НАЦІОНАЛЬНА АКАДЕМІЯ НАУК УКРАЇНИ

ІНСТИТУТ МАТЕМАТИКИ

Препринт 2023.1

ПОКУТНИЙ ОЛЕКСАНДР ОЛЕКСІЙОВИЧ

DICHOTOMY AND BOUNDED SOLUTIONS OF EVOLUTION EQUATIONS IN THE BANACH AND HILBERT SPACES

КИЇВ-2023

УДК 517.9

Дихотомія та обмежені розв'язки еволюційних рівнянь у банахових та гільбертових просторах / Покутний О. - Київ, 2023. - 19 с. - (Препр./ НАН України. Ін-т математики; 2023)

Препринт 2023.1

У рамках науково-дослідної роботи проведено дослідження з теорії різницевих еволюційних рівнянь у банахових та гільбертових просторах. Отримано умови наявності слабкого гомоклінічного хаосу.

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Dichotomy and bounded solutions of evolution equations in the Banach and Hilbert spaces

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Abstract. For a general discrete dynamics on a Banach and Hilbert spaces we give necessary and sufficient conditions of the existence of bounded solutions under assumption that the homogeneous difference equation admits an discrete dichotomy on the semi-axes. We consider the so called resonance (critical) case when the uniqueness of solution is disturbed. We show that admissibility can be reformulated in the terms of generalized or pseudoinvertibility. As an application we consider the case when the corresponding dynamical system is e-trichotomy

Keywords: exponential dichotomy, Moore-Penrose pseudoinverse operator, ill-posed problems, e-trichotomy, dynamical systems

$MSC[2020], 39A70, 47B39, 39A22$

Introduction. The diffeomorphism of $\mathbb{R}^n$ into itself with a homoclinic point is chaotic (see [30], [31]). Using the Melnikov method [32] or the theory of Noetherian operators [37], planar periodic systems with the Poincare map were constructed, which have transversal homoclinic points and, therefore, chaotic. Shilnikov examined an autonomous system in space with an orbit homoclinic up to the saddle-node and showed, under certain conditions, the presence of chaos. In the article [26] Shilnikov again considered autonomous systems in space with orbits homoclinic to hyperbolic periodic orbits and showed, under transversality conditions, the existence of chaos. For finite-dimensional systems and periodic mappings, Palmer rigorously proved Smale's horseshoe theorem in 1988 using the shadowing lemma (see [27]). For infinite-dimensional mappings and periodic systems, the proof was completed by Steinlein and Walther (see [28]). For finite-dimensional autonomous systems, such a proof was performed by Palmer in 1996. For infinite-dimensional systems in [29]. It should also be noted recent monographs on chaos [38], [39] (see also [42]). Proposed in the article method gives possibility to observe so called "weak" chaos in discrete systems. A separate article will be devoted to this question. The concept of exponential dichotomy plays a key role in the qualitative theory of dynamical systems. It is worth to mention here the papers [4–9,11,13,18,19,24,25,35,43] where dichotomy for difference equations was studied. Exponential-dichotomous systems represent themselves a class of trajectories which can either grow exponentially or decrease with the exponential rate at infinity. It generalizes the well-known notion of hyperbolicity of an operator to the non-autonomous case (see [11]). A lot of scientific papers study difference equations that allow an exponential dichotomy on the whole integer axis $\mathbb{Z}$ or on the semiaxis $\mathbb{Z}_+$. It should be noted that the condition of the exponential dichotomy on $\mathbb{Z}$ is usually equivalent to the regularity of the respective system (it guarantees the existence of solution condition which is unique and bounded on the entire axis $\mathbb{Z}$). In contrast, the dichotomy condition on the semiaxes can guarantee only normal or strong generalized normal solvability. Such class of problems belongs to the irregular (resonant) class. Solutions to this kind of problems do not exist with all heterogeneity, and their number can be infinite. In the finite-dimensional case sufficiently full study of systems, that allow exponential dichotomy on the semiaxes, has been done in the book [9] where the Noetherian of the respective problem was shown. Note that the theory of generalized inverse and Moore-Penrose pseudo-inverse operators was used while studying the

[1]The work is supported by the grant of the National Research Fund of Ukraine (project 2020.02.0089)

underlying problem in irregular case (see [9,20,22]). In this paper, we study this class of systems in infinite-dimensional spaces. It is shown that in contrast to the finite-dimensional case, for the generated operator of the corresponding equation, there are possibly much more options. Namely, the generated operator can be strong generalized normally solvable, normally solvable, n-normal, d-normal, Noetherian or Fredholm.

Statement of the problem. Consider the following equation

$$x_{n+1} = A_n x_n + h_n, n \in \mathbb{Z}, \tag{1}$$

where $A_n : B \to B$ - is a set of bounded operators, from the Banach space B into itself. Assume that

$$A = (A_n)_{n\in\mathbb{Z}} \in l_\infty(\mathbb{Z}, \mathcal{L}(B)), h = (h_n)_{n\in\mathbb{Z}} \in l_\infty(\mathbb{Z}, B).$$

It means that

$$|||A||| = \sup_{n\in\mathbb{Z}} ||A_n|| < +\infty, |||h||| = \sup_{n\in\mathbb{Z}} ||h_n|| < +\infty.$$

Let us give the conditions for the existence of bounded solutions of the equation (1). The corresponding homogeneous difference equation has the following form:

$$x_{n+1} = A_n x_n. \tag{2}$$

It should be noted that an arbitrary solution of a homogeneous equation can be represented as: $x_m = \Phi(m, n)x_n, m \geq n$, where

$$\Phi(m, n) = \begin{cases} A_{m-1}A_{m-2}...A_n, \text{ if } m > n \\ I, \text{ if } m = n. \end{cases}$$

It is clear, that $\Phi(m, 0) = A_{m-1}A_{m-2}...A_0$. Also, we denote $U(m) := \Phi(m, 0)$ and $U(0) = I$.

Traditionally [11], the mapping $\Phi(m, n)$ is called the **evolutionary operator** of the problem (2). Suppose that the equation (2) is exponentially dichotomous on the semiaxes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projectors P and Q in the space B respectively, which means that there are projectors $P(P^2 = P)$ and $Q(Q^2 = Q)$, constants $k_{1,2} \geq 1$, $0 < \lambda_{1,2} < 1$ such that

$$||U(n)PU^{-1}(m)|| \leq k_1\lambda_1^{n-m}, n \geq m$$

$$||U(n)(I - P)U^{-1}(m)|| \leq k_1\lambda_1^{m-n}, m \geq n$$

for arbitrary $m, n \in \mathbb{Z}_+$ (dichotomy on $\mathbb{Z}_+$);

$$||U(n)QU^{-1}(m)|| \leq k_2\lambda_2^{n-m}, n \geq m$$

$$||U(n)(I - Q)U^{-1}(m)|| \leq k_2\lambda_2^{m-n}, m \geq n$$

for arbitrary $m, n \in \mathbb{Z}_-$ (dichotomy on $\mathbb{Z}_-$).

