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Moduli of Continuity of the Derivatives of Invariant Tori for Linear Extensions of Dynamical Systems

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Numerous papers [1–11] dealt with the study of invariant tori of dynamical systems. The introduction [6] of the Green function $G_0(\tau,\varphi)$ of the problem on an invariant torus allowed exposing perturbation theory for differentiable and continuous invariant manifolds in a unified manner and necessitated studying the smoothness properties of that function [7–11]. The present paper also deals with this problem.

We consider the system of differential equations

$$d\varphi/dt = a(\varphi), \qquad dx/dt = A(\varphi)x + f(\varphi), \tag{1}$$

where $x \in \mathbf{R}^n$, $\varphi \in \mathscr{T}_m$, $\mathscr{T}_m$ is the m-dimensional torus, $a(\varphi), A(\varphi), f(\varphi) \in C^0(\mathscr{T}_m)$, and $C^0(\mathscr{T}_m)$ is the space of functions jointly continuous in φ and 2π-periodic in each φ_j, $j = 1,\dots,m$. By $\varphi_t(\varphi)$ one usually denotes the solution of the Cauchy problem

$$d\varphi/dt = a(\varphi), \qquad \varphi|_{t=0} = \varphi, \tag{2}$$

and $\Omega_\tau^t(\varphi)$ stands for the Cauchy matrix of the linear system

$$dx/dt = A(\varphi_t(\varphi))\,x \tag{3}$$

normalized at the point $t = \tau$ by the condition $\Omega_\tau^\tau(\varphi) = I_n$, where I_n is the $n \times n$ identity matrix. We also introduce the following notation: $C^q(\mathscr{T}_m) \subset C^0(\mathscr{T}_m)$, $q \ge 1$, is the subspace of functions $F(\varphi)$ with continuous partial derivatives $D_\varphi^p F(\varphi)$, $|p| = \sum_{i=1}^m p_m$, $|p| = 1,\dots,q$; $C'(\mathscr{T}_m; a) \subset C^0(\mathscr{T}_m)$ is the subspace of functions $F(\varphi)$ such that the composition $F(\varphi_t(\varphi))$ treated as a function of the variable t is continuously differentiable and $dF(\varphi_t(\varphi))/dt|_{t=0} := \dot F(\varphi) \in C^0(\mathscr{T}_m)$; for an $n \times n$ matrix $B(\varphi)$, we set $\|B\|_0 = \max_{\varphi\in\mathscr{T}_m}\|B(\varphi)\|$ and $\|B\| = \max_{\|x\|=1}\|Bx\|$; here $\|y\| = \langle y, y\rangle$, where $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is the inner product in $\mathbf{R}^n$.

Suppose that system (1) has an invariant torus $x = u(\varphi)$ or a Green function. Then we can naturally ask how continuous these functions are with respect to φ depending on the continuity of the matrix function $A(\varphi)$ and the vector functions $a(\varphi)$ and $f(\varphi)$. The dependence is by no means obvious. For example, even if the right-hand sides of system (1) are continuously differentiable, the invariant torus need not satisfy the Lipschitz condition with respect to φ. It also turns out that the Green function $G_0(\tau,\varphi)$ has better continuity properties than the invariant torus.

In the present paper, we study the behavior of the moduli of continuity for higher-order derivatives of the Green function and the invariant torus of system (1) and obtain estimates and convergence conditions for these derivatives.

According to [6], the homogeneous system

$$d\varphi/dt = a(\varphi), \qquad dx/dt = A(\varphi)x, \qquad x \in \mathbf{R}^n, \qquad \varphi \in \mathscr{T}_m, \tag{4}$$

corresponding to (1) has a Green function of the invariant torus problem if there exists an $n \times n$ matrix function $C(\varphi) \in C^0(\mathscr{T}_m)$ such that the function

$$G_0(\tau,\varphi) = \begin{cases} \Omega_\tau^0(\varphi)C(\varphi_\tau(\varphi)) & \text{for } \tau \le 0, \\ \Omega_\tau^0(\varphi)\left[C(\varphi_\tau(\varphi)) - I_n\right] & \text{for } \tau > 0 \end{cases} \tag{5}$$

satisfies the estimate

$$\|G_0(\tau,\varphi)\| \le K\exp\{-\gamma|\tau|\}, \tag{6}$$

where K and γ are positive constants independent of $\varphi \in \mathcal{T}_m$ and $t \in \mathbf{R}$. The function $G_0(\tau,\varphi)$ is called the *Green function* of the invariant torus problem for system (4). The existence of this function implies the existence of an invariant torus of system (1) for each vector function $f(\varphi) \in C^0(\mathcal{T}_m)$; the invariant torus is given by

$$x = u(\varphi) = \int_{-\infty}^{+\infty} G_0(\tau,\varphi) f(\varphi_\tau(\varphi))\, d\tau. \tag{7}$$

Recall that a relation of the form $x = u(\varphi)$ determines an invariant torus of system (1) if $u(\varphi) \in C'(\mathcal{T}_m; a)$ and $\dot u(\varphi) \equiv A(\varphi)u(\varphi) + f(\varphi)$ for all $\varphi \in \mathbf{R}^m$.

Note that the estimate (6) for the Green function (5) is equivalent to the estimate

$$\|G_t(\tau,\varphi)\| \le K\exp\{-\gamma|t-\tau|\} \tag{8}$$

for the function $G_t(\tau,\varphi) = \Omega_0^t(\varphi)G_0(\tau,\varphi)$. This follows from the identity $\Omega_\tau^t(\varphi_\theta(\varphi)) \equiv \Omega_{\tau+\theta}^{t+\theta}(\varphi)$, which holds for the Cauchy matrix of system (3) for any $t,\tau,\theta \in \mathbf{R}$.

Let $\Phi(\varphi) \in C^0(\mathcal{T}_m)$ be a matrix or vector function. The scalar function

$$\omega(\Phi;\sigma) = \sup_{\|\varphi-\bar\varphi\|\le\sigma} \|\Phi(\varphi) - \Phi(\bar\varphi)\|$$

is called the modulus of continuity of Φ. If $\Phi(\varphi) \in C^q(\mathcal{T}_m)$, $q \ge 1$, then we set

$$\omega_p(\Phi;z) = \begin{cases} \max\left\{\omega\left(D_\varphi^p\Phi;z\right), L_{q-1}(\Phi)z\right\} & \text{for } \ |p| = q, \\ L_{|p|}(\Phi)z & \text{for } \ |p| = 1,\dots,q-1, \end{cases}$$

where $L_p(\Phi) = \max_{|i|=0,\dots,|p|} L_i(\Phi)$ and $L_i(\Phi)$ is the Lipschitz constant of the corresponding $|i|$th-order partial derivative of $\Phi(\varphi)$, i.e., a constant such that $\left\|D_\varphi^i\Phi(\varphi) - D_\varphi^i\Phi(\bar\varphi)\right\| \le L_i(\Phi)\|\varphi-\bar\varphi\|$. We also write $\omega_0(A;z) = \omega(A;z)$, $\omega_0(f;z) = \omega(f;z)$, and $\omega_0(a;z) \equiv 0$.

For an arbitrary function $\Phi(\varphi)$ in the above-mentioned class, we set

$$J_\nu(\Phi;p;z) = \int_{-\infty}^{+\infty} \exp\{-\nu|\sigma|\}\omega_p\left(\Phi;F^{-1}(F(z)+|\sigma|)\right) d\sigma, \qquad |p| = 0,\dots,q,$$

where ν is an arbitrary positive constant and $F(z) = \int_\eta^z \omega(a;z)^{-1}d\sigma$, $\eta = \text{const} > 0$.

