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# COLLECTIVE DYNAMICS AND BIFURCATIONS IN SYMMETRIC NETWORKS OF PHASE OSCILLATORS. I

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The present paper is a brief survey of the history and development of the famous Kuramoto model of coupled phase oscillators. We consider several systems generalizing the classical Kuramoto model and given on symmetric oscillatory networks with different functions of interaction between the elements. We describe the collective dynamics and bifurcations of transitions between different modes of interacting elements, namely: partial and complete synchronization, global antiphase mode, slow switching, and chimera states. We show the relationship between the symmetries of networks and the existence of invariant manifolds of the system, cluster states, and more complicated collective behaviors. In part II, we plan to consider several models with nonglobal symmetric coupling.

## 1. Introduction

The first experiments revealing the antiphase synchronization of two clock pendula were carried out by the famous physicist Ch. Huygens in the 17th century [1, 2]. After this, for more than 200 years, no significant achievements were attained in the fields of accumulation of experimental data and theoretical substantiation of the phenomenon of synchronization. As an exception, we can mention the effects of synchronization of the flares of fireflies and circadian rhythms of the plants discovered by biologists. Rapid intensification of scientific interest in the phenomenon of synchronization occurred at the end of the 19th century and the beginning of the 20th century, which was explained by the appearance of electrodynamics and successes in the investigation of the interaction of neurons. This can be explained by the self-oscillating nature of the collective behavior of interacting objects for which a theoretical comprehension had not been yet available at that time. The subsequent advances in the development of the theory of synchronization and collective dynamics were connected with the appearance of computers, cybernetics, radiophysics, theory of oscillations, theory of bifurcations, achievements in the investigation of human brain, creation of mathematical neural models (first of all, of the Hodgkin–Huxley model), diverse biological observations of the collective dynamics of living organisms, discovery of oscillatory chemical reactions, lasers, devices stimulating the operation of heart and brain, and computerized and artificial neural networks. Among the researchers who made significant contributions to the formation of the theory of synchronization, we can especially mention Rayleigh, van der Pol, Andronov, Witt, and Wiener. In the middle of the last century, the researchers understood that the collective dynamics of interacting objects is quite complicated and, thus, it is necessary to develop mathematical models capable of description of the common features of the phenomenon of synchronization even independently of the nature and complexity of the interacting objects. At the same time, these models should be sufficiently simple and convenient for subsequent analytic investigations. The serious weighty attempt in this direction was made by A. Winfree [4, 5] in the middle of the 1960s. He proposed a model of coupled oscillators. As another significant achievement in the theory of synchronization, we can mention the model of coupled phase oscillators proposed by Y. Kuramoto from Japan in 1975 [6]. The *Kuramoto model* (KM) became widely recognized by the researchers working in various fields of science and extremely popular after the appearance of the book

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[7] in 1984. The KM became so popular due to its simplicity, convenience for investigations, and the possibility of description of structurally very different modes of collective dynamics, such as the complete synchronization, partial synchronization, antiphase synchronization, waves of rotation, slow switching, chimera states, etc. Indeed, in the classical KM, every oscillator is described by a single equation, which reflects the way of its interaction with the other oscillators. Another important aspect of the model is its variability, namely, the possibility of clear description of various interacting networks, evident complication of the types of interaction, e.g., by complicating the coupling function, by generalizing the description of eigenfrequencies of oscillators, or by introducing the adaptation or plasticity (i.e., by introducing additional equations for the description of the dependence of an element on its current phase).

In my opinion, an important contribution to the development and popularity of the KM was made by S. Strogatz who was a co-author of several theories based on the indicated model (in particular, of the Watanabe–Strogatz theory) and the author of numerous articles and monographs devoted to the problem of synchronization [8]. It is also necessary to mention the book by A. Pikovsky, M. Rosenblum, and J. Kurths [2] who presented the comprehensive history of synchronization, its physical motivation, mathematical description, and prospects of its application in various branches of natural science.

The investigations of KM and its applications led to the appearance of new types of models of collective dynamics and new mathematical theories. In particular, we can mention the KM with delay [9–14], the oscillatory model with central element [15–19], the oscillatory networks with hubs [20–23], the KM with plasticity [24–26], the KM with adaptation [27–29], the difference KM [14, 30], the infinite-dimensional KM [31–35], the chain and ring oscillatory models [36–43], the KM with random distributions of frequencies and noise [31, 35, 44–46], the KM with nonlinear phase shifts [47–49], the KM with feedback [10, 50, 51], the KM with inertia [9, 31, 52–54], and the KM on networks given by specific graphs [55, 57–59]. In addition to the above-mentioned Watanabe–Strogatz theory, it is also worth mentioning the Ott–Antonsen ansatz [60], the Hong–Strogatz theory [61, 62], and numerous studies (which can be regarded as a separate theory) of slow switching and heteroclinic structures in systems of the Kuramoto type [11, 55, 57, 63–65].

An important role in the theory of synchronization is played by the coherent-incoherent states discovered by Y. Kuramoto and D. Battogtokh [39] and later called "chimera states" or simply "chimeras" by D. Abrams and S. Strogatz [36, 66]. In addition to their specific structure, chimera states are of interest due to the fact that they were first described by mathematical equations and only later (in about ten years) observed in numerous physical experiments.

Note that, with the help of KM and similar models, significant results were obtained in the investigation of various scenarios, such as "winnerless competition," "winner-takes-all," and "competition for synchronization." Moreover, within the framework of the Kuramoto model, it is possible to simulate various situations in the theories of games and conflicts.

It is known that the modifications of KM enable one to easily construct systems with *a priori* specified symmetries and to determine the relationship between the symmetries of the system and its invariant sets. Moreover, these modifications make it possible to find and study new types of bifurcations. In addition, the Kuramoto-type models are widely used for the selection and investigation of the phenomenon of phase synchronization in systems with more complicated structures in which one node is described by numerous differential equations. These methods prove to be especially efficient for the analysis of various types of synchronization in neural networks.

At present, there are thousands of articles devoted to various modifications and generalizations of the KM and to the investigation of synchronization and collective dynamics of coupled oscillators and neurons. The extensive surveys of works published in this field can be found, e.g., in [2, 31, 35, 42, 55, 59, 67–74]. In this connection, we also especially mention the monographs [2, 5, 8, 75–79].

Note that the KM (and its generalizations) of a network of one-dimensional phase oscillators often plays the role of transient stage to the investigation of networks formed by more complex many-dimensional elements. In particular, systems of this kind can be obtained as a result of averaging of a more complicated system with preservation of some properties of the collective behavior [80–85]. The standard KM appears as the phase component of the Stuart–Landau coupled complex-valued systems [7]. In addition, the development of KM was stimulated by the results of observations of the behaviors of chemical and biological oscillators. After its appearance, the KM has found extensive applications in the studies of neural networks, superconducting Josephson junctions, and laser arrays, as well as in the simulation of various other complex systems of interacting elements. Despite the fact that multidimensional systems with very complicated individual behaviors are used to model individual elements of networks in the major part of applied cases, their collective behavior can be simulated by analyzing solely their phase (in a certain sense) synchronization. As one of the methods that can be used to study the synchronization of data, we can mention the separation of the phases of signals performed by using the Hilbert transformation [86]. This opens a possibility of their subsequent investigation in the form of oscillatory systems.

In the present survey, we briefly consider the models of interacting oscillators based on the KM. In addition, by using mainly our own results obtained earlier, we show the influence of various types of couplings on the formation of different forms of collective dynamics in the systems and on the bifurcation transitions between different behaviors. It is also demonstrated how the symmetry of interaction between the elements and coupling functions in the models affect the formation of various types of synchronization.

In the first part of the survey, we consider the most popular models of identical oscillators with global coupling, namely, the standard KM, the Kuramoto–Sakaguchi model (with one-harmonic coupling and phase shift), the Pikovsky–Rosenblum model (with nonlinear phase shift that depends on the order parameter), the Hansel–Mato–Meunier model (with two-harmonic coupling), and the generalized Daido/Hong–Strogatz model (with attractive and repulsive elements and phase shift).

In the second part of the present survey, we consider models of identical oscillators with symmetric interaction graphs, namely, the model with central element, the model with circulant coupling, the model of indistinguishable oscillators, and modular oscillator networks. In the second part, our main aim is to reveal the presence of collective dynamics with certain specific features (the interaction between the elements is specially constructed to guarantee the presence of certain particular types of solutions).

## 2. Kuramoto-Type Models of Phase Oscillators

***2.1. Description of the System.*** Consider a model of $N$ coupled phase oscillators of the Kuramoto type:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\Gamma_{ij}(\theta_i - \theta_j), \quad i = 1, \ldots, N, \tag{1}$$

where $\theta_i \in [0, 2\pi) = \mathbb{T}^1$ are phase variables, $\omega_i$ are the eigenfrequencies of oscillators, $K_{ij}$ are the parameters (strengths) of coupling between the oscillators, and $\Gamma_{ij}(x)$ are smooth $2\pi$-periodic coupling functions. We emphasize that each variable $\theta_i$ runs over a one-dimensional circle. Hence, the phase space of the system is a torus $\mathbb{T}^N$. For the sake of clarity, we can imagine $N$ points moving in a certain direction along the circle with their own angular velocities $\omega_i$ in the case where the system has no coupling (all $K_{ij} = 0$) but exhibit a common collective dynamics in the presence of couplings. The presence of couplings in the system and the dependence of individual phase points on the other points may result in a quite complicated general behavior. In particular, in the presence of mutual (collective) influence, the indicated phase points may not only move in a single direction but also oscillate in various ways, gather in groups (clusters) and move together for a certain time or permanently, move in a certain group for a relatively long time and then pass, for a short time, into a different group, chaotically oscillate, etc. The parameter $K_{ij}$ (which can also be variable: $K_{ij}(t)$) reflects the intensity of influence of the $j$th oscillator upon the $i$th oscillator. The equality of some parameters $K_{ij}$ to zero means that there is no coupling between the corresponding oscillators. This system is quite general. As a rule, it is customary to use a single coupling function

$\Gamma_{ij}(x) = g(x)$. The standard KM (starting point of investigations) describes a system of globally coupled oscillators with one coupling function $\Gamma_{ij}(x) = -\sin x$ and identical coupling strengths $K_{ij} = K$ for all indices. It is natural to start the analysis of the Kuramoto system from the case of *identical phase oscillators*, i.e., from the case where $\omega_i = \omega$, $i = 1, \ldots, N$, because this system is maximally symmetric. The case of identical oscillators is most extensively studied for all types of interaction because it enables one to obtain significant analytic results, unlike the case of nonidentical oscillators. Quite unexpected results are obtained (as a rule, first, with the help of computer simulations) for various conditions that look very asymmetric under symmetric conditions of interaction (such as switching between the clusters or chimera states). By varying the elements of the matrix of interaction between oscillators $K_{ij}$, the coupling functions $\Gamma_{ij}(x)$, and the eigenfrequencies $\omega_i$, it is possible to create the situation in which system (1) has a certain symmetry specified in advance. Quite often the symmetry properties of these systems are well hidden and can be revealed only as a result of detailed investigations. It turns out that the symmetries detected in the system frequently lead to the appearance of invariant manifolds corresponding, in turn, to various clusters (to a partial synchronization of several oscillators). In addition, the presence of a large number of symmetries may lead to the existence of manifolds completely filled either with fixed points or with a family of periodic solutions, which disappear for any (even insignificant) symmetry breaking. We also note that some bifurcations are possible in a system only in the presence of certain symmetries, including, in particular, local transcritical and pitchfork bifurcations (instead of more typical saddle-node bifurcations in asymmetric situations), and various heteroclinic bifurcations.