We give the conditions on the existence of solutions of the equation (1) that are bounded on the entire axis under the assumption, that the homogeneous equation (2) admits the exponential dichotomy on the semiaxes $\mathbb{Z}_+$, $\mathbb{Z}_-$ with the projectors P and Q introduced above. We use this definition for simplicity. The main result can be proved in more general case (see [13]) in the same way.

Recall some facts from the theory of generalized inverse and Moore-Penrose pseudo-inverse operators (see [9, 12, 15, 16]) which we will use a little bit later.

Definition. (see [9]). *A linear bounded operator D from one Banach space B_1 into another Banach space B_2 is called* **normally-solvable***, if its image $ImD = R(D)$ is closed $(\overline{R(D)} = R(D))$.*

Definition (see [9, 15]). *Operator $D \in \mathcal{L}(B_1, B_2)$ is called* **generalized invertible***, if there exists operator denoted by $D^- \in \mathcal{L}(B_1, B_2)$, such that*

$$D^- D D^- = D^-, D D^- D = D.$$

Note, that if the operator D has inverse D^{-1}, left inverse D_l^{-1} or right inverse D_r^{-1} then it coincides with D^-: $D^{-1} = D^-, D_l^{-1} = D^-, D_r^{-1} = D^-$ (see [9, 12, 15, 16]).

The following criterion of generalized invertibility is well-known (see [9, 15]).

Theorem 1. *Operator D has a generalized inverse if and only if the following conditions hold:*

i) D is a normally-solvable operator;

ii) The kernel $kerD = N(D)$ of the operator D is a complemented subspace of B_1 (there exists a subspace X of the Banach space B_1 such that $N(D) \oplus X = B_1$);

iii) The image $ImD = R(D)$ of the operator D is a complemented subspace of B_2 (there exists a subspace Y of the Banach space B_2 such that $R(D) \oplus Y = B_2$).

If we consider the operator $D \in \mathcal{L}(H_1, H_2)$ from one Hilbert space H_1 into another Hilbert space H_2, then from the set of all generalized inverse operators we can choose the operator D^+ which has the following properties:

$$D^+ D D^+ = D^+, \qquad D D^+ D = D,$$

$$(D D^+)^* = D D^+, \qquad (D^+ D)^* = D^+ D.$$

The operator $D^+ \in \mathcal{L}(H_2, H_1)$ is called the **Moore-Penrose pseudo-inverse** to the operator D (see [20, 22]). It follows from the Theorem 1 that the operator D in a Hilbert space has Moore-Penrose pseudo-inverse if and only if it is normally-solvable (which means the set of its values is closed) (see [9, 12]).

1 Main Results. Linear case

1.1 Banach space case

We state and prove the main result of the existence of bounded solutions of the inhomogeneous equation (1) defined in Banach space.

Theorem 2. *Suppose that a homogeneous equation is exponential dichotomous on the semi-axes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projectors P and Q respectively, and the operator*

$$D = P - (I - Q) : B \to B$$

is generalized invertible. The solutions of the equation (1) bounded on the entire axis $\mathbb{Z}$ exist if and only if the following condition is satisfied

$$\sum_{k=-\infty}^{+\infty} H(k+1)h_k = 0. \tag{3}$$

If the condition (3) holds, the set of bounded solutions has the following view:

$$x_n(c) = U(n)P\mathcal{P}_{N(D)}c + (G[h])(n), \qquad c \in B \tag{4}$$

where

$$G[h](n) = \begin{cases} \sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k - \sum_{k=n}^{+\infty} U(n)(I-P)U^{-1}(k+1)h_k + \\ \quad +U(n)PD^{-}[\sum_{k=0}^{+\infty}(I-P)U^{-1}(k+1)h_k + \\ \quad + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \;\; n \geq 0 \\ \sum_{k=-\infty}^{n-1} U(n)QU^{-1}(k+1)h_k - \sum_{k=n}^{-1} U(n)(I-Q)U^{-1}(k+1)h_k + \\ \quad +U(n)(I-Q)D^{-}[\sum_{k=0}^{+\infty}(I-P)U^{-1}(k+1)h_k + \\ \quad + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \;\; n \leq 0, \end{cases}$$

is generalized inverse Green's operator on $\mathbb{Z}$ with the following properties:

$$(G[h])(0+0) - (G[h])(0-0) = -\sum_{k=-\infty}^{+\infty} H(k+1)h_k = 0,$$

$$(LG[h])(n) = h_n, n \in \mathbb{Z},$$

where

$$(Lx)(n) := x_{n+1} - A_n x_n : l_\infty(\mathbb{Z}, B) \to l_\infty(\mathbb{Z}, B),$$

$H(k+1) = \mathcal{P}_{B_D}QU^{-1}(k+1) = \mathcal{P}_{B_D}(I-P)U^{-1}(k+1), D^{-}$ *is generalized inverse to the operator D, projectors $\mathcal{P}_{N(D)} = I - D^{-}D$ and $\mathcal{P}_{B_D} = I - DD^{-}$ (see [9, 12]), which project space B on the kernel $N(D)$ of the operator D and the subspace $B_D := B \ominus R(D)$ respectively ($B = B_D \oplus R(D) = (B \ominus R(D)) \oplus R(D)$).*

Proof. The general solution of the equation (1), bounded on the semiaxes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ has the form:

$$x_n(\xi) = \begin{cases} U(n)P\xi + \sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k - \\ \quad - \sum_{k=n}^{+\infty} U(n)(I-P)U^{-1}(k+1)h_k, \;\; n \geq 0 \\ U(n)(I-Q)\xi + \sum_{k=-\infty}^{n-1} U(n)QU^{-1}(k+1)h_k - \\ \quad - \sum_{k=n}^{-1}(I-Q)U^{-1}(k+1)h_k, \;\; n \leq 0. \end{cases} \tag{5}$$

