Ukrainian Mathematical Journal, Vol. 48, No. 1, 1996

SEPARATION OF VARIABLES IN LINEAR EXTENSIONS OF DYNAMICAL SYSTEMS ON TORI

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UDC 517.928

We study the problem of separation of variables in linear extensions of dynamical systems on tori.

Consider a system of differential equations

$$\frac{d\varphi}{dt} = a(\varphi), \qquad \frac{dx}{dt} = A(\varphi)x, \tag{1}$$

where $\varphi \in \mathcal{T}_m$, $x \in R^n$, and the vector function $a(\varphi)$ and matrix function $A(\varphi)$ are continuous in the collection of variables $\varphi_1, \ldots, \varphi_m$ on an m-dimensional torus $\mathcal{T}_m$. We also assume that the vector function $a(\varphi)$ is such that the Cauchy problem

$$\frac{d\varphi}{dt} = a(\varphi), \qquad \varphi\big|_{t=0} = \varphi_0$$

has a unique solution $\varphi_t(\varphi_0)$ for every fixed $\varphi_0 \in \mathcal{T}_m$.

Theorems that guarantee the separation of variables x into three blocks by the change of variables $x = L(\varphi)y$ were proved in [1–3]. In the present paper, we consider more general theorems that guarantee the separation of variables into $k+1$ blocks, where $k \geq 2$.

Below, we use the notation from [1].

Theorem 1. *Suppose that, for some scalar functions* $\lambda_i(\varphi) \in C^0(\mathcal{T}_m)$, *there exist nondegenerate symmetric matrices* $S_i(\varphi) \in C'(\mathcal{T}_m, a)$, $i = 1, \ldots, k$, *that satisfy the inequality*

$$\left\langle \left[\dot{S}_i[\varphi] + S_i(\varphi)A(\varphi) + A^*(\varphi)S_i(\varphi) + \lambda_i(\varphi)S_i(\varphi)\right]x,\ x\right\rangle \leq -\|x\|^2. \tag{2}$$

Also assume that the quadratic forms $\langle S_i(\varphi)x, x\rangle$ *reduce to algebraic sums of squares, i.e., there exist nondegenerate matrices* $Q_i(\varphi) \in C'(\mathcal{T}_m, a)$, $i = 1, \ldots, k$, *such that*

$$Q_i^*(\varphi)S_i(\varphi)Q_i(\varphi) = \operatorname{diag}\{I_{r_i}, -I_{n-r_i}\}. \tag{3}$$

Furthermore, we assume that $r_1 < r_2 < \ldots < r_k$. *Then the validity of the inequality*

$$m < \min_{2 \leq i \leq k} \{r_i - r_{i-1}\}, \tag{4}$$

where m *is the number of variables* $\varphi_1, \ldots, \varphi_m$, *is a sufficient condition for the existence of a nondegenerate matrix* $L(\varphi) \in C'(\mathcal{T}_m, a)$ *such that*

Institute of Mathematics, Ukrainian Academy of Sciences, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 48, No. 1, pp. 129–132, January, 1996. Original article submitted July 5, 1994.

 0041-5995/96/4801-0146 $15.00

$$L^{-1}(\varphi)[A(\varphi)L(\varphi)-\dot{L}(\varphi)] = \operatorname{diag}\{B_1(\varphi), B_2(\varphi), \ldots, B_{k+1}(\varphi)\}, \tag{5}$$

where $B_1(\varphi), B_2(\varphi), \ldots, B_{k+1}(\varphi)$ *are matrices of dimensions* $r_1 \times r_1$, $(r_2 - r_1)\times(r_2 - r_1)$, $\ldots$, $(n - r_k)\times(n - r_k)$, *respectively.*

Proof. According to [1, Sec. 3], for system (1) to be decomposable into k subsystems with respect to the variables x, it is sufficient that a decomposition of the n-dimensional identity matrix into a sum of variable projection matrices satisfying the invariance condition can be transformed into another decomposition into a sum of constant projection matrices.

For every $i = 1, \ldots, k$, the validity of inequality (2) guarantees the exponential dichotomy of the torus $x = 0$ of the system

$$\dot{\varphi} = a(\varphi), \qquad \dot{x} = \left[A(\varphi) + \frac{1}{2}\lambda_i(\varphi) I_n\right]x. \tag{6}$$

For these systems, the matrices $C_i(\varphi)$, $i = 1, \ldots, k$, of projection onto the subspaces $E_i^+(\varphi)$ along $E_i^-(\varphi)$ continuously depend on the parameter φ while their ranks are independent of φ, namely, $\operatorname{rank} C_i(\varphi) = r_i$.