Using the above-introduced notation, we estimate the difference between the values of a derivative of the solution of the Cauchy problem (2) at points $\varphi, \bar\varphi \in \mathcal{T}_m$, $\varphi \ne \bar\varphi$.

Lemma. *If $a(\varphi) \in C^q(\mathcal{T}_m)$, $q \ge 1$, then*

$$\left\|D_\varphi^p\varphi_t(\varphi) - D_\varphi^p\varphi_t(\bar\varphi)\right\| \le M_p\exp\{(\alpha|p|+\nu)|t|\}J_\nu(a;p;\|\varphi-\bar\varphi\|), \qquad |p| = 1,\dots,q, \tag{9}$$

for an arbitrary positive constant ν, where the M_p are positive constants and α is determined by the inequality $\alpha \ge \max_{\|\xi\|=1}\|(\partial a/\partial\varphi)\xi\|$.

Proof. Since $a(\varphi)$ is a periodic solution, it follows that $\omega(a;\sigma) = \omega(a;\eta)$ for sufficiently large σ and η. Let us represent the difference of solutions of the Cauchy problem (2) for $t \ge 0$ in the integral form $\varphi_t(\varphi) - \varphi_t(\bar\varphi) = \varphi - \bar\varphi + \int_0^t (a(\varphi_\sigma(\varphi)) - a(\varphi_\sigma(\bar\varphi)))\, d\sigma$. Then this difference admits the estimate $\|\varphi_t(\varphi) - \varphi_t(\bar\varphi)\| \le \|\varphi-\bar\varphi\| + \int_0^t \omega(a; \|\varphi_\sigma(\varphi) - \varphi_\sigma(\bar\varphi)\|)\, d\sigma$, and so

$$\|\varphi_t(\varphi) - \varphi_t(\bar\varphi)\| \le F^{-1}(F(\|\varphi-\bar\varphi\|) + |\sigma|), \tag{10}$$

where F^{-1} is the inverse function of F. In a similar way, we can show that the estimate (10) remains valid for $t < 0$.

We substitute the solution of system (2) for different $\varphi, \bar{\varphi} \in \mathscr{T}_m$ into the system itself and write out the difference

$$(d/dt)\left(\varphi_t(\varphi) - \varphi_t(\bar{\varphi})\right) = a\left(\varphi_t(\varphi)\right) - a\left(\varphi_t(\bar{\varphi})\right). \tag{11}$$

Since $a(\varphi) \in C^q(\mathscr{T}_m)$, it follows that the solution of this system also belongs to this class for any $t \in \mathbf{R}$; therefore, we can differentiate identity (11) q times with respect to any of the variables φ_i, $i = 1, \ldots, m$. In particular, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i} - \frac{\partial\varphi_t(\bar{\varphi})}{\partial\varphi_i}\right) &= \frac{\partial a(\varphi)}{\partial\varphi}\bigg|_{\varphi=\varphi_t(\varphi)} \frac{\partial\varphi_t(\varphi)}{\partial\varphi_i} - \frac{\partial a(\varphi)}{\partial\varphi}\bigg|_{\varphi=\varphi_t(\bar{\varphi})} \frac{\partial\varphi_t(\bar{\varphi})}{\partial\varphi_i} \\ &= \left(\frac{\partial a(\varphi)}{\partial\varphi}\bigg|_{\varphi=\varphi_t(\varphi)} - \frac{\partial a(\varphi)}{\partial\varphi}\bigg|_{\varphi=\varphi_t(\bar{\varphi})}\right)\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i} + \left(\frac{\partial a(\varphi)}{\partial\varphi}\bigg|_{\varphi=\varphi_t(\bar{\varphi})}\right)\left(\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i} - \frac{\partial\varphi_t(\bar{\varphi})}{\partial\varphi_i}\right).\end{aligned} \tag{12}$$

Treating system (12) as a system of linear nonhomogeneous equations for the difference of derivatives of solutions of problem (2), we have

$$\begin{aligned}\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i} - \frac{\partial\varphi_t(\bar{\varphi})}{\partial\varphi_i} &= \left(\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i} - \frac{\partial\varphi_t(\bar{\varphi})}{\partial\varphi_i}\right)\bigg|_{t=0} \\ &\quad + \int_0^t \Omega_\sigma^t\left(\frac{\partial a}{\partial\varphi}\right)\left(\frac{\partial a(\varphi)}{\partial\varphi}\bigg|_{\varphi=\varphi_\sigma(\varphi)} - \frac{\partial a(\varphi)}{\partial\varphi}\bigg|_{\varphi=\varphi_\sigma(\bar{\varphi})}\right)\frac{\partial\varphi_\sigma(\bar{\varphi})}{\partial\varphi_i}d\sigma\end{aligned} \tag{13}$$

for $t \geq 0$, where $\Omega_\tau^t(\partial a/\partial\varphi)$ is the Cauchy matrix of the system $dy/dt = \left(\partial a(\varphi)/\partial\varphi|_{\varphi=\varphi_t(\varphi)}\right) y$, $y \in \mathbf{R}^m$.

Next, using the estimate [7, p. 190]

$$\left\|D_\varphi^p\varphi_t(\varphi)\right\| \leq C_p \exp\{\alpha|p|\,|t|\}, \qquad |p| = 1, \ldots, q, \tag{14}$$

where the C_p are some positive constants, we obtain

$$\begin{aligned}\left\|\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i} - \frac{\partial\varphi_t(\bar{\varphi})}{\partial\varphi_i}\right\| &\leq \int_0^1 C_1\exp\{\alpha|t-\sigma|\}\omega\left(\frac{\partial a}{\partial\varphi}; \|\varphi_\sigma(\varphi) - \varphi_\sigma(\bar{\varphi})\|\right)\exp\{\alpha|\sigma|\}d\sigma \\ &\leq C_1^2\exp\{\alpha|t|\}\int_0^t \exp\{\nu(|t|-|\sigma|)\}\omega\left(\frac{\partial a}{\partial\varphi}; F^{-1}\left(F(\|\varphi-\bar{\varphi}\|) + |\sigma|\right)\right)\exp\{\alpha|\sigma|\}d\sigma \\ &\leq \frac{1}{2}C_1^2\exp\{(\alpha+\nu)|t|\}\int_{-\infty}^{+\infty}\exp\{-\nu|\sigma|\}\omega_1\left(a; F^{-1}\left(F\left(\|\varphi-\bar{\varphi}\|\right) + |\sigma|\right)\right)d\sigma.\end{aligned} \tag{15}$$

We have thereby proved the assertion of the lemma for $|p| = 1$. Further, we assume that the estimate (9) is valid for all p with $|p| \leq |l|$ and prove it for p such that $|p| = |l| + 1$. Let us differentiate identity (11) $|l|$ times with respect to arbitrary variables φ_i, $i = 1, \ldots, m$, assuming that $|l| + 1 \leq q$. We have

$$\begin{aligned}&\frac{d}{dt}\left(D_\varphi^l\left(\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i}\right) - D_\varphi^l\left(\frac{\partial\varphi_t(\bar{\varphi})}{\partial\varphi_i}\right)\right) \\ &= \frac{\partial a\left(\varphi_t(\varphi)\right)}{\partial\varphi_t}\left[D_\varphi^l\left(\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i}\right) - D_\varphi^l\left(\frac{\partial\varphi_t(\bar{\varphi})}{\partial\varphi_i}\right)\right] + \left[\frac{\partial a\left(\varphi_t(\varphi)\right)}{\partial\varphi_t(\varphi)} - \frac{\partial a\left(\varphi_t(\bar{\varphi})\right)}{\partial\varphi_t(\bar{\varphi})}\right]D_\varphi^l\left(\frac{\partial\varphi_t(\bar{\varphi})}{\partial\varphi_i}\right) \\ &\quad + R_l\left(\varphi_t(\varphi)\right) - R_l\left(\varphi_t(\bar{\varphi})\right),\end{aligned}$$

where

$$R_l\left(\varphi_t(\varphi)\right) = D^l_\varphi \left[\frac{\partial a\left(\varphi_t(\varphi)\right)}{\partial \varphi_t}\frac{\partial \varphi_t(\varphi)}{\partial \varphi_i}\right] - \frac{\partial a\left(\varphi_t(\varphi)\right)}{\partial \varphi_t}\left(D^l_\varphi\left(\frac{\partial \varphi_t(\varphi)}{\partial \varphi_i}\right)\right).$$