Let us prove that the expression (5) defines bounded on the semi-axes $(\mathbb{Z}_+, \mathbb{Z}_-)$ solutions. It is easy to see that for any $n \geq 0$ the expression $x_n = U(n)P\xi$ determines the bounded solution of the homogeneous equation (2). Moreover

$$||U(n)P\xi|| \leq ||U(n)P||||\xi|| \leq k_1\lambda_1^n||\xi||.$$

Similarly, the expression $U(n)(I-Q)\xi$ defines a bounded solution of the homogeneous equation (2) on $\mathbb{Z}_-$. It is easy to check that the expression

$$x_n := \sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k - \sum_{k=n}^{+\infty} U(n)(I-P)U^{-1}(k+1)h_k,$$

defines a solution of the nonhomogeneous equation (1) on $\mathbb{Z}_+$. Now we prove the boundedness on the semiaxes. To do this, we estimate the sums:

$$||\sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k|| \leq |||h|||\sum_{k=0}^{n-1}||U(n)PU^{-1}(k+1)|| \leq$$

$$\leq |||h|||\sum_{k=0}^{n-1} k_1\lambda_1^{n-k-1} = \frac{k_1(1-\lambda_1^n)}{1-\lambda_1}|||h||| < \infty,$$

and

$$||\sum_{k=n}^{+\infty} U(n)(I-P)U^{-1}(k+1)h_k|| \leq |||h|||\sum_{k=n}^{+\infty} k_1\lambda_1^{k+1-n} =$$

$$= k_1\lambda_1^{-n+1}|||h|||\sum_{k=n}^{+\infty}\lambda_1^k = \frac{k_1\lambda_1}{1-\lambda_1}|||h||| < \infty.$$

Finally, we get

$$\sup_{n\in\mathbb{Z}_+} ||x_n|| \leq k_1||\xi|| + k_1\frac{1+\lambda_1}{1-\lambda_1}|||h|||.$$

The boundness of the solution on $\mathbb{Z}_-$ is proved similarly and we obtain

$$\sup_{n\in\mathbb{Z}_-} ||x_n|| \leq k_2||\xi|| + k_2\frac{1+\lambda_2}{1-\lambda_2}|||h|||.$$

Let us find the condition, which guarantees, that the solution (5) will be bounded on the whole axis. This holds if and only if

$$x_{0+}(\xi) = x_{0-}(\xi). \tag{6}$$

Thus, uniting the solutions at zero, we get the desired result.

Substituting the corresponding expressions in (6), we obtain the operator equation

$$P\xi - \sum_{k=0}^{+\infty}(I-P)U^{-1}(k+1)h_k = (I-Q)\xi + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k.$$

Let us introduce the following element

$$g = \sum_{k=0}^{+\infty}(I-P)U^{-1}(k+1)h_k + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k.$$

We obtain the following equation

$$D\xi = g. \tag{7}$$

Since D is normally-solvable, then as it is known from [9], the necessary and sufficient condition for the solvability of the equation (7) is the following:

$$\mathcal{P}_{B_D} g = (I - DD^-)g = 0. \tag{8}$$

Since $D\mathcal{P}_{N(D)} = 0$, we obtain that $P\mathcal{P}_{N(D)} = (E-Q)\mathcal{P}_{N(D)}$. Based on the equality $\mathcal{P}_{B_D}D = (I-DD^-)D = D - DD^-D = 0$ we obtain that $\mathcal{P}_{B_D}Q = \mathcal{P}_{B_D}(I-P)$. Finally, the condition (8) can be rewritten as

$$\sum_{k=-\infty}^{+\infty} \mathcal{P}_{B_D}QU^{-1}(k+1)h_k = 0,$$

or

$$\sum_{k=-\infty}^{+\infty} \mathcal{P}_{B_D}(I-P)U^{-1}(k+1)h_k = 0.$$

Thus, we have proved that if (3) is satisfied, then the set of the operator equation solutions (7) has the following view [9]:

$$\xi = D^{-}g + P\mathcal{P}_{N(D)}c, \text{ for any } c \in B.$$

Direct substitution into the equation (5) gives us that the set of solutions, bounded on the entire axis $\mathbb{Z}$, in the form (4).

Remark 1. *We have the following estimates for the norm of the solution*

$$||x_n|| \le k_1\lambda_1^n||\mathcal{P}_{N(D)}c|| + k_1\lambda_1^n||D^-||\left(\frac{k_1\lambda_1}{1-\lambda_1} + \frac{k_2\lambda_2}{1-\lambda_2}\right)|||h|||+$$

$$+k_1\frac{(1+\lambda_1-\lambda_1^n)}{1-\lambda_1}|||h|||, \quad n \ge 0,$$

$$||x_n|| \le k_2\lambda_2^{-n}||\mathcal{P}_{N(D)}c|| + k_2\lambda_2^{-n}||D^-||\left(\frac{k_1\lambda_1}{1-\lambda_1} + \frac{k_2\lambda_2}{1-\lambda_2}\right)|||h|||+$$

$$+k_2\frac{1+\lambda_2-\lambda_2^{-n+1}}{1-\lambda_2}|||h|||, \quad n \le 0.$$

From these inequalities follows the estimate

$$|||x||| \le K||\mathcal{P}_{N(D)}c|| + K||D^-||\left(\frac{k_1\lambda_1}{1-\lambda_1} + \frac{k_2\lambda_2}{1-\lambda_2}\right)|||h||| + K\frac{1+\Lambda_1}{1-\Lambda_2}|||h|||,$$

where $K = \max\{k_1, k_2\}$, $\Lambda_1 = \max\{\lambda_1, \lambda_2\}$, $\Lambda_2 = \min\{\lambda_1, \lambda_2\}$.

Remark 2. *It should also be noted that if bounded solutions are united together at zero as follows*

$$x_{0+}(\xi) = x_{0-}(\xi) + c,$$

where c is an element of Banach space, then we obtain a bounded solution for problem with a jump.

Remark 3. *It should be noted that the condition of solvability (3)*

$$\mathcal{P}_{B_D}g = 0,$$

of theorem 1.1 is equivalent to

$$\mathcal{P}_{N(D^*)}g = 0,$$

where $\mathcal{P}_{N(D^*)}$ is projector onto the kernel of the operator D^* adjoint to D. We also have that $^{\perp}N(D^*) = \{x \in B : \phi(x) = 0, \phi \in N(D^*)\} = R(D)$ *(see [16]).*

It is well known that d-normal and n-normal operators play important role in the theory of boundary-value problems (see [9, 16]). For such classes of operators, the similar theorems with refinements will be valid. Recall that a linear and bounded operator $D : B_1 \to B_2$ from one Banach space B_1 into another space B_2 is *n-normal* if it is normally-solvable and $n =$

$n(D) = dimN(D) < \infty$. The operator $D : B_1 \to B_2$ is *d-normal* if it is normally solvable and $d = d(D) = dimN(D^*) < \infty$. Operator D is called *Noetherian (Fredholm non-zero index operator)*, if it is simultaneously n-normal and d-normal. Operator D is called *topologically Noetherian* if it has complemented kernel and image (see [2]) Operator D is generalized invertible if it is topologically Noetherian.

