psi, \tau, \mu_2, \frac{\tau}{n})_{\mathcal{S}^p}} = \left(\frac{\tau}{2^{\alpha p} \int_0^\tau \sin^{\alpha p} \frac{t}{2} dt} \right)^{1/p} \nu(n). \quad (3.17)$$

Proof. As shown in [29], for arbitrary numbers $\tau \in (0, \frac{3\pi}{4}]$ and $\gamma \geq 1$

$$\inf_{\substack{k \geq n \\ k \in \mathbb{N}}} \int_0^\tau \left| \sin \frac{\nu t}{2n} \right|^\gamma dt = \int_0^\tau \sin^\gamma \frac{t}{2} dt.$$

Therefore, for $\gamma = \alpha p$ and $\tau \in (0, \frac{3\pi}{4}]$, we have

$$\begin{aligned} I_{n, \alpha, p}(\tau, \mu_2) &= 2^{\frac{\alpha p}{2}} \inf_{\substack{k \geq n \\ k \in \mathbb{N}}} \int_0^\tau \left(1 - \cos \frac{kt}{n}\right)^{\frac{\alpha p}{2}} dt = 2^{\alpha p} \inf_{\substack{k \geq n \\ k \in \mathbb{N}}} \int_0^\tau \left| \sin \frac{kt}{2n} \right|^{\alpha p} dt \\ &= 2^{\alpha p} \int_0^\tau \left| \sin \frac{t}{2} \right|^{\alpha p} dt = 2^{\frac{\alpha p}{2}} \int_0^\tau (1 - \cos t)^{\frac{\alpha p}{2}} dt. \end{aligned}$$

Thus, relation (3.17) follows from equality (3.4') of Corollary 3.2 with $\mu(t) = t$ and $\tau \in (0, \frac{3\pi}{4}]$. \square

Note that in the case where $p = 2$, $\psi(k) = (ik)^{-r}$, $r \geq 0$ and $k = 1$ or $r \geq 1/2$ and $k \in \mathbb{N}$, equality (3.17) follows from the results of Taikov [21], [22], (see also [10]).

3.2 Widths of the classes $L^\psi(\varphi, \mu, \tau, n)_{\mathcal{S}^p}$

In this subsection, the values of Kolmogorov, Bernstein, linear, and projection widths are found for the the classes $L^\psi(\varphi, \mu, \tau, n)_{\mathcal{S}^p}$ in the case when the sequences $\psi(k)$ satisfy some natural restrictions. To state these results, denote by Ψ the set of arbitrary sequences $\{\psi(k)\}_{k \in \mathbb{Z}}$ of complex numbers such that $|\psi(k)| = |\psi(-k)| \geq |\psi(k+1)|$ for $k \in \mathbb{N}$.

Theorem 3.2. *Assume that $1 \leq p < \infty$, $\psi \in \Psi$, $\tau > 0$, the function $\varphi \in \Phi$ is non-decreasing on the interval $[0, \tau]$ and $\mu \in M(\tau)$. Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$ the following inequalities are true:*

$$\begin{aligned} \left(\frac{\mu(\tau) - \mu(0)}{\int_0^\tau \varphi^p(t) d\mu(t)} \right)^{1/p} |\psi(n)| &\leq P_N(L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p}, \mathcal{S}^p) \\ &\leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)} \right)^{1/p} |\psi(n)|, \end{aligned} \quad (3.18)$$

where the quantity $I_{n, \varphi, p}(\tau, \mu)$ is defined by (3.2), and P_N is any of the widths b_N , d_N , λ_N or π_N . If, in addition, condition (3.3) holds, then

$$P_N(L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p}, \mathcal{S}^p) = \left(\frac{\mu(\tau) - \mu(0)}{\int_0^\tau \varphi^p(t) d\mu(t)} \right)^{1/p} |\psi(n)|. \quad (3.19)$$

Proof. Based on Theorem 3.1, taking into account the definition of the set Ψ , for an arbitrary function $f \in L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p}$, we have

$$\begin{aligned} E_n(f)_{\mathcal{S}^p} &\leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)} \right)^{1/p} \Omega_\varphi\left(f^\psi, \tau, \mu, \frac{\tau}{n}\right)_{\mathcal{S}^p} |\psi(n)| \\ &\leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)} \right)^{1/p} |\psi(n)|. \end{aligned} \quad (3.20)$$

