Kolmogorov widths and entropy numbers of the classes of periodic functions in the space $B_{q,1}$

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L_p-space

Let $\mathbb{R}^d$ be a d-dimensional space with elements $\boldsymbol{x} = (x_1, \dots, x_d)$ and $(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + \dots + x_d y_d$.

By $L_p(\mathbb{T}^d)$, $\mathbb{T}^d = \prod_{j=1}^d [0, 2\pi)$, we denote the space of 2π-periodic in each variable functions $f(\boldsymbol{x})$, for which

$$\|f\|_p := \|f\|_{L_p(\mathbb{T}^d)} = \left((2\pi)^{-d} \int_{\mathbb{T}^d} |f(\boldsymbol{x})|^p \, d\boldsymbol{x}\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{T}^d)} = \operatorname{ess\,sup}_{\boldsymbol{x} \in \mathbb{T}^d} |f(\boldsymbol{x})| < \infty.$$

Let further

$$L_p^0 := L_p^0(\mathbb{T}^d) := \left\{ f \in L_p(\mathbb{T}^d) \colon \int_0^{2\pi} f(\boldsymbol{x}) dx_j = 0 \ \text{ a.e.}, \ j = 1, \dots, d \right\}.$$

$B_{q,1}$-space

Let $V_l(t)$, $t \in \mathbb{R}$, $l \in \mathbb{N}$, denotes the de la Vallée-Poussin kernel of the form

$$V_l(t) = 1 + 2 \sum_{k=1}^{l} \cos kt + 2 \sum_{k=l+1}^{2l-1} \left(1 - \frac{k-l}{l}\right) \cos kt\,,$$

where for $l = 1$ we assume that the third term equals to zero.

We associate each vector $\boldsymbol{s} \in \mathbb{N}^d$ with the polynomial

$$A_{\boldsymbol{s}}(\boldsymbol{x}) = \prod_{j=1}^{d} (V_{2^{s_j}}(x_j) - V_{2^{s_j-1}}(x_j)),$$

and for $f \in L_p^0$, $1 \le p \le \infty$, set

$$A_{\boldsymbol{s}}(f) := A_{\boldsymbol{s}}(f, \boldsymbol{x}) := (f * A_{\boldsymbol{s}})(\boldsymbol{x}),$$

where "$*$" denotes the operation of convolution.

For trigonometric polynomials t w.r.t. the trigonometric system, we define the norm $\|t\|_{B_{q,1}} := \sum_{\boldsymbol{s} \in \mathbb{N}^d} \|A_{\boldsymbol{s}}(t)\|_q$, $1 \le q \le \infty$ (where the sum contains a finite number of terms). Similarly, for functions $f \in L_q^0$, such that the series $\sum_{\boldsymbol{s} \in \mathbb{N}^d} \|A_{\boldsymbol{s}}(f)\|_q$ is convergent, we set

$$\|f\|_{B_{q,1}} := \sum_{\boldsymbol{s} \in \mathbb{N}^d} \|A_{\boldsymbol{s}}(f)\|_q, \quad 1 \le q \le \infty.$$

For $f \in B_{q,1}$, $1 \le q \le \infty$, the following relations hold:

$$\|f\|_q \ll \|f\|_{B_{q,1}}; \qquad \|f\|_{B_{1,1}} \ll \|f\|_{B_{q,1}} \ll \|f\|_{B_{\infty,1}}.$$

Order relations

For positive quantities a and b, the notation $a \asymp b$ means that there exist positive constants C_1 and C_2, that do not depend on an essential parameter in the values of a, b, and such that $C_1 a \le b$ (in this case, we write $a \ll b$) and $C_2 a \ge b$ (denoted by $a \gg b$).

