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ON THE SMOOTHNESS OF THE GREEN FUNCTION FOR THE PROBLEM OF BOUNDED INVARIANT MANIFOLDS

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We investigate the smoothness of the Green function for the problem of bounded invariant manifolds of linear extensions of dynamical systems.

We consider the system of differential equations

$$\frac{d\psi}{dt} = a(\psi), \quad \frac{dx}{dt} = A(\psi)x, \tag{1}$$

where $\psi \in \boldsymbol{R}^m$, $x \in \boldsymbol{R}^n$, and the vector function $a(\psi)$ and matrix function $A(\psi)$ are defined for all $\psi \in \boldsymbol{R}^m$ and are continuous in the collection of variables $\psi_1, \ldots, \psi_m$. For the vector function $a(\psi)$, we additionally assume that the Cauchy problem

$$\frac{d\psi}{dt} = a(\psi), \quad \psi|_{t=0} = \psi_0 \tag{2}$$

has a unique solution $\psi_t(\psi_0)$ defined for all $t \in \boldsymbol{R}$ and continuously dependent on ψ_0. Denote by $C^0(\boldsymbol{R}^m)$ the space of functions $F(\psi)$ continuous in the collection of variables $\psi_1, \ldots, \psi_m$ and bounded on $\boldsymbol{R}^m$, by $C^q(\boldsymbol{R}^m)$, $q \geq 1$, the space of functions having continuous partial derivatives up to the qth order inclusive with respect to each variable ψ_j, $j = \overline{1, m}$ (we denote by $D^p_\psi F(\psi)$ any derivative of order $|p| = \sum_{i=1}^m p_m$ of a function $F(\psi)$ with respect to the variables $(\psi_1, \ldots, \psi_m) = \psi$), and by $C'(\boldsymbol{R}^m; a)$ the subspace of the space $C^0(\boldsymbol{R}^m)$ of functions $F(\psi)$ such that the function $F(\psi_t(\psi))$ is continuously differentiable with respect to t for all $t \in \boldsymbol{R}$ and $\psi_0 \in \boldsymbol{R}^m$ and, in addition,

$$\left.\frac{d}{dt} F(\psi_t(\psi))\right|_{t=0} := \dot{F}(\psi) \in C^0(\boldsymbol{R}^m).$$

Numerous works (see [1–6]) are devoted to the investigation of bounded invariant manifolds of dynamical systems. The introduction of the Green function of the problem on an invariant torus [2] gave the possibility of presenting from the general point of view the perturbation theory of differentiable and continuous invariant manifolds of dynamical systems, which explains the necessity of investigating the properties of this function (smoothness, roughness, etc.). In particular, the properties of smoothness of the functions $G_0(\tau, \psi)$ and $u(\psi)$ are fairly well studied in the case where the matrix function $A(\psi)$ and the vector function $a(\psi)$ are 2π-periodic in each variable ψ_j, $j = \overline{1, m}$, i.e., they are given on an m-dimensional torus $\mathcal{T}_m$ [4]. Although many properties of this system are preserved in the case where these functions are given not on a compact manifold but in the entire space $\boldsymbol{R}^m$ [4, 5], the investigation of the problem of smoothness in this case is quite different and encounters substantial difficulties. This

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is explained, first of all, by the fact that, in the general case, the functions on the right-hand sides of system (1) and their derivatives may be unbounded. In the present work, we indicate certain classes of functions such that, for the functions $a(\psi)$, $A(\psi)$, and $f(\psi)$ from these classes, the unique Green function and the bounded invariant manifold of system (1) are smooth.

It is known [2] that system (1) has the Green function of the problem of bounded invariant manifolds if there exists an $n \times n$ matrix $C(\psi) \in C^0(\boldsymbol{R}^m)$ such that the function

$$G_0(\tau,\psi) = \begin{cases} \Omega_\tau^0(\psi)C(\psi_\tau(\psi)), & \tau \le 0, \\ \Omega_\tau^0(\psi)[C(\psi_\tau(\psi)) - I_n], & \tau > 0, \end{cases} \tag{3}$$

satisfies the estimate

$$\| G_0(\tau,\psi)\| \le K\exp\{-\gamma\,|\tau|\}, \tag{4}$$

where the positive constants K and γ are independent of $\psi \in \boldsymbol{R}^m$ and $t \in \boldsymbol{R}$, and $\Omega_0^t(\psi)$ is the matriciant of the linear system

$$\frac{dx}{dt} = A(\psi_t(\psi))x, \qquad \Omega_0^t(\psi)|_{t=0} = I_n.$$

In this case, a function of the form $G_0(\tau,\psi)$ is called the Green function of the problem of bounded invariant manifolds of system (1). The existence of this function leads to the existence of a bounded invariant manifold of the system of equations

$$\frac{d\psi}{dt} = a(\psi), \qquad \frac{dx}{dt} = A(\psi)x + f(\psi) \tag{5}$$

for every vector function $f(\psi) \in C^0(\boldsymbol{R}^m)$. This manifold can be represented as follows:

$$x = u(\psi) = \int_{-\infty}^{\infty} G_0(\tau,\psi)\, f(\psi_\tau(\psi))d\tau. \tag{6}$$

Recall that a bounded invariant manifold of system (5) is given by the equality $x = u(\psi)$ if $u(\psi) \in C'(\boldsymbol{R}^m; a)$ and the identity $\dot u(\psi) \equiv A(\psi)u(\psi) + f(\psi)$ holds for all $\psi \in \boldsymbol{R}^m$.

Note that estimate (4) for the Green function (3) is equivalent to the estimate

$$\| G_t(\tau,\psi)\| \le K\exp\{-\gamma\,|t-\tau|\} \tag{7}$$

for the function $G_t(\tau,\psi) = \Omega_0^t(\psi)G_0(\tau,\psi)$. The latter follows from the identity $\Omega_\tau^t(\psi_\theta(\psi)) \equiv \Omega_{\tau+\theta}^{t+\theta}(\psi)$ for the matriciant of system (1), which is true for any $t, \tau, \theta \in \boldsymbol{R}$.

To prove our principal results, we need necessary conditions of convergence and an estimate for the integral

$$J(\tau,t,\mu) = \int_{-\infty}^{\infty} \exp\{-\gamma(|t-\sigma| + |\sigma-\tau|) + \mu_1|\sigma| + \mu_2\max\{|\sigma|,|\tau|\} + \mu_3\max\{|\sigma|,|t|\}\}\, d\sigma, \tag{8}$$

where $t, \tau \in \boldsymbol{R}$ are parameters, γ and μ_1 are positive constants, and μ_2 and μ_3 are nonnegative constants. Let us formulate a preliminary result.