Then, taking into account the definition of the projection width π_N , and relations (2.8) and (3.20), we conclude that

$$\begin{aligned} \pi_{2n-1}(L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p}, \mathcal{S}^p) &\leq E_n(L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p})_{\mathcal{S}^p} \\ &= \sup_{f \in L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p}} E_n(f)_{\mathcal{S}^p} \leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)} \right)^{1/p} |\psi(n)|. \end{aligned} \quad (3.21)$$

Since the widths b_N , d_N , λ_N and π_N do not increase with increasing N and

$$b_N(K, X) \leq d_N(K, X) \leq \lambda_N(K, X) \leq \pi_N(K, X) \quad (3.22)$$

(see, for example, [23, Ch. 4]), then by virtue of (3.21), we get the estimate from above in (3.18).

To obtain the necessary lower estimate, it suffices to show that

$$b_{2n}(L^\psi(k, \mu, \tau, n)_{\mathcal{S}^p}, \mathcal{S}^p) \geq \left(\frac{\mu(\tau) - \mu(0)}{\int_0^\tau \varphi^p(u) d\mu(u)} \right)^{1/p} |\psi(n)| =: R_n. \quad (3.23)$$

In the $(2n+1)$ -dimensional space \mathcal{T}_{2n+1} of trigonometric polynomials of order n , consider ball B_{2n+1} , whose radius is equal to the number R_n defined in (3.23), that is,

$$B_{2n+1} = \left\{ t_n \in \mathcal{T}_{2n+1} : \|t_n\|_{\mathcal{S}^p} \leq R_n \right\}.$$

and prove the embedding $B_{2n+1} \subset L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p}$.

For an arbitrary polynomial $T_n \in B_{2n+1}$, due to (2.1) and the parity of the function φ , we have

$$\omega_\varphi^p(T_n^\psi, t)_{\mathcal{S}^p} = \sup_{0 \leq v \leq t} \sum_{|k| \leq n} \varphi^p(kv) |\widehat{T}_n^\psi(k)|^p.$$

Then, taking into account relation (2.4) and the nondecreasing of the function φ on $[0, a]$, for $\tau \in (0, a]$ we get

$$\begin{aligned} & (\mu(\tau) - \mu(0)) \Omega_\varphi^p\left(T_n^\psi, \tau, \mu, \frac{\tau}{n}\right)_{\mathcal{S}^p} = \int_0^\tau \omega_\varphi^p\left(T_n^\psi, \frac{t}{n}\right)_{\mathcal{S}^p} d\mu(t) \\ &= \int_0^\tau \sup_{0 \leq v \leq \frac{t}{n}} \sum_{|k| \leq n} \varphi^p(kv) |\widehat{T}_n^\psi(k)|^p d\mu(t) = \int_0^\tau \sup_{0 \leq v \leq t} \sum_{|k| \leq n} \varphi^p\left(\frac{kv}{n}\right) \left| \frac{\widehat{T}_n(k)}{\psi(k)} \right|^p d\mu(t) \\ &\leq \frac{1}{|\psi(n)|^p} \int_0^\tau \sum_{|k| \leq n} \varphi^p(t) |\widehat{T}_n(k)|^p d\mu(t) \leq \frac{\|T_n\|_{\mathcal{S}^p}^p}{|\psi(n)|^p} \int_0^\tau \varphi^p(t) d\mu(t). \end{aligned}$$

Therefore, given the inclusion $T_n \in B_{2n+1}$ it follows that $\Omega_\varphi(T_n^\psi, \tau, \mu, \frac{\tau}{n})_{\mathcal{S}^p} \leq 1$. Thus, $T_n \in L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p}$, the embedding $B_{2n+1} \subset L^\psi(\varphi, \mu, \tau, n)_{\mathcal{S}^p}$ is true. By the definition of Bernstein width, the inequality (3.23) holds. Thus, relation (3.18) is proved. It is easy to see that, under the condition (3.3), the upper and lower bounds for the quantities $P_N(L^\psi(\varphi, \tau, \mu, n)_{\mathcal{S}^p}, \mathcal{S}^p)$ coincide and, therefore, equalities (3.19) hold. \square

In the case where the function $\varphi(t) = 2^{\frac{\alpha}{2}}(1 - \cos t)^{\frac{\alpha}{2}}$, we obtain the following statement:

