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JACKSON-TYPE INEQUALITIES AND WIDTHS OF FUNCTIONAL CLASSES IN THE MUSIELAK–ORLICZ TYPE SPACES

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In the Musielak–Orlicz-type spaces $\mathcal{S}_{\mathbf{M}}$, exact Jackson-type inequalities are obtained in terms of best approximations of functions and the averaged values of their generalized moduli of smoothness. The values of Kolmogorov, Bernstein, linear, and projective widths in $\mathcal{S}_{\mathbf{M}}$ are found for classes of periodic functions defined by certain conditions on the averaged values of the generalized moduli of smoothness.

1. Introduction

Let $\mathbf{M} = \{M_k(t)\}_{k\in\mathbb{Z}},\ t \ge 0$, be a sequence of Orlicz functions. In other words, for every $k \in \mathbb{Z}$, the function $M_k(t)$ is a nondecreasing convex function for which $M_k(0) = 0$ and $M_k(t) \to \infty$ as $t \to \infty$. The modular space (or Musielak–Orlicz-type space) $\mathcal{S}_{\mathbf{M}}$ is the space of 2π-periodic complex-valued Lebesgue summable functions f $(f \in L)$ such that the following quantity (which is also called the Luxemburg norm of f) is finite:

$$\|f\|_{\mathbf{M}} := \|\{\hat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_{\mathbf{M}}(\mathbb{Z})} := \inf\Big\{a > 0 : \sum_{k\in\mathbb{Z}} M_k(|\hat{f}(k)|/a) \le 1\Big\}, \tag{1}$$

where $\hat{f}(k) := [f]\widehat{\ }(k) = (2\pi)^{-1}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,\ k \in \mathbb{Z}$, are the Fourier coefficients of the function f.

The spaces $\mathcal{S}_{\mathbf{M}}$ defined in this way are Banach spaces. Functional spaces of this type have been studied by mathematicians since the 1930s (see, for example, the monographs [16], [17], [19]). If all functions M_k are identical (namely, $M_k(t) \equiv M(t),\ k \in \mathbb{Z}$), the spaces $\mathcal{S}_{\mathbf{M}}$ coincide with the ordinary Orlicz-type spaces $\mathcal{S}_M$ [9]. If $M_k(t) = \mu_k t^{p_k},\ p_k \ge 1,\ \mu_k \ge 0$, then $\mathcal{S}_{\mathbf{M}}$ coincide with the weighted spaces $\mathcal{S}_{\mathbf{p},\,\mu}$ with variable exponents [1]. If all $M_k(t) = t^p,\ p \ge 1$, then the spaces $\mathcal{S}_{\mathbf{M}}$ are the known spaces $\mathcal{S}^p$ (see, for example, [25, Chapter 11]), which in the case $p = 2$ coincide with ordinary Lebesgue spaces $\mathcal{S}^2 = L_2$.

In this paper, we study the approximative properties of the spaces $\mathcal{S}_{\mathbf{M}}$. In particular, exact Jackson-type inequalities in $\mathcal{S}_{\mathbf{M}}$ are obtained in terms of the best approximations of functions and the averaged values of their generalized moduli of smoothness. The values of Kolmogorov, Bernstein, linear, and projective

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widths in the spaces $\mathcal{S}_{\mathbf{M}}$ are found for classes of periodic functions defined by certain conditions on the averaged values of the generalized moduli of smoothness.

Jackson-type (or Jackson–Stechkin-type) inequalities are inequalities that estimate the values of the best approximations of functions via the value of their modulus of continuity (smoothness) at a certain point. The first exact Jackson-type inequality for the best uniform approximations of 2π-periodic continuous functions by trigonometric polynomials was obtained by Korneichuk [14] in 1962. A similar result for the best uniform approximations of continuous functions given on the real axis by entire functions of the exponential type was obtained by Dzyadyk in [11]. In 1967, Chernykh [35; 34] proved two unimprovable inequalities for 2π-periodic functions from the Lebesgue spaces L_2. In [34], it was shown in particular that the averaged values of the moduli of smoothness can be more effective for characterizing the structural and approximative properties of the functions f than the moduli themselves. In [27; 28] (see also [18, Chapter 4]), Taikov originated systematic investigations of the problem of exact inequalities that estimate the values of the best approximations of functions via the averaged values of their moduli of smoothness. He first considered the functional classes of 2π-periodic functions defined by certain conditions on the averaged values of their moduli of smoothness and found the exact values of the widths of such classes in the spaces L_2. Later, similar topics were studied by numerous mathematicians in various functional spaces (see, for example, [26; 30; 13; 4; 5; 24; 6; 3; 2]). For more detailed information on the results obtained in this direction, see also [21; 32; 31; 6].

2. Preliminaries

2.1. Orlicz norm. In addition to the Luxemburg norm (1) of the space $\mathcal{S}_{\mathbf{M}}$, consider the Orlicz norm that is defined as follows. Let $\mathbf{M}^* = \{M_k^*(v)\}_{k\in\mathbb{Z}},\ v \ge 0$, be the sequence of functions defined by the relations

$$M_k^*(v) := \sup\{uv - M_k(u) : u \ge 0\}, \quad k \in \mathbb{Z}.$$

Consider the set $\Lambda = \Lambda(\mathbf{M}^*)$ of sequences of positive numbers $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}} M_k^*(\lambda_k) \le 1$. For any function $f \in \mathcal{S}_{\mathbf{M}}$, define its Orlicz norm by the equality

$$\|f\|_{\mathbf{M}}^* := \|\{\hat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_{\mathbf{M}}^*(\mathbb{Z})} := \sup\left\{\sum_{k\in\mathbb{Z}} \lambda_k |\hat{f}(k)| :\ \lambda \in \Lambda(\mathbf{M}^*)\right\}. \tag{2}$$

Further, we will mainly use the Orlicz norm for functions $f \in \mathcal{S}_{\mathbf{M}}$. However, taking into account Lemma 1, some corollaries can also be formulated from the results obtained when considering the Luxemburg norm.