Lemma 1. *If*

$$2\gamma > \mu, \tag{9}$$

where $\mu = \mu_1 + \mu_2 + \mu_3$, *then the integral* $J(\tau, t, \mu)$ *converges for all* $t, \tau \in \boldsymbol{R}$, *and the following estimate is true:*

$$J(\tau, t, \mu) \le K \exp\{-\gamma |t-\tau| + \mu \max\{|t|, |\tau|\},$$

where

$$K = \frac{2(2\gamma + \max\{\gamma, \mu\})}{\mu_1(2\gamma - \mu)}.$$

Proof. We decompose the coordinate plane $O\tau t$ by the straight lines $t = 0$, $\tau = 0$, $t = \tau$, and $t = -\tau$ into eight regions. On each of these regions, we consider the integral $J(t, \tau, \mu)$ as a sum of integrals whose limits are chosen so as to remove the moduli in the integrands. Let $0 \le t \le \tau$. Then we can write

$$\begin{aligned} J(t, \tau, \mu) = & \int_{-\infty}^{-\tau} \exp\{-\gamma(t-\tau) - \gamma(\tau-\sigma) - \mu_1\sigma - \mu_2\sigma - \mu_3\sigma\} d\sigma \\ & + \int_{-\tau}^{-t} \exp\{-\gamma(t-\sigma) - \gamma(\tau-\sigma) - \mu_1\sigma + \mu_2\tau - \mu_3\sigma\} d\sigma \\ & + \int_{-t}^{0} \exp\{-\gamma(t-\sigma) - \gamma(\tau-\sigma) - \mu_1\sigma + \mu_2\tau + \mu_3 t\} d\sigma \\ & + \int_{0}^{t} \exp\{-\gamma(t-\sigma) - \gamma(\tau-\sigma) + \mu_1\sigma + \mu_2\tau + \mu_3 t\} d\sigma \\ & + \int_{t}^{\tau} \exp\{-\gamma(\sigma-t) - \gamma(\tau-\sigma) + \mu_1\sigma + \mu_2\tau + \mu_3\sigma\} d\sigma \\ & + \int_{\tau}^{+\infty} \exp\{-\gamma(\sigma-t) - \gamma(\sigma-\tau) + \mu_1\sigma + \mu_2\sigma + \mu_3\sigma\} d\sigma. \end{aligned} \tag{10}$$

Condition (9) guarantees the convergence of each integral in expansion (10). The condition $0 \le t \le \tau$ also implies that $-\gamma(t+\tau) < -\gamma|\tau - t|$. Therefore, we have

$$\begin{aligned} J(t, \tau, \mu) = & \frac{1}{2\gamma - \mu} \exp\{-\gamma(t+3\tau) + \mu\tau\} \\ & + \frac{1}{2\gamma - \mu_1 - \mu_3} \exp\{-\gamma(t+\tau) + \mu_2\tau\}(\exp\{-(2\gamma - \mu_1 - \mu_3)t\} \end{aligned}$$

$$- \exp\{-(2\gamma - \mu_1 - \mu_3)\tau\}) + \frac{1}{2\gamma - \mu_1} \exp\{-\gamma(t-\tau) + \mu_2 \tau + \mu_3 t\}(1 - \exp\{-(2\gamma - \mu_1)t\})$$

$$+ \frac{1}{2\gamma + \mu_1} \exp\{-\gamma(t-\tau) + \mu_2 \tau + \mu_3 t\}(\exp\{(2\gamma + \mu_1)t\} - 1)$$

$$+ \frac{1}{\mu_1 + \mu_3} \exp\{-\gamma(\tau - t) + \mu_2 \tau\}(\exp\{(\mu_1 + \mu_3)\tau\} - \exp\{(\mu_1 + \mu_3)t\})$$

$$+ \frac{1}{2\gamma - \mu} \exp\{-\gamma(\tau - t) + \mu\tau\} \le K_1 \exp\{-\gamma(\tau - t) + \mu\tau\} = K_1 \exp\{-\gamma|\tau - t| + \mu|\tau|\},$$

where

$$K_1 = \frac{2(2\gamma + \mu)}{\mu_1(2\gamma - \mu)}.$$

Consider one of the cases where the parameters t and τ lie on different sides of the origin, namely, let $-t \le \tau \le 0$. Then we get

$$J(t, \tau, \mu) = \int_{-\infty}^{-t} \exp\{-\gamma(t-\sigma) - \gamma(\tau-\sigma) - \mu_1\sigma - \mu_2\sigma - \mu_3\sigma\} d\sigma$$

$$+ \int_{-t}^{-\tau} \exp\{-\gamma(t-\sigma) - \gamma(\tau-\sigma) - \mu_1\sigma - \mu_2\sigma + \mu_3 t\} d\sigma$$

$$+ \int_{\tau}^{0} \exp\{-\gamma(t-\sigma) - \gamma(\sigma-\tau) - \mu_1\sigma - \mu_2\tau + \mu_3 t\} d\sigma$$

$$+ \int_{0}^{-\tau} \exp\{-\gamma(t-\sigma) - \gamma(\sigma-\tau) + \mu_1\sigma - \mu_2\tau + \mu_3 t\} d\sigma$$

$$+ \int_{-\tau}^{t} \exp\{-\gamma(t-\sigma) - \gamma(\sigma-\tau) + \mu_1\sigma + \mu_2\sigma + \mu_3 t\} d\sigma$$

$$+ \int_{t}^{+\infty} \exp\{-\gamma(\sigma-t) - \gamma(\sigma-\tau) + \mu_1\sigma + \mu_2\sigma + \mu_3\sigma\} d\sigma$$

$$= \frac{1}{2\gamma - \mu} \exp\{-\gamma(3t + \tau) + \mu t\}$$

$$+ \frac{1}{2\gamma - \mu_1 - \mu_2} (\exp\{(2\gamma - \mu_1 - \mu_2)\tau\} - \exp\{-(2\gamma - \mu_1 - \mu_2)t\})$$

$$+ \frac{2}{\mu_1} \exp\{-\gamma(t-\tau) - \mu_2\tau + \mu_3 t\}(\exp\{-\mu_1\tau\} - 1)$$

$$+ \frac{1}{\mu_1+\mu_2} \exp\{-\gamma(t-\tau) + \mu_3 t\}(\exp\{(\mu_1+\mu_2)t\} - \exp\{-(\mu_1+\mu_2)\tau\})$$

$$+ \frac{1}{2\gamma-\mu} \exp\{-\gamma(t-\tau) + \mu t\} \le K_2 \exp\{-\gamma|t-\tau| + \mu|t|\},$$

where

$$K_2 = \frac{3\gamma}{\mu_1(2\gamma-\mu)}.$$

Here, we have also used condition (9).