In the present work, we briefly describe several cases in which the presence of symmetries in system (1) leads to various types of synchronization and certain types of bifurcations.

***2.2. System in Phase Differences.*** First, we note that the right-hand sides of the general system (1) depend on the phase differences $\theta_i - \theta_j$, which reveals the presence of symmetry of the phase shift $\mathbb{T}^1$ along a circle in the system specified by the action $\theta_i \mapsto \theta_i + \varepsilon$. This means that, in the study of various types of collective dynamics, it is possible to fix one oscillator and then consider a system with phase variables whose dimension is smaller by 1. In particular, we fix the first oscillator and introduce the phase differences as follows:

$$\varphi_i = \theta_1 - \theta_{i+1}, \quad i = 1, \ldots, N-1. \tag{2}$$

In this way, we pass from system (1) to the following auxiliary system in phase differences:

$$\frac{d\varphi_i}{dt} = \Delta_i + \frac{1}{N} \sum_{j=1}^{N} \big( K_{1,j}\Gamma_{1,j}(\varphi_{j-1}) - K_{i+1,j}\Gamma_{i+1,j}(\varphi_{j-1} - \varphi_{i-1}) \big), \quad i = 1, \ldots, N-1, \tag{3}$$

where $\Delta_i = \omega_1 - \omega_{i+1}$, $i = 1, \ldots, N-1$, and $\varphi_0 = 0$. The last system "absorbs" the indicated symmetry. The entire one-dimensional invariant manifold $\theta_i = \theta$, $i = 1, \ldots, N$, of synchronization in system (1) shrinks into a single point $\varphi_i = 0$, $i = 1, \ldots, N-1$, for the new system. The drawbacks of the new system are the absence (or latency) of one symmetry and a more complicated right-hand side. As its advantages, we can mention the fact that the certain specific sets decrease their dimension by 1 [not in all cases (depending on the situation)]. This observation has common features with the investigation of Poincaré sections in the description of the solutions of a dynamical system. Thus, in particular, instead of the analysis of periodic solutions or quasiperiodic trajectories, we can often restrict ourselves to the investigation of equilibria or periodic orbits, respectively. In these cases, the analyses of stability and bifurcations become much simpler. For the sake of convenience of our investigations, in order to reduce system (1), it is possible to use, parallel with the phase differences (2), any collection of $N - 1$ linearly independent phase differences $\varphi_{ij} = \varphi_i - \varphi_j$. This can be done because the dynamics of different reduced systems are topologically equivalent.

In the description of the dynamics of system (1), it is sometimes convenient to pass to the new variables (2) and preserve the $N$th equation in the system. In particular, a nondegenerate substitution

$$\overline{\Phi}^T = \begin{pmatrix} \theta_1 \\ \Phi^T \end{pmatrix} = S_N \Theta^T, \qquad S_N = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & -1 & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 1 & \vdots & \ddots & -1 & 0 \\ 1 & 0 & \dots & 0 & -1 \end{pmatrix},$$

reduces system (1) written in the vector form, i.e.,

$$\frac{d\Theta}{dt} = F(\Theta)$$

to the system

$$\frac{d\overline{\Phi}}{dt} = \overline{G}(\overline{\Phi}),$$

which is an extension of the system

$$\frac{d\Phi}{dt} = G(\Phi), \quad \overline{\Phi} = (\theta_1, \Phi),$$

obtained with the help of the equation

$$\frac{d\theta_1}{dt} = F_1(\theta_1, \dots, \theta_N) = \overline{F}_1(\varphi_1, \dots, \varphi_N).$$

The relationship between the eigenvalues $\lambda$ of the linearized system of *identical oscillators* (1) and the linearization of its reduction (3) can be represented in the form [37]:

$$\det(A - \lambda I_N) = -\lambda \det(B - \lambda I_{N-1}),$$

where

$$A = A(\Theta_0) = \left.\frac{\partial F(\Theta)}{\partial \Theta}\right|_{\Theta=\Theta_0}, \qquad B = B(\Phi_0) = \left.\frac{\partial G(\Phi)}{\partial \Phi}\right|_{\Phi=\Phi_0},$$

and $I_N$ is the identity square matrix of dimension $N$.

***2.3. Notation and Definitions.*** One of the concepts required for the comprehension and evaluation of the degree of synchronization in systems of coupled phase oscillators is the concept of Kuramoto *order parameter*. A complex mean field

$$Z(t) = R(t)e^{\imath\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{\imath\theta_j(t)},$$

where $\imath = \sqrt{-1}$, is called the complex order parameter and its amplitude $R(t)$ is called the order parameter (in different works, the term “order parameter” is used either for $Z$ or for $R$). We denote

$$\Theta = (\theta_1, \dots, \theta_N), \quad \overline{\Theta}_{ij} = \lim_{t\to\infty} \left|\theta_i(t) - \theta_j(t)\right|, \quad \Omega_{ij} = \lim_{t\to\infty} \frac{1}{t} \left|\theta_i(t) - \theta_j(t)\right|.$$

We now present the definitions (sometimes with short explanations) of different (most important) types of synchronization in system (1).

Two oscillators $\theta_i$ and $\theta_j$ are *phase synchronized*, if $\overline{\Theta}_{ij} = 0$.

Two oscillators $\theta_i$ and $\theta_j$ are *frequency synchronized*, if $\Omega_{ij} = 0$.

Two oscillators $\theta_i$ and $\theta_j$ are *desynchronized* (in any sense), if $\Omega_{ij} \neq 0$.

The system exhibits *full synchronization* $\Theta_{\text{sync}}$ if all its oscillators are synchronized, i.e.,

$$\Theta_{\text{sync}} = (\theta, \dots, \theta). \tag{4}$$

It is easy to see that full synchronization is equivalent to the equality

$$R = R(\Theta_{\text{sync}}) = 1.$$

The simultaneous synchronization of $m$ oscillators ($m \geq 2$) in the system is called either an *m-cluster* or simply a *cluster*.

A *k-cluster state* is defined as a partition of the set of oscillators $(\theta_1, \dots, \theta_N)$ for which the entire set of their indices can be split into subsets in the following way:

$$\mathcal{A} = \{1, \dots, N\} = \bigcup_{j=1}^{k} \mathcal{A}_j, \quad \mathcal{A}_j \cap \mathcal{A}_i = \varnothing \quad \text{for} \quad j \neq i,$$

where any oscillators $\theta_i$ and $\theta_m$ are identical if their indices belong to the same subset $\mathcal{A}_j$ and are not identical if their indices belong to different subsets. The size of each cluster $a_j$ is defined as the number of indices in the corresponding set

$$a_j = |\mathcal{A}_j|, \qquad \sum_{j=1}^{k} a_j = N.$$

Two oscillators are called *phase-locked* if

$$\left|\theta_i(t) - \theta_j(t)\right| \leq C < 2\pi.$$

The oscillators $\theta_i$ and $\theta_j$ are in the *antiphase* if $\Theta_{ij} = \pi$.

The system possesses a splay state $\Theta_{\text{splay}}$ (or *rotation waves*) if $\Theta_{i,i+1} = m \cdot 2\pi/N$, $i = 1, \dots, N$ (the index $i$ is taken modulo $N$), or (in a different notation)

$$\Theta_{\text{splay}} = \left(\theta, \theta + m\frac{2\pi}{N}, \theta + m\frac{2\cdot 2\pi}{N}, \dots, \theta + m\frac{(N-1)2\pi}{N}\right), \quad m = 1, \dots, N-1. \tag{5}$$

The order parameter

$$R(\Theta_{\text{splay}}) = 0.$$

The system has the state of *global antiphase* if $R = 0$ (the previous state is a partial case of this state). By the definition of the order parameter the state of full antiphase is specified by the $(N-2)$-dimensional set $\mathcal{M}^{(N)} \in \mathbb{T}^N$:

$$\mathcal{M}^{(N)} = \left\{ (\theta_1, \ldots, \theta_N) \colon \sum_{j=1}^{N} e^{\imath \theta_j} = 0 \right\}. \tag{6}$$

We now assume that the original system (1) has the state of *slow (heteroclinic) switching* between the clusters when the corresponding system (3) in phase differences has a heteroclinic cycle whose saddle points correspond to the cluster states. As a variation of the state of slow switching, we can consider the state corresponding to the limit cycle of system (3), which is close to the indicated heteroclinic cycle and to the cycle formed when this cycle disappears as a result of bifurcation. Here, we do not give more accurate mathematical definitions in order not to make this concept narrower because it is used by numerous researchers in somewhat different contexts for the description of some physical or neural phenomena (see, e.g., [63, 64, 85, 87–90]). The idea of this definition is based on the fact that the phase trajectory stays for a very long time in the neighborhood of a saddle (saddle-node), which corresponds to the clusterization of a certain number of oscillators (but not all). Then the analyzed trajectory "passes" to a different saddle for a very short time, which also corresponds to a cluster but with a different collection of oscillators. The concept of slow switching can be even more general in view of the fact that the nodes of heteroclinic cycles can be not only saddles but also saddle limit cycles or saddle chaotic orbits.

The presented list of synchronization conditions does not contain the so-called "chimera state" or the coherent-incoherent state, which is possibly one of the most interesting and well-studied states of collective interaction in systems of coupled elements. The chimera states cannot appear in the systems of globally coupled phase oscillators. Therefore, these states are not considered in the first part of the present survey. Actually, they are discussed in Part II dealing with the investigation of symmetric nonglobal interacting networks. Furthermore, the history of appearance of chimeras, their definition, and networks in which they may exist require additional and more detailed discussions.