Corollary 3.5. *Assume that $1 \leq p < \infty$, $\psi \in \Psi$, $\tau \in (0, \pi]$, $\alpha \in \mathbb{N}$ and $\mu \in M(\tau)$. Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$ the following inequalities are true:*

$$\left(\frac{\mu(\tau) - \mu(0)}{2^{\alpha p} \int_0^\tau \sin^{\alpha p} \frac{t}{2} d\mu(t)} \right)^{1/p} |\psi(n)| \leq P_N(L^\psi(\alpha, \tau, \mu, n)_{\mathcal{S}^p}, \mathcal{S}^p)$$

$$\leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n,\alpha,p}(\tau, \mu)} \right)^{1/p} |\psi(n)|,$$

where the quantity $I_{n,\alpha,p}(\tau, \mu)$ is defined by (3.2) with $\varphi(t) = 2^{\frac{\alpha}{2}}(1 - \cos kt)^{\frac{\alpha}{2}}$, and P_N is any of the widths b_N , d_N , λ_N or π_N . If, in addition, condition (3.3') holds, then

$$P_N(L^\psi(\alpha, \tau, \mu, n)_{\mathcal{S}^p}, \mathcal{S}^p) = \left(\frac{\mu(\tau) - \mu(0)}{2^{\alpha p} \int_0^\tau \sin^{\alpha p} \frac{t}{2} d\mu(t)} \right)^{1/p} |\psi(n)|.$$

For the weight functions $\mu_1(t) = 1 - \cos t$ and $\mu_2(t) = t$, Corollary 3.5 yields the following statements:

Corollary 3.6. Assume that $1 \leq p < \infty$, $\psi \in \Psi$, $\alpha \in \mathbb{N}$ and $\mu_1(t) = 1 - \cos t$. Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$

$$\frac{(\frac{\alpha p}{2} + 1)^{1/p}}{2^\alpha} |\psi(n)| \leq P_N(L^\psi(\alpha, \pi, \mu_1, n)_{\mathcal{S}^p}, \mathcal{S}^p) \leq \left(\frac{2}{I_{n,\alpha,p}(\pi, \mu_1)} \right)^{1/p} |\psi(n)|,$$

where $I_{n,\alpha,p}(\pi, \mu_1)$ is the quantity of the form (3.13), and P_N is any of the widths b_N , d_N , λ_N or π_N . If, in addition, the number $\frac{\alpha p}{2} \in \mathbb{N}$, then

$$P_N(L^\psi(\alpha, \pi, \mu_1, n)_{\mathcal{S}^p}, \mathcal{S}^p) = \frac{(\frac{\alpha p}{2} + 1)^{1/p}}{2^\alpha} |\psi(n)|.$$

Corollary 3.7. Assume that $\psi \in \Psi$, $0 < \tau \leq \frac{3\pi}{4}$, $\mu_2(t) = t$, the numbers $\alpha > 0$ and $1 \leq p < \infty$ such that $\alpha p \geq 1$. Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$

$$P_N(L^\psi(\alpha, \tau, \mu_2, n)_{\mathcal{S}^p}, \mathcal{S}^p) = \left(\frac{\tau}{2^{\alpha p} \int_0^\tau \sin^{\alpha p} \frac{t}{2} dt} \right)^{1/p} |\psi(n)|,$$

where P_N is any of the widths b_N , d_N , λ_N or π_N .

3.3 Widths of the classes $L^\psi(\varphi, \mu, \tau, \Omega)_{\mathcal{S}^p}$

Let us find the widths of the classes $L^\psi(\varphi, \mu, \tau, \Omega)_{\mathcal{S}^p}$ that are defined by a majorant Ω of the averaged values of generalized moduli of smoothness.

Theorem 3.3. Let $1 \leq p < \infty$, $\psi \in \Psi$, the function $\varphi \in \Phi$ be non-decreasing on a certain interval $[0, a]$, $a > 0$, and $\varphi(a) = \sup\{\varphi(t) : t \in \mathbb{R}\}$. Let also $\tau \in (0, a]$, the function $\mu \in M(\tau)$ and for all $\xi > 0$ and $0 < u \leq a$, the function Ω satisfies the condition

$$\Omega\left(\frac{u}{\xi}\right) \left(\int_0^{\xi\tau} \varphi_*^p(t) d\mu\left(\frac{t}{\xi}\right) \right)^{1/p} \leq \Omega(u) \left(\int_0^\tau \varphi^p(t) d\mu(t) \right)^{1/p}, \quad (3.24)$$

where

$$\varphi_*(t) := \begin{cases} \varphi(t), & 0 \leq t \leq a, \\ \varphi(a), & t \geq a. \end{cases} \quad (3.25)$$

Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$ the following inequalities are true:

$$\begin{aligned} \left(\frac{\mu(\tau) - \mu(0)}{\int_0^\tau \varphi^p(t) d\mu(t)} \right)^{1/p} |\psi(n)| \Omega\left(\frac{\tau}{n}\right) &\leq P_N(L^\psi(\varphi, \tau, \mu, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p) \\ &\leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)} \right)^{1/p} |\psi(n)| \Omega\left(\frac{\tau}{n}\right), \end{aligned} \quad (3.26)$$

where the quantity $I_{n, \varphi, p}(\tau, \mu)$ is defined by (3.2), and P_N is any of the widths b_N , d_N , λ_N or π_N . If, in addition, condition (3.3) holds, then

$$P_N(L^\psi(\varphi, \tau, \mu, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p) = \left(\frac{\mu(\tau) - \mu(0)}{\int_0^\tau \varphi^p(t) d\mu(t)} \right)^{1/p} |\psi(n)| \Omega\left(\frac{\tau}{n}\right). \quad (3.27)$$

Proof. The proof of the theorem basically repeats the proof of Theorem 3.5. Based on inequality (3.1), for an arbitrary function $f \in L^\psi(\varphi, \tau, \mu, \Omega)_{\mathcal{S}^p}$

$$E_n(f)_{\mathcal{S}^p} \leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)} \right)^{1/p} |\psi(n)| \Omega_\varphi\left(\frac{\tau}{n}\right), \quad (3.28)$$

whence, taking into account the definition of the width π_N and relation (2.8), we obtain

$$\begin{aligned} \pi_{2n-1}(L^\psi(\varphi, \mu, \tau, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p) &= E_n(L^\psi(\varphi, \mu, \tau, \Omega)_{\mathcal{S}^p})_{\mathcal{S}^p} \\ &= \sup_{f \in L^\psi(\varphi, \mu, \tau, \Omega)_{\mathcal{S}^p}} E_n(f)_{\mathcal{S}^p} \leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)} \right)^{1/p} |\psi(n)| \Omega_\varphi\left(\frac{\tau}{n}\right). \end{aligned} \quad (3.29)$$

To obtain the necessary lower estimate, let us show that

$$b_{2n}(L^\psi(\varphi, \mu, \tau, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p) \geq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \varphi, p}(\tau, \mu)} \right)^{1/p} |\psi(n)| \Omega_\varphi\left(\frac{\tau}{n}\right) =: R_n^*. \quad (3.30)$$

For this purpose, in the $(2n+1)$ -dimensional space \mathcal{T}_{2n+1} of trigonometric polynomials of order n , consider ball B_{2n+1} , whose radius is equal to the number R_n defined in (3.30), that is,

$$B_{2n+1}^* = \left\{ T_n \in \mathcal{T}_{2n+1} : \|T_n\|_{\mathcal{S}^p} \leq R_n^* \right\}$$

and prove the validity of the embedding $B_{2n+1}^* \subset L^\psi(\varphi, \mu, \tau, \Omega)_{\mathcal{S}^p}$.

Assume that $T_n \in B_{2n+1}^*$. Taking into account the non-decrease of the function φ on $[0, a]$ and relations (2.4) and (3.25), we have

$$(\mu(\tau) - \mu(0)) \cdot \Omega_\varphi^p(T_n^\psi, \tau, \mu, u)_{\mathcal{S}^p} = \int_0^u \omega_\varphi^p(T_n^\psi, t)_{\mathcal{S}^p} d\mu\left(\frac{\tau t}{u}\right)$$