Consider the other three cases where the parameters lie in the first and third quadrants of the plane $O\tau t$. Carrying out the change of variables $t \to \tau$ and $\tau \to t$, or $t \to -t$ and $\tau \to -\tau$, or $t \to -\tau$ and $\tau \to -t$, respectively, we establish that these cases reduce to the first case considered above. The cases where the parameters lie in the other two quadrants of this coordinate plane can be reduced to the second case considered above by the same changes of the variables. Thus, we obtain estimates of integrals (9) for the parameters that belong to each of eight regions of the plane $O\tau t$. By setting $K = \max\{K_1, K_2\}$, we complete the proof of the lemma.

Denote by $C^q_\nu(\boldsymbol{R}^m)$ a class of matrix or vector functions $F(\psi) \in C^q(\boldsymbol{R}^m)$ such that, for certain positive constants α, α_p, α'_p, and ν, the estimates

$$\|F(\psi)\| \le \alpha, \quad \|D^p_\psi F(\psi)\| \le \alpha_p \|\psi\|^{\nu|p|} + \alpha'_p \tag{11}$$

are true for all integer-valued vectors $p = (p_1, \ldots, p_m)$ such that $|p| = p_1 + \ldots + p_m \le q$. For example, $F(\psi) = \sin\psi^2$. For the vector function $a(\psi)$, we assume that

$$\|a(\psi)\| \le \alpha_1 \|\psi\| + \alpha_2, \tag{12}$$

$$\sup_{\psi \in \boldsymbol{R}^m} \|D^p_\psi a(\psi)\| < +\infty, \quad |p| = \overline{1, q}, \tag{13}$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$. As an example of such functions, we can mention $a(\psi) = \psi \sin \ln(1 + \psi^2)$, etc. We prove the following statement:

Theorem 1. *Suppose that the system of equations (1) has the unique Green function (3) satisfying estimate (4), a function $a(\psi) \in C^q(\boldsymbol{R}^m)$ is such that inequalities (12) and (13) are satisfied, and $A(\psi) \in C^q_\nu(\boldsymbol{R}^m)$. If the inequality*

$$2\gamma > q(\alpha_0 + \alpha_1\nu), \tag{14}$$

where

$$\alpha_0 = \sup_{\psi \in \boldsymbol{R}^m} \left(\max_{\|\eta\|=1} \left| \left\langle \frac{\partial a(\psi)}{\partial \psi} \eta, \eta \right\rangle \right| \right), \tag{15}$$

is satisfied, then the Green function $G_t(\tau, \psi)$ *has all continuous partial derivatives up to the order* q *inclusive and the following estimates are true:*

$$\|D^p_\psi G_t(\tau,\psi)\| \le \exp\{-\gamma|t-\tau| + |p|(\alpha_0+\alpha_1\nu)\max\{|t|,|\tau|\}\}\ (K_p\|\psi\|^{\nu|p|} + K'_p), \quad |p| = \overline{1,q}, \tag{16}$$

where K_p *and* K'_p *are certain positive constants independent of* ψ, t, *and* τ.

Proof. By virtue of the uniqueness of the Green function, according to [4], the difference $G_t(\tau,\psi) - G_t(\tau,\overline{\psi})$ can be represented as follows:

$$G_t(\tau,\psi) - G_t(\tau,\overline{\psi}) = \int_{-\infty}^{+\infty} G_t(\sigma,\psi)[A(\psi_\sigma(\psi)) - A(\psi_\sigma(\overline{\psi}))]G_\sigma(\tau,\overline{\psi})\,d\sigma. \tag{17}$$

Since the constants K and γ in estimate (7) are independent of ψ and τ, the boundedness of the matrix function $A(\psi)$ in the entire space R^m guarantees the uniform convergence of integral (17) in the parameters ψ and $\overline{\psi}$. Therefore, the right-hand side of the inequality

$$\begin{aligned} Z(\psi,\overline{\psi}) &:= \|G_t(\tau,\psi) - G_t(\tau,\overline{\psi})\| \\ &\le K^2\int_{-\infty}^{+\infty}\exp\{-\gamma(|t-\sigma| + |\sigma-\tau|)\}\|A(\psi_\sigma(\psi)) - A(\psi_\sigma(\overline{\psi}))\|\,d\sigma \\ &\le K^2\exp\{-\gamma(|t-\tau| + \delta\max\{|t|,|\tau|\}\}\int_{-\infty}^{+\infty}\exp\{-\delta|\sigma|\}\|A(\psi_\sigma(\psi)) - A(\psi_\sigma(\overline{\psi}))\|\,d\sigma \end{aligned}$$

is a continuous function in the collection of variables ψ, $\overline{\psi}$; here, δ is a certain fixed number that belongs to the interval $(0, 2\gamma)$ and, furthermore, for $\psi = \overline{\psi}$, we have $Z(\overline{\psi},\overline{\psi}) = 0$. This allows us to conclude that the Green function $G_0(\tau,\psi)$ continuously depends on the variables ψ.

Let $\psi - \overline{\psi} = (0,\ldots,0,\psi_i - \overline{\psi}_i, 0,\ldots,0)$ and let $\psi_i \ne \overline{\psi}_i$ for certain $i = \overline{1,m}$. Dividing relation (17) by $\psi_i - \overline{\psi}_i$, we obtain

$$\frac{G_t(\tau,\psi) - G_t(\tau,\overline{\psi})}{\psi_i - \overline{\psi}_i} = \int_{-\infty}^{+\infty} G_t(\sigma,\psi)\frac{A(\psi_\sigma(\psi)) - A(\psi_\sigma(\overline{\psi}))}{\psi_i - \overline{\psi}_i}G_\sigma(\tau,\overline{\psi})\,d\sigma.$$

Taking into account the continuous dependence of the Green function on the variable ψ, we formally pass to the limit as $\overline{\psi}_i \to \psi_i$ in the last equality. This yields

$$\frac{\partial G_t(\tau,\psi)}{\partial\psi_i} = \int_{-\infty}^{+\infty} N_1(t,\tau,\sigma,\psi)\,d\sigma, \tag{18}$$

where

$$N_1(t,\tau,\sigma,\psi) = G_t(\sigma,\psi)\left[\sum_{k=1}^{m}\frac{\partial A(\psi_\sigma(\psi))}{\partial\psi_{\sigma_k}}\frac{\partial\psi_{\sigma_k}}{\partial\psi_i}\right]G_\sigma(\tau,\psi). \tag{19}$$

This limit transition is correct whenever the integral on the right-hand side converges uniformly in $\psi \in D$ in every bounded domain $D \in \boldsymbol{R}^m$.