Lemma 1. *For any function $f \in \mathcal{S}_{\mathbf{M}}$, the following relation holds*:

$$\|f\|_{\mathbf{M}} \le \|f\|_{\mathbf{M}}^* \le 2\,\|f\|_{\mathbf{M}}. \tag{3}$$

Relation (3) follows from the similar relation for corresponding norms in the modular Orlicz sequence spaces (see, for example [16, Chapter 4]).

2.2. Generalized moduli of smoothness and their averaged values. Let $\omega_\alpha(f,\delta)$ be the modulus of smoothness of a function $f\in\mathcal{S}_{\mathbf{M}}$ of order $\alpha>0$, i.e.,

$$\omega_\alpha(f,t)^*_{\mathbf{M}} := \sup_{|h|\le t}\|\Delta_h^\alpha f\|^*_{\mathbf{M}} = \sup_{|h|\le t}\left\|\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}f(\cdot-jh)\right\|^*_{\mathbf{M}}, \tag{4}$$

where $\binom{\alpha}{j}=\frac{1}{j!}\alpha(\alpha-1)\cdots(\alpha-j+1)$ for $j\in\mathbb{N}$ and $\binom{\alpha}{j}=1$ for $j=0$. By the definition, for any $k\in\mathbb{Z}$, we have

$$|[\Delta_h^\alpha f]\widehat{\ }(k)| = |1-\mathrm{e}^{-\mathrm{i}kh}|^\alpha|\hat f(k)| = 2^{\frac{\alpha}{2}}(1-\cos kh)^{\frac{\alpha}{2}}|\hat f(k)|. \tag{5}$$

Consider the set Φ of all continuous bounded nonnegative pair functions φ such that $\varphi(0)=0$ and the Lebesgue measure of the set $\{t\in\mathbb{R}:\ \varphi(t)=0\}$ is equal to zero. For a fixed function $\varphi\in\Phi$, $h\in\mathbb{R}$ and for any $f\in\mathcal{S}_{\mathbf{M}}$, we denote by $\{[\Delta_h^\varphi f]\widehat{\ }(k)\}_{k\in\mathbb{Z}}$ the sequence of numbers such that for any $k\in\mathbb{Z}$,

$$[\Delta_h^\varphi f]\widehat{\ }(k) = \varphi(kh)\hat f(k). \tag{6}$$

If there exists a function $\Delta_h^\varphi f\in L$ whose Fourier coefficients coincide with the numbers $[\Delta_h^\varphi f]\widehat{\ }(k)$, $k\in\mathbb{Z}$, then, as above, the expression $\|\Delta_h^\varphi f\|^*_{\mathbf{M}}$ denotes the Orlicz norm of the function $\Delta_h^\varphi f$. If such a function does not exist, then the notation $\|\Delta_h^\varphi f\|^*_{\mathbf{M}}$ denotes the norm $\|\cdot\|_{l^*_{\mathbf{M}}(\mathbb{Z})}$ of the sequence $\{[\Delta_h^\varphi f]\widehat{\ }(k)\}_{k\in\mathbb{Z}}$.

As in [23], [8], [7], define the generalized modulus of smoothness of a function $f\in\mathcal{S}_{\mathbf{M}}$ by the equality

$$\omega_\varphi(f,t)^*_{\mathbf{M}} = \sup_{|h|\le t}\|\Delta_h^\varphi f\|^*_{\mathbf{M}}. \tag{7}$$

It follows from (5) that $\omega_\alpha(f,t)^*_{\mathbf{M}}=\omega_\varphi(f,t)^*_{\mathbf{M}}$ when $\varphi(t)=\varphi_\alpha(t)=2^{\frac{\alpha}{2}}(1-\cos kt)^{\frac{\alpha}{2}}$. In the general case, such modules were considered, in particular, in [33], [15], [31], [6].

Further, let $\mathcal{M}(\tau)$, $\tau>0$, be the set of all functions μ, bounded nondecreasing and nonconstant on the segment $[0,\tau]$. By $\Omega_\varphi(f,\tau,\mu,u)^*_{\mathbf{M}}$, $u>0$, denote the average value of the generalized modulus of smoothness $\omega_\varphi(f,t)^*_{\mathbf{M}}$ of the function f with the weight $\mu\in\mathcal{M}(\tau)$, that is,

$$\Omega_\varphi(f,\tau,\mu,u)^*_{\mathbf{M}} := \frac{1}{\mu(\tau)-\mu(0)}\int_0^u \omega_\varphi(f,t)^*_{\mathbf{M}}\mathrm{d}\mu\left(\frac{\tau t}{u}\right). \tag{8}$$

Note that for any $f\in\mathcal{S}_{\mathbf{M}}$, $\tau>0$, $\mu\in\mathcal{M}(\tau)$ and $u>0$ the functionals $\Omega_\varphi(f,\tau,\mu,u)^*_{\mathbf{M}}$ do not exceed the value $\omega_\varphi(f,u)^*_{\mathbf{M}}$, and therefore in a number of questions they can be more effective for characterizing the structural and approximative properties of the function f.

2.3. Definition of ψ-integrals, ψ-derivatives and functional classes. Let $\psi=\{\psi(k)\}_{k\in\mathbb{Z}}$ be an arbitrary sequence of complex numbers. If, for a given function $f\in L$ with the Fourier series $\sum_{k\in\mathbb{Z}}\hat f(k)\mathrm{e}^{\mathrm{i}kx}$, the series $\sum_{k\in\mathbb{Z}}\psi(k)\hat f(k)\mathrm{e}^{\mathrm{i}kx}$ is the Fourier series of a certain function $F\in L$, then F is called (see, for example, [25, Chapter 11]) ψ-integral of the function f and is denoted as $F=\mathcal{J}^\psi(f,\cdot)$. In turn, the function f is called the ψ-derivative of the function F and is denoted as $f=F^\psi$. The Fourier coefficients of functions f and f^ψ are related by the equalities

$$\hat f(k)=\psi(k)\hat f^\psi(k),\quad k\in\mathbb{Z}. \tag{9}$$

The set of ψ-integrals of functions f of L is denoted as L^ψ. If $\mathfrak{N} \subset L$, then $L^\psi\mathfrak{N}$ denotes the set of ψ-integrals of functions $f \in \mathfrak{N}$. In particular, $L^\psi\mathcal{S}_{\mathbf{M}}$ is the set of ψ-integrals of functions $f \in \mathcal{S}_{\mathbf{M}}$.