$$\begin{aligned}
&= \int_0^u \sup_{0 \leq v \leq t} \sum_{|k| \leq n} \varphi^p(kv) |\widehat{T}_n^\psi(k)|^p d\mu\left(\frac{\tau t}{u}\right) = \int_0^u \sup_{0 \leq v \leq t} \sum_{|k| \leq n} \varphi^p(kv) \left| \frac{\widehat{T}_n(k)}{\psi(k)} \right|^p d\mu\left(\frac{\tau t}{u}\right) \\
&\leq \frac{1}{|\psi(n)|^p} \int_0^u \sum_{|k| \leq n} \varphi_*^p(nt) |\widehat{T}_n(k)|^p d\mu\left(\frac{\tau t}{u}\right) \leq \frac{\|T_n\|_{\mathcal{S}^p}^p}{|\psi(n)|^p} \int_0^u \varphi_*^p(nt) d\mu\left(\frac{\tau t}{u}\right). \\
&\leq \frac{\|T_n\|_{\mathcal{S}^p}^p}{|\psi(n)|^p} \int_0^{nu} \varphi_*^p(t) d\mu\left(\frac{\tau t}{nu}\right).
\end{aligned}$$

From the inclusion of $T_n \in B_{2n+1}^*$ and relation (3.24) with $\xi = \frac{nu}{\tau}$, it follows that

$$\Omega_\varphi(T_n^\psi, \tau, \mu, u)_{\mathcal{S}^p} \leq \left(\frac{\int_0^{nu} \varphi_*^p(t) d\mu\left(\frac{\tau t}{nu}\right)}{\int_0^\tau \varphi^p(t) d\mu(t)} \right)^{1/p} \Omega_\varphi\left(\frac{\tau}{n}\right) \leq \Omega_\varphi(u).$$

Therefore, indeed $B_{2n+1}^* \subset L^\psi(\varphi, \tau, \mu, \Omega)_{\mathcal{S}^p}$ and by definition of Bernstein width, relation (3.30) is true. Combining relations (3.22), (3.28) and (3.30), and taking into account monotonic non-increase of each of the widths b_N , d_N , λ_N and π_N on N , we get (3.26). Under the additional condition (3.3), the upper and lower estimates of the quantities $P_N(L^\psi(\varphi, \tau, \mu, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p)$ coincide in relation (3.26) and hence, equalities (3.27) are true. \square

In the case $\varphi(t) = \varphi_\alpha(t) = 2^{\frac{\alpha}{2}}(1 - \cos t)^{\frac{\alpha}{2}}$, the following statement is true:

Corollary 3.8. *Let $1 \leq p < \infty$, $\psi \in \Psi$, $\tau \in (0, \pi]$, $\alpha > 0$ and $\mu \in M(\tau)$. Let also for all $\xi > 0$ and $0 < u \leq \pi$, the function Ω satisfies the condition*

$$\Omega\left(\frac{u}{\xi}\right) \left(\int_0^{\xi\tau} (1 - \cos t)_*^{\frac{\alpha p}{2}} d\mu\left(\frac{t}{\xi}\right) \right)^{1/p} \leq \Omega(u) \left(\int_0^\tau (1 - \cos t)^{\frac{\alpha p}{2}} d\mu(t) \right)^{1/p}, \quad (3.24')$$

where

$$(1 - \cos t)_* := \begin{cases} 1 - \cos t, & 0 \leq t \leq \pi, \\ 2, & t \geq \pi. \end{cases} \quad (3.25')$$

Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$ the following inequalities are true:

$$\begin{aligned}
&\left(\frac{\mu(\tau) - \mu(0)}{2^{\alpha p} \int_0^\tau \sin^{\alpha p} \frac{t}{2} d\mu(t)} \right)^{1/p} |\psi(n)| \Omega\left(\frac{\tau}{n}\right) \leq P_N(L^\psi(\alpha, \tau, \mu, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p) \\
&\leq \left(\frac{\mu(\tau) - \mu(0)}{I_{n, \alpha, p}(\tau, \mu)} \right)^{1/p} |\psi(n)| \Omega\left(\frac{\tau}{n}\right),
\end{aligned}$$

where the quantity $I_{n, \alpha, p}(\tau, \mu)$ is defined by (3.2) with $\varphi(t) = 2^{\frac{\alpha}{2}}(1 - \cos kh)^{\frac{\alpha}{2}}$, and P_N is any of the widths b_N , d_N , λ_N or π_N . If, in addition, condition (3.3') holds, then

$$P_N(L^\psi(\alpha, \tau, \mu, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p) = \left(\frac{\mu(\tau) - \mu(0)}{2^{\alpha p} \int_0^\tau \sin^{\alpha p} \frac{t}{2} d\mu(t)} \right)^{1/p} |\psi(n)| \Omega\left(\frac{\tau}{n}\right).$$

Note that for specific weighted functions $\mu \in M(\tau)$ and some restrictions on other parameters, the question of the existence of functions Ω satisfying conditions of the form (3.24) and (3.24'), investigated in [21], [22], [3], [31], etc.