Note that the definitions of all states described above do not require their stability or the stability of the corresponding trajectory of the system corresponding to specific conditions. The problem of stability of each of these states can be the object of separate investigations. However, unless otherwise specified, we assume that the state exists, if it is asymptotically stable. Moreover, some other states related to the elements of synchronization in the system may also exist (e.g., chaotic synchronization). However, their definitions are quite complicated and ambiguous.

We now make several remarks concerning some definitions. The phase synchronization is a weaker concept, than the frequency one. The oscillators in the antiphase are frequency synchronized. The system may have several clusters (groups of synchronized oscillators) that can be either phase-locked between themselves or not phase-locked (rotating over the circle so that their difference passes through a barrier equal to $2\pi$). The order parameter $R$ can be either a constant (e.g., in the state of full synchronization or in the full antiphase state) or a function of the time $R = R(t)$ (as for any state with at least two desynchronized oscillators). In a certain sense, complex collective dynamical states can be regarded as superpositions of simpler collective behaviors, as, e.g., chaotic synchronization [91, 92], heteroclinic chimeras [93–95], or chaotic chimeras [96, 97]. All definitions introduced for phase oscillators have their natural analogs for more complicated coupled dynamical systems (e.g., for the models of neural networks).

## 3. Systems of Globally Coupled Identical Oscillators

The mathematical definition of the networks of elements (oscillators, neurons, etc.) dynamically interacting with each other always includes the following *three main components*:

1. The description of the dynamics of an *individual node* (not coupled with the other node) with the help of equations or systems of equations.

2. The description of the *architecture of interaction between separate elements of the network*, which is most often realized with the help of a graph of couplings.

3. The description of the *influence of one element upon another element in each link of the connection* realized either with the help of additional terms on the right-hand sides in the system of equations characterizing an individual element or with the help of additional equations or systems of equations.

A complex system of interacting elements is always characterized by important specific features of the collective dynamics if one of components 1–3 described above possesses certain symmetry properties. In particular, as a characteristic feature of symmetric networks, we can mention the presence of various clusters and invariant sets of the dynamical models. The presence of symmetry properties often significantly simplifies the analytic investigation of the corresponding systems and opens the possibilities of their various reductions. It is clear that the best possibilities of reduction are observed for the systems in which symmetry properties are observed for each of the indicated three components. In the present work, we consider systems with symmetries in the description of the individual nodes and couplings between the nodes, as well as the cases of complete or partial symmetry of the interacting network. In what follows, it is shown how the presence of these symmetries in the oscillatory systems leads to the existence of invariant regions or closed sets containing solutions of a certain type. The presence of symmetries in a system is frequently a necessary condition for the existence of heteroclinic cycles or multiparameter families of periodic or quasiperiodic solutions.

Consider the most common case of a model of *globally coupled oscillators with the same coupling function* $\Gamma_{ij}(x) = g(x)$

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} g(\theta_i - \theta_j), \quad i = 1, \ldots, N. \tag{7}$$

We say that the system describes *identical oscillators* if all eigenfrequencies are identical:

$$\omega_i = \omega, \quad i = 1, \ldots, N. \tag{8}$$

Condition (8) means that all nodes of the system are absolutely identical (component 1 in the description of interaction presented above). Since the of oscillatory nodes are identical, system (7) is maximally symmetric because it has the global graph of identical couplings for all pairs of elements and the same influence of any $i$th element upon the $j$th element specified by the expression $Kg(\theta_i - \theta_j)/N$. In what follows, we break only the symmetry of the interaction graph specified by the matrix of coupling strengths

$$K = (K_{ij})_{i,j=1}^{N}.$$

Moreover, as shown in Subsection 3.5, the symmetry of the coupling function $g(x)$ noticeably affects the dynamics of the system and can turn it either into a gradient dynamics or into a Hamilton-like dynamics.

The system of identical oscillators can be immediately simplified. By the change of variables $\theta_i \to \theta_i - \omega t$, we can remove the parameter $\omega$ from the system. Then, with the help of time scaling, we can make the coupling parameter $K$ equal to a constant (without loss of generality). In particular, we set $K = N$. Then system (7) takes the form

$$\frac{d\theta_i}{dt} = \sum_{j=1}^{N} g(\theta_i - \theta_j), \quad i = 1, \ldots, N. \tag{9}$$

System (7) corresponds to a reduced system with variables (2) of the form

$$\frac{d\varphi_i}{dt} = \Delta_i + g(0) - g(-\varphi_i) + \sum_{j=1}^{N-1} (g(\varphi_j) - g(\varphi_j - \varphi_i)), \quad i = 1, \ldots, N = 1. \tag{10}$$

Moreover, the condition of identity of oscillators (2) corresponds to the condition

$$\Delta_i = 0, \quad i = 1, \ldots, N - 1. \tag{11}$$

System (7), (8) and, hence, (9) possess the permutation symmetry $\mathbf{S}_N$. This leads to the invariance of any clusters $\theta_i = \theta_j$, $i \neq j$, because

$$\sum_{j=1}^{N} g(\theta_j - \theta_j) = 0.$$

Any cluster may contain from two to $N$ oscillators. A cluster does not break if the initial conditions are specified for this cluster. The $N$-cluster corresponds to the complete synchronization of the system. The clusters can be either attractors, or repellers, or may have the saddle structure of stability. For the system in phase differences, the indicated clusters correspond to the invariant straight lines or hyperplanes. The invariant hyperplanes of dimension $N-2$ split the entire phase space $\mathbb{T}^{N-1}$ of system (10), (11) into $(N-1)!$ invariant symmetric regions containing phase-locked trajectories (a trajectory cannot leave the region of its origin). One of the invariant regions of system (10) is described by the formula

$$\mathcal{C} = \{(\varphi_1, \ldots, \varphi_{N-1} : 0 < \varphi_1 < \varphi_2 < \cdots < \varphi_{N-1} < 2\pi\}. \tag{12}$$

The other regions can be obtained by the permutations of indices [80, 98]. The violation of the identity of eigenfrequencies (e.g., due to perturbations) leads to the loss of symmetry of the system and destruction of the indicated invariant regions. In the last case, one may observe the appearance of phase-unlocked trajectories (e.g., the extremely high sensitivity of trajectories to the indicated perturbations was studied in [99, 100]). In addition to the invariant manifolds corresponding to clusters, the presence of symmetry $\mathbf{S}_N$ may serve as a cause of slow switching, which corresponds to heteroclinic trajectories in the phase space $\mathbb{T}^{N-1}$ of system (3). These heteroclinic trajectories may have different intrinsic symmetries that are subsets of the symmetry $\mathbf{S}_N$. Thus, one may observe the existence of $\mathbf{Z}_N$-symmetric heteroclinic cycles or heteroclinic cycles based on the saddles with $\mathbf{S}_{N/2} \times \mathbf{S}_{N/2}$ isotropy. The existence of these cycles depends on the coupling function of the system and (frequently) on the parity of the number of oscillators [55, 80, 89, 99, 101].

For a coupling function

$$g(x) = -\sin(x - \alpha),$$

where $\alpha$ is a parameter, system (7) is called the *Kuramoto–Sakaguchi model* [56]. This model is studied by using the Watanabe–Strogatz theory [49, 83], which gives a fairly good description of the dynamics of the system inside the invariant regions by reducing it to the three-dimensional dynamics. The theory presented in [102] makes it also possible to describe the cluster solutions on the boundaries of the invariant regions. The Watanabe–Strogatz theory admits a generalization and can be applied to a more complicated system with nonlinear phase shift $\alpha = \alpha(\mathrm{R}, \beta)$ in the coupling function. In the presence of the second harmonic in the function $g(x)$, the dynamics of the system can be much more complicated. Thus, in particular, in [94, 103], it was shown that the system of four identical oscillators possesses *chaotic trajectories* in the case where $g(x)$ has the third and fourth harmonics. This is an example of chaos with maximum possible number of symmetries. The dynamics, stability, and bifurcations of system (7) were analyzed in detail in [99, 101, 104] (and many other works) for the case where

$$g(x) = -\sin(x - \alpha) + r \sin(2x - \beta),$$

where $\alpha$, $\beta$, and $r$ are parameters, i.e., for the *Hansel–Mato–Meunier model* [64, 105]. Note that various slow switching conditions are typical of models of this kind. The general properties of solutions of system (7) are well described in the case of an arbitrary number of harmonics of the coupling function $g(x)$ but under the condition that this function is either even or odd. In these cases, the system can be either gradient or conservative. Note that breaking of the global symmetry of couplings in the system leads to the destruction of some cluster states and the corresponding invariant sets of the system. In this case, the conservation of partial symmetries of the permutations of elements in the system also preserves its cluster states (this is shown in Subsection 3.6). We also note that the symmetrically coupled systems of even dimensions have a more diverse set of collective behaviors and a larger number of various bifurcations as compared with the systems of odd dimensions.

***3.1. Standard Kuramoto Model.*** It is natural first to consider the classical model with a very simple coupling proposed by Kuramoto in [6, 7]. In what follows, it is called the *standard KM*. In other words, we consider a system of globally coupled phase oscillators (7) with a positive (attractive) coupling strength $K > 0$ and a coupling function

$$g(x) = -\sin x. \tag{13}$$

Both in the analyzed case of a system with very simple function (13) and in the other cases with more complicated functions and more nontrivial interaction graphs, to describe specific states of system (7) [or (9)] with variables $\theta_i$, we find the corresponding solutions of system (3) in phase differences with variables $\varphi_i$. System (9) has the following states:

*Stable full synchronization* $\Theta_{\text{sync}}$, which corresponds to a stable fixed point $\Phi_{\text{sync}} = (0, \ldots, 0)$ of system (10).

*The state of full antiphase* $\mathcal{M}^{(N)}$ given by relation (6). The $(N-2)$-dimensional invariant manifold $\mathcal{M}^{(N)}$ corresponds to the $(N-3)$-dimensional invariant manifold

$$\overline{\mathcal{M}}^{(N)} = \left\{ (\varphi_1, \ldots, \varphi_{N-1}) \colon \sum_{j=1}^{N-1} e^{\imath \varphi_j} = -1 \right\},$$

which is completely formed by fixed points of system (10). Every point of the invariant manifold $\overline{\mathcal{M}}^{(N)}$ is stable inside this manifold. Moreover, it is repulsive in two directions transverse to $\overline{\mathcal{M}}^{(N)}$. In other words, the analyzed manifold is a repeller of the system.