For arbitrary fixed $\varphi \in \Phi$, $\tau > 0$ and $\mu \in M(\tau)$, define the functional classes

$$L^\psi(\varphi, \tau, \mu, n)^*_{\mathbf{M}} := \Big\{ f \in L^\psi\mathcal{S}_{\mathbf{M}} : \Omega_\varphi\Big(f^\psi, \tau, \mu, \frac{\tau}{n}\Big)^*_{\mathbf{M}} \le 1,\ n \in \mathbb{N} \Big\}, \tag{10}$$

$$L^\psi(\varphi, \tau, \mu, \Omega)^*_{\mathbf{M}} := \big\{ f \in L^\psi\mathcal{S}_{\mathbf{M}} : \Omega_\varphi(f^\psi, \tau, \mu, u)^*_{\mathbf{M}} \le \Omega(u),\ 0 \le u \le \tau \big\}, \tag{11}$$

where $\Omega(u)$ is a fixed continuous monotonically increasing function of the variable $u \ge 0$ such that $\Omega(0) = 0$.

Note that in the Lebesgue spaces L_2, for $\psi(k) = k^{-r}$, $r \in \mathbb{N}$, $\varphi(t) = 2^{\frac{\alpha}{2}}(1-\cos kt)^{\frac{\alpha}{2}}$, and the weight function $\mu(t) = t$, Taikov [27; 28] (see also [18, Chapter 4]) first considered the functional classes similar to (10) and (11). He found the exact values of the widths of such classes in the spaces L_2 in the case when the majorants Ω of the averaged values of the moduli of smoothness satisfied some constraints. The problem of finding the exact values of the widths in different spaces of functional classes of this kind was also studied in [36; 22; 12; 20; 30; 31; 2].

2.4. Best approximations and widths of functional classes. Let $\mathcal{T}_{2n+1}$, $n = 0, 1, \ldots$, be the set of all trigonometric polynomials $T_n(x) = \sum_{|k|\le n} c_k \mathrm{e}^{\mathrm{i}kx}$ of the order n, where c_k are arbitrary complex numbers.

For any function $f \in \mathcal{S}_{\mathbf{M}}$ denote by $E_n(f)^*_{\mathbf{M}}$ its best approximation by the trigonometric polynomials $T_{n-1} \in \mathcal{T}_{2n-1}$ in the space $\mathcal{S}_{\mathbf{M}}$ with respect to the norm $\|\cdot\|^*_{\mathbf{M}}$, i.e.,

$$E_n(f)^*_{\mathbf{M}} := \inf_{T_{n-1}\in\mathcal{T}_{2n-1}} \|f - T_{n-1}\|^*_{\mathbf{M}}. \tag{12}$$

From (2), it follows [10, Lemma 2] that for any $f \in \mathcal{S}_{\mathbf{M}}$ and all $n = 0, 1, \ldots,$

$$E_n(f)^*_{\mathbf{M}} = \|f - S_{n-1}(f)\|^*_{\mathbf{M}} = \sup\Big\{ \sum_{|k|\ge n} \lambda_k |\hat{f}(k)| : \lambda \in \Lambda \Big\}, \tag{13}$$

where $S_{n-1}(f) = S_{n-1}(f, \cdot) = \sum_{|k|\le n-1} \hat{f}(k)\mathrm{e}^{\mathrm{i}k\cdot}$ is the partial Fourier sum of the order $n-1$ of the function f.

Further, let K be a convex centrally symmetric subset of $\mathcal{S}_{\mathbf{M}}$ and let $B^*_{\mathbf{M}}$ be a unit ball of the space $\mathcal{S}_{\mathbf{M}}$ with respect to the norm $\|\cdot\|^*_{\mathbf{M}}$. Let also F_N be an arbitrary N-dimensional subspace of the space $\mathcal{S}_{\mathbf{M}}$, $N \in \mathbb{N}$, and $\mathcal{L}(\mathcal{S}_{\mathbf{M}}, F_N)$ be a set of linear operators from $\mathcal{S}_{\mathbf{M}}$ to F_N. By $\mathcal{P}(\mathcal{S}_{\mathbf{M}}, F_N)$ denote the subset of projection operators of the set $\mathcal{L}(\mathcal{S}_{\mathbf{M}}, F_N)$, that is, the set of the operators A of linear projection onto the set F_N such that $Af = f$ when $f \in F_N$. The following quantities are called Bernstein, Kolmogorov, linear, and projection N-widths of the set K in the space $\mathcal{S}_{\mathbf{M}}$, respectively:

$$b_N(K, \mathcal{S}_{\mathbf{M}}) = \sup_{F_{N+1}} \sup\{\varepsilon > 0 : \varepsilon B^*_{\mathbf{M}} \cap F_{N+1} \subset K\},$$

$$d_N(K, \mathcal{S}_{\mathbf{M}}) = \inf_{F_N} \sup_{f\in K} \inf_{u\in F_N} \|f - u\|^*_{\mathbf{M}},$$

$$\lambda_N(K, \mathcal{S}_{\mathbf{M}}) = \inf_{F_N} \inf_{A\in\mathcal{L}(\mathcal{S}_{\mathbf{M}}, F_N)} \sup_{f\in K} \|f - Af\|^*_{\mathbf{M}},$$

$$\pi_N(K, \mathcal{S}_{\mathbf{M}}) = \inf_{F_N} \inf_{A\in\mathcal{P}(\mathcal{S}_{\mathbf{M}}, F_N)} \sup_{f\in K} \|f - Af\|^*_{\mathbf{M}}.$$