Since $R_l\left(\varphi_t(\varphi)\right)$ is a differential expression containing the terms

$$\sum_{|\theta|=1}^{|l-j|} D^\theta_{\varphi_t}\left(\frac{\partial a\left(\varphi_t(\varphi)\right)}{\partial \varphi_t}\right)\sum_\zeta C_{\theta\zeta}\left(D_\varphi\varphi_t(\varphi)\right)^{\zeta_1}\left(D^2_\varphi\varphi_t(\varphi)\right)^{\zeta_2}\cdots\left(D^{l-j}_\varphi\varphi_t(\varphi)\right)^{\zeta_{l-j}} D^j_\varphi\left(\frac{\partial\varphi_t(\varphi)}{\partial\varphi_s}\right),$$
$$|j| = 0,\ldots,|i-1|, \qquad s = 1,\ldots,m,$$

with constant coefficients, where $\zeta_1+\zeta_2+\cdots+\zeta_{l-j} = |\theta|$ and $\zeta_1 + 2\zeta_2 + \cdots + |l-j|\zeta_{l-j} = |l-j|$, we can estimate the difference of $(l+1)$st-order partial derivatives of the solutions of the Cauchy problem (2), use an argument similar to (13), and take into account the relations $\left[D^p_\varphi\varphi_t(\varphi)\right]\Big|_{t=0} = 0$ for $|p| = 2,\ldots,q$, thus obtaining

$$\begin{aligned}
\left\|D^{l+1}_\varphi\varphi_t(\varphi) - D^{l+1}_\varphi\varphi_t(\bar\varphi)\right\| &= \left\|D^l_\varphi\left(\frac{\partial\varphi_t(\varphi)}{\partial\varphi_i}\right) - D^l_\varphi\left(\frac{\partial\varphi_t(\bar\varphi)}{\partial\varphi_i}\right)\right\| \\
&\le 0 + \int_0^t \left\|\Omega^t_\sigma\left(\frac{\partial a}{\partial\varphi}\right)\left[\frac{\partial a\left(\varphi_\sigma(\varphi)\right)}{\partial\varphi_\sigma(\varphi)} - \frac{\partial a\left(\varphi_\sigma(\bar\varphi)\right)}{\partial\varphi_\sigma(\bar\varphi)}\right] D^{l+1}_\varphi\varphi_\sigma(\varphi)\right\| d\sigma \\
&\quad + \int_0^t \left\|\Omega^t_\sigma\left(\frac{\partial a}{\partial\varphi}\right)\right\| \left\|R_l\left(\varphi_\sigma(\varphi)\right) - R_l\left(\varphi_\sigma(\bar\varphi)\right)\right\| d\sigma := I_1 + I_2.
\end{aligned}$$

Just as in (15), using inequality (14), we estimate the first integral as follows:

$$\begin{aligned}
I_1 &\le \int_0^1 C_1\exp\{\alpha|t-\sigma|\}\omega\left(\frac{\partial a}{\partial\varphi};\|\varphi_\sigma(\varphi)-\varphi_\sigma(\bar\varphi)\|\right) C_{l+1}\exp\{\alpha(l+1)|\sigma|\}d\sigma \\
&\le (c)\exp\{\alpha|t|\}\int_0^t \exp\{\alpha l|\sigma|\}\omega\left(a;F^{-1}\left(|\sigma| + F\left(\|\varphi-\bar\varphi\|\right)\right)\right)d\sigma \\
&\le (c)\exp\{(l+1)\alpha|t|\}\int_0^t \exp\{\nu(|t|-|\sigma|)\}\omega_1\left(a;F^{-1}\left(|\sigma| + F\left(\|\varphi-\bar\varphi\|\right)\right)\right)d\sigma \\
&\le (c)\exp\{(\alpha(l+1)+\nu)|t|\}\int_0^{+\infty} \exp\{-\nu|\sigma|\}\omega_{l+1}\left(a;F^{-1}\left(|\sigma| + F\left(\|\varphi-\bar\varphi\|\right)\right)\right)d\sigma.
\end{aligned}$$

Here and in the following, the symbols (c) and (K) stand for various positive constants.

Let us estimate the second integral:

$$\begin{aligned}
I_2 &\le \int_0^t C_1\exp\{\alpha|t-\sigma|\}\left\|D^{l-j}_\varphi\left(\frac{\partial a\left(\varphi_\sigma(\varphi)\right)}{\partial\varphi_\sigma}\right) - D^{l-j}_\varphi\left(\frac{\partial a\left(\varphi_\sigma(\bar\varphi)\right)}{\partial\varphi_\sigma}\right)\right\|\left\|D^{l-j}_\varphi\left(\frac{\partial\varphi_\sigma(\varphi)}{\partial\varphi_s}\right)\right\| d\sigma \\
&\quad + \int_0^t C_1\exp\{\alpha|t-\sigma|\}\left\|D^{l-j}_\varphi\left(\frac{\partial a\left(\varphi_\sigma(\bar\varphi)\right)}{\partial\varphi_\sigma}\right)\right\|\left\|D^j_\varphi\frac{\partial\varphi_\sigma(\varphi)}{\partial\varphi_s} - D^j_\varphi\frac{\partial\varphi_\sigma(\bar\varphi)}{\partial\varphi_s}\right\| d\sigma := J_1 + J_2.
\end{aligned}$$

For the integrand of J_1, we have

$$\begin{aligned}
&\left\| D_{\varphi}^{l-j}\left(\partial a\left(\varphi_{\sigma}(\varphi)\right)/\partial\varphi_{\sigma}(\varphi)\right) - D_{\varphi}^{l-j}\left(\partial a\left(\varphi_{\sigma}(\bar\varphi)\right)/\partial\varphi_{\sigma}(\bar\varphi)\right)\right\| \\
&\quad\le (K)\Big[\left\| D_{\varphi_{\sigma}}^{\theta}\left(\partial a\left(\varphi_{s}(\varphi)\right)/\partial\varphi_{\sigma}(\varphi)\right) - D_{\varphi_{\sigma}}^{\theta}\left(\partial a\left(\varphi_{s}(\bar\varphi)\right)/\partial\varphi_{\sigma}(\bar\varphi)\right)\right\| \\
&\qquad\times \left\| D_{\varphi}\varphi_{\sigma}(\varphi)\right\|^{\zeta_1}\left\| D_{\varphi}^{2}\varphi_{\sigma}(\varphi)\right\|^{\zeta_2}\cdots\left\| D_{\varphi}^{l-j}\varphi_{\sigma}(\varphi)\right\|^{\zeta_{l-j}} \\
&\qquad+\left\| D_{\varphi_{\sigma}}^{\theta}\left(\partial a\left(\varphi_{\sigma}(\bar\varphi)\right)/\partial\varphi_{\sigma}\right)\right\| \\
&\qquad\times\Big[\left\| D_{\varphi}\varphi_{\sigma}(\varphi)-\varphi_{\sigma}(\bar\varphi)\right\|\left\|\left(D_{\varphi}^{2}\left(\varphi_{\sigma}(\varphi)\right)^{\zeta_2}\right)\cdots\left(D_{\varphi}^{l-j}\left(\varphi_{\sigma}(\varphi)\right)^{\zeta_{l-j}}\right)\right\|+\cdots \\
&\qquad+\left\| D_{\varphi}\varphi_{\sigma}(\varphi)\cdots\left(D_{\varphi}^{l-j-1}\right)^{l-j-1}\right\|\left\|\left(D_{\varphi}^{l-j}\varphi_{\sigma}(\varphi)\right)^{l-j}-\left(D_{\varphi}^{l-j}\varphi_{\sigma}(\bar\varphi)\right)^{l-j}\right\|\Big]\Big] := D.
\end{aligned}$$