*The saddle points $S_{ij}$ of system (10)* in phase differences have $i$ coordinates equal to 0 and $j$ coordinates equal to $\pi$, $i \neq 0$. Clearly, the system has $2^N - 1$ saddle points of this kind. In the case of even $N$, the saddles $S_{ij}$ belong to $\overline{\mathcal{M}}^{(N)}$ and are degenerate for $i = N/2 - 1$ and $j = N/2$. The unstable manifolds $W^u(S_{ij})$ approach $\Phi_{\text{sync}}$, whereas the stable manifolds $W^s(S_{ij})$ approach $\overline{\mathcal{M}}^{(N)}$ in the inverse time.

The system in phase differences (10) has no other special sets (e.g., limit cycles). All its nonspecial trajectories originate on $\overline{\mathcal{M}}^{(N)}$ and terminate in $\Phi_{\text{sync}}$. Note that, for $K < 0$, the dynamics of the system is described in the same way for the inverse time. In this case, the only attractor is the set of asynchronous states and the state of full synchronization is the sole repeller. The case $K = 0$ corresponds to a bifurcation.

Although the present work is devoted to the description of systems with symmetric couplings, we now present brief information about globally coupled *nonidentical* oscillators. In other words, we assume that equality (8) is not true and $\omega_i = \omega + \delta_i$, $i = 1, \dots, N$, where $\delta_i$ are perturbations of the eigenfrequencies from the state of their total identity. For the reduced system (3), we denote the differences of eigenfrequencies by $\Delta_i = \omega_1 - \omega_{i+1}$, $i = 1, \dots, N-1$. For the reduced system, it is clear that the case of identical frequencies for (1) corresponds to $\Delta_i = 0$, $i = 1, \dots, N-1$. First, we fix the parameter $K > 0$ and vary the eigenfrequencies of the original system [at least, some of them (with $\delta_i \neq 0$)], which corresponds to the deviations of $\Delta_i$ from zero for system (10). What happens with the above-mentioned specific states of the system of identical oscillators? The invariant $(N-3)$-dimensional manifold $\overline{\mathcal{M}}^{(N)}$ "disintegrates" for the weakest deviation of any parameter $\Delta_i$ from zero. Almost all points of this manifold disappear and only $N-1$ isolated fixed points are preserved. These points are close to the splay state points

$$\Phi_{\text{splay}} = \left(\frac{2m\pi}{N}, \frac{4\pi}{N}, \dots, \frac{2(N-1)m\pi}{N}\right), \quad m = 1, \dots, N-1, \tag{14}$$

where the indices are taken modulo $N$ for system (3), (11). In other words, the case $\Delta_i = 0$, $i = 1, \dots, N-1$, corresponds to a bifurcation and this bifurcation is degenerate. It is clear that the other $2^N$ hyperbolic points $\Phi_{\text{sync}}$ and $S_{ij}$ do not change their stability under small perturbations of $\Delta_i$.

Hence, for the variations of the frequency differences, we observe the motion of the above-mentioned $2^N + N - 1$ points in the phase space. For subsequent variations of the parameters, the fixed points approach each other, bifurcate, and disappear. Most often, we encounter saddle-node bifurcations of two points or, in the presence of additional symmetry, pitchfork bifurcations. The general pattern of bifurcations can be quite complicated because we do not fix specific distributions of frequencies or various their variations. The sequence of bifurcations (possibly, not only local) leads to a sequence of the disappearances of fixed points. The point that disappears in the last turn is a stable point whose motion started at the origin for $\Delta_i = 0$. The $(N-1)$-dimensional space of parameters $\Delta_i$ contains an $(N-2)$-dimensional sphere-like surface corresponding to a bifurcation of disappearance of a stable point.

Note that the order parameter $R$ of a stable point uniformly decreases from 1 in the process of its motion along an arbitrary straight line in the parametric space from the origin. If this point (frequency synchronization) exists, then its order parameter $R$ is constant for any parameters. As soon as the last equilibrium position disappears, the remaining states correspond to clusterizations of various kinds, with the exception of full synchronization. The last bifurcation leads to the formation of at least two clusters with different average frequencies $\Omega_i$. As the frequencies undergo subsequent changes, larger clusters split into smaller groups up to the total decomposition into a system of $N$ individual oscillators with different average frequencies. In the process of motion along a straight line from the origin in the parametric $\Delta_i$-space, we observe a branched tree of average frequencies (like, e.g., in Fig. 1 from [106]).

A more traditional approach to the investigation of bifurcation transitions between cluster states is reduced to the fixation of a specific distribution of frequencies (by using certain natural-science, probabilistic, or symmetry-based arguments) and variations of a single parameter of coupling strength $K$ [7, 35, 107, 108]. In this case, the crucial problem is to determine the bifurcation value of $K_c = K_c(\omega_1, \dots, \omega_N)$ (depending on the distribution) such that, for $K = K_c$, the frequency synchronization of all oscillators disappears, and we observe the onset of clusterization (frequency-splitting bifurcation). It is quite popular to use either the representation of the bifurcation diagram in the $(K, R)$-parametric plane (see, e.g., Fig. 3 in [35]) or the $(K, \overline{\omega}_i)$ bifurcation diagram, where $\overline{\omega}_i$ are the average frequencies of oscillators attained under the action of the vector field of the system (see, e.g., Fig. 1 from [106]).

For $K \geq K_c$, there exists the state of full frequency synchronization with average frequency $\Omega = \overline{\omega}_i$, $i = 1, \dots, N$, whose order parameter $R$ increases with $K$. For $K < K_c$, there exist only clusters (oscillators whose average frequencies coincide). Their number increases as the value of $K$ decreases (as a result of frequency-splitting bifurcations). For $K = 0$, all oscillators have individual behaviors and their average frequencies coincide with the eigenfrequencies: $\overline{\omega}_i = \omega_i$, $i = 1, \dots, N$. The presence of symmetries in the distribution of eigenfrequencies (e.g., the frequencies equidistant from the average value or uniformly distributed over the segment) also leads to the existence of quasiperiodic invariant manifolds and phase chaos [109, 110]. Note that an important role in comprehension of the collective behavior of oscillators in clusters and different clusters is played by the rotation numbers (winding ratios) [106, 111, 112].

***3.2. Kuramoto–Sakaguchi Model.*** A system of globally coupled phase oscillators (7) with coupling function

$$g(x) = -\sin(x - \alpha), \tag{15}$$

where $\alpha \in \mathbb{T}$ is a parameter of phase shift, was introduced and studied in detail by Sakaguchi and Kuramoto [56]. Model (7), (15) attracts attention of numerous researchers because it has a broad spectrum of applications in various branches of science. The investigation of the phenomena of the collective dynamics can be used for the description of the interaction of simple particles, arrays of Josephson junctions, electrochemical oscillators, arrays of lasers, neural networks, and living organisms. The systems of coupled oscillators with phase shifts on nonglobal (especially symmetric) networks are very useful for numerous applications and, therefore, quite popular. Significant attention is given to models of ring networks, circulant networks, and networks of indistinguishable elements because they serve as sources of appearance of very interesting conditions of collective interaction, such as chimera states and conservative-dissipative space conditions [36, 37, 39, 93, 113].

In [82, 83], Watanabe and Strogatz showed how to describe the dynamics of superconducting arrays of Josephson junctions with the help of the Kuramoto–Sakaguchi model and developed a theory (Watanabe–Strogatz theory), which enables one to significantly reduce the analyzed system for the purposes of subsequent investigation. The main result of this theory is the possibility of reduction of the $N$-dimensional system of globally coupled identical oscillators (9), (15) to the following three-dimensional system:

$$\begin{aligned}
\frac{d\widetilde{\rho}}{dt} &= K\sin\alpha \frac{(1-\widetilde{\rho}^2)^{3/2}}{N}\sum_{i=1}^{N}\frac{\sin(\psi_i - \widetilde{\Psi})}{1-\widetilde{\rho}\cos(\psi_i-\widetilde{\Psi})} + K\cos\alpha\frac{1-\widetilde{\rho}^2}{N}\sum_{i=1}^{N}\frac{\widetilde{\rho}-\cos(\psi_i-\widetilde{\Psi})}{1-\widetilde{\rho}\cos(\psi_i-\widetilde{\Psi})},\\
\widetilde{\rho}\frac{d\widetilde{\Psi}}{dt} &= K\sin\alpha \frac{(1-\widetilde{\rho}^2)^{1/2}}{N}\sum_{i=1}^{N}\frac{\widetilde{\rho}-\cos(\psi_i - \widetilde{\Psi})}{1-\widetilde{\rho}\cos(\psi_i-\widetilde{\Psi})} - K\cos\alpha\frac{1-\widetilde{\rho}^2}{N}\sum_{i=1}^{N}\frac{\sin(\psi_i-\widetilde{\Psi})}{1-\widetilde{\rho}\cos(\psi_i-\widetilde{\Psi})},\\
\widetilde{\rho}\frac{d\widetilde{\Phi}}{dt} &= K\sin\alpha \frac{1}{N}\sum_{i=1}^{N}\frac{\widetilde{\rho}-\cos(\psi_i - \widetilde{\Psi})}{1-\widetilde{\rho}\cos(\psi_i-\widetilde{\Psi})} - K\cos\alpha\frac{(1-\widetilde{\rho}^2)^{1/2}}{N}\sum_{i=1}^{N}\frac{\sin(\psi_i-\widetilde{\Psi})}{1-\widetilde{\rho}\cos(\psi_i-\widetilde{\Psi})},
\end{aligned}\tag{16}$$

where $\widetilde{\Psi}$ and $\widetilde{\Phi}$ are the "global phases" and $\widetilde{\rho} \in [0, 1]$ is the "amplitude." The indicated reduction is realized by the change of variables

$$\theta_i(t) = \widetilde{\Phi}(t) + 2\arctan\left[\sqrt{\frac{1+\widetilde{\rho}(t)}{1-\widetilde{\rho}(t)}}\tan\left(\frac{1}{2}\psi_i - \widetilde{\Psi}(t)\right)\right], \quad i = 1, \ldots, N, \tag{17}$$

where $\psi_i$ are constants satisfying the conditions

$$\sum_{i=1}^{N}\cos\psi_i = \sum_{i=1}^{N}\sin\psi_i = 0.$$

The theory states that the set of constants $\psi_i$, together with the solutions of system (16), describe the solutions of system (9) with the help of transformation (17). In [82, 83], it was shown how the initial conditions $\widetilde{\rho}(0)$, $\widetilde{\Psi}(0)$, and $\widetilde{\Phi}(0)$ and constants $\psi_i$ of the new system are connected with the initial conditions $\theta_i(0)$ of the original system.