3. Main results

3.1. Jackson-type inequalities. In this subsection, Jackson-type inequalities are obtained in terms of the best approximations and the averaged values of generalized moduli of smoothness in the spaces $\mathcal{S}_{\mathbf{M}}$. To state these results, denote by Ψ the set of arbitrary sequences $\psi = \{\psi(k)\}_{k\in\mathbb{Z}}$ of complex numbers such that $|\psi(k)| = |\psi(-k)| \geq |\psi(k+1)|$ for $k \in \mathbb{N}$. Here and below, we also assume that the sequence $\mathbf{M}^* = \{M_k^*(v)\}_{k\in\mathbb{Z}}$ satisfies the condition

$$M_k^*(v) > M_k^*(1) = 1, \quad v > 1,\ k \in \mathbb{Z}. \tag{14}$$

Theorem 2. *Assume that $f \in L^{\psi}\mathcal{S}_{\mathbf{M}}$, condition (14) holds, $\varphi \in \Phi$, $\tau > 0$, $\mu \in \mathcal{M}(\tau)$ and $\psi \in \Psi$. Then for any $n \in \mathbb{N}$,*

$$E_n(f)^*_{\mathbf{M}} \leq \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)} |\psi(n)|\, \Omega_{\varphi}\Big(f^{\psi}, \tau, \mu, \frac{\tau}{n}\Big)^*_{\mathbf{M}}, \tag{15}$$

where

$$I_{n,\varphi}(\tau,\mu) := \inf_{\substack{k\geq n\\ k\in\mathbb{N}}} \int_0^{\tau} \varphi\Big(\frac{kt}{n}\Big) \mathrm{d}\mu(t). \tag{16}$$

If, in addition, the function φ is nondecreasing on the interval $[0,\tau]$ and the condition

$$I_{n,\varphi}(\tau,\mu) = \int_0^{\tau} \varphi(t)\mathrm{d}\mu(t), \tag{17}$$

holds, then inequality (15) cannot be improved and therefore,

$$\sup_{\substack{f\in L^{\psi}\mathcal{S}_{\mathbf{M}}\\ f\neq \mathrm{const}}} \frac{E_n(f)^*_{\mathbf{M}}}{\Omega_{\varphi}(f^{\psi}, \tau, \mu, \frac{\tau}{n})^*_{\mathbf{M}}} = \frac{\mu(\tau)-\mu(0)}{\int_0^{\tau}\varphi(t)\mathrm{d}\mu(t)} |\psi(n)|. \tag{18}$$

Proof. Let $f \in L^{\psi}\mathcal{S}_{\mathbf{M}}$. By virtue of (9) and (13), we have

$$\begin{aligned} E_n(f)^*_{\mathbf{M}} &= \sup\Big\{\sum_{|k|\geq n} \lambda_k |\hat{f}(k)| :\ \lambda \in \Lambda\Big\} \leq \sup\Big\{\sum_{|k|\geq n} \lambda_k \Big|\frac{\psi(n)}{\psi(k)}\Big| |\hat{f}(k)| : \lambda \in \Lambda\Big\} \\ &= |\psi(n)| \sup\Big\{\sum_{|k|\geq n} \lambda_k \Big|\frac{\hat{f}(k)}{\psi(k)}\Big| : \lambda \in \Lambda\Big\} = |\psi(n)| E_n(f^{\psi})^*_{\mathbf{M}}. \end{aligned} \tag{19}$$

As shown in [10, Proof of Theorem 1], for any $g \in \mathcal{S}_{\mathbf{M}}$, $\tau > 0$, $\varphi \in \Phi$, $\mu \in \mathcal{M}(\tau)$ and $n \in \mathbb{N}$,

$$E_n(g)^*_{\mathbf{M}} \leq \frac{1}{I_{n,\varphi}(\tau,\mu)} \int_0^{\tau} \omega_{\varphi}\Big(g, \frac{t}{n}\Big)^*_{\mathbf{M}} \mathrm{d}\mu(t). \tag{20}$$

Let us set $g = f^{\psi}$ in (20). Then

$$E_n(f^{\psi})^*_{\mathbf{M}} \le \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)} \frac{\int_0^{\tau} \omega_{\varphi}(f^{\psi}, \frac{t}{n})^*_{\mathbf{M}} \mathrm{d}\mu(t)}{\mu(\tau)-\mu(0)} \le \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)} \Omega_{\varphi}\Big(f^{\psi}, \tau, \mu, \frac{\tau}{n}\Big)^*_{\mathbf{M}}. \tag{21}$$

Combining inequalities (19) and (21), we obtain (15).

Now assume that the function φ is nondecreasing on the interval $[0, \tau]$ and condition (17) holds. Then by virtue of (15), we have

$$\sup_{\substack{f\in L^{\psi}\mathcal{S}_{\mathbf{M}}\\ f\neq \mathrm{const}}} \frac{E_n(f)^*_{\mathbf{M}}}{\Omega_{\varphi}(f^{\psi}, \tau, \mu, \frac{\tau}{n})^*_{\mathbf{M}}} \le \frac{\mu(\tau)-\mu(0)}{\int_0^{\tau} \varphi(u)\mathrm{d}\mu(u)} |\psi(n)|. \tag{22}$$

To prove the unimprovability of (22), consider the function

$$f_n(x) = \gamma + \varepsilon_{-n}\delta \mathrm{e}^{-\mathrm{i}nx} + \varepsilon_n \delta \mathrm{e}^{\mathrm{i}nx},$$

where γ and δ are arbitrary complex numbers, and ε_k, $k\in\{-n, n\}$, are integers such that $|\varepsilon_n|+|\varepsilon_{-n}|=1$. Taking into account (6), (9) and (14), we have

$$\|\Delta_h^{\varphi} f_n^{\psi}\|^*_{\mathbf{M}} = |\delta| \frac{\varphi(nh)}{|\psi(n)|} \sup\{|\varepsilon_{-n}|\lambda_{-n} + |\varepsilon_n|\lambda_n : \lambda \in \Lambda\} = |\delta| \frac{\varphi(nh)}{|\psi(n)|}.$$

Since the function $\varphi(nh)$ is nondecreasing on the interval $\big[0, \frac{\tau}{n}\big]$, then for $0 \le t \le \tau$,

$$\omega_{\varphi}(f_n^{\psi}, t) = |\delta| \frac{\varphi(nt)}{|\psi(n)|}. \tag{23}$$