Since

$$\begin{aligned}
&\left\|\left(D_{\varphi}^{p}\varphi_{\sigma}(\varphi)\right)^{\zeta_p}-\left(D_{\varphi}^{p}\varphi_{\sigma}(\bar\varphi)\right)^{\zeta_p}\right\| \\
&\quad\le\zeta_p\left(\max\left\{\left\|D_{\varphi}^{p}\varphi_{\sigma}(\varphi)\right\|,\ \left\|D_{\varphi}^{p}\varphi_{\sigma}(\bar\varphi)\right\|\right\}\right)^{\zeta_{p-1}}\left\|D_{\varphi}^{p}\varphi_{\sigma}(\varphi)-D_{\varphi}^{p}\varphi_{s}(\bar\varphi)\right\|,
\end{aligned}\tag{16}$$

and $|\theta|$ varies from 1 to $|l-j|$, we have

$$\begin{aligned}
D&\le(K)\exp\{\alpha|\theta|\,|\sigma|\}\sum_{s=1}^{|l-j|}\omega_s\left(a;F^{-1}\left(|\sigma|+F\left(\|\varphi-\bar\varphi\|\right)\right)\right)\exp\left\{\left(|l-j|-\zeta_s\right)|\sigma|\right\} \\
&\le(K)\exp\{\alpha|l-j|\,|\sigma|\}\omega_{l-j}\left(a;F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right)+|\sigma|\right)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_1&\le(K)\int\limits_0^t\exp\{\alpha(|t-\sigma|+|l-j|\,|\sigma|)\}\omega_{|l-j|}\left(a;F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right)+|\sigma|\right)\right)\left\|D_{\varphi}^{j+1}\varphi_{\sigma}(\varphi)\right\|d\sigma \\
&\le\exp\{(a(l+1)+\nu)|t|\}\int\limits_{-\infty}^{+\infty}\exp\{-\nu|\sigma|\}\omega_{l+1}\left(a;F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right)+|\sigma|\right)\right)d\sigma.
\end{aligned}$$

Finally, since $|j|+1\le|l|$, it follows that

$$\begin{aligned}
J_2&\le\int\limits_0^t\exp\{\alpha|t-\sigma|\}(c)\left\|D_{\varphi}^{j+1}\varphi_{\sigma}(\varphi)-D_{\varphi}^{j+1}\varphi_{\sigma}(\bar\varphi)\right\|d\sigma \\
&\le(K)\int\limits_0^t\exp\{\alpha|t-\sigma|+(\alpha(j+1)+\nu)|\sigma|\}\int\limits_{-\infty}^{+\infty}\exp\{-\nu|\tau|\}\omega_{j+1}\left(a;F^{-1}\left(|\sigma|+F\left(\|\varphi-\bar\varphi\|\right)\right)\right)d\tau\,d\sigma \\
&\le(K)\exp\{\alpha|t|\}\int\limits_0^t\exp\{(\alpha j+\nu)|\sigma|\}\int\limits_{-\infty}^{+\infty}\exp\{\nu|\tau|\}\omega_{j+1}\left(a;F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right)+|\tau|\right)\right)d\tau\,d\sigma \\
&\le(K)\exp\{(\alpha(l+1)+\nu)|t|\}\int\limits_{-\infty}^{+\infty}\exp\{-\nu|\tau|\}\omega_l\left(a;F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right)+|\tau|\right)\right)d\tau.
\end{aligned}$$

Therefore, we have proved the estimate (9) for $|p|=|l|+1$ as well. The case $t<0$ can be considered in a similar way. This completes the proof of the lemma.

By analyzing the moduli of continuity of the Green function of system (4), one can obtain the following assertion.

Theorem 1. *Let $a(\varphi), A(\varphi) \in C^q(\mathscr{T}_m)$, and let system (4) have a unique Green function (5) satisfying the estimate (6) for the invariant torus problem. If*

$$2\gamma > \alpha q, \tag{17}$$

then there exist all partial derivatives of order $\leq q$ of the function $G_t(\tau, \varphi)$ with respect to φ, and

$$\begin{aligned}&\left\| D_\varphi^p G_t(\tau, \varphi) - D_\varphi^p G_t(\tau, \bar\varphi)\right\| \\ &\quad \leq \exp\{-\gamma|t-\tau| + (\alpha|p| + \nu)\max\{|t|, |\tau|\}\}\left(K_p J_\nu(a; p; \|\varphi - \bar\varphi\|) + \bar K_p J_\nu(A; p; \|\varphi - \bar\varphi\|)\right), \\ &\quad |p| = 0, \dots, q,\end{aligned}$$

for each $\nu \in (0, 2\gamma - \alpha|p|)$, where K_p and $\bar K_p$ are some positive constants.

Proof. As was mentioned above, the existence of a Green function (5) satisfying the estimate (6) provides the existence of $G_t(\tau, \varphi)$ and the validity of the estimate (8). By [7, p. 126], the difference of values of this function at points $\varphi \neq \bar\varphi$ admits the representation

$$G_t(\tau, \varphi) - G_t(\tau, \bar\varphi) = \int_{-\infty}^{+\infty} G_t(\sigma, \varphi)\left[A(\varphi_\sigma(\varphi)) - A(\varphi_\sigma(\bar\varphi))\right] G_\sigma(\tau, \bar\varphi)\, d\sigma. \tag{18}$$

Moreover, it was shown in [11] that inequality (17) guarantees the qth-order differentiability of the Green function and the validity of the estimates

$$\left\| D_\varphi^p G_t(\tau, \varphi)\right\| \leq K_p' \exp\{-\gamma|t-\tau| + \alpha|p|\max\{|t|, |\tau|\}\}, \qquad |p| = 1, \dots, q, \tag{19}$$

where the K_p' are positive constants. Therefore, we can differentiate both sides of inequality (18). Since condition (17) provides the convergence of the integral occurring on the right-hand side in the representation (32) in [11], we have

$$\begin{aligned}&D_\varphi^p G_t(\tau, \varphi) - D_\varphi^p(\tau, \bar\varphi) \\ &= \int_{-\infty}^{+\infty} \sum_{|\lambda_1| + |\lambda_2| + |\lambda_3| = |p|} C_{\lambda_1, \lambda_2, \lambda_3} D_\varphi^{\lambda_1} G_t(\sigma, \varphi)\left[D_\varphi^{\lambda_2} A(\varphi_\sigma(\varphi)) - D_\varphi^{\lambda_2}(\varphi_\sigma(\bar\varphi))\, D^{\lambda_3} G_\sigma(\tau, \bar\varphi)\right] d\sigma,\end{aligned} \tag{20}$$