Let us estimate integrand (19). First, we estimate the solution $\psi_t(\psi)$ of Cauchy problem (2) and its derivatives. Assuming that $t \geq 0$, we represent the solution in the integral form as follows:

$$\psi_t(\psi) = \psi + \int_0^t a(\psi_\tau(\psi))\, d\tau. \tag{20}$$

Then, by virtue of (12), we can write

$$\| \psi_t(\psi) \| \leq \|\psi\| + \int_0^t (\alpha_1 \|\psi_\sigma(\psi)\| + \alpha_2)\, d\sigma,$$

whence it follows that [17]

$$\| \psi_t(\psi) \| \leq \|\psi\| \exp\{\alpha_1 |t|\} + \alpha_2 \alpha_1^{-1}(\exp\{\alpha_1 |t|\} - 1) \leq (\|\psi\| + \alpha_2 \alpha_1^{-1}) \exp\{\alpha_1 |t|\}. \tag{21}$$

In a similar way, we can verify that estimate (21) is also true for $t < 0$.

The fact that the vector function $a(\psi)$ belongs to the space $C^q(\boldsymbol{R}^m)$ implies that the solution $\psi_t(\psi)$ of system (2) also belongs to this space for any $t \in \boldsymbol{R}$. We substitute the solution $\psi_t(\psi)$ into system (2). The identity obtained can be differentiated q times with respect to each variable ψ_j, $j = \overline{1, m}$. We have

$$\frac{d}{dt}(D_\psi^p \psi_t(\psi)) \equiv D_\psi^p a(\psi(\psi)), \quad |p| = \overline{1, q}. \tag{22}$$

Denote by $\Omega_\tau^t \left(\frac{\partial a}{\partial \psi} \right)$ the normal fundamental matrix of solutions of the linear system of differential equations

$$\frac{dy}{dt} = \left(\frac{\partial a(\psi)}{\partial \psi} \bigg|_{\psi = \psi_t(\psi)} \right) y, \quad y \in \boldsymbol{R}^m.$$

Then the boundedness of the first derivatives of the function $a(\psi)$ yields

$$\left\| \Omega_\tau^t \left(\frac{\partial a}{\partial \psi} \right) \right\| \leq C_1 \exp\{\alpha_0 |t - \tau|\}, \tag{23}$$

where $C_1 = \text{const} > 0$ and α_0 is defined by equality (15). Thus, we have obtained an estimate for the first derivatives of $\psi_t(\psi)$. For $|p| = 2$, equality (23) has the form

$$\frac{d}{dt} \left(\frac{\partial^2 \psi_t(\psi)}{\partial \psi_i \partial \psi_j} \right) = \left(\frac{\partial a(\psi)}{\partial \psi} \bigg|_{\psi = \psi_t(\psi)} \right) \frac{\partial^2 \psi_t(\psi)}{\partial \psi_i \partial \psi_j} + R_2(\psi_t(\psi)), \tag{24}$$

where

$$R_2(\psi_t(\psi)) = \operatorname{colon}\left(\sum_{l=1}^{m}\sum_{k=1}^{m}\frac{\partial^2 a_1(\psi_t(\psi))}{\partial\psi_{t_l}\partial\psi_{t_k}}\frac{\partial\psi_{t_k}}{\partial\psi_j}\frac{\partial\psi_{t_l}}{\partial\psi_i},\ldots,\sum_{l=1}^{m}\sum_{k=1}^{m}\frac{\partial^2 a_m(\psi_t(\psi))}{\partial\psi_{t_l}\partial\psi_{t_k}}\frac{\partial\psi_{t_k}}{\partial\psi_j}\frac{\partial\psi_{t_l}}{\partial\psi_i}\right), \qquad i,j=\overline{1,m}.$$

Since

$$\left.\frac{\partial\psi_t(\psi)}{\partial\psi}\right|_{t=0} = \operatorname{colon}\big(\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{m-i}\big),$$

we have $[D_\psi^p\psi_t(\psi)]\big|_{t=0} = 0$ for $|p| = \overline{2,q}$. Writing the second derivative of $\psi_t(\psi)$ as a solution of the linear inhomogeneous system (24) with the inhomogeneity $R_2(\psi_t(\psi))$, for $t\geq 0$ we have

$$\frac{\partial^2\psi_t(\psi)}{\partial\psi_i\partial\psi_j} = \int_0^t \Omega_\sigma^t\left(\frac{\partial a}{\partial\psi}\right)R_2(\psi_\sigma(\psi))\,d\sigma.$$

Taking into account estimate (23) for the first derivatives and the boundedness of derivatives of $a(\psi)$ (13), we get

$$\left\|\frac{\partial^2\psi_t(\psi)}{\partial\psi_i\partial\psi_j}\right\| \leq \left\|\frac{\partial^2 a}{\partial\psi^2}\right\|_0 mC_1^3\int_0^t \exp\{\alpha_0|t-\sigma| + 2\alpha_0|\sigma|\}\,d\sigma = C_2\exp\{2\,\alpha_0|t|\}. \tag{25}$$

By analogy, we can obtain the same estimate for the second derivatives for $t<0$.

Further, by using the method of mathematical induction, we prove the general estimate

$$\|D_\psi^p\psi_t(\psi)\| \leq C_p\exp\{\alpha_0|p||t|\}, \qquad |p| = \overline{1,q}. \tag{26}$$

Beginning with the second derivative, this estimate is equivalent to the following:

$$\left\|D_\psi^{p_1}\left(\frac{\partial\psi_t(\psi)}{\partial\psi_i}\right)\right\| \leq C_p\exp\{\alpha_0(|p_1|+1)|t|\}, \qquad |p_1| = \overline{1,q-1}, \tag{27}$$

where the vector p_1 is such that $|p_1|+1 = |p|$. As proved above, inequality (27) holds for $|p_1| = 1$ [inequality (25)]. Assume that inequality (27) is true for all $|p_1| \leq |l|-1$. Let us prove that it is also valid for $|p_1| = |l|$. For $p_1 = l$, equality (22) can be written as follows:

$$\frac{d}{dt}\left(D_\psi^l\left(\frac{\partial\psi_t(\psi)}{\partial\psi_i}\right)\right) \equiv \frac{\partial\psi_t(\psi)}{\partial\psi}\left(D_\psi^l\left(\frac{\partial\psi_t(\psi)}{\partial\psi_i}\right)\right) + R_l(\psi_t(\psi)), \tag{28}$$

where

$$R_l(\psi_t(\psi)) = D_\psi^l\left[\frac{\partial a(\psi_t(\psi))}{\partial\psi_t}\frac{\partial\psi_t(\psi)}{\partial\psi_i}\right] - \frac{\partial a(\psi_t(\psi))}{\partial\psi_t}\left(D_\psi^l\left(\frac{\partial\psi_t(\psi)}{\partial\psi_i}\right)\right).$$