Despite evident beauty and efficiency of the Watanabe–Strogatz theory, it has a certain restriction in applications caused by the singularity of the change of variables and the conditions imposed on $\psi_i$. The indicated restriction is connected with the fact that the theory cannon be applied on a set of measure zero that contains all solutions corresponding to the cluster states $\theta_i = \theta_j$, $i \neq j$. In other words, the Watanabe–Strogatz theory can be efficiently used only inside the canonical invariant regions $\mathcal{C}$. Note that the specific states and their local bifurcations occur just on the cluster solutions. Hence, it became necessary to develop a theory alternative to the Watanabe–Strogatz theory in order to supplement the latter and remove its disadvantages. Thus, in particular, a new method was proposed in [102]. According to this method, it is necessary to pass from the original system to a system in phase differences and then describe all possible equilibria, as well as their stability and bifurcations. Moreover, this method describes not only local bifurcations of the equilibria but also the global bifurcations (the appearance of heteroclinic cycles). Combining both theories, it is possible to show that the special states of the Kuramoto–Sakaguchi system written in phase differences are as follows:

*The origin* $\Phi_{\text{sync}} = (0, \ldots, 0)$ corresponding to the *fully synchronous state* $\Theta_{\text{sync}}$ of the original system, which is stable for $\alpha \in (-\pi/2, \pi/2)$. If $\alpha = \pm\pi/2$, then this point loses its stability as a result of $N$ simultaneous transcritical bifurcations. The order parameter $R(\Phi_{\text{sync}}) = 1$.

The $(N-3)$-dimensional *antiphase invariant manifold* $\mathcal{M}^{(N)}$ *completely formed by fixed points*. Every point of the manifold is neutrally stable in $N-2$ directions inside the manifold for any $\alpha$. In two directions transverse to the manifold, the point is stable for $\alpha \in (\pi/2, 3\pi/2)$. The manifold loses its stability as a result of the Andronov–Hopf degenerate bifurcation for $\alpha = \pm\pi/2$. The order parameter $R(\mathcal{M}^{(N)}) = 0$.

*Two-cluster states* $\Theta_{p,N-p}$. For the system in phase differences, these states belong to one-dimensional invariant manifolds described by the relations

$$\begin{aligned}\mathcal{P}_2 = \big\{(\varphi_1, \ldots, \varphi_N)\colon \varphi_1 = \varphi_2 = \ldots = \varphi_p \\ \neq \varphi_{p+1} = \varphi_{p+2} = \ldots = \varphi_{N-1} = 0,\ p = 1, \ldots, N-1\big\}.\end{aligned} \tag{18}$$

The two-cluster states are saddle points on the invariant manifolds for any parameters $\alpha$ and have the following isotropy: $\mathbf{S}_p \times \mathbf{S}_{N-p}$, $p \neq N/2$. These saddle points have the following coordinates on the indicated manifolds:

$$\varphi_j = \begin{cases}\overline{\varphi}(p,\alpha), & \alpha \in [0, \pi/2) \cup [\pi, 3\pi/2), \\ -\overline{\varphi}(p,\alpha), & \alpha \in [\pi/2, \pi) \cup [3\pi/2, 2\pi),\end{cases} \tag{19}$$

where

$$\overline{\varphi}(p,\alpha) = \arccos\left(-\frac{2p(N-p) + \bigl(N^2 - 2p(N-p)\bigr)\cos(2\alpha)}{N^2 + 2p(N-p)(\cos(2\alpha)-1)}\right).$$

The order parameter of a two-cluster state depends on the location of the corresponding point on the manifold and is given by the formula

$$R(\mathcal{P}_2, \varphi_j) = \frac{1}{N}\sqrt{N^2 + 2p(p-N)(1-\cos\varphi_j)}.$$

The phase of the mean field is, respectively,

$$\psi = \psi(\mathcal{P}_2, \varphi_j) = \varphi_j/2.$$

The *invariant manifold with symmetry* $\mathbf{S}_{N/2} \times \mathbf{S}_{N/2}$ for a system with even number of oscillators. It is completely formed by fixed points for the critical values of the parameter $\alpha = \pm\pi/2$. Every point of the manifold is neutrally stable inside the manifold and is a saddle point in the transverse directions. Every two symmetric saddle points form heteroclinic cycles that are also formed by stable and unstable one-dimensional invariant manifolds of these saddles.

The $(N-2)$-parameter *sets of limit cycles* filling the canonical invariant regions $\mathcal{C}$ for $\alpha = \pm\pi/2$. The order parameter of each orbit is a variable function

$$R = R(t) = R(\xi(t)) \in (0,1),$$

where $\xi(t)$ is a periodic function specifying the location of a phase point in the periodic orbit. The order parameter $R(t)$ depends on the amplitude of the periodic orbit which, in turn, depends on the degree of proximity to the points of the manifold $\mathcal{M}^{(N)}$ and oscillates near zero if the orbit is located in the vicinity of a point from $\mathcal{M}^{(N)}$. The order parameter $R(t)$ takes its maximum values close to 1 for large amplitudes and in the case of proximity of parts of the orbit to the cluster manifolds.

The continuous *families of heteroclinic trajectories* on the boundaries of the invariant regions $\mathcal{C}$, which jointly form heteroclinic cycles for $\alpha = \pm\pi/2$. The order parameters of these cycles are also functions of $t$: $R = R(t) \in (0,1]$. The function $R(t)$ attains the value 1 only at the point $\Phi_{\text{sync}}$ of the largest heteroclinic cycle, which is formed by $N$ two-cluster invariant manifolds $\mathcal{P}_2$ and passes through the indicated point (at this time, this is a degenerate saddle).

It follows from the analysis presented above that the system has no $k$-clusters with $k \geq 3$. For $\alpha = \pm\pi/2$, we observe transcritical bifurcations of two-cluster saddles with stable (unstable) node permanently located at the origin (i.e., corresponding to the synchronous state). These bifurcations change the stability of synchronous state into the opposite one. Parallel with transcritical bifurcations, we also observe the Andronov–Hopf degenerate bifurcations at points of the antiphase manifold $\mathcal{M}^{(N)}$. These bifurcations also change the stability of the manifold into the opposite one. In this case, the Andronov–Hopf bifurcation is degenerate and does not lead to the formation of limit cycles after (or prior to) bifurcations. The stabilities of the synchronous and antiphase states are opposite for $\alpha \neq \pm\pi/2$. For $\alpha = \pm\pi/2$, when the indicated two bifurcations occur simultaneously, each of $(N-1)!$ invariant regions (12) is completely filled with periodic orbits and the $(N-2)$-dimensional boundaries of these regions $\partial\mathcal{C}$ are filled with heteroclinic trajectories, which jointly form heteroclinic cycles. Hence, different invariant regions are separated by $(N-2)$-parameter families of heteroclinic cycles.

***3.3. System with Nonlinear Coupling Function.*** We now consider a generalization of the Kuramoto–Sakaguchi model with nonlinear phase shift proposed by Pikovsky and Rosenblum [47–49]. In other words, we consider systems (9) and (10) with the following coupling function:

$$g(x-\alpha) = -\sin(x-\alpha(R,\beta)). \tag{20}$$

It has a phase shift $\alpha$, which is no longer a constant. In this case, it is a smooth function of the order parameter $R$ and a vector parameter $\beta = (\beta_1, \ldots, \beta_m)$, $m \geq 1$. In the cited works, Pikovsky and Rosenblum not only proposed a physical motivation for the application of models of this type but also generalized the Watanabe–Strogatz theory for this system. According to the generalized theory, the system with nonlinear phase shift can be reduced to a three-dimensional system of the form (16) with $N-3$ constants of motion $\psi_j$. It is clear that the reduced system depending on the nonlinear phase shift $\alpha(R,\beta)$ has a more complicated dynamics than the analogous reduction of the Kuramoto–Sakaguchi system. Note that the Pikovsky–Rosenblum theory is correct for both the finite-dimensional and infinite-dimensional systems. As for the Kuramoto–Sakaguchi model, the analyzed model encounters certain difficulties in its application to cluster manifolds. Therefore, the Pikovsky–Rosenblum system requires additional investigations with the help of alternative methods. The equilibria of system (10) in phase differences were described in [102]. Based on the cited work and the results of the Watanabe–Strogatz theory [47, 49, 83], we now describe all possible attractors of the original system (9), (20) and the bifurcations of transitions between them.

Thus, the analyzed system has the following attractors:

*Full synchronization* $\Theta_{\mathrm{sync}}$, which is stable for

$$\alpha(1,\beta) \in (-\pi/2 + 2\pi l,\ \pi/2 + 2\pi l), \quad l \in \mathbb{Z}.$$

This state corresponds to the origin $\Phi_{\mathrm{sync}}$ of the system in phase differences. For $\alpha(1,\beta) = \pm\pi/2$, the point $\Phi_{\mathrm{sync}}$ loses its stability as a result of simultaneous transcritical bifurcations that occur along the invariant two-cluster manifolds with the following isotropy: $\mathbf{S}_p \times \mathbf{S}_{N-p}$, $p = 1, \ldots, [N/2]-1$. In the case of even number of oscillators $N = 2p$, parallel with the transcritical bifurcation, we observe a pitchfork bifurcation along manifolds with isotropy $\mathbf{S}_{N/2} \times \mathbf{S}_{N/2}$.

The *manifold* $\mathcal{M}^{(N)}$ (6) that describes the global antiphase conditions. The corresponding invariant manifold for system (10), (20) is an $(N-3)$-dimensional set in the phase space $\mathbb{T}^{N-1}$ which is completely formed by equilibrium positions. The invariant manifold is neutral in $N-3$ internal directions for any values of parameters of the system. The manifold is stable in two directions transverse to this manifold if

$$\alpha(0,\beta) \in (\pi/2 + 2\pi l,\ 3\pi/2 + 2\pi l), \quad l \in \mathbb{Z}.$$

In the cases where $\alpha(0,\beta) = \pi/2 + 2\pi l$, $l \in \mathbb{Z}$, the Andronov–Hopf bifurcation occurs at all points of the manifold simultaneously. As a result of this bifurcation, we observe the formation of $(N-3)$-parameter families (manifolds) of the limit cycles (for the illustrations of bifurcations, see the figures presented in [102]).

The *two-cluster states* $\Theta_{p,N-p}$ characterized by the isotropy $S_p \times S_{N-p}$, $p \neq N/2$. In [102], it was proved that any $k$-clusters with $k \geq 3$ are impossible in this system for arbitrary functions $\alpha(R,\beta)$. The two-cluster states $\Theta_{p,N-p}$ of the original system (9) correspond to the equilibria of system (10) located on one-dimensional invariant manifolds $\mathcal{P}_2$ (18). The number of equilibria on each of these manifolds depends on the number of zeros of the function $\alpha(R,\beta)$ in the case where the variables $\varphi_i$ belong to the above-mentioned manifold. The indicated attractors (equilibria) appear and disappear due to the presence of saddle-node bifurcations on the indicated manifolds and also for the transcritical bifurcations of two-cluster equilibria with synchronous equilibrium at the origin.