Let us show that the matrices $[C_i(\varphi) - C_{i-1}(\varphi)]$, $i = 2, \ldots, k$, are projection matrices. Inequalities (2) yield

$$\begin{aligned}
&\exp\left\{\frac{1}{2}\int_\tau^t \lambda_i(\varphi_\sigma(\varphi))\,d\sigma\right\}\left\|\Omega_\tau^t(\varphi)C_i(\varphi_\tau(\varphi))\right\| \le K\exp\{-\gamma(t-\tau)\}, \qquad \tau \le t,\\
&\exp\left\{\frac{1}{2}\int_\tau^t \lambda_i(\varphi_\sigma(\varphi))\,d\sigma\right\}\left\|\Omega_\tau^t(\varphi)[C_i(\varphi_\tau(\varphi)) - I_n]\right\| \le K\exp\{-\gamma(t-\tau)\}, \qquad t < \tau.
\end{aligned} \tag{7}$$

Note that the matrices $C_i(\varphi)$, $i = 1, \ldots, k$, possess the invariance property

$$C_i(\varphi_t(\varphi))\Omega_0^t(\varphi) \equiv \Omega_0^t(\varphi)C_i(\varphi). \tag{8}$$

Assume that $1 \le i < j \le k$. Since

$$[C_j(\varphi) - C_i(\varphi)]C_i(\varphi) \equiv [C_j(\varphi) - I_n]C_i(\varphi),$$

$$C_i(\varphi)[C_j(\varphi) - C_i(\varphi)] \equiv C_i(\varphi)[C_j(\varphi) - I_n],$$

taking into account the relation $\dim E_i^+(\varphi) = r_i < r_j = \dim E_j^+(\varphi)$ and inequality (7) for fixed i and j, we get

$$[C_j(\varphi) - C_i(\varphi)]C_i(\varphi) \equiv 0, \quad C_i(\varphi)[C_j(\varphi) - C_i(\varphi)] \equiv 0. \tag{9}$$

It follows from (9) that, for any $i, j = 1, \ldots, k$,

$$C_i(\varphi)C_j(\varphi) \equiv C_j(\varphi)C_i(\varphi) \equiv C_l(\varphi), \tag{10}$$

where $l = \min\{i, j\}$. Taking (10) into account, we obtain

$$[C_i(\varphi) - C_{i-1}(\varphi)]^2 \equiv C_i^2(\varphi) - C_i(\varphi)C_{i-1}(\varphi) - C_{i-1}(\varphi)C_i(\varphi) + C_{i-1}^2(\varphi) \equiv C_i(\varphi) - C_{i-1}(\varphi),$$

$$[C_i(\varphi) - C_{i-1}(\varphi)][C_j(\varphi) - C_{j-1}(\varphi)]$$

$$\equiv C_i(\varphi)C_j(\varphi) - C_i(\varphi)C_{j-1}(\varphi) - C_{i-1}(\varphi)C_j(\varphi) + C_{i-1}(\varphi)C_{j-1}(\varphi) \equiv 0 \quad \forall\, i \neq j.$$

Finally, for every $i, j = 2, \ldots, k,$ we get

$$[C_i(\varphi) - C_{i-1}(\varphi)][C_j(\varphi) - C_{j-1}(\varphi)] \equiv \begin{cases} C_i(\varphi) - C_{i-1}(\varphi), & i = j, \\ 0, & i \neq j. \end{cases}$$

It follows from (8) that each matrix $[C_i(\varphi) - C_{i-1}(\varphi)],\ i = 2, \ldots, k,$ possesses the invariance property

$$[C_i(\varphi_t(\varphi)) - C_{i-1}(\varphi_t(\varphi))]\Omega_0^t(\varphi) \equiv \Omega_0^t(\varphi)[C_i(\varphi) - C_{i-1}(\varphi)]. \tag{11}$$

Thus, the identity matrix can be represented as the sum of $k+1$ projection matrices

$$I_n \equiv C_1(\varphi) + [C_2(\varphi) - C_1(\varphi)] + \ldots + [I_n - C_k(\varphi)], \tag{12}$$

and each projection matrix satisfies (11).

It follows from identity (3) with $i = 1$ that there exists a nondegenerate matrix $T(\varphi) \in C'(\mathcal{T}_m, a)$ such that

$$T^{-1}(\varphi)C_1(\varphi)T(\varphi) \equiv \operatorname{diag}\{I_{r_1}, 0\},$$

and

$$T_1^{-1}(\varphi)[C_2(\varphi) - C_1(\varphi)]\, T(\varphi) \equiv \operatorname{diag}\{0, P_1(\varphi)\},$$

where $P_1(\varphi)$ is the $(n - r_1) \times (n - r_1)$-dimensional projection matrix.