The quantity $R_l(\psi_t(\psi))$ has the form of a differential expression with terms in the form of the products

$$D_\psi^{l-j}\left(\frac{\partial a(\psi_t(\psi))}{\partial\psi_t}\right)D_\psi^j\left(\frac{\partial\psi_t(\psi)}{\partial\psi_\nu}\right),$$

where j is an integer-valued vector $|j| = \overline{0, |l|-1}$, $\nu = \overline{1, m}$, with constant coefficients. Here,

$$D_\psi^{l-j}\left(\frac{\partial a(\psi_t(\psi))}{\partial \psi_t}\right) = \sum_{\sigma=1}^{|l-j|} D_{\psi_t}^{\sigma}\left(\frac{\partial a(\psi_t(\psi))}{\partial \psi_t}\right) \sum_{\eta} C_{\sigma\eta} (D_\psi \psi_t(\psi))^{\eta_1} (D_\psi^2 \psi_t(\psi))^{\eta_2} \ldots (D_\psi^{l-j} \psi_t(\psi))^{\eta_{l-j}},$$

where

$$\eta_1 + \eta_2 + \ldots + \eta_{l-j} = \sigma, \quad \eta_1 + 2\eta_2 + \ldots + |l-j|\eta_{l-j} = |l-j|,$$

and $C_{\sigma\eta}$ are certain positive constants. Then, by virtue of inequality (13) and the induction hypothesis, we obtain the estimate

$$\left\| D_\psi^{l-j}\left(\frac{\partial a(\psi_t(\psi))}{\partial \psi_t}\right) D_\psi^{j}\left(\frac{\partial \psi_t}{\partial \psi}\right)\right\| \le M \exp\{\alpha_0(|l-j||t| + (|j|+1)\alpha_0|t|\}$$

$$= M \exp\{\alpha_0(|l|+1)|t|\} \quad \forall t \in \boldsymbol{R}, \tag{29}$$

where

$$M = C_{j+1} \sum_{\sigma=1}^{|l-j|} \sup_{\psi \in \boldsymbol{R}^m} \|D_\psi^{\sigma+1} a(\psi)\| \sum_{\eta} C_{\sigma\eta} \sum_{k=1}^{|l-j|} C_k^{\eta_k}.$$

Regarding (28) as an inhomogeneous system of equations, we obtain the following equality for the partial derivatives $D_\psi^l\left(\frac{\partial \psi_t(\psi)}{\partial \psi_i}\right)$ for $t \ge 0$:

$$D_\psi^l\left(\frac{\partial \psi_t(\psi)}{\partial \psi_i}\right) = \int_0^t \Omega_\tau^t\left(\frac{\partial a}{\partial \psi}\right) R_l(\psi_\tau(\psi))\, d\tau.$$

Hence, by virtue of (23) and (29), we get estimate (27) for $|p_1| = |l|$. Similarly, we obtain an estimate for $t < 0$. Therefore, inequality (26) is true for all $|p| = \overline{1, q}$.

By using estimates (11) and (21), we obtain the following inequality:

$$\begin{aligned}
\|D_{\psi_t(\psi)}^p A(\psi_t(\psi))\| &\le \alpha_p \|\psi_t(\psi)\|^{\nu|p|} + \alpha_p' \\
&\le \alpha_p(\|\psi\| + \alpha_2\alpha_1^{-1})^{\nu|p|} \exp\{\alpha_1 \nu |p||t|\} + \alpha_p' \\
&\le \alpha_p' K(\nu|p|)(\|\psi\|^{\nu|p|} + (\alpha_2\alpha_1^{-1})^{\nu|p|}) \exp\{\alpha_1 \nu |p||t|\} + \alpha_p' \\
&= (\bar{\alpha}_p \|\psi\|^{\nu|p|} + \bar{\beta}_p) \exp\{\alpha_1 \nu|p||t| + \alpha_p'\}, \quad |p| = \overline{1, q},
\end{aligned} \tag{30}$$

where $\bar{\alpha}_p = \alpha_p K(\nu|p|)$, $\bar{\beta}_p = \alpha_p K(\nu|p|)\alpha_2^{\nu|p|}\alpha_1^{-\nu|p|}$, and

$$K(z) = \begin{cases} 2^{z-1}, & z > 1, \\ 1, & z \in [0; 1]. \end{cases}$$

By using estimates (26) and (30) and inequality (7), we can now estimate integrand (19) as follows:

$$\|N_1(t, \tau, \sigma, \psi)\| \le K^2 C_1 \sqrt{m}\,[(\bar{\alpha}_1 \|\psi\|^{\nu} + \bar{\beta}_1) \exp\{\alpha_1 \nu |\sigma|\} + \alpha_1']$$

$$\times \exp\{-\gamma(|t-\sigma| + |\sigma - \tau|) + \alpha_0 |\sigma|\}$$

$$\le (K_1 \|\psi\|^{\nu} + \bar{K}_1) \exp\{-\gamma(|t-\tau| + |\sigma-\tau|) + (\alpha_0 + \alpha_1 \nu)|\sigma|\}, \tag{31}$$

where $K_1 = K^2 C_1 \sqrt{m}\,\bar{\alpha}_1$ and $\bar{K}_1 = K^2 C_1 \sqrt{m}\,(\bar{\beta}_1 + \alpha_1')$. By using the lemma with $\mu_1 = \alpha_0 + \alpha_1 \nu$ and $\mu_2 = \mu_3 = 0$, we conclude that inequality (14) with $q = 1$ guarantees the uniform convergence of integral (18). This implies that all partial derivatives of the Green function $\dfrac{\partial G_t(\tau, \psi)}{\partial \psi_i}$, $i = \overline{1, m}$, exist. Inequality (31) also yields estimate (16) for $|p| = 1$.

Thus, we have established the statement of the theorem for $q = 1$. By using the method of mathematical induction, we complete the proof of the theorem. Assume that the Green function has all continuous partial derivatives up to the order $q \le |l| - 1$, where

$$l = (l_1, \ldots, l_m), \qquad |l| = \sum_{i=1}^{m} l_i.$$

Also assume that estimates (16) hold for all p such that $|p| = \overline{1, |l|-1}$. Let us prove that inequality (14) with $q = |l|$ guarantees that $G_t(\tau, \psi)$ belongs to $C^l(R^m)$ and estimate (16) holds for the indicated value of q.