where the $C_{\lambda_1,\lambda_2,\lambda_3}$ are some constants, $\lambda_i = (\lambda_{i1}, \dots, \lambda_{im})$, $\lambda_{1j} \geq 0$, $\lambda_{2j} \geq 1$, $\lambda_{3j} \geq 0$, $|\lambda_i| = \sum_{j=1}^m \lambda_{ij}$, and

$$D_\varphi^{\lambda_2} A(\varphi_t(\varphi)) = \sum_{|\theta|=1}^{|\lambda_2|} D_\varphi^\theta A(\varphi)\Big|_{\varphi = \varphi_t(\varphi)} \sum_\varrho C_{\theta\varrho}(D_\varphi \varphi_t(\varphi))^{\varrho_1}\left(D_\varphi^2 \varphi_t(\varphi)\right)^{\varrho_2} \cdots \left(D_\varphi^{\lambda_2}\varphi_t(\varphi)\right)^{\varrho_{\lambda_2}}, \tag{21}$$

$\varrho_1 + \varrho_2 + \dots + \varrho = |\theta|$, $\varrho_1 + 2\varrho_2 + \dots + |\lambda_2|\, \varrho_{\lambda_2} = |\lambda_2|$.

In the following, we estimate the difference of derivatives of the unique Green function on the basis of the representation (20).

We can readily see that

$$-(|t-\sigma| + |\sigma - \tau|) \leq -2|\sigma| + |t+\tau| \qquad \forall t, \tau, \sigma \in \mathbf{R}, \tag{22}$$

$$|t-\tau| + |t+\tau| \equiv \max\{|t|, |\tau|\} \qquad \forall t, \tau \in \mathbf{R}, \tag{23}$$

and

$$-\gamma(|t-\sigma| + |\sigma-\tau|) \leq -(\gamma - \delta)|t-\tau| - \delta(|t-\sigma| + |\sigma - \tau|) \tag{24}$$

for any $t, \tau, \sigma \in \mathbf{R}$ and γ, δ such that $\gamma > \delta > 0$. Using (22)–(24), for each $\beta \in (0, 2\gamma - \nu)$ we can write out the chain of inequalities

$$\begin{aligned} -\gamma(|t-\sigma| + |\sigma - \tau|) + \beta|\sigma| &\le -(\gamma - (\nu+\beta)/2)|t-\tau| - ((\nu+\beta)/2)(|t-\sigma| + |\sigma-\tau|) + \beta|\sigma| \\ &\le -(\gamma - (\nu+\beta)/2)|t-\tau| - ((\gamma+\beta)/2)(2|\sigma| - |t+\tau|) + \beta|\sigma| \\ &= -\gamma|t-\tau| + ((\nu+\beta)/2)(|t-\tau| + |t+\tau|) - (\gamma-\beta)|\sigma| + \beta|\sigma| \\ &= -\gamma|t-\tau| + (\nu+\beta)\max\{|t|, |\tau|\} - \nu|\sigma|. \end{aligned} \tag{25}$$

Therefore, for any positive constants γ, β_1, β_2, β_3, and ν such that $2\gamma > \beta_1 + \beta_2 + \beta_3 + \nu$ and for any $t, \tau, \sigma \in \mathbf{R}$, we have

$$\begin{aligned} &-\gamma(|t-\sigma| + |\sigma-\tau|) + \beta_1|\sigma| + \beta_2 \max\{|t|, |\sigma|\} + \beta_3 \max\{|\tau|, |\sigma|\} \\ &\qquad \le -\gamma|t-\tau| + (\beta_1 + \beta_2 + \beta_3 + \nu)\max\{|t|, |\tau|\} - \nu|\sigma|. \end{aligned} \tag{26}$$

Further, using inequality (25) and the representation (18), we write out the estimate

$$\begin{aligned} &\|G_t(\tau, \varphi) - G_t(\tau, \bar{\varphi})\| \le K^2 \int_{-\infty}^{+\infty} \exp\{-\gamma(|t-\sigma| + |\sigma-\tau|)\}\omega\left(A; \|\varphi_\sigma(\varphi) - \varphi_\sigma(\bar{\varphi})\|\right) d\sigma \\ &\le K^2 \exp\{-\gamma|t-\tau| + \nu\max\{|t|, |\tau|\}\} \int_{-\infty}^{+\infty} \exp\{-\nu|\sigma|\}\omega\left(A; F^{-1}\left(F\left(\|\varphi - \bar{\varphi}\|\right) + |\sigma|\right)\right) d\sigma. \end{aligned} \tag{27}$$

The norm of the difference of the values of the function (21) at different points φ can be estimated with the help of inequality (16) and the assertion of the lemma as follows:

$$\begin{aligned} &\left\| D_\varphi^{\lambda_2} A\left(\varphi_\sigma(\varphi)\right) - D_\varphi^{\lambda_2} A\left(\varphi_\sigma(\bar{\varphi})\right) \right\| \\ &\le \sum_{\theta=1}^{|\lambda_2|} \sum_{\varrho} C_{\theta\varrho} \left[\left\| D_\varphi^\theta A(\varphi)\Big|_{\varphi=\varphi_\sigma(\varphi)} - D_\varphi^\theta A(\varphi)\Big|_{\varphi=\varphi_\sigma(\bar{\varphi})} \right\| \right. \\ &\quad \times \left\| \left(D_\varphi \varphi_\sigma(\varphi)\right)^{\varrho_1} \left(D_\varphi^2 \varphi_\sigma(\varphi)\right)^{\varrho_2} \cdots \left(D^{\lambda_2} \varphi_\sigma(\varphi)\right)^{\varrho_{\lambda_2}} \right\| \\ &\quad + \left\| D_\varphi^\theta A(\varphi)\Big|_{\varphi=\varphi_\sigma(\varphi)} \right\| \left(\left\| \left(D_\varphi \varphi_\sigma(\varphi)\right)^{\varrho_1} - \left(D_\varphi \varphi_\sigma(\bar{\varphi})\right)^{\varrho_1} \right\| \right. \\ &\quad \times \left\| \left(D_\varphi^2 \varphi_\sigma(\varphi)\right)^{\varrho_2} \cdots \left(D_\varphi^{\lambda_2} \varphi_\sigma(\varphi)\right)^{\varrho_{\lambda_2}} \right\| + \cdots \\ &\quad \left. \left. + \left\| D_\varphi \varphi_\sigma(\bar{\varphi}) \cdots \left(D^{\lambda_2 - 1} \varphi_\sigma(\bar{\varphi})\right)^{\varrho_{\lambda_2 - 1}} \right\| \left\| \left(D_\varphi^{\lambda_2} \varphi_\sigma(\varphi)\right)^{\varrho_{\lambda_2}} - \left(D_\varphi^{\lambda_2} \varphi_\sigma(\bar{\varphi})\right)^{\varrho_{\lambda_2}} \right\| \right) \right] \\ &\le \sum_{\theta=1}^{|\lambda_2|} \sum_{\varrho} \prod_{\varrho=1}^{|\lambda_2|} C_\varrho \exp\left\{(\alpha|\lambda_2| + \nu)|\sigma|\right\} \left[\max_{\varphi \in \mathcal{T}_m} \left\| D_\varphi^\theta A(\varphi) \right\| \int_{-\infty}^{+\infty} \exp\{-\nu|\sigma|\} \sum_{i=1}^{|\lambda_2|} M_i \varrho_i C_i^{-1} \right. \\ &\quad \left. \times \omega_i\left(a; F^{-1}\left(|u| + F\left(\|\varphi - \bar{\varphi}\|\right)\right)\right) du + \exp\{-\nu|\sigma|\}\omega\left(D_\varphi^\theta A; F^{-1}\left(F\left(\|\varphi - \bar{\varphi}\|\right) + |\sigma|\right)\right) \right]. \end{aligned} \tag{28}$$