Taking into account (8), (23) and the equality $E_n(f_n)^*_{\mathbf{M}} = |\delta|$, we see that

$$\sup_{\substack{f\in L^{\psi}\mathcal{S}_{\mathbf{M}}\\ f\neq \mathrm{const}}} \frac{E_n(f)^*_{\mathbf{M}}}{\Omega_{\varphi}\big(f^{\psi}, \tau, \mu, \frac{\tau}{n}\big)^*_{\mathbf{M}}} \ge \frac{E_n(f_n)^*_{\mathbf{M}}}{\Omega_{\varphi}\big(f_n^{\psi}, \tau, \mu, \frac{\tau}{n}\big)^*_{\mathbf{M}}} = \frac{|\delta|(\mu(\tau)-\mu(0))|\psi(n)|}{\int_0^{\tau/n} |\delta|\varphi(nt)\mathrm{d}\mu(nt)} = \frac{\mu(\tau)-\mu(0)}{\int_0^{\tau} \varphi(u)\mathrm{d}\mu(u)} |\psi(n)|. \tag{24}$$

Relations (22) and (24) yield (18). □

3.2. Widths of the classes $L^{\psi}(\varphi, \mu, \tau, n)^*_{\mathbf{M}}$. In this subsection, the values of Kolmogorov, Bernstein, linear, and projection widths are found for the classes $L^{\psi}(\varphi, \mu, \tau, n)^*_{\mathbf{M}}$ in the case when the sequences $\psi(k)$ satisfy some natural restrictions.

Theorem 3. *Assume that $\psi \in \Psi$, $\tau > 0$, condition (14) holds, the function $\varphi \in \Phi$ is nondecreasing on the interval $[0, \tau]$ and $\mu \in \mathcal{M}(\tau)$. Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$,*

$$\frac{\mu(\tau)-\mu(0)}{\int_0^{\tau} \varphi(t)\mathrm{d}\mu(t)} |\psi(n)| \le P_N(L^{\psi}(\varphi, \tau, \mu, n)^*_{\mathbf{M}}, \mathcal{S}_{\mathbf{M}}) \le \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)} |\psi(n)|, \tag{25}$$

where the quantity $I_{n,\varphi}(\tau, \mu)$ is defined by (16), and P_N is any of the widths b_N, d_N, λ_N or π_N. If, in addition, condition (17) holds, then

$$P_N(L^{\psi}(\varphi, \tau, \mu, n)^*_{\mathbf{M}}, \mathcal{S}_{\mathbf{M}}) = \frac{\mu(\tau)-\mu(0)}{\int_0^{\tau} \varphi(t)\mathrm{d}\mu(t)} |\psi(n)|. \tag{26}$$

Proof. The proof of Theorems 3 and 4 basically repeats the proof of corresponding theorems in the spaces $\mathcal{S}^p$ (see [20; 2]) and is adapted in accordance with the properties of the spaces $\mathcal{S}_{\mathbf{M}}$. Based on Theorem 2, taking into account the definition of the set Ψ, for an arbitrary function $f \in L^{\psi}(\varphi, \tau, \mu, n)^*_{\mathbf{M}}$,

$$E_n(f)^*_{\mathbf{M}} \le \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)}\Omega_{\varphi}\Big(f^{\psi}, \tau, \mu, \frac{\tau}{n}\Big)^*_{\mathbf{M}}|\psi(n)| \le \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)}|\psi(n)|. \tag{27}$$

Then, taking into account the definition of the projection width π_N, and relations (13) and (27), we conclude that

$$\pi_{2n-1}(L^{\psi}(\varphi, \tau, \mu, n)^*_{\mathbf{M}}, \mathcal{S}_{\mathbf{M}}) \le \sup_{f \in L^{\psi}(\varphi,\tau,\mu,n)^*_{\mathbf{M}}} E_n(f)^*_{\mathbf{M}} \le \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)}|\psi(n)|. \tag{28}$$

Since the widths b_N, d_N, λ_N and π_N do not increase with increasing N and

$$b_N(K,X) \le d_N(K,X) \le \lambda_N(K,X) \le \pi_N(K,X) \tag{29}$$

(see, for example, [29, Chapter 4]), then by virtue of (28), we get the upper estimate in (25).

To obtain the necessary lower estimate, it suffices to show that

$$b_{2n}(L^{\psi}(\varphi, \mu, \tau, n)^*_{\mathbf{M}}, \mathcal{S}_{\mathbf{M}}) \ge \frac{\mu(\tau)-\mu(0)}{\int_0^{\tau}\varphi(u)\mathrm{d}\mu(u)}|\psi(n)| =: R_n. \tag{30}$$

In the $(2n+1)$-dimensional space $\mathcal{T}_{2n+1}$ of trigonometric polynomials of order n, consider the ball B_{2n+1}, whose radius is equal to the number R_n defined in (30), that is,

$$B_{2n+1} = \{t_n \in \mathcal{T}_{2n+1} : \|t_n\|^*_{\mathbf{M}} \le R_n\},$$

and prove the embedding $B_{2n+1} \subset L^{\psi}(\varphi, \tau, \mu, n)^*_{\mathbf{M}}$.

For an arbitrary polynomial $T_n \in B_{2n+1}$, due to (7) and the parity of the function φ, we have

$$\omega_{\varphi}(T_n^{\psi}, t)^*_{\mathbf{M}} = \sup_{0\le v\le t}\sup\Big\{\sum_{|k|\le n}\lambda_k\varphi(kv)|\widehat{T}_n^{\psi}(k)| : \lambda \in \Lambda\Big\}.$$