By analogy with (18), we represent the derivatives of the Green function of the $|l|$th order as follows:

$$D_\psi^l G_t(\tau, \psi) = \int_{-\infty}^{+\infty} \sum_{|\lambda_1| + |\lambda_2| + |\lambda_3| = |l|} C_{\lambda_1, \lambda_2, \lambda_3} D_\psi^{\lambda_1} G_t(\sigma, \psi)\, D_\psi^{\lambda_2} A(\psi_\sigma(\psi))\, D_\psi^{\lambda_3} G_\sigma(\tau, \psi)\, d\sigma, \tag{32}$$

where $C_{\lambda_1, \lambda_2, \lambda_3}$ are certain arbitrary constants,

$$\lambda_i = (\lambda_{i1}, \ldots, \lambda_{im}), \quad \lambda_{1j} \ge 0, \quad \lambda_{2j} \ge 1, \quad \lambda_{3j} \ge 0, \quad |\lambda_i| = \sum_{j=1}^{m} \lambda_{ij},$$

$$D_\psi^{\lambda_2} A(\psi_t(\psi)) = \sum_{\theta=1}^{|\lambda_2|} D_\psi^{\theta} A(\psi)|_{\psi = \psi_\sigma(\psi)}$$

$$\times \sum_{\omega} C_{\theta\omega} (D_\psi \psi_\sigma(\psi))^{\omega_1} (D_\psi^2 \psi_\sigma(\psi))^{\omega_2} \ldots (D_\psi^{\lambda_2} \psi_\sigma(\psi))^{\omega_{\lambda_2}}, \tag{33}$$

$$\omega_1 + \omega_2 + \ldots + \omega_{\lambda_2} = \theta, \qquad \omega_1 + 2\omega_2 + \ldots + |\lambda_2| \omega_{\lambda_2} = |\lambda_2|.$$

Since $|\lambda_1| \le |l|-1$ and $|\lambda_3| \le |l|-1$, according to the induction hypothesis, the derivatives of the Green function in the integrand satisfy inequalities (16). By virtue of (26) and (30), we get

$$\|D_\psi^{\lambda_2} A(\psi_\sigma(\psi))\| \le \sum_{\theta=1}^{|\lambda_2|} [(\alpha_\theta \|\psi\|^{\nu\theta} + \beta_\theta)\exp\{\alpha_1 \nu\theta|\sigma|\}]$$

$$\times \sum_{\omega} \overline{C}_{\theta\omega} \exp\{\alpha_0(\omega_1 + 2\omega_2 + \ldots + |\lambda_2|\omega_{\lambda_2})|\sigma|\}$$

$$\le (M_{\lambda_2}\|\psi\|^{\nu|\lambda_2|} + \overline{M}_{\lambda_2})\exp\{|\lambda_2|(\alpha_0 + \alpha_1\nu)|\sigma|\},$$

where M_{λ_2} and $\overline{M}_{\lambda_2}$ are certain sufficiently large positive constants independent of ψ and σ. Since $|\lambda_1| + |\lambda_2| + |\lambda_3| = |l|$, taking into account the estimates for $D_\psi^{\lambda_1} G_t(\sigma,\psi)$ and $D_\psi^{\lambda_3} G_t(\sigma,\psi)$ and the estimate proved above with the use of the lemma for $\mu_1 = |\lambda_2|\xi$, $\mu_2 = |\lambda_1|\xi$, $\mu_3 = |\lambda_3|\xi$, and $\xi = \alpha_0 + \alpha_1\nu$, we establish the uniform convergence of the integral on the right-hand side of (32). Furthermore, by using inequality (14), we obtain estimate (16) for the Green function for $q = |l|$. Thus, Theorem 1 is proved.

Remark 1. Under the conditions of Theorem 1, the projection matrix $C(\psi)$ belongs to the same class as $A(\psi)$.

Indeed, since $C(\psi) = G_0(0,\psi)$, by virtue of Theorem 1, there exist all derivatives $D_\psi^p C(\psi)$, $|p| = \overline{1,q}$. By setting $t = \tau = 0$ in inequality (16), we obtain

$$\|D_\psi^p C(\psi)\| \le K_p\|\psi\|^{\nu|p|} + \overline{K}_p, \quad |p| = \overline{1,q},$$

whence it follows that $C(\psi) \in C_\nu^q(\boldsymbol{R}^m)$.

Theorem 2. *Suppose that the conditions of Theorem 1 are satisfied. If the inequality*

$$\gamma > q(\alpha_0 + \alpha_1\nu) \tag{34}$$

is true, then, for any fixed vector function $f(\psi) \in C_\nu^q(\boldsymbol{R}^m)$, *the inhomogeneous system (5) has a unique bounded invariant manifold* $x = u(\psi)$ *defined by equality (6) and, moreover,* $u(\psi) \in C_\nu^q(\boldsymbol{R}^m)$.

Proof. It is obvious that if inequality (34) holds, then inequality (14) is also satisfied. Then, by virtue of Theorem 1, $G_t(\tau,\psi) \in C^q(\boldsymbol{R}^m)$ and inequalities (16) hold for all p such that $|p| = \overline{1,q}$. Taking into account that the vector function belongs to the class $C_\nu^q(\boldsymbol{R}^m)$, we obtain the estimate

$$\|D_{\psi_t(\psi)}^p f(\psi_t(\psi))\| \le (L_p^{(1)}\|\psi\|^{\nu|p|} + L_p^{(2)})\exp\{\alpha_1\nu|p||t|\}, \quad t \in \boldsymbol{R}, \quad |p| = \overline{1,q}, \tag{35}$$

with certain positive constants $L_p^{(1)}$ and $L_p^{(2)}$ independent of ψ and t.

As mentioned above, the existence of the unique Green function (3), (4) of system (1) guarantees the existence of a unique bounded invariant manifold of the form (6) for system (5) for every $f(\psi) \in C_\nu^q(\boldsymbol{R}^m)$. Let us show that $u(\psi) \in C_\nu^q(\boldsymbol{R}^m)$. Differentiating the integrand in (6) $|p|$ times with respect to the parameters $\psi_1, \ldots, \psi_m$, we obtain

$$D_\psi^p G_0(\tau,\psi) f(\psi_\tau(\psi)) = \sum_{|s|+|r|=|l|} C_{sr} D_\psi^s G_0(\tau,\psi) D_\psi^r f(\psi_\tau(\psi)) := \bar{N}_p(\tau,\psi), \tag{36}$$

where C_{sr} are certain positive constants, and the vector indices $s = (s_1, \dots, s_m)$ and $r = (r_1, \dots, r_m)$ are such that

$$s_i \geq 0, \quad r_j \geq 0, \quad |s| = \sum_{i=1}^{m} s_i, \quad |r| = \sum_{j=1}^{m} r_j.$$

In (36), we have

$$D_\psi^r f(\psi_t(\psi)) = \sum_{\theta=1}^{|r|} D_{\psi_\tau}^\theta f(\psi_\tau(\psi)) \sum_\omega C_{\theta\omega} (D_\psi \psi_\tau(\psi))^{\omega_1} (D_\psi^2 \psi_\tau(\psi))^{\omega_2} \dots (D_\psi^r \psi_\tau(\psi))^{\omega_r}$$

with the indices of summation

$$\omega_1 + \omega_2 + \dots + \omega_r = \theta, \quad \omega_1 + 2\omega_2 + \dots + |r|\omega_r = |r|.$$