For even dimensions, transcritical bifurcations occur parallel with pitchfork bifurcations of saddle points on one-dimensional invariant manifolds with the following symmetries: $S_{N/2} \times S_{N/2}$. It is possible to study stability and bifurcations inside each invariant manifold $\mathcal{P}_2$ by using relation (19) for specific values of the function $\alpha(R, \beta)$.

*Heteroclinic cycles* of system (10), (20) corresponding to the conditions of long-term synchronization of clusters with switching. Heteroclinic cycles (that can be not only stable) are formed from saddle two-cluster points and their one-dimensional invariant manifolds. In the analyzed system, one may encounter three types of heteroclinic and homoclinic cycles. These cycles may have either the symmetry $Z_N$ or $S_{N/2} \times S_{N/2}$ depending on the basic saddle points. For the indicated transcritical bifurcation, the point $\Phi_{\text{sync}}$ is a degenerate (semistable in every direction) saddle and a base for $N$ symmetric homoclinic cycles (each of these cycles contains the point $\Phi_{\text{sync}}$ and a loop formed by the invariant manifold $\varphi_i = 0, i \neq j$). Since the phase space is a torus $\mathbb{T}^{N-1}$, every described invariant manifold connects the point $\Phi_{\text{sync}}$ with itself along the variable $\varphi_j$: $W^u(\Phi_{\text{sync}}) = W^s(\Phi_{\text{sync}})$. Note that, as a result of bifurcations, for

$$\alpha(1f\beta) = \pi/2 + 2\pi l, \quad l \in \mathbb{Z},$$

the heteroclinic cycles form $(N-3)$-parametric families in the cases where $N \geq 4$.

The *families of periodic orbits* for the system in phase differences, which form $(N-2)$-dimensional invariant manifolds inside the invariant region $\overline{C} \setminus \overline{\mathcal{M}}^{(N)}$. Every trajectory is neutrally stable along $N-3$, $N \geq 4$, directions inside the manifold and may have a different stability in the direction transverse to the manifold. By $\mathcal{L}_s$, $\mathcal{L}_u$, and $\mathcal{L}_n$, we denote the stable, unstable, and neutral cycles in the direction transverse to the set, respectively. The existence of these trajectories is well described by the already mentioned Watanabe–Strogatz theory. Under certain conditions, in particular, for $\alpha(R, \beta) = \pm\pi/2$, the *entire phase space,* except the $(N-2)$-dimensional hyperplanes $\varphi_i = 0$, $\varphi_i = \varphi_j$, and the invariant manifold $\overline{\mathcal{M}}^{(N)}$, is filled with periodic orbits (the $(N-2)$-dimensional family of orbits fills $\mathbb{T}^{N-1}$). Depending on the function $\alpha(R, \beta)$, the families of periodic orbits $\mathcal{L}$ appear and disappear as a result of the following three basic types of bifurcations:

(i) the Andronov–Hopf bifurcation of all points of the invariant manifold $\overline{\mathcal{M}}^{(N)}$ in two directions transverse to this manifold for $\alpha(0, \beta) = \pm\pi/2$;

(ii) the heteroclinic bifurcations (of several types) on the boundary $\partial\overline{C}$ of the canonical invariant region $\overline{C}$, which lead to the appearance (disappearance) of a heteroclinic cycle or a set of heteroclinic cycles;

(iii) bifurcations of the limit sets of limit cycles $\mathcal{L}_s$, $\mathcal{L}_u$ inside the invariant region $\overline{C}$ (most often, this is a saddle-node bifurcation of cycles but sometimes may have the form of their pitchfork bifurcation).

As shown above, the Pikovsky–Rosenblum model, as a generalization of the Kuramoto–Sakaguchi model, has a more complicated dynamics. We now briefly describe the specific features of the additional dynamics acquired by the system when the phase shift becomes nonconstant. These are:

(i) the appearance of a larger number (depending on $\alpha(R, \beta)$) of stable and unstable points on the invariant two-cluster manifolds $\mathcal{P}_2$ (for $\alpha = \text{const}$, this is only one saddle);

(ii) the appearance of manifolds of limit cycles $\mathcal{C}$ inside the invariant region $\overline{C}$ (not only of the set of periodic orbits that fill this region and not only for $\alpha = \pm\pi/2$);

(iii) the appearance of heteroclinic cycles of various structures for different bifurcations (not only for the transcritical/heteroclinic bifurcation).

The detailed description of bifurcations with presentation of the phase portraits and bifurcation diagrams for system (9), (20) in the case of a quadratic nonlinear phase shift

$$\alpha = \alpha(R, \beta) = \beta_1 + \beta_2 R^2$$

can be found in [47, 49, 102].

***3.4. System with Two-Harmonic Coupling Function.*** In Section 3.2, we have described the dynamics of the Kuramoto–Sakaguchi model, i.e., the case where the coupling function in system (9) has only one harmonic: $g(x) = -\sin(x - \alpha)$. It is natural to expect that, in the presence of other harmonics of this function, the dynamical and bifurcation properties of the model become more complicated. It turns out that, even in the presence of the second harmonic, i.e., for

$$g(x) = -\sin(x - \alpha) + r\sin(2x - \beta), \tag{21}$$

system (9) may have a great variety of new nontrivial solutions [55, 64, 114, 115]. For $\beta = 0$, this system is called the Hansel–Mato–Meunier model (see the works [64, 105], where this model was studied in detail for the first time). Even for low dimensions $N = 3, 4, 5$, system (9), (21) may exhibit a quite complicated collective dynamics [11, 99–101, 116]. The full description of this system in the case of arbitrarily many oscillators still remains an open problem. The Watanabe–Strogatz theory cannot be used in this case unlike the Kuramoto–Sakaguchi model. As the simplest important new property appearing in system (9) in the presence of the second harmonics [i.e., for $r \neq 0$ in (21)], we can mention the formation of $k$-cluster states with $k \geq 3$. At the same time, the system with a one-harmonic function $g(x)$ possesses solely the two-cluster states $\Theta_{p,N-p}$. Note that, in view of the symmetry of permutations $\mathbf{S}_N$ [observed for a system with any $g(x)$], the phase space of the system in phase differences (10), (11) is split by $(N-2)$-dimensional invariant manifolds $\varphi_i = 0$ and $\varphi_i = \varphi_j$ into $(N-1)!$ canonical invariant regions $\overline{\mathcal{C}}$. Despite this fact, the inner parts of these regions are no longer free of fixed points, as in the case where $r = 0$. This specific feature immediately implies the presence of significant variability of the local bifurcations of these equilibria and also leads to a large number of global bifurcations based on the indicated local bifurcations. The presence of symmetric cluster states (that can be saddles $S_i$ for some values of the parameters), as well as the possibility of connection between them with the help of (also symmetric) one-dimensional invariant manifolds $W^u(S_i) = W^s(S_j)$, lead to the formation of various heteroclinic cycles.

In what follows, we describe several states typical of the system with two-harmonic coupling.

The *full synchronization* $\Theta_{\text{sync}}$ (4) corresponding to the origin $\Phi_{\text{sync}}$ of system (10), (11) in phase differences. This state is stable for

$$g'(0) = 2r\cos\beta - \cos\alpha < 0.$$

The stability of $\Phi_{\text{sync}}$ can be lost as a result of simultaneous transcritical bifurcations along two-cluster invariant manifolds with isotropy $\mathbf{S}_p \times \mathbf{S}_{N-p}$, $p \neq N/2$, and pitchfork bifurcations along two-cluster manifolds with isotropy $\mathbf{S}_{N/2} \times \mathbf{S}_{N/2}$ if the dimension of the system is even .

The *splay state* $\Theta_{\text{splay}}$ (5). This state corresponds to $(N-1)!$ equilibria of $\Phi_{\text{splay}}$ (14) for system (10), (11) each of which is located inside a separate invariant region $\overline{\mathcal{C}}$. The analyzed state is stable whenever

$$\sum_{j=1}^{N-1} g'\left(\frac{2\pi}{N} j\right)\left(1 - \cos\left(\frac{2pj\pi}{N}\right)\right) < 0, \quad p = 1, \ldots, N-1,$$

and loses its stability as a result of the Andronov–Hopf bifurcations.

The *completely asynchronous state* $\mathcal{M}^{(N)}$ (6), which corresponds to the set $\overline{\mathcal{M}}^{(N)}$ for system (10), (11). Unlike the Kuramoto–Sakaguchi system, this set is not invariant for any $N$ and $r \neq 0$. The set $\overline{\mathcal{M}}^{(N)}$ is completely formed by the equilibria of system (10), (11) for $N = 2, 3$. It is invariant for $N = 4$. In the general case, it is not invariant for $N \geq 5$. As already indicated,

$$\Phi_{\text{splay}} \subset \overline{\mathcal{M}}^{(N)}.$$

The *two-cluster states* $\Theta_{p,N-p}$ corresponding to $\Phi_{p,N-p}$ on one-dimensional invariant manifolds $\mathcal{P}_2$ (18) for system (10), (11). The bifurcations of these states may occur along the invariant manifolds $\mathcal{P}_2$ and in the directions transverse to these manifolds (unlike the Kuramoto–Sakaguchi system). In this case, the bifurcation curves are described by quite complicated relations [101].

The *multicluster states* are possible for $r \neq 0$. The problem of detection of the presence of these states is nontrivial even for systems with low dimensions [99, 101, 116].

*Slow switching* between the clusters corresponding to the heteroclinic cycles of the system in phase differences (10), (11). The presence of a large number of various heteroclinic cycles is one of the main specific features of the system. The appearance and disappearance of these cycles are connected with the presence of various global heteroclinic bifurcations or even chains of bifurcations of this kind. Among other bifurcations, we can mention saddle-node/heteroclinic bifurcations, transcritical/homoclinic bifurcations $\mathbf{S}_N$, heteroclinic bifurcations $\mathbb{Z}_N$, pitchfork/heteroclinic bifurcations, saddle-connected/heteroclinic bifurcations, and transcritical-pitchfork/heteroclinic bifurcations. Note that a stable heteroclinic trajectory in a system of coupled elements can also be interpreted as the "winnerless competition" [55, 117, 118].

*Multiparameter families of heteroclinic trajectories*. Starting from the dimension $N = 3$, the system may also possess families of heteroclinic cycles located both on the boundaries of the invariant regions $\overline{\mathcal{C}}$ and inside them [99].