Sometimes, especially in the Russian-language literature (see [16], [9]), Fredholm operators are called Noetherian operators. The Noetherian operators are operators for which all conditions of Fredholm with zero index operator are satisfied but $n(L)$ may differ from $m(L)$. This class of operators is named after F. Noether who studied a class of singular integral equations with operators of this sort for the first time as early as in 1920 (see [21]).

Corollary. *Suppose that the conditions of Theorem 1.1 hold and the bounded operator*

$$D = P - (E - Q) : B \to B$$

is d - normal. Bounded on the entire axis solutions of the equation (1) exist if and only if d-linear independent conditions are satisfied

$$\sum_{k=-\infty}^{+\infty} H_d(k+1)h_k = 0. \tag{9}$$

When the conditions (9) hold, solutions bounded on the entire axis have the form (4), where

$$H_d(n) = [\mathcal{P}_{B_D}Q]_d U^{-1}(n) = [\mathcal{P}_{B_D}(E - P)]_d U^{-1}(n),$$

$$d \le m \ (m = dimcokerD < \infty), d = dim(\mathcal{P}_{B_D}Q).$$

Proof. It should be noted that the operator $\mathcal{P}_{B_D}$ is finite dimensional (since it is d - normal), and then the operator $\mathcal{P}_{B_D}Q$ is also finite dimensional $(R(\mathcal{P}_{B_D}Q) \subset R(\mathcal{P}_{B_D}))$. The rest of the corollary is obtained in the same way as Theorem 1.1.

Corollary. *Suppose that conditions of Theorem 1.1 are satisfied and the operator*

$$D = P - (E - Q) : B \to B$$

is n-normal. Then bounded on the entire axis solutions of the equation (1) exist if and only if the conditions (3) are satisfied. Under conditions (3) equation (1) has a r - parametric set of the bounded solutions

$$x_n(c_r) = U(n)[P\mathcal{P}_{N(D)}]_r c_r + (G[h])(n), \tag{10}$$

where $r \le n$ $(n = dimker(D))$.

Proof. Since the operator D is n - normal operator, then its kernel has finite dimension. This means that the operator $\mathcal{P}_{N(D)}$ is also finite dimensional and the operator $P\mathcal{P}_{N(D)}$ is finite dimensional $((R(P\mathcal{P}_{N(D)})) \subset R(\mathcal{P}_{N(D)}))$. If the dimension of the kernel $dim\ N(D) = n$, then

$$dim(P\mathcal{P}_{N(D)}) = dim((E - Q)P\mathcal{P}_{N(D)}) = r \le n.$$

The rest of the corollary is obtained in the same way as Theorem 1.1.

Corollary. *Suppose that the condition of Theorem 1.1 is satisfied and the operator*

$$D = P - (E - Q) : B \to B$$

is Noetherian. Then bounded on the entire axis solutions of the equations exist (1) if and only if d-conditions (9) hold. When the conditions (9) hold, the equation (1) has r-parametric set of the bounded solutions

$$x_n(c_r) = U(n)[P\mathcal{P}_{N(D)}]_r c_r + (G[h])(n), \tag{11}$$

where $r \leq n$ $(n = dimker(D))$, $d \leq m$ $(m = dimcoker D)$.

Remark 4. *If the operator D is Fredholm (Fredholm with zero index), then under the conditions of the corollary $r = d$.*

Corollary. Suppose that under condition of Theorem 1.1

$$[P, Q] = PQ - QP = 0, \quad PQ = Q.$$

This is the so-called case of exponential trichotomy of equation (2) on $\mathbb{Z}$. In this case, inhomogeneous equation (1) has at least one solution on $\mathbb{Z}$ for any element $h \in l_\infty(\mathbb{Z}, B)$.

Proof. From the equality $\mathcal{P}_{B_D} D = 0$ and $DP = (P - (E - Q))P = QP = Q$ follows that $\mathcal{P}_{B_D} Q = \mathcal{P}_{B_D} DP = 0$. From this equality we obtain the automatic fulfilment of the conditions (3) and the solvability for an arbitrary $h \in l_\infty(\mathbb{Z}, B)$. The rest follows from Theorem 2.

Corollary. *If the conditions of the Theorem 1.1 and additional conditions such as following hold*

$$[P, Q] = PQ - QP = 0, PQ = Q = P$$

then the nonhomogeneous equation (1) has unique bounded on $\mathbb{Z}$ solution for arbitrary $h \in l_\infty(\mathbb{Z}, B)$.

Remark 5. *In this case, the considered system is exponentially-dichotomous on the entire axis $\mathbb{Z}$. In the finite dimensional case, this result is well known (see [11]). Theorem 1.1 under less restrictive assumptions, allows to find a set of bounded solutions.*

1.2 The case of Hilbert spaces.

Let us show, that in the case of Hilbert spaces it is possible to obtain a more general result. Using the technique, given in [10] it is possible to introduce the concept of a strong pseudo-inverse operator to an operator D and weaken the assumption of the Theorem 1.1 on the generalized invertibility of the operator D. In this case, the original equation (1) can always be made solvable in one of the meanings, defined below. Let us explain this approach in more details.

Let us describe the construction of a strong pseudo-inverse according to Moore-Penrose operator, which is used to represent the solutions of the operator equation (7).

We distinguish three types of solutions (7)

$$D\xi = g, \quad D : \mathcal{H} \to \mathcal{H},$$

$\mathcal{H}$ - is a Hilbert space.

1) Classical solutions.

If the operator D is normally solvable ($R(D) = \overline{R(D)}$) and, therefore, pseudo-invertible, then the element g belongs to the set of values ($g \in R(D)$) of the operator D if and only if $P_{B_D} g = 0$ or $P_{N(D^*)} g = 0$ (see [9]); P_{B_D}, $P_{N(D^*)}$ – are orthoprojectors onto the subspace B_D and cokernel of the operator D. In this case, there is Moore-Penrose pseudo-inverse operator D^+ and the set of solutions of the equation (7) has the form

$$\xi = D^+ g + P_{N(D)} c, \text{ for any } c \in \mathcal{H},$$

where $P_{N(D)}$ - is orthoprojector onto the kernel of the operator D.

2) Strong generalized solutions.