Taking (26) and (35) into account, we obtain the estimate

$$\|D_\psi^r f(\psi_\tau(\psi))\| \leq (T_r^{(1)} \|\psi\|^{\nu|r|} + T_r^{(2)}) \exp\{|r|(\alpha_0 + \alpha_1 \nu)|\tau|\}, \quad \tau \in \boldsymbol{R}, \tag{37}$$

where

$$T_r^{(i)} = \sum_{\theta=1}^{|r|} L_\theta^{(i)} \sum_\omega C_{\theta\omega} \prod_{\eta=1}^{\omega_r} (C_\eta)^\eta, \quad i = \overline{1,2}.$$

Taking estimates (4), (16), and (37) into account, we can conclude that inequality (34) guarantees the uniform convergence of the integrals

$$\int_{-\infty}^{+\infty} \bar{N}_p(\tau,\psi) d\tau, \quad |p| = \overline{1,q} \tag{38}$$

in $\psi \in D$ in every bounded domain $D \in \boldsymbol{R}^m$. This implies that all partial derivatives of the bounded invariant manifold $x = u(\psi)$ exist. Furthermore, estimating (38), we get

$$\|D_\psi^p u(\psi)\| \leq B_p^{(1)} \|\psi\|^{\nu|p|} + B_p^{(2)}, \quad |p| = \overline{1,q},$$

where

$$B_p^{(i)} = K^{|p|} \sum_{|s|+|r|=|p|} C_{sr} T_r^{(i)}, \quad i = \overline{1,2},$$

i.e., $u(\psi) \in C_\nu^q(\boldsymbol{R}^m)$. The theorem is proved.

Imposing stronger conditions on the vector function $a(\psi)$ and weaker conditions on $A(\psi)$ and $f(\psi)$, we can also obtain certain results concerning the smoothness of $G_t(\tau, \psi)$, $u(\psi)$, and $C(\psi)$. We introduce the class $C^q_{\nu,\beta}(R^m)$, $0<\beta<1$, of matrix or vector functions $F(\psi) \in C^q(R^m)$ satisfying the conditions

$$\|D^p_\psi F(\psi)\| \leq L_p \exp\{\nu |p| \, \|\psi\|^{1-\beta}\} + L'_p, \tag{39}$$

where α, ν, L_p, and L'_p are positive constants, and

$$p = (p_1, \ldots, p_m), \quad |p| = \sum_{i=1}^{m} p_i, \quad p_i \in N.$$

Theorem 3. *Suppose that there exists a unique Green function of the problem of bounded invariant manifolds, $A(\psi) \in C^q_{\nu,\beta}(R^m)$, and a function $a(\psi) \in C^q(R^m)$ satisfies the condition*

$$\|a(\psi)\| \leq \alpha_1 \|\psi\|^\beta, \quad 0<\beta<1, \tag{40}$$

and inequality (13). Then the inequality

$$2\gamma > q(\alpha_0 + \alpha_1 \nu(1-\beta)) \tag{41}$$

is a sufficient condition for the validity of the inclusions $G_t(\tau, \psi) \in C^q(R^m)$ and $C(\psi) \in C^q_{\nu,\beta}(R^m)$ and the estimate

$$\|D^p_\psi G_t(\tau, \psi)\| \leq (W_p \exp\{\nu |p| \|\psi\|^{1-\beta}\} + \overline{W}_p)$$
$$\times \exp\{-\gamma(|t-\tau| + |p|(\alpha_0 + \alpha_1 \nu(1-\beta)) \max\{|t|, |\tau|\}\}, \quad |p| = \overline{1, q}. \tag{42}$$

Proof. By virtue of the uniqueness of the Green function $G_t(\tau, \psi)$, we can write the difference of the values of this function for different values of the parameter ψ in the form (17) and verify that $G_t(\tau, \psi) \in C^0(R^m)$ (see the proof of Theorem 1). Then we can write the formal equality (18) and prove that, under the conditions of Theorem 3, the integral on the right-hand side of this equality is uniformly convergent in ψ in every bounded domain $D \subset R^m$. By virtue of condition (13), we obtain estimate (26) for $D^p_\psi \psi_t(\psi)$.

Writing the integral representation (20) of the unique solution of system (2) for $t \geq 0$ and using (40), we obtain the inequality

$$\|\psi_t(\psi)\| \leq \|\psi\| + \int_0^t \|\psi_\sigma(\psi)\|^\beta d\sigma.$$

By solving this integral inequality (see Theorem 1.12 in [7]), we get

$$\|\psi_t(\psi)\| \leq (\|\psi\|^{1-\beta} + \alpha_1(1-\beta)|t|)^{1/(1-\beta)}.$$

Analogously, we establish that the last estimate is also true for $t<0$. Thus, with regard for (39), we write

$$\|D^p_{\psi_t(\psi)} A(\psi_t(\psi))\| \le L_p \exp\{\nu|p|\,\|\psi_t(\psi)\|^{1-\beta}\} + L'_p$$
$$\le L_p \exp\{\nu|p|(\|\psi\|^{1-\beta} + \alpha_1(1-\beta)|t|)\} + L'_p$$
$$\le (L_p \exp\{\nu|p|\,\|\psi\|^{1-\beta}\} + L'_p)\exp\{\alpha_1\nu(1-\beta)|p|\,|t|\}, \qquad |p| = \overline{1,q}.$$

Therefore, we can estimate the integrand in (18) as follows:

$$\|N_1(t,\tau,\sigma,\psi)\| \le K^2 C_1\sqrt{m}\,(L_p \exp\{\nu\|\psi\|^{1-\beta}\} + L'_p)$$
$$\times \exp\{-\gamma(|t-\sigma| + |\sigma-\tau|) + (\alpha_0 + \alpha_1\nu(1-\beta))|\sigma|\}.$$

Inequality (41) enables us to use Lemma 1 and thus prove the uniform convergence of integral (18) in ψ and establish estimate (42) for $q = 1$. Hence, $G_t(\tau,\psi) \in C^1(R^m)$.