*Periodic and quasiperiodic states*. The system in phase differences has periodic orbits inside the invariant regions $\overline{\mathcal{C}}$, which may also correspond to quasiperiodic solutions of the original system in the case of incommensurability of the eigenfrequencies of individual oscillators with the frequency of mean field of the system. The most typical bifurcations connected with the appearance of periodic orbits are the Andronov–Hopf bifurcations inside invariant regions and various heteroclinic bifurcations mentioned above. The limit cycles "repeat" the structure of heteroclinic cycles for minor changes in the parameter after the corresponding bifurcation and also, to a certain extent, describe the slow switching conditions.

More detailed information about the slow switching conditions, chaotic states, and bifurcation transitions in symmetric oscillator models with generalized coupling function can be found in [11, 55, 57, 67, 94, 103, 104, 119].

***3.5. System with Even and Odd Couplings.*** The dynamics of system (9) can be fairly completely described in the cases where the coupling function $g(x)$ has an arbitrary number of harmonics and is even or odd. In the case where $g(x)$ is *odd*, system (9) becomes *gradient* and has the following potential:

$$V(\Theta) = -\frac{1}{2N} \sum_{k,j=1}^{N} h(\theta_k - \theta_j), \quad h'(x) = g(x).$$

In the case where $g(x)$ is *even*, system (9) is *divergence free*, i.e.,

$$\sum_{j=1}^{N} \left( \frac{\partial F_j}{\partial \theta_j} \right) = 0,$$

where $F = (F_1, \dots, F_N)$, $F_j = F_j(\Theta)$, is the vector of the right-hand sides of system (9). In addition, the system with even coupling function is also *time reversible*. Recall that an involution $\mathcal{R}: \mathbb{T}^N \to \mathbb{T}^N$ is a time-reversible symmetry if

$$\frac{d}{dt}\mathcal{R}(\Theta) = -F(\mathcal{R}(\Theta)).$$

In other words, $\mathcal{R}$ maps any solution of the system into another solution of this system with the opposite direction of the phase flow [120, 121]. System (9) has the time-reversible symmetry $\mathcal{R}$, where $\mathcal{R}(\Theta) = -\Theta$, $t \mapsto -t$. The indicated time-reversible symmetry is preserved under the transition to the system in phase differences (10). A fixed subspace of the time-reversible symmetry has the form

$$\mathbf{Fix}\,\mathcal{R} = \left\{\Theta \in \mathbb{T}^N : \mathcal{R}\Theta = \Theta\right\}.$$

The time-reversible symmetry $\mathcal{R}$, together with the permutation symmetry $\mathbf{S}_N$, determine the structure of the phase space and enable us to describe the general properties of trajectories of the system. For an even coupling function, system (9) has a Hamilton-like structure of the phase space. The system of three oscillators ($N = 3$) possesses the first integral

$$V(\theta) = h(\theta_1 - \theta_2) + h(\theta_2 - \theta_3) + h(\theta_3 - \theta_2).$$

It is possible to assume that the analyzed system has the first integrals in a similar case for $N \geq 4$. In more detail, the dynamics of globally coupled systems with even and odd coupling functions was considered in the works [3, 55, 80, 101, 103].

***3.6. Oscillatory Model with Attractive and Repulsive Couplings.*** As already indicated, the standard KM [with $g(x) = -\sin(x)$] of globally coupled identical oscillators has only one attractor for a positive coupling strength: $K > 0$. This system also has a repeller in the form of an $(N-2)$-dimensional invariant manifold $\mathcal{M}^{(N)}$. This means that, for the negative values of the coupling strength, $K < 0$, the role of attractor in the KM is played just by the manifold $\mathcal{M}^{(N)}$. Hence, for $K > 0$, all oscillators try to synchronize with each other (with the order parameter $R = 1$). These couplings are called *attractive*. For $K < 0$, all oscillators try to arrange along a circle in order to form a global antiphase state (with $R = 0$). These couplings are called *repulsive*. Daido proposed an oscillatory model, which has both attractive and repulsive couplings [122]. The detailed analysis of this model can be found in the works by Hong and Strogatz [61, 123]. We can also mention the works [124–127] devoted to the study of collective dynamics in the oscillatory models with two types of coupling. Hong and Strogatz also advanced a witty sociological interpretation of their model. They split the set of all oscillators into the following two groups:

(i) *conformists* who try both to synchronize with each other and, moreover, to force the oscillators from a different group to synchronize with them (with the order parameter $R = 1$) and

(ii) *contrarians* who try to attain the global antiphase state (with $R = 0$) or, at least, to bring their own group in the antiphase state relative to the other group.

We now consider a generalization of the model developed in the above-mentioned works. Thus, we study a model with phase shift $\alpha \in \mathbb{T}^1$. By $\mathcal{H}_1$ we denote the set of oscillators-conformists containing $N_1$ elements. Moreover, by $\mathcal{H}_2$ we denote the set of oscillators-contrarians formed by $N_2 = N - N_1$ elements. Assume that each set $\mathcal{H}_i$ is nonempty, i.e., $N_1 > 0$ and $N_2 > 0$. Without loss of generality (due to the permutation symmetry), the oscillators of subpopulations $\mathcal{H}_1$ and $\mathcal{H}_2$ can be indexed as $\{1, 2, \dots, N_1\} = J_1$ and $\{N_1 + 1, \dots, N\} = J_2$, respectively. As a result, the model of identical oscillators with attractive and repulsive elements and phase shift takes the form [128]

$$\frac{d\theta_i}{dt} = \omega - \frac{K_1}{N} \sum_{j=1}^{N} \sin(\theta_i - \theta_j - \alpha), \quad i \in J_1, \tag{22}$$

$$\frac{d\theta_i}{dt} = \omega - \frac{K_2}{N} \sum_{j=1}^{N} \sin(\theta_i - \theta_j - \alpha), \quad i \in J_2, \tag{23}$$

where $K_1 > 0$, $K_2 < 0$. In other words, Eqs. (22) with attractive interaction describe the dynamics of conformists, whereas Eqs. (23) with repulsive interaction describe the dynamics of contrarians. The oscillators in this model are identical only conventionally because their permutations can be performed only within each of the sets $\mathcal{H}_i$. By using the Watanabe–Strogatz theory [47, 49] and the Ott–Antonsen ansatz [60], Hong and Strogatz developed a theory, which enables one to reduce a *system with large number of oscillators* with interaction of two types (and without phase shift: $\alpha = 0$) first to a system of six equations and then, by passing to the thermodynamical limit, to the following system of three equations:

$$\frac{dr_1}{dt} = c(1 - r_1^2)(q_1 r_1 + q_2 r_2 \cos\delta),$$

$$\frac{dr_2}{dt} = -(1 - c)(1 - r_2^2)(q_1 r_1 \cos\delta + q_2 r_2),$$

$$\frac{d\delta}{dt} = \sin\delta\left[q_1(1 - c)\left(\frac{r_1}{r_2} + r_1 r_2\right) + q_2 c\left(\frac{r_2}{r_1} + r_1 r_2\right)\right],$$

where $q_1 = N_1/N$ and $q_2 = N_2/N$ are the relative fractions of oscillators of two types, $c = K_1/(K_1 - K_2)$ is the relative intensity of coupling of the conformists, and the role of variables is played by two partial order parameters $r_s$ (the amplitudes of the mean fields of the groups $\mathcal{H}_s$),

$$z_s = r_s e^{\imath\psi_s} = \frac{1}{N_s} \sum_{j \in J_s} e^{\imath\theta_j}, \quad s = 1, 2,$$

and the phase difference of the mean fields $\delta = \psi_1 - \psi_2$. In order to understand the relationship between the conformists and contrarians, it is necessary to know the relationship between the global and local order parameters: $Z = q_1 z_1 + q_2 z_2$. By using the indicated reduction, Hong and Strogatz demonstrated the existence of four stable dynamical states, which are considered in what follows. The analyzed theory describes systems of infinitely many oscillators and does not take into account some specific features of finite-dimensional systems. Therefore, the theory was supplemented with an alternative method of investigations [128], which made it possible to show the existence of two more stable states in the finite-dimensional case (one state is a special case of the other state with certain specific features). All other results are in complete agreement with the Hong–Strogatz theory. The indicated method can be described as the reduction of a system to phase variables with subsequent investigation of the stationary and nonstationary states of this system and their bifurcations. Moreover, this method enables one to obtain results for the case of nonzero phase shift $\alpha$. By analyzing both approaches, we now present several results.

System (22), (23) can be reduced with the help of phase differences (2) to the following system:

$$\frac{d\varphi_i}{dt} = -\sin(\varphi_i + \alpha) - \sum_{j=1}^{N-1} \sin(\varphi_j - \alpha) - \sum_{j=1, j\neq i}^{N-1} \sin(\varphi_i - \varphi_j + \alpha), \qquad i = 1, \ldots, N_1 - 1, \tag{24}$$

$$\frac{d\varphi_i}{dt} = -k\sin(\varphi_i + \alpha) - \sum_{j=1}^{N-1} \sin(\varphi_j - \alpha) - k\sum_{j=1, j\neq i}^{N-1} \sin(\varphi_i - \varphi_j + \alpha) - (k-1)\sin\alpha, \qquad i = N_1, \ldots, N - 1, \tag{25}$$

where $k = K_2/K_1$ is a new parameter. System (22), (23) has two types of natural invariant manifolds

$$\mathcal{P}^1_{i,j} = \left\{(\theta_1, \ldots, \theta_N)\colon \theta_i = \theta_j\right\}, \quad i, j \in J_1, \quad i \neq j,$$

$$\mathcal{P}^2_{i,j} = \left\{(\theta_1, \ldots, \theta_N)\colon \theta_i = \theta_j\right\}, \quad i, j \in J_2, \quad i \neq j,$$

corresponding to clusters with arbitrarily many oscillators from the first $\mathcal{H}_1$ or second $\mathcal{H}_2$ group. It is easy to see that the oscillators from different groups do not form clusters and, hence, the corresponding invariant manifolds do not exist. This implies that the original system (22), (23) and the corresponding reduced system (24), (25) do not have any canonical invariant regions of the form (12). More precisely, unlike the systems of globally coupled identical oscillators with a common coupling strength $K$ whose phase space is decomposed into closed regions, the phase space of system (24), (25) is also split into invariant regions by the invariant hyperplanes $\varphi_i = 0$ or $\varphi_i = \varphi_j$ but these regions are no longer bounded in the entire phase space. As already indicated, all solutions of the systems in phase differences described above are *phase-locked*. However, the solutions of system (24), (25) should not necessarily be phase locked. Thus, the periodic trajectories of the analyzed system can be either homologic to zero in the space $\mathbb{T}^{N-1}$ or not homologic to zero.