Then, taking into account (9) and the nondecrease of the function φ on $[0,a]$, for $\tau \in (0,a]$ we get

$$\begin{aligned}
&(\mu(\tau)-\mu(0))\Omega_{\varphi}\Big(T_n^{\psi}, \tau, \mu, \frac{\tau}{n}\Big)^*_{\mathbf{M}} \\
&\qquad = \int_0^{\tau}\omega_{\varphi}\Big(T_n^{\psi}, \frac{t}{n}\Big)^*_{\mathbf{M}}\mathrm{d}\mu(t) = \int_0^{\tau}\sup_{0\le v\le \frac{t}{n}}\sup\Big\{\sum_{|k|\le n}\lambda_k\varphi(kv)|\widehat{T}_n^{\psi}(k)| : \lambda\in\Lambda\Big\}\mathrm{d}\mu(t) \\
&\qquad = \int_0^{\tau}\sup_{0\le v\le \frac{t}{n}}\sup\Big\{\sum_{|k|\le n}\lambda_k\varphi(kv)\Big|\frac{\widehat{T}_n(k)}{\psi(k)}\Big| : \lambda\in\Lambda\Big\}\mathrm{d}\mu(t) \\
&\qquad \le \frac{1}{|\psi(n)|}\int_0^{\tau}\varphi(t)\sup\Big\{\sum_{|k|\le n}\lambda_k|\widehat{T}_n(k)| : \lambda\in\Lambda\Big\}\mathrm{d}\mu(t) = \frac{\|T_n\|^*_{\mathbf{M}}}{|\psi(n)|}\int_0^{\tau}\varphi(t)\mathrm{d}\mu(t).
\end{aligned}$$

Thus, given the inclusion $T_n \in B_{2n+1}$ it follows that $\Omega_\varphi\big(T_n^\psi, \tau, \mu, \frac{\tau}{n}\big)^*_{\mathbf{M}} \le 1$. So, $T_n \in L^\psi(\varphi, \tau, \mu, n)^*_{\mathbf{M}}$ and $B_{2n+1} \subset L^\psi(\varphi, \mu, \tau, n)^*_{\mathbf{M}}$. By the definition of Bernstein width, (30) holds. Thus, (25) is proved. It is easy to see that, under condition (17), the upper and lower bounds for the quantities $P_N(L^\psi(\varphi, \tau, \mu, n)^*_{\mathbf{M}}, \mathcal{S}_{\mathbf{M}})$ coincide and, therefore, (26) holds. ☐

3.3. Widths of the classes $L^\psi(\varphi, \mu, \tau, \Omega)^*_{\mathbf{M}}$. Let us find the widths of the classes $L^\psi(\varphi, \mu, \tau, \Omega)^*_{\mathbf{M}}$ that are defined by a majorant Ω of the averaged values of generalized moduli of smoothness.

Theorem 4. *Let $\psi \in \Psi$, condition (14) hold, the function $\varphi \in \Phi$ be nondecreasing on a certain interval $[0, a]$, $a > 0$, and $\varphi(a) = \sup\{\varphi(t) : t \in \mathbb{R}\}$. Let also $\tau \in (0, a]$, the function $\mu \in \mathcal{M}(\tau)$ and $\Omega(u)$ be a fixed continuous monotonically increasing function of the variable $u \ge 0$ such that $\Omega(0) = 0$ and for all $\xi > 0$ and $0 < u \le a$, the condition*

$$\Omega\Big(\frac{u}{\xi}\Big) \int_0^{\xi\tau} \varphi_*(t)\mathrm{d}\mu\Big(\frac{t}{\xi}\Big) \le \Omega(u) \int_0^{\tau} \varphi(t)\mathrm{d}\mu(t), \tag{31}$$

is satisfied, where

$$\varphi_*(t) := \begin{cases} \varphi(t), & 0 \le t \le a, \\ \varphi(a), & t \ge a. \end{cases} \tag{32}$$

Then for any $n \in \mathbb{N}$ and $N \in \{2n-1, 2n\}$,

$$\frac{\mu(\tau)-\mu(0)}{\int_0^\tau \varphi(t)\mathrm{d}\mu(t)} |\psi(n)| \Omega\Big(\frac{\tau}{n}\Big) \le P_N(L^\psi(\varphi, \tau, \mu, \Omega)^*_{\mathbf{M}}, \mathcal{S}_{\mathbf{M}}) \le \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau, \mu)} |\psi(n)|\ \Omega\Big(\frac{\tau}{n}\Big), \tag{33}$$

where the quantity $I_{n,\varphi}(\tau, \mu)$ is defined by (16), and P_N is any of the widths b_N, d_N, λ_N or π_N. If, in addition, condition (17) holds, then

$$P_N(L^\psi(\varphi, \tau, \mu, \Omega)^*_{\mathbf{M}}, \mathcal{S}_{\mathbf{M}}) = \frac{\mu(\tau)-\mu(0)}{\int_0^\tau \varphi(t)\mathrm{d}\mu(t)} |\psi(n)| \Omega\Big(\frac{\tau}{n}\Big). \tag{34}$$

Proof. Based on (15), for an arbitrary function $f \in L^\psi(\varphi, \tau, \mu, \Omega)^*_{\mathbf{M}}$,

$$E_n(f)^*_{\mathbf{M}} \le \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau, \mu)} |\psi(n)| \Omega_\varphi\Big(\frac{\tau}{n}\Big), \tag{35}$$

whence, taking into account the definition of the width π_N and (13), we obtain

$$\pi_{2n-1}(L^\psi(\varphi, \mu, \tau, \Omega)^*_{\mathbf{M}}, \mathcal{S}_{\mathbf{M}}) = \sup_{f \in L^\psi(\varphi,\tau,\mu,\Omega)^*_{\mathbf{M}}} E_n(f)^*_{\mathbf{M}} \le \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau, \mu)} |\psi(n)| \Omega_\varphi\Big(\frac{\tau}{n}\Big). \tag{36}$$

To obtain the necessary lower estimate, let us show that

$$b_{2n}(L^\psi(\varphi, \mu, \tau, \Omega)^*_{\mathbf{M}}, \mathcal{S}_{\mathbf{M}}) \ge \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau, \mu)} |\psi(n)| \Omega_\varphi\Big(\frac{\tau}{n}\Big) =: R^*_n. \tag{37}$$

For this purpose, in the $(2n+1)$-dimensional space $\mathcal{T}_{2n+1}$ of trigonometric polynomials of order n, consider the ball B_{2n+1}, whose radius is equal to the number R^*_n defined in (37), that is,

$$B^*_{2n+1} = \{T_n \in \mathcal{T}_{2n+1} : \|T_n\|^*_{\mathbf{M}} \le R^*_n\}$$

and prove the validity of the embedding $B^*_{2n+1} \subset L^\psi(\varphi, \mu, \tau, \Omega)^*_{\mathbf{M}}$.