Consider a case, when the set of values of the operator D is not closed, i.e. $R(D) \neq \overline{R(D)}$. We show that in this case D can be extended to the operator $\overline{D}$ in the way that the operator $\overline{D}$ is normally-solvable.

Since the operator D is linear and bounded, the decompositions of the space $\mathcal{H}$ into direct sums take place

$$\mathcal{H} = N(D) \oplus X, \mathcal{H} = \overline{R(D)} \oplus Y.$$

Here $X = N(D)^\perp$, $Y = \overline{R(D)}^\perp$. It can be stated, that there are operators of orthogonal projection $P_{N(D)}$, P_X onto the corresponding spaces $P_{\overline{R(D)}}$, P_Y. Denote by $\mathcal{H}_1$ the quotient space of the space $\mathcal{H}$ with respect to the kernel $N(D)$ ($\mathcal{H}_1 = \mathcal{H}/N(D)$). Then, as it is known (see [3,23]) there exist a continuous bijection $p : X \to \mathcal{H}_1$ and projection $j : \mathcal{H} \to \mathcal{H}_1$. The triple $(\mathcal{H}, \mathcal{H}_1, j)$ is a locally trivial bundle with a typical layer $P_{N(D)}\mathcal{H}$. Now we define the operator

$$\mathcal{D} = P_{\overline{R(D)}} D j^{-1} p : X \to R(D) \subset \overline{R(D)}.$$

It is easy to verify that the operator which is defined in this way is linear, injective and continuous. Now using the completion process (see [14]) by norm $||x||_{\overline{X}} = ||\mathcal{D}x||_F$, where $F = \overline{R(D)}$, we obtain a new space $\overline{X}$ and an extended operator $\overline{\mathcal{D}}$. Then

$$\overline{\mathcal{D}} : \overline{X} \to \overline{R(D)}, \quad X \subset \overline{X}$$

and the operator constructed in this way is a homeomorphism between the spaces $\overline{X}$ and $\overline{R(D)}$. Consider the extended operator $\overline{D} = \overline{\mathcal{D}} P_{\overline{X}} : \overline{\mathcal{H}} \to \mathcal{H}$,

$$\overline{\mathcal{H}} = N(D) \oplus \overline{X}, \mathcal{H} = R(\overline{D}) \oplus Y.$$

It is clear that $\overline{D}x = Dx, x \in \mathcal{H}$ and operator $\overline{D}$ is normally-solvable (in this case $R(\overline{D}) = \overline{R(\overline{D})}$), therefore is pseudo-invertible with pseudo-inverse $\overline{D}^+$ (see [9]).

Definition. *The operator* $\overline{D}^+$ *is called* **strong pseudo-inverse** *to the operator* D.

Then the set of strong generalized solutions of the equation (7) has the form:

$$\xi = \overline{D}^+ g + P_{N(\overline{D})} c, \quad \text{for any } c \in \overline{\mathcal{H}}.$$

Note also, that if in this case $g \in R(D)$, then strong generalized solution becomes the usual classic solution.

3) Generalized pseudo-solutions.

Let us consider the case, when $g \notin \overline{R(D)}$, which for the element g is equivalent to the condition $P_{N(D^*)}g \neq 0$. In this case, there are elements of $\overline{\mathcal{H}}$, which minimize the norm $||\overline{D}\xi - g||_{\mathcal{H}}$ for $\xi \in \overline{\mathcal{H}}$,

$$\xi = \overline{D}^{+} g + P_{N(\overline{D})}c, \quad c \in \overline{\mathcal{H}}.$$

These elements will be called **generalized pseudo-solutions** of equation (7) by analogy with pseudo-solutions (see [9]). Note that if $\overline{\mathcal{H}} = \mathcal{H}$ (the operator D has a closed set of values), then generalized pseudo-solutions are actually ordinary pseudo-solutions.

Combining now the above with the results obtained earlier we can formulate the Theorem.

Theorem 3. *Suppose that a homogeneous equation is exponentially-dichotomous on the semiaxes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projectors P and Q respectively.*

a1) Strong generalized solutions of the equation (1), bounded on the entire axis $\mathbb{Z}$ exist if and only if the following condition is satisfied

$$\sum_{k=-\infty}^{+\infty} \overline{H}(k+1)h_k = 0. \tag{12}$$

a2) Under the condition (12), the set of bounded strong generalized solutions has the following form:

$$x_n^0(c) = U(n)PP_{N(D)}c + \overline{(G[h])}(n), \qquad c \in \overline{\mathcal{H}} \tag{13}$$

where $\overline{G[h]}(n)$ - is the extension of the generalized Green operator on $\overline{\mathcal{H}}$,

$$\overline{G[h]}(n) = \begin{cases} \sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k - \sum_{k=n}^{+\infty} U(n)(E-P)U^{-1}(k+1)h_k + \\ \qquad +U(n)P\overline{D}^{+}[\sum_{k=0}^{+\infty}(E-P)U^{-1}(k+1)h_k + \\ \qquad + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \ \ n \geq 0 \\ \sum_{k=-\infty}^{n-1} U(n)QU^{-1}(k+1)h_k - \sum_{k=n}^{-1} U(n)(E-Q)U^{-1}(k+1)h_k + \\ \qquad +U(n)(E-Q)\overline{D}^{+}[\sum_{k=0}^{+\infty}(E-P)U^{-1}(k+1)h_k + \\ \qquad + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \ \ n \leq 0, \end{cases}$$

b1) Generalized pseudo-solutions of equation (1) bounded on the entire axis $\mathbb{Z}$ exist if and only if the following condition is satisfied

$$\sum_{k=-\infty}^{+\infty} \overline{H}(k+1)h_k \neq 0. \tag{14}$$

b2) Under the condition (14), the set of bounded generalized pseudo-solutions has the following form:

$$x_n^0(c) = U(n)PP_{N(D)}c + \overline{(G[h])}(n), \qquad c \in \overline{\mathcal{H}},$$

where $\overline{H}(n+1) = P_{\overline{\mathcal{H}}_{\overline{D}}}QU^{-1}(n+1) = P_{\overline{\mathcal{H}}_{\overline{D}}}(E-P)U^{-1}(n+1), \overline{D}^{+}$ is strong pseudo-inverse according to Moore-Penrose operator to the operator D, $P_{N(\overline{D})} = I - \overline{D}^{-}\overline{D}$ and $P_{\overline{\mathcal{H}}_{\overline{D}}} = I - \overline{D}\overline{D}^{-}$ are projectors that project the space $\mathcal{H}$ onto the kernel $N(\overline{D})$ of the operator $\overline{D}$ and the subspace $\overline{\mathcal{H}}_{\overline{D}} := \overline{\mathcal{H}} \ominus R(\overline{D})$ respectively ($\overline{\mathcal{H}} = \overline{\mathcal{H}}_{\overline{D}} \oplus R(\overline{D}) = (\overline{\mathcal{H}} \ominus R(\overline{D})) \oplus R(\overline{D})$).