Further, by using the method of mathematical induction, we prove the theorem for any q. Assuming that the statement of the theorem is true for all

$$|q| = \overline{1, |l|-1}, \qquad |l| = \sum_{i=1}^{m} l_i \ge 2,$$

we prove that it is also true for $q = |l|$. We write the $|l|$th-order partial derivatives of the function $G_t(\tau,\psi)$ in the form of the formal equality (32). According to (33), we get

$$\|D^{\lambda_2}_{\psi} A(\psi_\sigma(\psi))\| \le \sum_{\theta=1}^{|\lambda_2|} (L_{\lambda_2}\exp\{\nu\theta\,\|\psi\|^{1-\beta}\} + L'_{\lambda_2})\exp\{\alpha_1\nu\theta(1-\beta)|\sigma|\}$$
$$\times \sum_{\omega} C_{\theta\omega}\prod_{\eta=1}^{\omega_{\lambda_2}} (C_\eta)^\eta \exp\{\alpha_0(\omega_1 + \omega_2 + \ldots + \omega_{\lambda_2})|\sigma|\}$$
$$\le (S_{\lambda_2}\exp\{\nu|\lambda_2|\,\|\psi\|^{1-\beta} + \bar{S}_{\lambda_2}\})\exp\{(\alpha_0 + \alpha_1\nu(1-\beta))|\lambda_2|\,|\sigma|\},$$

where S_{λ_2} and $\bar{S}_{\lambda_2}$ are certain positive constants. According to the induction hypothesis, estimates (42) hold for the derivatives of the Green function in the integrand in (32). Therefore, by virtue of Lemma 1, inequality (41) implies that integral (32) is uniformly convergent and estimate (42) holds for $q = |l|$.

By setting $t = \tau = 0$ in estimate (42), we establish that $C(\psi) \in C^q_{\nu,\beta}(R^m)$. The theorem is proved.

Theorem 4. *Suppose that system (1) has the unique Green function (3) satisfying estimate (4),* $A(\psi) \in C^q_{\nu,\beta}(R^m)$, *and inequalities (40) and (13) hold for the vector function* $a(\psi)$. *If the inequality*

$$\gamma > q(\alpha_0 + \alpha_1\nu(1-\beta)) \tag{43}$$

is true, then, for a fixed vector function $f(\psi) \in C^q_{\nu,\beta}(R^m)$, *system (5) has a unique bounded invariant manifold* $x = u(\psi)$, *which also belongs to the class* $C^q_{\nu,\beta}(R^m)$.

Proof. Since condition (43) is stronger than (41), by virtue of Theorem 3, we can write estimates (42) for the derivatives of the Green function $G_t(\tau, \psi)$. By analogy with Theorem 2, we estimate the integrand of (38), taking into account that $f(\psi) \in C^q_{\nu,\beta}(R^m)$. This yields

$$\|\bar{N}_p(\tau,\psi)\| \le \sum_{|s|+|r|=|p|} C_{sr}(W_s \exp\{\nu|s|\,\|\psi\|^{1-\beta} + \overline{W}_s\})$$

$$\times \exp\{-\gamma|\tau| + |s|(\alpha_0 + \alpha_1\nu(1-\beta))|\tau|\}$$

$$\times \sum_{\theta=1}^{|r|} (L_r \exp\{\nu\theta\,\|\psi\|^{1-\beta}\} + L'_r)\exp\{\alpha_1\nu\theta(1-\beta)|\tau|\}$$

$$\times \sum_{\omega} C_{\theta\omega} \prod_{\eta=1}^{\omega_r} (C_\eta)^\eta \exp\{\alpha_0|r||\tau|\} \le (Q_p \exp\{\nu|p|\,\|\psi\|^{1-\beta}\} + \overline{Q}_p)$$

$$\times \exp\{-\gamma|\tau| + |p|(\alpha_0 + \alpha_1\nu(1-\beta))\}, \quad |p| = \overline{1,q}, \tag{44}$$

where Q_p and $\overline{Q}_p > 0$. Therefore, condition (41) guarantees the uniform convergence of integral (38) in ψ, which implies that there exist all partial derivatives of the bounded invariant manifold $u(\psi)$ up to order q inclusive. Estimate (44) also implies that $u(\psi) \in C^q_{\nu,\beta}(R^m)$. The theorem is proved.

Remark 2. Consider a linear extension of a dynamical system to an m-dimensional torus, i.e., the case where $\psi = \varphi \in \mathcal{T}_m$ in system (1). It is obvious that, under the conditions of Theorems 1–4, we have $\nu = 0$ and, therefore,

$$\|D^p_\varphi G_t(\tau,\varphi)\| \le K_p \exp\{-\gamma(|t-\tau| + \alpha_0|p|\max\{|t|,|\tau|\}\}, \quad |p| = \overline{1,q}, \quad t,\tau \in R.$$

By setting $t = 0$, we obtain known estimates for the derivatives of the Green function $G_0(\tau, \varphi)$ [4].

Remark 3. It turns out that the validity of inequalities (14) and (41) under the conditions of Theorem 2 and Theorem 4, respectively, is insufficient for the bounded invariant manifold to belong to the class $C^q(R^m)$, i.e., if the inequalities

$$\gamma \le q(\alpha_0 + \alpha_1\nu) < 2\gamma$$

and, correspondingly (Theorem 4),

$$\gamma \le q(\alpha_0 + \alpha_1\nu(1-\beta)) < 2\gamma$$

are satisfied, then the Green function $G_t(\tau, \psi)$ necessarily belongs to the class $C^q(R^m)$, whereas the invariant manifold $u(\psi)$ may not belong to this class.

Let us illustrate this statement by the following example:

Example 1. Consider the system

$$\frac{d\psi}{dt} = \tanh\psi, \quad \frac{dx}{dt} = x + \frac{2\sinh\psi}{\cosh^2\psi},$$

where $\psi,\ x \in \boldsymbol{R}$, for which $\alpha_0 = 1$ and $\nu = 0$. This system has a unique Green function $G_0(\tau, \psi)$ with the matrix $C(\psi) \equiv 0$ and index $\gamma = 1$; according to (3), this function has the form

$$G_0(\tau, \psi) = \begin{cases} 0, & \tau \le 0, \\ -e^{-\tau}, & \tau > 0. \end{cases}$$

The invariant manifold is defined by the expression

$$x = u(\psi) = \int_0^{+\infty} \frac{(-2)e^{-\tau}\sinh\psi_\tau(\psi)}{\cosh^2\psi_\tau(\psi)}\,d\tau = -2\int_0^{+\infty} \frac{\sinh\psi\, e^{-2\tau}}{e^{-2\tau} + \sinh^2\psi}\,d\tau = \sinh\psi\ln\tanh^2\psi.$$

In the case considered, we have $2\gamma > \alpha_0 + \alpha_1\nu = \gamma$, and $q = 1$, i.e., the conditions of Theorem 1 are satisfied, whereas the conditions of Theorem 2 are not satisfied. Although the Green function belongs to the space $C^1(\boldsymbol{R})$ (moreover, it is analytic in ψ), the invariant manifold $u(\psi)$ does not have a finite derivative at the point $\psi = 0$:

$$\lim_{\psi\to 0}\frac{du(\psi)}{d\psi} = \lim_{\psi\to 0}(\cosh\psi\ln\tanh^2\psi + 2\cosh^{-1}\psi) = -\infty;$$

this means that $u(\psi)$ does not belong to $C^1(\boldsymbol{R})$.

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