By using the condition $m < r_2 - r_1$, which follows from (4), we establish [4] that there exists an $(n - r_1) \times (n - r_1)$ matrix $T_1(\varphi) \in C'(\mathcal{T}_m, a)$ such that

$$T_1^{-1}(\varphi)P_1(\varphi)T_1(\varphi) \equiv \operatorname{diag}\{I_{r_2 - r_1}, 0\}.$$

Then there exists a nondegenerate matrix $L_1(\varphi) \in C'(\mathcal{T}_m, a)$ of the form $L_1(\varphi) = T(\varphi)\operatorname{diag}\{I_{r_1}, T_1(\varphi)\}$ such that

$$L_1^{-1}(\varphi)C_1(\varphi)L_1(\varphi) \equiv \operatorname{diag}\{I_{r_1}, 0, 0\},$$

$$L_1^{-1}(\varphi)[C_2(\varphi) - C_1(\varphi)]\, L_1(\varphi) \equiv \operatorname{diag}\{0, I_{r_2 - r_1}, 0\}.$$

This yields

$$L_1^{-1}(\varphi)C_2(\varphi)L_1(\varphi) \equiv L_1^{-1}(\varphi)C_1(\varphi)L_1(\varphi) + L_1^{-1}(\varphi)[C_2(\varphi) - C_1(\varphi)]\, L_1(\varphi) \equiv \operatorname{diag}\{I_{r_2}, 0\}.$$

Hence,

$$L_1^{-1}(\varphi)[C_3(\varphi)-C_2(\varphi)]L_1(\varphi) \equiv \operatorname{diag}\{0, P_2(\varphi)\},$$

where $P_2(\varphi)$ is the $(n-r_2)\times(n-r_2)$ projection matrix.

By repeating an analogous operation $k-2$ times, we prove that there exists a nondegenerate matrix $L(\varphi)\in C'(\mathcal{T}_m, a)$ of the form

$$L(\varphi) = T(\varphi)\operatorname{diag}\{I_{r_1}, T_1(\varphi)\}\ldots\operatorname{diag}\{I_{r_{k-1}}, T_{k-1}(\varphi)\}.$$

By using this matrix, we can transform identity (12) into the following identity:

$$I_n \equiv L^{-1}(\varphi)C_1(\varphi)L(\varphi) + L^{-1}(\varphi)[C_2(\varphi)-C_1(\varphi)]L(\varphi) + \ldots + L^{-1}(\varphi)[I_n - C_k(\varphi)]L(\varphi)$$

$$\equiv \operatorname{diag}\{I_{r_1}, 0, \ldots, 0\} + \operatorname{diag}\{0, I_{r_2-r_1}, \ldots, 0\} + \ldots + \operatorname{diag}\{0, 0, \ldots, I_{n-r_k}\}. \tag{13}$$

This result, according to [1], is sufficient for the validity of relation (5). Theorem 1 is proved.

If we omit condition (4) in Theorem 1, then the separation of variables can become impossible. Nevertheless, this condition can be omitted if we impose additional restrictions on the matrices $S_i(\varphi)$, $i=1,\ldots,k$. Let us formulate the corresponding statement as a theorem.

Theorem 2. *Suppose that, for some* $\lambda_i(\varphi)\in C^0(\mathcal{T}_m)$, $i=1,\ldots,k$, *there exist nondegenerate symmetric matrices* $S_i(\varphi)\in C'(\mathcal{T}_m, a)$, $i=1,\ldots,k$, *that satisfy inequality (5) and have the form*

$$S_1(\varphi) = \operatorname{diag}\left\{\check{S}_1, -\hat{S}_1(\varphi)\right\},$$

$$S_i(\varphi) = \operatorname{diag}\left\{\check{S}_i, -\hat{S}_i(\varphi)\right\}, \quad i=2,\ldots,k,$$

where $\check{S}_i(\varphi)$ *are* $r_1\times r_1$ *matrices,* $\hat{S}_1(\varphi)$ *and* $\check{S}_i(\varphi)$, $i=1,\ldots,k$, *are positive definite, and* $\hat{S}_i(\varphi)$, $i=2,\ldots,k$, *have* r_i-r_1 *positive eigenvalues and* $n-r_i$ *negative ones. Then for the existence of a nondegenerate matrix* $L(\varphi)\in C'(\mathcal{T}_m, a)$ *satisfying (5), it is necessary and sufficient that the quadratic forms* $(\hat{S}_i(\varphi)z, z)$, $i=2,\ldots,k$, $z\in R^{n-r_1}$, *be reducible to an algebraic sum of squares, i.e., that there exist* $(n-r_1)\times(n-r_1)$ *matrices* $\overline{Q}_i(\varphi)\in C'(\mathcal{T}_m, a)$, $i=2,\ldots,k$, *such that*

$$\overline{Q}_i^*(\varphi)\hat{S}_1(\varphi)\overline{Q}_i(\varphi) = \operatorname{diag}\{I_{r_i-r_1}, -I_{n-r_i}\}.$$

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