By a lemma in [11], the validity of the inequality $2\gamma > \mu$, where $\mu = \mu_1 + \mu_2 + \mu_3$, provides the convergence of the integral

$$J(\tau, t, \mu) = \int_{-\infty}^{+\infty} \exp\left\{-\gamma(|t-\sigma| + |\sigma-\tau|) + \mu_1|\sigma| + \mu_2 \max\{|\sigma|, |\tau|\} + \mu_3 \max\{|\sigma|, |t|\}\right\} d\sigma$$

with respect to the parameters $t, \tau \in \mathbf{R}$, where γ, μ_1 are positive constants, and μ_2, μ_3 are non-negative constants, and the validity of the estimate $J(\tau, t, \mu) \le K \exp\{-\gamma|t-\tau| + \mu\max\{|t|, |\tau|\}\}$, where $K = 2\left(2\gamma + \max\{\gamma, \mu\}\right) / \left(\mu_1(2\gamma - \mu)\right)$. Using this inequality and relations (19) and (26), for the derivatives of the Green function (20) we obtain

$$
\begin{aligned}
&\left\| D_\varphi^p G_t(\tau,\varphi) - D_\varphi^p G_t(\tau,\bar\varphi) \right\| \\
&\le K'_{\lambda_1} K'_{\lambda_3} \int_{-\infty}^{+\infty} \exp\{-\gamma(|t-\sigma| + |\sigma-\tau|) + \lambda_1 \max\{|t|,|\sigma|\} + \lambda_3 \max\{|\sigma|,|\tau|\}\} \\
&\quad \times \left\| D_\varphi^{\lambda_2} A(\varphi_\sigma(\varphi)) - D_\varphi^{\lambda_2} A(\varphi_\sigma(\bar\varphi)) \right\| d\sigma \\
&\le K'_{\lambda_1} K'_{\lambda_3} |\lambda_2| \sum_{\mu} \prod_{\mu=1}^{\lambda_2} C_\mu \Bigg[\exp\{\gamma|t-\tau| + (\alpha|p|+\nu)\} \max\{|t|,|\tau|\} \int_{-\infty}^{+\infty} \exp\{-\nu|\sigma|\} \\
&\quad \times \omega\left(D_\varphi^{\lambda_2} A; F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right) + |\sigma|\right)\right) d\sigma \\
&\quad + \sum_{i=1}^{|\lambda_2|} M_i C_i^{-1} \max_{|j|=0,\ldots,|\lambda_2|} \left\| D_\varphi^j A \right\|_0 \int_{-\infty}^{+\infty} \exp\{-\gamma(|t-\sigma| + |\sigma-\tau|) + (\alpha|p|+\nu)|\sigma|\} \\
&\quad \times \int_{-\infty}^{+\infty} \exp\{-\nu|\sigma|\} \omega_p\left(a; F^{-1}\left(|u| + F\left(\|\varphi-\bar\varphi\|\right)\right)\right) du\, d\sigma \Bigg] \\
&\le \exp\{-\gamma|t-\tau| + (\alpha|p|+\nu)\max\{|t|,|\tau|\}\} \Bigg(K_p \int_{-\infty}^{+\infty} \exp\{-\nu|\sigma|\} \omega_p\left(a; F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right) + |\sigma|\right)\right) d\sigma \\
&\quad + \bar K_p \int_{-\infty}^{+\infty} \exp\{-\nu|\sigma|\} \omega_p\left(A; F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right) + |\sigma|\right)\right) d\sigma \Bigg)
\end{aligned}
$$

with some sufficiently large constants K_p and $\bar K_p$. This completes the proof of the theorem.

Remark 1. Under the assumptions of Theorem 1, the projection matrix $C(\varphi)$ occurring in the definition of the Green function and the derivatives of $C(\varphi)$ satisfy the estimates

$$
\left\| D_\varphi^p C(\varphi) - D_\varphi^p C(\bar\varphi) \right\| \le K_p J_\nu\left(a; p; \|\varphi-\bar\varphi\|\right) + \bar K_p J_\nu\left(A; p; \|\varphi-\bar\varphi\|\right), \qquad |p| = 0,\ldots,q.
$$

Remark 2. To investigate the moduli of continuity of a nonunique Green function of system (4), one can use Theorem 1 and the method of extending weakly regular systems to regular ones [7, p. 137].

As was shown in [11], the smoothness conditions for the function determining an invariant torus of system (1) are more restrictive than the smoothness conditions for the corresponding Green function. Therefore, the continuity properties of the invariant torus with respect to φ are worse than those of the Green function of system (4).

Theorem 2. *If the condition*

$$
\gamma > \alpha q \tag{29}
$$

is imposed under the assumptions of Theorem 1, then for any $\nu \in (0, \gamma - \alpha|p|)$ and for each vector function $f(\varphi) \in C^q(\mathscr{T}_m)$, there exists a unique invariant torus $x = u(\varphi)$ with all partial derivatives of order $\le q$, and

$$
\begin{aligned}
\left\| D_\varphi^p u(\varphi) - D_\varphi^p u(\bar\varphi) \right\| &\le N_p J_\nu\left(a; p; \|\varphi-\bar\varphi\|\right) + \bar N_p J_\nu\left(A; p; \|\varphi-\bar\varphi\|\right) \\
&\quad + \bar{\bar N}_p J_\nu\left(f; p; \|\varphi-\bar\varphi\|\right), \qquad |p| = 0,\ldots,q,
\end{aligned} \tag{30}
$$

where N_p, $\bar N_p$, and $\bar{\bar N}_p$ are positive constants.

Proof. The existence and uniqueness of the invariant torus follow from the assumption on the existence and uniqueness of the Green function. Let us prove the estimates (30). Using the representation (7), we can write out the relation

$$u(\varphi) - u(\bar{\varphi}) = \int_{-\infty}^{+\infty} \left([G_0(\tau, \varphi) - G_0(\tau, \bar{\varphi})] f(\varphi_\tau(\varphi)) + G_0(\tau, \bar{\varphi}) [f(\varphi_\tau(\varphi)) - f(\varphi_\tau(\bar{\varphi}))] \right) d\tau, \quad (31)$$

which, together with (27), yields

$$\begin{aligned} \|u(\varphi) - u(\bar{\varphi})\| &\le \int_{-\infty}^{+\infty} \Big[K^2 \exp\{(-\gamma + \nu)|\tau|\} J_\nu (A; 0; \|\varphi - \bar{\varphi}\|) \max_{\varphi \in \mathcal{T}_m} \|f(\varphi)\| \\ &\quad + K \exp\{-\gamma|\tau|\} \omega \left(f; F^{-1}(F(\|\varphi - \bar{\varphi}\|) + |\tau|)\right) \Big] d\tau \\ &\le \frac{2K^2 \|f\|_0}{\gamma - \nu} J_\nu (A; 0; \|\varphi - \bar{\varphi}\|) + K J_\gamma (f; 0; \|\varphi - \bar{\varphi}\|). \end{aligned}$$