The following corollary is also true.

Corollary. *Suppose that $[P, Q] = 0$ and equation (1) is considered in Hilbert space $B = \mathcal{H}$. Then the operator D has a generalized inverse D^{-}, which coincides with the operator D ($D^{-} = D$). In this situation, we have the following options:*

1 a.) Equation (1) has bounded solutions if and only if the following condition is satisfied

$$\sum_{k=-\infty}^{+\infty} H(k+1)h_k = 0; \tag{15}$$

1 b.) When the condition (15) is satisfied, bounded solutions have the form

$$x_n^0(c) = U(n)(P - PQ)c + (G[h])(n), \tag{16}$$

where

$$G[h](n) = \begin{cases} \sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k - \sum_{k=n}^{+\infty} U(n)(E-P)U^{-1}(k+1)h_k + \\ \qquad + U(n)PD[\sum_{k=0}^{+\infty}(E-P)U^{-1}(k+1)h_k + \\ \qquad + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \ \ n \geq 0 \\ \sum_{k=-\infty}^{n-1} U(n)QU^{-1}(k+1)h_k - \sum_{k=n}^{-1} U(n)(E-Q)U^{-1}(k+1)h_k + \\ \qquad + U(n)(E-Q)D[\sum_{k=0}^{+\infty}(E-P)U^{-1}(k+1)h_k + \\ \qquad + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \ \ n \leq 0, \end{cases}$$

is generalized Green's operator $H(n+1) = P_{\mathcal{H}_D}QU^{-1}(n+1) = P_{\mathcal{H}_D}(E-P)U^{-1}(n+1) = (Q - PQ)U^{-1}(n+1)$, $P_{N(D)} = I - D^2 = P_{\mathcal{H}_D}$ are projectors.

2 a.) Equation (1) has pseudo-solutions if and only if

$$\sum_{k=-\infty}^{+\infty} H(k+1)h_k = q \neq 0; \tag{17}$$

2 b.) Under condition (17) the set of bounded pseudo-solutions has this view

$$x_n^0(c) = U(n)(P - PQ)c + (G[h])(n), \tag{18}$$

where

$$G[h](n) = \begin{cases} \sum_{k=0}^{n-1} U(n)PU^{-1}(k+1)h_k - \sum_{k=n}^{+\infty} U(n)(E-P)U^{-1}(k+1)h_k + \\ \qquad + U(n)PD[\sum_{k=0}^{+\infty}(E-P)U^{-1}(k+1)h_k + \\ \qquad + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \ \ n \geq 0 \\ \sum_{k=-\infty}^{n-1} U(n)QU^{-1}(k+1)h_k - \sum_{k=n}^{-1} U(n)(E-Q)U^{-1}(k+1)h_k + \\ \qquad + U(n)(E-Q)D[\sum_{k=0}^{+\infty}(E-P)U^{-1}(k+1)h_k + \\ \qquad + \sum_{k=-\infty}^{-1} QU^{-1}(k+1)h_k], \ \ n \leq 0, \end{cases}$$

is generalized Green's operator.

Proof. If $[P, Q] = 0$, then it is easy to see that

$$DDD = D.$$

From this equality, it follows that $D = D^{-}$. The second part of the Theorem follows from the fact that the equation (7) under the condition

$$P_{B_D}g = P_{\mathcal{H}_D}g \neq 0,$$

has (see [9]) pseudo-solutions (in this case this condition is equivalent to $g \notin R(D)$) and then the set $\xi = D^{-}g + P_{N(D)}c, c \in \mathcal{H}$ is the set of those elements which minimize the residual norm $||D\xi - g||$. Substituting the obtained identities into representations (18) we obtain

$$PP_{N(D)} = P(I - D^2) = P(P + Q - 2PQ) = P^2 + PQ - 2P^2Q = P - PQ,$$

$$P_{B_D}Q = P_{\mathcal{H}_D}Q = (I - D^2)Q = (P + Q - 2PQ)Q =$$

$$= PQ + Q^2 - 2PQ^2 = Q - PQ.$$

Corollary. *If additional* $PQ = 0$ *then*

$$PP_{N(D)} = P, P_{B_D}Q = Q$$

Corollary. Suppose that $[P, Q] = 0$ and $P = P^*$, $Q = Q^*$. Then the operator D has a Moore-Penrose pseudoinverse, which coincides with the operator D ($D^+ = D$).

Proof. This follows from the fact that $DDD = D$ and $D^* = P^* - I + Q^* = D$.

2 Nonlinear case. Bifurcation of solutions

Consider the following weakly nonlinear equation

$$x_{n+1}(\varepsilon) = A_n x_n(\varepsilon) + \varepsilon Z(x_n(\varepsilon), n, \varepsilon) + h_n \tag{19}$$

in the Hilbert space $\mathcal{H}$ where the nonlinear vector-valued function $Z(x(n, \varepsilon), n, \varepsilon)$ satisfies the following conditions

$$Z(\cdot, n, \varepsilon) \in C[||x - x^0|| \leq q], Z(x(n, \varepsilon), \cdot, \varepsilon) \in l_\infty(\mathbb{Z}, \mathcal{H}), Z(x(n, \varepsilon), n, \cdot) \in C[0, \varepsilon_0]$$

in the neighborhood of solution $x_n^0(c)$ of the generating linear problem (1) (q is a small enough constant). We are looking for necessary and sufficient conditions for the existence of strong generalized solutions $x_n(\varepsilon) : \mathbb{Z} \to \mathcal{H}$ of (19) bounded on the entire integer axis

$$x_{\cdot}(\varepsilon) \in l_\infty(\mathbb{Z}, \mathcal{H}), \quad x_n(\cdot) \in C[0, \varepsilon_0],$$

which turn into one of the solutions $x_n^0(c)$ of the generating equation (1) for $\varepsilon = 0$: $x_n(0) = x_n^0(c)$ (in the form 13).