Assume that $T_n \in B^*_{2n+1}$. Taking into account the nondecrease of the function φ on $[0,a]$ and on (9) and (32), we have

$$\begin{aligned}
&(\mu(\tau)-\mu(0))\cdot\Omega_\varphi(T_n^\psi,\tau,\mu,u)^*_{\mathbf{M}}\\
&\qquad=\int_0^u \omega_\varphi(T_n^\psi,t)^*_{\mathbf{M}}\mathrm{d}\mu\Big(\frac{\tau t}{u}\Big)=\int_0^\tau \sup_{0\le v\le t}\sup\Big\{\sum_{|k|\le n}\lambda_k\varphi(kv)\Big|\frac{\widehat{T}_n(k)}{\psi(k)}\Big| : \lambda\in\Lambda\Big\}\mathrm{d}\mu\Big(\frac{\tau t}{u}\Big)\\
&\qquad\le\frac{1}{|\psi(n)|}\int_0^\tau \varphi_*(nt)\sup\Big\{\sum_{|k|\le n}\lambda_k|\widehat{T}_n(k)| : \lambda\in\Lambda\Big\}\mathrm{d}\mu\Big(\frac{\tau t}{u}\Big)\\
&\qquad=\frac{\|T_n\|^*_{\mathbf{M}}}{|\psi(n)|}\int_0^u \varphi_*(nt)\mathrm{d}\mu\Big(\frac{\tau t}{u}\Big)=\frac{\|T_n\|^*_{\mathbf{M}}}{|\psi(n)|}\int_0^{nu}\varphi_*(t)\mathrm{d}\mu\Big(\frac{\tau t}{nu}\Big).
\end{aligned}$$

From the inclusion of $T_n \in B^*_{2n+1}$ and relation (31) with $\xi=\frac{nu}{\tau}$, it follows that

$$\Omega_\varphi(T_n^\psi,\tau,\mu,u)^*_{\mathbf{M}}\le\frac{\int_0^{nu}\varphi_*(t)\mathrm{d}\mu\big(\frac{\tau t}{nu}\big)}{\int_0^\tau\varphi(t)\mathrm{d}\mu(t)}\Omega\Big(\frac{\tau}{n}\Big)\le\Omega(u).$$

Therefore, $B^*_{2n+1}\subset L^\psi(\varphi,\tau,\mu,\Omega)^*_{\mathbf{M}}$ and by the definition of Bernstein width, relation (37) is true. Combining (29), (35) and (37), and taking into account monotonic nonincrease of each of the widths b_N, d_N, λ_N and π_N on N, we get (33). Under the additional condition (17), the upper and lower estimates of the quantities $P_N(L^\psi(\varphi,\tau,\mu,\Omega)^*_{\mathbf{M}},\mathcal{S}_{\mathbf{M}})$ coincide in (33) and hence, (34) holds. □

3.4. Some corollaries and remarks. As mentioned above, the functional classes similar to the classes of the kind (10) and (11) were studied by many authors. To compare our results with the results of these studies, let us give some notation.

Consider the case where $M_k(t)=t^p(p^{-1/p}q^{-1/q})^p$, $p>1$, $1/p+1/q=1$. Here, for $M_k^*(v)=v^q$, the set $\Lambda(\mathbf{M}^*)$ is a set of all sequences of positive numbers $\lambda=\{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\|\lambda\|_{l_q(\mathbb{Z})}\le 1$. Then the spaces $\mathcal{S}_{\mathbf{M}}$ coincide with the mentioned above spaces $\mathcal{S}^p$ of functions $f\in L$ with the finite norm

$$\|f\|_p:=\|f\|_{\mathcal{S}^p}=\|\{\hat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_p(\mathbb{Z})}=\Big(\sum_{k\in\mathbb{Z}}|\hat{f}(k)|^p\Big)^{1/p},\quad 1\le p<\infty. \tag{38}$$

As shown in [10], for any $f\in\mathcal{S}^p$ and $p>1$, $\|f\|^*_{\mathbf{M}}=\|f\|_p$.

In the case $p=1$, the similar equality for norms $\|f\|^*_{\mathbf{M}}=\|f\|_1$ obviously can be obtained if we consider all $M_k(u)=u$, $k\in\mathbb{Z}$, and the set Λ is a set of all sequences of positive numbers $\lambda=\{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\|\lambda\|_{l_\infty(\mathbb{Z})}=\sup_{k\in\mathbb{Z}}\lambda_k\le 1$.

In particular, for $M_k(t)=\frac{t^2}{4}$, the space $\mathcal{S}_{\mathbf{M}}$ coincides with the Lebesgue space L_2 of functions $f\in L$ with finite norm

$$\|f\|_{L_2}=\Big(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\mathrm{d}t\Big)^{1/2}.$$

In the spaces $\mathcal{S}^p$, we denote the generalized modulus of smoothness of a function f by $\omega_\varphi(f,t)_p$, and the best approximation of f by the trigonometric polynomials $T_{n-1}\in\mathcal{T}_{2n-1}$ is denoted by $E_n(f)_p$. By $\Omega_\varphi(f,\tau,\mu,u)_p,\ u>0$, we denote the average value of the generalized modulus of smoothness $\omega_\varphi(f,t)_p$ of the function f with the weight $\mu\in\mathcal{M}(\tau)$. Theorems 2, 3 and 4 yields the following corollaries:

Corollary 5. *Assume that* $f\in\mathcal{S}^p,\ 1\le p<\infty,\ \varphi\in\Phi,\ \tau>0,\ \mu\in\mathcal{M}(\tau)$ *and* $\psi\in\Psi$. *Then for any* $n\in\mathbb{N}$,

$$E_n(f)_p\le\frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)}|\psi(n)|\,\Omega_\varphi\Big(f^\psi,\tau,\mu,\frac{\tau}{n}\Big)_p,\tag{39}$$

where the quantity $I_{n,\varphi}(\tau,\mu)$ *is defined by (16). If, in addition, the function* φ *is nondecreasing on* $[0,\tau]$ *and condition (17) holds, then (39) cannot be improved and therefore,*

$$\sup_{\substack{f\in L^\psi\mathcal{S}^p\\ f\ne\mathrm{const}}}\frac{E_n(f)_p}{\Omega_\varphi(f^\psi,\tau,\mu,\frac{\tau}{n})_p}=\frac{\mu(\tau)-\mu(0)}{\int_0^\tau\varphi(t)\mathrm{d}\mu(t)}|\psi(n)|.\tag{40}$$