By Theorem 2 in [11], inequality (29) provides the existence of continuous partial derivatives $D_\varphi^p u(\varphi)$, $|p| = 1, \ldots, q$. Therefore, we can differentiate relation (31) q times, and condition (29) provides the uniform convergence of the integral occurring on the right-hand side. We have

$$\begin{aligned} D_\varphi^p u(\varphi) - D_\varphi^p u(\bar{\varphi}) = \int_{-\infty}^{+\infty} \sum_{|s|+|r|=|p|} C_{sr} \Big[& \left(D_\varphi^s G_0(\tau, \varphi) - D_\varphi^s G_0(\tau, \bar{\varphi}) \right) D_\varphi^r f(\varphi_\tau(\varphi)) \\ & + D_\varphi^s G_0(\tau, \bar{\varphi}) \left(D_\varphi^r f(\varphi_\tau(\varphi)) - D_\varphi^r f(\varphi_\tau(\bar{\varphi})) \right) \Big] d\sigma, \end{aligned} \quad (32)$$

where $C_{sr} = \text{const}$, $s = (s_1, \ldots, s_m)$ and $r = (r_1, \ldots, r_m)$ are multiindices such that $s_i \ge 0$ and $r_i \ge 0$, and

$$|s| = \sum_{i=1}^m s_i, \qquad |r| = \sum_{i=1}^m r_i,$$

$$D_\varphi^r f(\varphi_t(\varphi)) = \sum_{|\theta|=1}^{|r|} D_{\varphi_t}^\theta f(\varphi_t(\varphi)) \sum_\zeta C_{\theta\zeta} (D_\varphi \varphi_t(\varphi))^{\zeta_1} \left(D_\varphi^2 \varphi_t(\varphi) \right)^{\zeta_2} \cdots \left(D_\varphi^r \varphi_t(\varphi) \right)^{\zeta_r},$$

$$\zeta_1 + \zeta_2 + \cdots + \zeta_r = |\theta|, \qquad \zeta_1 + 2\zeta_2 + \cdots + |r|\zeta_r = |r|.$$

By analogy with the estimate (28), for the difference of derivatives of the function $f(\varphi_t(\varphi))$ we can write

$$\begin{aligned} \left\| D_\varphi^r f(\varphi_\tau(\varphi)) - D_\varphi^r f(\varphi_\tau(\bar{\varphi})) \right\| &\le \sum_{\theta=1}^{|r|} \sum_\zeta \prod_{k=1}^{|r|} C_k \exp\{(\alpha|r| + \nu)|\tau|\} \\ \times \Big[\max_{\varphi \in \mathcal{T}_m} & \left\| D_\varphi^\theta f(\varphi) \right\| \int_{-\infty}^{+\infty} \exp\{-\nu|\tau|\} \sum_{i=1}^{|r|} M_i \zeta_i C_i^{-1} \omega_i \left(a; F^{-1}(|\xi| + F(\|\varphi - \bar{\varphi}\|))\right) d\xi \\ + \exp\{-\nu|\tau|\} \omega & \left(D_\varphi^\theta f; F^{-1}(F(\|\varphi - \bar{\varphi}\|) + |\tau|) \right) \Big]. \end{aligned} \quad (33)$$

Further, taking account of the choice of the constant ν and using (19), (32), (33), and the assertion of Theorem 1, we obtain the chain of inequalities

$$\left\|D^p_\varphi u(\varphi) - D^p_\varphi u(\bar\varphi)\right\| \le \sum_{|s|+|r|=|p|} C_{sr}\Bigg[\max_{\varphi\in\mathcal{T}_m}\left\|D^r_\varphi f(\varphi)\right\| \int\limits_{-\infty}^{+\infty} \exp\{(-\gamma+\alpha|s|+\nu)|\tau|\} \int\limits_{-\infty}^{+\infty} \exp\{-\nu|\sigma|\}$$

$$\times\left(K_p\omega_p\left(a; F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right)+|\sigma|\right)\right) + \bar K_p\omega_p\left(A; F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right)+|\sigma|\right)\right)\right) d\sigma\, d\tau$$

$$+\int\limits_{-\infty}^{+\infty} K'_r\sum_{\theta=1}^{|r|}\sum_{\zeta}\prod_{k=1}^{|r|} C_k\exp\{(-\gamma+\alpha(|s|+|r|))|\tau|\}\Bigg[\max_{\varphi\in\mathcal{T}_m}\left\|D^\theta_\varphi f(\varphi)\right\|\exp\{\nu|\sigma|\}\int\limits_{-\infty}^{+\infty}\exp\{-\nu|\xi|\}$$

$$\times\sum_{i=1}^{|r|} M_i\zeta_i C_i^{-1}\omega_i\left(a; F^{-1}\left(|\xi| + F\left(\|\varphi-\bar\varphi\|\right)\right)\right) d\xi\, \omega\left(D^\theta_\varphi f; F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right)+|\sigma|\right)\right)\Bigg] d\tau\Bigg]$$

$$\le \sum_{|s|+|r|=|p|} C_{sr}\Bigg[\frac{2}{\gamma-\alpha|s|-\nu}\Bigg(\Bigg(K_p\max_{\varphi\in\mathcal{T}_m}\left\|D^r_\varphi f\right\|_0 + K'_r\sum_{\theta=1}^{|r|}\sum_{\zeta}\prod_{k=1}^{|r|} C_k M_\theta\zeta_\theta C_\theta^{-1}\Bigg)$$

$$\times\int\limits_{-\infty}^{+\infty}\exp\{-\nu|\sigma|\}\omega_i\left(a; F^{-1}\left(|\sigma| + F\left(\|\varphi-\bar\varphi\|\right)\right)\right) d\sigma$$

$$+\bar K_p\max_{\varphi\in\mathcal{T}_m}\left\|D^r_\varphi f\right\|_0\int\limits_{-\infty}^{+\infty}\exp\{-\nu|\sigma|\}\omega_i\left(A; F^{-1}\left(|\sigma| + F\left(\|\varphi-\bar\varphi\|\right)\right)\right) d\sigma\Bigg)$$

$$+\int\limits_{-\infty}^{+\infty}\exp\{(-\gamma+\alpha|p|)|\tau|\}\omega_i\left(f; F^{-1}\left(F\left(\|\varphi-\bar\varphi\|\right)+|\tau|\right)\right) d\tau\Bigg]$$

$$\le N_p J_\nu\left(a; p; \|\varphi-\bar\varphi\|\right) + \bar N_p J_\nu\left(A; p; \|\varphi-\bar\varphi\|\right) + \bar{\bar N}_p J_\nu\left(f; p; \|\varphi-\bar\varphi\|\right).$$

This completes the proof of the theorem.

Suppose that $a(\varphi), A(\varphi), f(\varphi) \in C^0(\mathcal{T}_m)$. Then we can ask whether the Green functions $G_t(\tau,\varphi)$ and the invariant torus $u(\varphi)$ are continuous with respect to the variables φ. It follows from (27) that for the continuity of the Green function with respect to φ it is sufficient that $J_\nu\left(A; 0; \|\varphi-\bar\varphi\|\right) \to 0$ as $\bar\varphi\to\varphi$, and for the continuity of the invariant torus we must additionally require that $J_\nu\left(f; 0; \|\varphi-\bar\varphi\|\right) \to 0$ as $\bar\varphi\to\varphi$ (see Theorem 2). By Theorems 1 and 2, to investigate the continuity of derivatives of $G_t(\tau,\varphi)$ and $u(\varphi)$, we must impose similar conditions on $J_\nu(\cdot\,; p; \cdot)$. Hence finding a criterion for the convergence of the functions J_ν to zero is quite important.