For the weight functions $\mu_1(t) = 1 - \cos t$ and $\mu_2(t) = t$, Corollary 3.8 yields the following statements:

Corollary 3.9. *Let $1 \leq p < \infty$, $\psi \in \Psi$, $\mu_1(t) = 1 - \cos t$ and for all $\xi > 0$ and $0 < u \leq \pi$, the function Ω satisfies the condition*

$$\Omega\left(\frac{u}{\xi}\right) \left(\frac{1}{\xi} \int_0^{\pi\xi} (1 - \cos t)_*^{\frac{\alpha p}{2}} \sin \frac{t}{\xi} dt \right)^{1/p} \leq \Omega(u) \left(\int_0^\pi (1 - \cos t)^{\frac{\alpha p}{2}} \sin t dt \right)^{1/p}, \quad (3.31)$$

where the function $(1 - \cos t)_*$ is given by (3.25'). Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$

$$\frac{(\frac{\alpha p}{2} + 1)^{1/p}}{2^\alpha} |\psi(n)| \Omega\left(\frac{\tau}{n}\right) \leq P_N(L^\psi(\alpha, \pi, \mu_1, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p) \leq \frac{2^{1/p} |\psi(n)|}{I_{n,\alpha,p}^{1/p}(\pi, \mu_1)} \Omega\left(\frac{\tau}{n}\right),$$

where $I_{n,\alpha,p}(\pi, \mu_1)$ is the quantity of the form (3.13), and P_N is any of the widths b_N , d_N , λ_N or π_N . If, in addition, $\frac{\alpha p}{2} \in \mathbb{N}$, then

$$P_N(L^\psi(\alpha, \pi, \mu_1, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p) = \frac{(\frac{\alpha p}{2} + 1)^{1/p}}{2^\alpha} |\psi(n)| \Omega\left(\frac{\tau}{n}\right).$$

In the case where $p = 2$, $\psi(k) = (ik)^{-r}$, $r \in \mathbb{N}$, and $\alpha = 1$, the statement of Corollary 3.9 was obtained by Aynulloyev [3]. In [3], the existence of functions Ω satisfying condition (3.31) under the above restrictions on the parameters p and α was also proved.

Corollary 3.10. *Let $1 \leq p < \infty$, $\psi \in \Psi$, $0 < \tau \leq \frac{3\pi}{4}$, $\mu_2 = t$ and for all $\xi > 0$ and $0 < u \leq \pi$, the function Ω satisfies the condition*

$$\Omega\left(\frac{u}{\xi}\right) \left(\frac{1}{\xi} \int_0^{\xi\tau} (1 - \cos t)_*^{\frac{\alpha p}{2}} dt \right)^{1/p} \leq \Omega(u) \left(\int_0^\tau (1 - \cos t)^{\frac{\alpha p}{2}} dt \right)^{1/p}. \quad (3.32)$$

Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$

$$P_N(L^\psi(\alpha, \tau, \mu_2, \Omega)_{\mathcal{S}^p}, \mathcal{S}^p) = \left(\frac{\tau}{2^{\alpha p} \int_0^\tau \sin^{\alpha p} \frac{t}{2} dt} \right)^{1/p} |\psi(n)| \Omega\left(\frac{\tau}{n}\right),$$

where P_N is any of the widths b_N , d_N , λ_N or π_N .

Note that the statements of Corollaries 3.2 (case $\psi \in \Psi$), 3.5, 3.8 and was proved by Serdyuk [13].

In the case when $p = 2$, $\psi(k) = (ik)^{-r}$ and $r \geq 0$, $\alpha = 1$ or $r \geq 1/2$, $\alpha \in \mathbb{N}$, the statement of Corollary 3.10 follows from results of the papers [21], [22] (see also [11, Ch. 4]), where the existence of functions Ω satisfying (3.32) with the corresponding restrictions on p , α and r was also proved.

The question of establishing Jackson-type inequalities in the spaces \mathcal{S}^p , as well as finding exact values of the widths of classes generated by averaged values of moduli of smoothness of a form similar to (2.3), was considered in [25], [26], [29], etc.

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