We now mention another property of the Kuramoto–Sakaguchi system preserved for the analyzed model. Namely, system (22), (23) possesses the invariant manifold $\mathcal{M}^{(N)}$ and system (24), (25) possesses the invariant manifold $\overline{\mathcal{M}}^{(N)}$ completely formed by fixed points. The last property is preserved for both parameters $\alpha$ and $k$. In addition, as shown in [128], for $\alpha = 0$ and any $k$ except $k = 0$ and $k = -N_2/N_1$, system (24), (25) has equilibria only of the following two types:

(i) the points of the indicated manifold $\overline{\mathcal{M}}^{(N)}$ and

(ii) the points $\Phi = (\varphi_1, \ldots, \varphi_N)$ solely with the coordinates 0 or $\pi$.

Not all points of the sets (i) and (ii) are stable. Some points can be stable for certain values of the parameter $k$ but the other points are unstable for all values of this parameter. We now describe all possible stable (stationary and nonstationary) conditions of the interaction of oscillators from the analyzed two groups:

The *$\pi$-state*. This is the state of confrontation of conformists and contrarians in the antiphase. In this case, $r_1 = r_2 = 1$ and $\delta = \pi$. This state is stable if the following conditions are simultaneously satisfied: $k < -N_2/N_1$, and $N_1 > N_2$, $N \geq 3$. The second condition implies that the analyzed state is stable only if the conformists predominate over the contrarians. However, they cannot win over them.

The *noncoherent state*. This state corresponds to the antiphases of both groups of oscillators: $r_1 = r_2 = 0$. It is clear that the analyzed state corresponds to the complete victory of contrarians because the total order parameter $R$ is equal to zero. All equilibria that belong to this state form an $(N-3)$-dimensional invariant manifold which is a submanifold of $\overline{\mathcal{M}}^{(N)}$. All points of this manifold are neutral inside the manifold and have different dimensions in two transverse directions. Thus, there exists a range of values of the parameter $[k_{\min}(\Phi), k_{\max}(\Phi)]$, which depends on the location of a separate point in the phase space and determines its stability. As two evident subsets of the analyzed state, we can mention the cases where the system contains only contrarians ($N_2 = N$) and the system with even dimension and identical numbers of the conformists and contrarians ($N_1 = N_2$, $r_1 = r_2 = 1$, $\delta = \pi$).

The *fuzzy $\pi$-state*. This state corresponds to the confrontation of the groups of conformists and contrarians in the antiphase but without clear synchronization of elements in each group. To be more precise, for the analyzed state, we have $\delta = \pi$ and $N_1 r_1 = N_2 r_2$. The presented equalities imply that $R = 0$ and, hence, the investigated state is also a subset of the manifold $\overline{\mathcal{M}}^{(N)}$. As in the previous case, the property of stability of every point representing this state is individual and depends on the location of the point in the space.

The *renegade state*. This is the state of antiphase confrontation of all synchronized conformists with almost all contrarians, with the exception of a single contrarian-renegade who "passed to the opposite side." The analyzed state is formally described in the following way: $r_1 = 1$, $r_2 = (N_2 - 2)/N_2$, $\delta = \pi$. In this case,

$$R = (N_1 - N_2 + 2)/N = 2n_1 - 1 - 2/N.$$

It is easy to see that (unlike the previous states) this relation contains the number of oscillators $N$, which indicates the existence of a renegade state only in the presence of finitely many elements. The renegade state is stable if the following conditions are satisfied:

$$k \in \left(-\frac{N_2}{N_1} - \frac{N_2 - 2}{N_2}\right) \quad \text{for} \quad 2 \leq N_2 \leq N_1 + \frac{1 - (-1)^N}{2}.$$

Note that the states in which two or more contrarians are synchronized with conformists are unstable. To be more precise, they are saddle points of system (24), (25).

*Full synchronization*. This state is possible in two cases:

(i) the system includes only conformists ($N_1 = N$);

(ii) the system has only one contrarian ($N_2 = 1$, $r_1 = R = 1$, $\delta = 0$).

The second case is a subset of the states from the previous item but for the full synchronization. In both cases, the analyzed state is stable for $k > \overline{k}$, when the bifurcation value depends on the number of oscillators:

$$\overline{k} = \begin{cases} 0, & N_1 = 1, \ldots, N-2, \\ -\dfrac{1}{N-1}, & N_1 = N-1, \\ -\infty, & N_1 = N. \end{cases}$$

*Traveling waves*. Unlike four previous cases, this state is nonstationary. In this state, all conformists are synchronized ($r_1 = 1$) and the positions of contrarians vary with time (the local order parameter $r_2 = r_2(t)$ is not constant). Moreover, it is clear that the global order parameter $R = R(t)$ and the angle between the mean phases $\delta = \delta(t)$ are also functions of time. The state corresponds to the existence of stable phase-unlocked trajectories of system (24), (25). In addition, these trajectories can be periodic, quasiperiodic, or chaotic (for $N \geq 4$).

The dynamics of system (22), (23) becomes much more complicated in the presence of phase shift $\alpha \neq 0$ [128]. The presence of phase shift leads to the appearance of new stationary and nonstationary collective behaviors. We consider only some of these cases.

*Mutual rotation of two synchronous clusters* of conformists (with $r_1 = 1$) and contrarians (with $r_2 = 1$) with a free angle $\delta = \delta(t)$ between them. This state describes two-cluster rotational waves and is realized in the presence of at least two conformists and two contrarians.

*Slow switching.* These states are represented by various homoclinic and heteroclinic trajectories that are not homologic to zero.

The multidimensional sets of *neutral periodic orbits and heteroclinic cycles* that are possible only for $\alpha = \pm\pi/2$.

The *states without full synchronization of conformists* with $r_1 = r_1(t)$ specified by the phase-unlocked attractors of system (24), (25).

The system has numerous bifurcation lines whose codimension is equal to 1 in the plane of parameters $(\alpha, k)$. System (24), (25) has all possible local bifurcations (saddle-node, transcritical, pitchfork, and Andronov–Hopf bifurcations) and also various bifurcations of formation of the limit and heteroclinic cycles. In this case, the limit and heteroclinic cycles can be either phase-locked or phase-unlocked. The bifurcations observed in the Hong–Strogatz model (for $\alpha = 0$) always correspond to the bifurcation points of codimension 2 in the $(\alpha, k)$-plane independently of the number of oscillators in the groups. We now especially mention the case $\alpha = \pm\pi/2$ in which the system exhibits a Hamilton-like dynamics. In this case, the system has additional time-reversible symmetries and may exhibit (for $N \geq 4$) the phenomenon of conservative chaos similar to the *ABC*-flows [129, 130]. It is also worth noting that the system is sensitive to perturbations of the phase shift $\alpha$ from zero when the stable states of the Hong–Strogatz model are destroyed, and we observe the appearance of the states of new types [128].

In this section, we consider a system with couplings that can be represented by the matrix $K = (K_{ij})_{i,j=1}^{N}$, where $K_{ij} = K_1 > 0$ for $i = 1, \ldots, N_1$ and $K_{ij} = K_2 < 0$ for $i = N_1 + 1, \ldots, N$. As a natural generalization of this situation, we can consider a system with a more general matrix of couplings with rows of identical elements: $K_{ij} = K_i$, $i = 1, \ldots, N$ (similar systems were considered, e.g., in [131, 132]). Then the oscillators for which the action of the other oscillators is positive (i.e., $K_i > 0$) can be called conformists, whereas the oscillators subjected to the negative influence of the other oscillators ($K_i < 0$) can be called contrarians. The dynamics of this system is clearly more complicated than for Eqs. (22), (23) but preserves numerous properties described above. In particular, the corresponding system in phase differences with $\alpha = 0$ has the same equilibria as system (24), (25). Hence, by using the methods presented in [61, 123, 128], we can give a fairly complete description of the collective behavior of the system.

## 4. Discussion

We present a brief survey of investigations of the phenomenon of synchronization in the natural sciences and give a motivation for the development of mathematical models aimed at description of this phenomenon. The KM of coupled phase oscillators is a quite simple mathematical system, which enables one not only to describe the extremely complicated and diverse dynamics of interaction between the coupled elements but also to analyze this interaction with the help of rigorous mathematical terminology. The KM can be regarded as the starting point in the investigation of various more complicated natural models (such as, e.g., dynamical Hodgkin–Huxley networks of neurons). In the present work, we also discuss some results obtained for various types of collective dynamics and bifurcation transitions between different states for the networks of phase oscillators, which have the symmetries of individual nodes, graphs, and functions of coupling between the elements. It is shown how the type of coupling function affects the presence and formation of various synchronous states and in what way the presence of a certain number of harmonics, parameters, and symmetries of the coupling function causes the appearance of various bifurcations.

Despite the great number of available publications (the complete list of works dealing with the study of similar models is several times longer than the list of references presented in our work), numerous important problems connected with the investigation of collective dynamics in Kuramoto-type models remain open in the general case. In what follows, we formulate some of these problems.

1. Full description of the model of globally coupled identical phase oscillators of the form (9) with a coupling function $g(x)$, which has arbitrarily many harmonics in the Fourier series.

2. Determination of the integrals of motion of system (9) for even functions $g(x)$ with any number of harmonics. Description of the families of periodic, quasiperiodic, heteroclinic, and chaotic trajectories for this system.

3. Description of the invariant sets and solutions of nonglobally coupled systems with different symmetries in the graphs of interaction between the elements, i.e., for different symmetric coupling matrices

$$K = (K_{ij})_{i,j=1}^{N}.$$

4. Description of various synchronous states and the trees of bifurcation transitions between different conditions of phase synchronization for various symmetric distributions of eigenfrequencies $\omega_i$ and the variations of coupling strength/strengths.

5. Problems formulated in the previous items for the infinite-dimensional systems obtained as a result of thermodynamic limit transitions.

We have formulated simple and obvious statements of the problems connected with the investigation of phase oscillators. These problems often lead either to the discovery of new and very interesting phenomena (e.g., slow switching or chimera states) or to the creation of new profound mathematical theories (such as the Watanabe–Strogatz or Ott–Antonsen theory).

The results described in the first part of the survey give a partial answer to the problems formulated in items 1 and 2.

In the second part of the work, we give a partial answer to the problems described in item 3, i.e., consider some phenomena of collective behavior in the nonglobal networks of coupled identical Kuramoto-type oscillators. It is shown in what way the architecture of couplings between the elements of a network (it is no longer global and, hence, more diverse) leads to the appearance of various collective dynamical states.

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