Theorem 4. (necessary condition). *Suppose the equation (2) admits a dichotomy on the semi-axes* $\mathbb{Z}_+$ *and* $\mathbb{Z}_-$ *with projectors* P *and* Q *respectively. Let the equation (19) has a strong generalized solution* $x_n(\varepsilon)$ *bounded on* $\mathbb{Z}$, *which turns into one of the generating solutions* $x_n^0(c)$ *(13) of the equation (1) with element* $c = c^* \in \overline{\mathcal{H}}$. *Then the element* c^* *satisfies the equation*

$$F(c^*) = \sum_{k=-\infty}^{+\infty} \overline{H}(k+1) Z(U(k) P P_{N(D)} c^* + \overline{(G[h])}(k), k, 0) = 0. \tag{20}$$

Proof. We can consider the nonlinearity Z as an inhomogeneity and apply theorem (1.2). Then we obtain

$$\sum_{k=-\infty}^{+\infty} \overline{H}(k+1)Z(x_k(\varepsilon), k, \varepsilon) = 0.$$

Passing to the limit as $\varepsilon \to 0$ we obtain the required condition of solvability (20).

Remark 6. *Equation (20) will be called the* **equation for generating elements** *by analogy with the case of periodic problem [9], [40].*

In order to obtain a sufficient condition for the existence of strong generalized bounded solutions, we additionally assume that the nonlinear vector-function Z is strongly differentiable $Z(\cdot, n, \varepsilon) \in C^1[||x - x^0|| \leq q]$ in a neighborhood of the generating solution $x_n^0(c^*)$ (13) (c^* is the root of the equation for generating elements (20)). Making a change of variables $x_n(\varepsilon) = y_n(\varepsilon) + x_n^0(c^*)$ we obtain the following equation

$$y_{n+1}(\varepsilon) = A_n y_n(\varepsilon) + \varepsilon Z(y_n(\varepsilon) + x_n^0(c^*), n, \varepsilon), \quad y_n(0) = 0. \tag{21}$$

Due to strong differentiability of Z, we obtain the following expansion

$$Z(y_n(\varepsilon) + x_n^0(c^*), n, \varepsilon) = Z(x_n^0(c^*), n, 0) + A_1(n)y_n(\varepsilon) + \mathcal{R}(y_n(\varepsilon), n, \varepsilon)$$

where

$$A_1(n) = \frac{\partial Z(x, n, 0)}{\partial x}|_{x = x_n^0(c^*)}$$

is Frechet derivative and nonlinearity $\mathcal{R}$ such that $\mathcal{R}(0, n, 0) = 0$. Under condition of strong generalized solvability (13)

$$\sum_{k=-\infty}^{+\infty} \overline{H}(k+1) \left(Z(x_k^0(c^*), k, 0) + A_1(k)y_k(\varepsilon) + \mathcal{R}(y_k(\varepsilon), k, \varepsilon) \right) = 0 \tag{22}$$

the set of strong generalized solutions of the equation (21) has the following form

$$y_n(\varepsilon) = U(n)PP_{N(D)}c(\varepsilon) + \overline{y}_n(\varepsilon), \tag{23}$$

where

$$\overline{y}_n(\varepsilon) = \varepsilon \overline{G[Z(y_\cdot(\varepsilon) + x_\cdot^0(c^*))]}(n).$$

Substituting representation (23) into the generalized solvability condition (22) we obtain the following operator equation

$$B_0 c(\varepsilon) = -\sum_{k=-\infty}^{+\infty} \overline{H}(k+1)\left(A_1(k)\overline{y}_k(\varepsilon) + \mathcal{R}(y_k(\varepsilon), k, \varepsilon)\right) \tag{24}$$

with operator B_0 in the form $B_0 = \sum_{k=-\infty}^{+\infty} \overline{H}(k+1)U(k)PP_{N(D)}$. Since the operator B_0 in a Hilbert space $\mathcal{H}$ always has a strong Moore-Penrose pseudoinverse $\overline{B}_0^+$ [10] the condition for the strong generalized solvability of equation (24) takes the following form

$$P_{\overline{\mathcal{H}}_{B_0}} \sum_{k=-\infty}^{+\infty} \overline{H}(k+1)\left(A_1(k)\overline{y}_k(\varepsilon) + \mathcal{R}(y_k(\varepsilon), k, \varepsilon)\right) = 0$$

where $P_{\overline{\mathcal{H}}_{B_0}}$ is orthoprojector onto the subspace $\overline{\mathcal{H}}_{B_0}$ ($\overline{\mathcal{H}}_{B_0} = \widetilde{\overline{\mathcal{H}}}_{\overline{D}} \ominus R(\overline{B}_0)$, here $\widetilde{\overline{\mathcal{H}}}_{\overline{D}}$ is the completion of $\overline{\mathcal{H}}_{\overline{D}}$ according to the corresponding norm). Since we have $\overline{H}(k+1) = P_{\overline{\mathcal{H}}_{\overline{D}}} QU^{-1}(k+1)$, then a sufficient condition for the strong generalized solvability of a nonlinear equation (21) is the following condition

$$P_{\overline{\mathcal{H}}_{B_0}} P_{\overline{\mathcal{H}}_{\overline{D}}} Q = 0. \tag{25}$$

Under condition (25) the set of strong generalized solutions of the equation (24) has the following form

$$c(\varepsilon) = -\overline{B}_0^+ \sum_{k=-\infty}^{+\infty} \overline{H}(k+1) \left(A_1(k)\overline{y}_k(\varepsilon) + \mathcal{R}(y_k(\varepsilon), k, \varepsilon)\right) + P_{N(B_0)} c_\rho(\varepsilon), \tag{26}$$

here $c_\rho(\varepsilon) \in \mathcal{H}$. Thus, the problem of the existence of a strong generalized solution $y_n(\varepsilon)$ of the equation (21) bounded on the entire $\mathbb{Z}$ axis reduces to the strong generalized solvability of the following operator system

$$\begin{cases} y_n(\varepsilon) = U(n)PP_{N(D)}c + \overline{y}_n(\varepsilon), \\ c(\varepsilon) = -\overline{B}_0^+ \sum_{k=-\infty}^{+\infty} \overline{H}(k+1) \left(A_1(k)\overline{y}_k(\varepsilon) + \mathcal{R}(y_k(\varepsilon), k, \varepsilon)\right) + P_{N(B_0)} c_\rho(\varepsilon), \\ \overline{y}_n(\varepsilon) = \varepsilon \overline{G[Z(y_\cdot(\varepsilon) + x^0_\cdot(c^*))]}(n), \end{cases} \tag{27}$$

or in the form

$$\begin{pmatrix} y_n(\varepsilon) \\ c(\varepsilon) \\ \overline{y}_n(\varepsilon) \end{pmatrix} = \begin{pmatrix} 0 & U(n)PP_{N(D)} & I \\ 0 & 0 & L_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_n(\varepsilon) \\ c(\varepsilon) \\ \overline{y}_n(\varepsilon) \end{pmatrix} + g_n(\varepsilon), \tag{28}$$

where

$$L_1 * = -\overline{B}_0^+ \sum_{k=-\infty}^{+\infty} \overline{H}(k+1) A_1(k) *,$$

$$g_n(\varepsilon) = \begin{pmatrix} 0 \\ -\overline{B}_0^+ \sum_{k=-\infty}^{+\infty} \overline{H}(k+1)\mathcal{R}(y_k(\varepsilon), k, \varepsilon) + P_{N(B_0)} c_\rho(\varepsilon) \\ \varepsilon \overline{G[Z(y_\cdot(\varepsilon) + x^0_\cdot(c^*))]}(n))^T \end{pmatrix}$$