Theorem 3. *Let $a(\varphi) \in C^0(\mathcal{T}_m)$ and $\Phi(\varphi) \in C^q(\mathcal{T}_m)$. The integrals $J_\nu\left(\Phi; p; \|\varphi-\bar\varphi\|\right)$, $|p| = 0,\dots,q$, converge to zero as $\bar\varphi\to\varphi$ if and only if $F(z)$ is a divergent integral, i.e.,*

$$\lim_{z\to+0} F(z) = -\infty. \tag{34}$$

Proof. Let condition (34) be satisfied. Since $\Phi(\varphi)$ is a periodic function, it follows that the modulus of continuity has the following property: $\omega\left(D^p_\varphi\Phi;\sigma\right) \equiv \omega\left(D^p_\varphi\Phi;\eta\right)$, $|p| = 0,\dots,q$, are constant for all $\sigma\ge\eta$, where η is a sufficiently large number (for example, $\eta = 2\pi\sqrt{m}$). Hence we write out the integral $J_\nu\left(\Phi; p; \|\varphi-\bar\varphi\|\right)$ as a sum of integrals, in the first of which we perform the change of variables $F^{-1}(\sigma + F(z)) = \xi$ and set $z = \|\varphi-\bar\varphi\|$. We have

$$\int\limits_{-\infty}^{+\infty}\exp\{-\nu|\sigma|\}\omega_p\left(\Phi; F^{-1}(F(z)+|\sigma|)\right) d\sigma$$

$$= 2\left(\int\limits_{0}^{-F(z)}\exp\{-\nu\sigma\}\omega_p\left(\Phi; F^{-1}(\sigma+F(z))\right) d\sigma + \int\limits_{-F(z)}^{+\infty}\exp\{-\nu\sigma\}\omega_p(\Phi;\eta) d\sigma\right) \tag{35}$$

$$= 2\exp\{\nu F(z)\}\left(\frac{\omega_p(\Phi;\eta)}{\nu} + J_\nu(z)\right),$$

where $J_\nu(z) \equiv \int\limits_z^\eta \exp\{-\nu F(\sigma)\}\dfrac{\omega_p(\Phi;\sigma)}{\omega(a;\sigma)}d\sigma$. Suppose that

$$\lim_{z\to+0} J_\nu(z) = C < \infty.$$

Then it follows from condition (34) that

$$\lim_{z\to+0}\exp\{\nu F(z)\}\left(\frac{\omega_p(\Phi;\eta)}{\nu}+J_\nu(z)\right)=\left(\frac{\omega_p(\Phi;\eta)}{\nu}+C\right)\lim_{z\to+0}\exp\{\nu F(z)\}=0.$$

If

$$\lim_{z\to+0} J_\nu(z) = \infty,$$

then, using l'Hôpital's rule, we obtain

$$\begin{aligned}\lim_{z\to+0} J_\nu(\Phi;p;z) &= \lim_{z\to+0}\left(\left(\frac{\omega_p(\Phi;\eta)}{\nu}+J_\nu(z)\right)\Big/\exp\{-\nu F(z)\}\right)\\ &= \lim_{z\to+0}\left(-\exp\{-\nu F(z)\}\frac{\omega_p(\Phi;z)}{\omega(a;z)}\Big/\left((-\nu)\exp\{-\nu F(z)\}\frac{1}{\omega(a;z)}\right)\right)\\ &= \frac{1}{\nu}\lim_{z\to+0}\omega_p(\Phi;z)=0.\end{aligned}$$

Let us show that the convergence of any of the integrals $J_\nu(\Phi;p;\|\varphi-\bar\varphi\|)$, $|p|=0,\ldots,q$, to zero leads to the validity of condition (34). Indeed, if this is not the case, i.e., $F(\|\varphi-\bar\varphi\|) > -c > -\infty$ (where c is a positive constant), then

$$J_\nu(\Phi;p;\|\varphi-\bar\varphi\|) \ge 2\exp\{-c\nu\}\left(\omega_p(\Phi;\eta)/\nu + J_\nu(z)\right) \ge \left(2\omega_p(\Phi;\eta)/\nu\right)\exp\{-c\nu\},$$

which contradicts the convergence of the integral $J_\nu(\Phi;p;\|\varphi-\bar\varphi\|)$ to zero as $\bar\varphi\to\varphi$ and completes the proof of the theorem.

Note that condition (34) provides the uniqueness of the solution of the Cauchy problem (2) (the well-known Osgood theorem). In particular, the functions $L\sigma$, $L\sigma|\ln\sigma|$, $L\sigma|\ln\sigma|\ln|\ln\sigma|$, ..., where L is a positive constant, satisfy this condition.

Let us consider the case in which the right-hand sides of (4) satisfy the Lipschitz condition, i.e., $a(\varphi)$, $A(\varphi)\in C_{\mathrm{Lip}}(\mathscr{T}_m)$. Then

$$\omega(a;\sigma)=\begin{cases} L\sigma & \text{for } \sigma\in[0,\eta],\\ L\eta & \text{for } \sigma\in[\eta,\infty),\end{cases}\qquad \omega(A;\sigma)=\begin{cases} \bar L\sigma & \text{for } \sigma\in[0,\eta],\\ \bar L\eta & \text{for } \sigma\in[\eta,\infty),\end{cases}$$

and the function $F(z)$ has the form

$$F(z)=\begin{cases} L^{-1}\ln(z/\eta) & \text{for } z\in[0,\eta],\\ (L\eta)^{-1}(z-\eta) & \text{for } z\in[\eta,\infty).\end{cases}$$

Therefore, with regard for (25), we obtain

$$J_\nu(A;0;z)=2z^{\nu/L}\left(\frac{\omega(A;\eta)}{\nu\eta^{\nu/L}}+\frac{\bar L}{L}\int\limits_z^\eta\sigma^{-\nu/L}d\sigma\right)$$

for $z\in[0,\eta]$, since we are interested only in z close to 0. The integral $J_\nu(A;0;z)$ converges to zero as $z\to+0$.

Since, by (27), the unique Green function satisfies the inequality

$$\|G_0(\tau,\varphi)-G_0(\tau,\bar\varphi)\| \le 2K^2\exp\{(-\gamma+\nu)|\tau|\}\,\|\varphi-\bar\varphi\|^{\nu/L}\left(\frac{\omega(A;\eta)}{\nu\eta^{\nu/L}}+\frac{\bar L}{L}\int\limits_{\|\varphi-\bar\varphi\|}^{\eta}\sigma^{-\nu/L}d\sigma\right)$$

for any $\nu \in (0,\gamma)$, it follows that the condition $a(\varphi), A(\varphi) \in C_{\mathrm{Lip}}(\mathscr{T}_m)$ is sufficient for $G_t(\tau,\varphi) \in C^0(\mathscr{T}_m)$. Moreover, choosing $\nu \in (0, \min\{L,\gamma\})$, we find that under the above assumptions the Green function satisfies the Hölder condition

$$\|G_0(\tau,\varphi) - G_0(\tau,\bar{\varphi})\| \leq \bar{\bar{K}}\, \|\varphi - \bar{\varphi}\|^{\beta}, \qquad \varphi, \bar{\varphi} \in \mathscr{T}_m,$$

with exponent $\beta = \nu/L$ and constant $\bar{\bar{K}}$ independent of $\varphi \in \mathscr{T}_m$. In a similar way, using the estimates established in Theorem 2, we find that the invariant torus also satisfies the Hölder condition with the same exponent β if, in addition to the above-mentioned conditions, we require that the vector function $f(\varphi)$ also belongs to the class $C_{\mathrm{Lip}}(\mathscr{T}_m)$.

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