Corollary 6. *Assume that* $1\le p<\infty,\ \psi\in\Psi,\ \tau>0$, *the function* $\varphi\in\Phi$ *is nondecreasing on the interval* $[0,\tau]$ *and* $\mu\in\mathcal{M}(\tau)$. *Then for any* $n\in\mathbb{N}$ *and* $N\in\{2n-1,2n\}$,

$$\frac{\mu(\tau)-\mu(0)}{\int_0^\tau\varphi(t)\mathrm{d}\mu(t)}|\psi(n)|\le P_N(L^\psi(\varphi,\tau,\mu,n)_p,\mathcal{S}^p)\le\frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)}|\psi(n)|,\tag{41}$$

where the quantity $I_{n,\varphi}(\tau,\mu)$ *is defined by (16), and* P_N *is any of the widths* b_N, d_N, λ_N *or* π_N. *If, in addition, condition (17) holds, then*

$$P_N(L^\psi(\varphi,\tau,\mu,n)_p,\mathcal{S}^p)=\frac{\mu(\tau)-\mu(0)}{\int_0^\tau\varphi(t)\mathrm{d}\mu(t)}|\psi(n)|.\tag{42}$$

Corollary 7. *Let* $1\le p<\infty,\ \psi\in\Psi$, *the function* $\varphi\in\Phi$ *be nondecreasing on a certain interval* $[0,a]$, $a>0$, *and* $\varphi(a)=\sup\{\varphi(t):\ t\in\mathbb{R}\}$. *Let also* $\tau\in(0,a]$, *the function* μ *be in* $\mathcal{M}(\tau)$ *and for all* $\xi>0$ *and* $0<u\le a$, $\Omega(u)$ *be a fixed continuous monotonically increasing function of the variable* $u\ge0$ *such that* $\Omega(0)=0$ *and condition (31) is satisfied. Then for any* $n\in\mathbb{N}$ *and* $N\in\{2n-1,2n\}$,

$$\frac{\mu(\tau)-\mu(0)}{\int_0^\tau\varphi(t)\mathrm{d}\mu(t)}|\psi(n)|\ \Omega\Big(\frac{\tau}{n}\Big)\le P_N(L^\psi(\varphi,\tau,\mu,\Omega)_p,\mathcal{S}^p)\le\frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)}|\psi(n)|\ \Omega\Big(\frac{\tau}{n}\Big),\tag{43}$$

where the quantity $I_{n,\varphi}(\tau,\mu)$ *is defined by (16), and* P_N *is any of the widths* b_N, d_N, λ_N *or* π_N. *If, in addition, condition (17) holds, then*

$$P_N(L^\psi(\varphi,\tau,\mu,\Omega)_p,\mathcal{S}^p)=\frac{\mu(\tau)-\mu(0)}{\int_0^\tau\varphi(t)\mathrm{d}\mu(t)}|\psi(n)|\Omega\Big(\frac{\tau}{n}\Big).\tag{44}$$

Let us note that in $\mathcal{S}^p$, statements similar to Corollaries 5-7 were also obtained in [2] and [20]. In [2], instead of the average values $\Omega_\varphi(f,\tau,\mu,u)_p$ of the generalized moduli of smoothness $\omega_\varphi(f,t)_p$, the authors considered the quantities

$$\Omega_\varphi(f,\tau,\mu,s,u)_p=\bigg(\frac{1}{\mu(\tau)-\mu(0)}\int_0^u\omega_\varphi^s(f,t)_p\mathrm{d}\mu\Big(\frac{\tau t}{u}\Big)\bigg)^{1/s}\tag{45}$$

in the case where $s=p$. In [20], the quantities $\Omega_\varphi(f,\tau,\mu,p,u)_p$ were considered when $\varphi(t)=\varphi_\alpha(t)=2^{\frac{\alpha}{2}}(1-\cos kt)^{\frac{\alpha}{2}}$. In the case $s=p=1$ and $\psi\in\Psi$, the results of Corollaries 5-7 coincide with the corresponding results of [2].

In the case $p=2$, $\psi(k)=(\mathrm{i}k)^{-r}$, $r=0,1,\dots$ and $\mu_1(t)=1-\cos t$, equalities of the kind in (39) with the quantities $\Omega_{\varphi_\alpha}(f,\tau,\mu_1,2,u)_2$ follow from the result of Chernykh [34] when $\alpha=1$, and from results of Yussef [36] when $\alpha=k\in\mathbb{N}$ and $n\in\mathbb{N}$. For the weight function $\mu_2(t)=t$ and $\psi(k)=(\mathrm{i}k)^{-r}$, $r=0,1,\dots$, equalities of the kind in (39) with the quantities $\Omega_{\varphi_\alpha}(f,\tau,\mu_2,2,u)_2$ were obtained by Taikov [27; 28] ($k=1$ or $r\ge 1/2$ and $k\in\mathbb{N}$).

The widths of the classes

$$L^\psi(\varphi,\tau,\mu,s,\Omega)_p=\{f\in L^\psi\mathcal{S}^p:\ \Omega_\varphi(f^\psi,\tau,\mu,s,u)_p\le\Omega(u),\ u\in[0,\tau]\},$$

when $p=s=2$, $\mu(t)=\mu_2(t)=t$, $\varphi(t)=\varphi_\alpha(t)=2^{\frac{\alpha}{2}}(1-\cos kt)^{\frac{\alpha}{2}}$, $\psi(k)=(\mathrm{i}k)^{-r}$, for $r\ge 0$ and $\alpha=1$ or $r\ge 1/2$ and $\alpha\in\mathbb{N}$, were obtained in [27; 28] (see also [18, Chapter 4]), where the existence of functions Ω satisfying conditions similar to (31) was also proved.

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