If we denote the operator on the right hand side of the system (28) and the vector as

$$S(n) = \begin{pmatrix} 0 & U(n)PP_{N(D)} & I \\ 0 & 0 & L_1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z_n(\varepsilon) = \begin{pmatrix} y_n(\varepsilon) \\ c(\varepsilon) \\ \overline{y}_n(\varepsilon) \end{pmatrix},$$

then we obtain the operator system

$$(I - S(n)) z_n(\varepsilon) = g_n(\varepsilon). \tag{29}$$

It is easy to see that the operator $(I - S(n))$ has a bounded inverse in the following form

$$(I - S(n))^{-1} = \begin{pmatrix} I & U(n)PP_{N(D)} & U(n)PP_{N(D)}L_1 + I \\ 0 & I & L_1 \\ 0 & 0 & I \end{pmatrix}. \tag{30}$$

Thus, the operator system (29) can be represented as follows

$$z_n(\varepsilon) = (I - S(n))^{-1} F_n(\varepsilon) z_n(\varepsilon), \tag{31}$$

where $g_n(\varepsilon) = F_n(\varepsilon)z_n(\varepsilon)$. By choosing the parameter $\varepsilon \in [0, \varepsilon_*]$ and the element c_ρ, one can achieve that the operator $(I - S(n))^{-1}F_n(\varepsilon)$ on the right-hand side of the operator system (31) is contractive, and applying the contraction mapping principle [41] , one can obtain a sufficient condition for the existence of strong generalized solutions of the nonlinear equation (19).

Theorem. (sufficient condition). Suppose the equation (2) admits a dichotomy on the semi-axes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projectors P and Q respectively and the considered linear equation (1) has strong generalized bounded solutions $x_n(c)$ in the form (13). Assume that

$$P_{\overline{\mathcal{H}}_{B_0}} P_{\overline{\mathcal{H}_D}} Q = 0.$$

Then for each element $c = c^*$ satisfying the equation for generating elements (20) there are strong generalized solutions $x_n(\varepsilon)$ of the nonlinear equation (19) bounded on the entire $\mathbb{Z}$ axis, turns for $\varepsilon = 0$ into the generating solution $x_n^0(c^*)$: $x_n(0) = x_n^0(c^*)$. These solutions can be found using a convergent iterative process for $\varepsilon \in [0, \varepsilon_*] \subset [0, \varepsilon_0]$

$$y_n^{l+1}(\varepsilon) = U(n)PP_{N(D)}c^{l+1}(\varepsilon) + \overline{y}_n^{l+1}(\varepsilon),$$

$$c^{l+1}(\varepsilon) = -\overline{B}_0^+ \sum_{k=-\infty}^{+\infty} \overline{H}(k+1)\left(A_1(k)\overline{y}_k^{l+1}(\varepsilon) + \mathcal{R}(y_k^l(\varepsilon), k, \varepsilon)\right) + \mathcal{P}_{N(B_0)}c_\rho(\varepsilon),$$

$$\overline{y}_n(\varepsilon) = \varepsilon\overline{G[Z(y_\cdot(\varepsilon) + x_\cdot^0(c^*))]}(n),$$

$$x_n^l(\varepsilon) = y_n^l(\varepsilon) + x_n^0(c^*), y_n^0(\varepsilon) = 0, l = \overline{0, \infty}.$$

Remark 7. *It should be noted that if we choose the element $c_\rho(\varepsilon) \in \mathcal{H}$ in the form $c_\rho(\varepsilon) = \varepsilon\tilde{c}_\rho$ then we can always achieve that the operator $(I - S(n))^{-1}F_n(\varepsilon)$ is contractive.*

Remark 8. *The number of strong generalized bounded solutions of the nonlinear equation (19) depends on the dimension of the space $N(B_0)$. If $P_{N(B_0)} = 0$ then we have the unique strong generalized bounded solution of the equation (19). If $P_{N(B_0)} \neq 0$ then we get that new solutions $x_n(\varepsilon)$ appear from the point $\varepsilon = 0$ ($x_n(0) = x_n^0(c^*)$).*

Remark 9. *We can consider a more general boundary value problem with some boundary conditions and choose among the set of bounded solutions those that are needed (for example periodic, homoclinic or heteroclinic).*

Corollary. Suppose that operator $F(c)$ has a Frechet derivative for an element $c = c^*$ that satisfies the equation for generating elements (20). If $F'(c^*)$ has bounded inverse then equation (19) has a unique bounded solution.

Proof. It is easy to check that in this case $F'(c^*) = B_0$. It follows from this equality that the conditions of theorem 2 are satisfied. Conditions for the invertibility of the operator B_0 connect the necessary and sufficient conditions of the existence of bounded solutions.

Remark 10. *In the finite-dimensional case, the condition for the invertibility of the operator B_0 is the condition for the simplicity of the root $c = c^*$ and we can obtain a discrete analogue of the well-known Palmer theorem [34], [11, p.408] and the Melnikov conditions of the existence of chaos (in the sense of Bernoulli). In more general case we obtain the weak conditions of solvability.*

3 Conclusions

Proposed in the article statements gives us possibility to investigate the question of the existence of bounded solutions of the linear and weakly nonlinear nonhomogeneous equation in the Banach and Hilbert spaces in general. Developed in the paper method allows us to investigate boundary value problems on the whole axis with conditions at infinity. As application we can consider the countable system of difference equations. Obtained in the work the necessary condition of solvability is an analogue of Fredholm's alternative.

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Наукове видання

О.ПОКУТНИЙ

DICHOTOMY AND BOUNDED SOLUTIONS OF EVOLUTION EQUATIONS

IN THE BANACH AND HILBERT SPACES

Підп. до друку 2023. Папір тип.
Тираж 30 пр. Зам. 1.

Інститут математики НАН України
01601, Київ, вул. Терещенківська, 3