Function classes

For a function $f \in L_p^0$, $1 \le p \le \infty$, we consider its first difference in the jth variable with step $h \in \mathbb{R}$:

$$\Delta_{h,j} f(\boldsymbol{x}) = f(x_1, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_d) - f(\boldsymbol{x})$$

and define its lth difference, $l \in \mathbb{N}$, by

$$\Delta_{h,j}^l f(\boldsymbol{x}) = \overbrace{\Delta_{h,j} \cdots \Delta_{h,j}}^{l} f(\boldsymbol{x}).$$

Assume that the vectors $\boldsymbol{k} \in \mathbb{N}^d$ and $\boldsymbol{h} \in \mathbb{R}^d$ are given. Then the mixed difference of order $\boldsymbol{k}$ with a vector step $\boldsymbol{h}$ is defined by the equality

$$\Delta_{\boldsymbol{h}}^{\boldsymbol{k}} f(\boldsymbol{x}) = \Delta_{h_1,1}^{k_1} \Delta_{h_2,2}^{k_2} \cdots \Delta_{h_d,d}^{k_d} f(\boldsymbol{x}).$$

The spaces $B_{p,\theta}^{\boldsymbol{r}}(\mathbb{T}^d)$, $1 \le p, \theta \le \infty$, where $\boldsymbol{r} \in \mathbb{R}^d$ is a given vector with the elements $r_j > 0$, $j = 1, \dots, d$, are defined as follows:

$$B_{p,\theta}^{\boldsymbol{r}} := B_{p,\theta}^{\boldsymbol{r}}(\mathbb{T}^d) := \{ f \in L_p^0 \colon \ \|f\|_{B_{p,\theta}^{\boldsymbol{r}}} < \infty \},$$

and the respective norm is specified by

$$\|f\|_{B_{p,\theta}^{\boldsymbol{r}}} := \left(\int_{\mathbb{T}^d} \|\Delta_{\boldsymbol{h}}^{\boldsymbol{k}} f\|_p^\theta \prod_{j=1}^{d} \frac{dh_j}{h_j^{1+r_j\theta}} \right)^{\frac{1}{\theta}}, \qquad 1 \le \theta < \infty,$$

$$\|f\|_{H_p^{\boldsymbol{r}}} \equiv \|f\|_{B_{p,\infty}^{\boldsymbol{r}}} := \sup_{\boldsymbol{h}} \|\Delta_{\boldsymbol{h}}^{\boldsymbol{k}} f\|_p \prod_{j=1}^{d} h_j^{-r_j}.$$

Here we assume that the components of the vectors $\boldsymbol{k}$ and $\boldsymbol{r}$ satisfy the conditions $k_j > r_j$, $1, \dots, d$.

In this form, the definition of the spaces $H_p^{\boldsymbol{r}}$ was considered in [Temlyakov 1986], and of $B_{p,\theta}^{\boldsymbol{r}}$, $1 \le \theta < \infty$, in [Nikol'skii, Lizorkin 1989].

These spaces belong to the scale of spaces of mixed smoothness, introduced in [Nikol'skii 1963] and [Amanov 1965]. In addition, they are generalizations of the well-known isotropic Besov spaces [Besov 1961], and the Nikol'skii spaces [Nikol'skii 1951] for the case $\theta = \infty$.

We keep the notation $B_{p,\theta}^{\boldsymbol{r}}$ also for the respective classes (the unit balls in the spaces $B_{p,\theta}^{\boldsymbol{r}}$).

With a growth of the parameter θ the classes $B_{p,\theta}^{\boldsymbol{r}}$ are expanding, i.e.,

$$B_{p,1}^{\boldsymbol{r}} \subset B_{p,\theta_1}^{\boldsymbol{r}} \subset B_{p,\theta_2}^{\boldsymbol{r}} \subset B_{p,\infty}^{\boldsymbol{r}} \equiv H_p^{\boldsymbol{r}}, \quad 1 \le \theta_1 \le \theta_2 \le \infty.$$

We assume that coordinates of the vector $\boldsymbol{r} \in \mathbb{R}^d$ are ordered such that

$$0 < r_1 = r_2 = \dots = r_\nu < r_{\nu+1} \le \dots \le r_d.$$

Approximation characteristics

- **The Kolmogorov width**
  Let Y be a normed space with the norm $\|\cdot\|_Y$, $L_M(Y)$ be a set of subspaces of dimension at most M in the space Y, and W be a centrally-symmetric set in Y.

  The quantity
  $$d_M(W, Y) := \inf_{L_M \in L_M(Y)} \sup_{w \in W} \inf_{u \in L_M} \|w - u\|_Y$$
  is called the Kolmogorov M-width of the set W in the space Y (see [Kolmogorov 1936]).

- **The entropy number**
  Let X be a Banach space and let $B_X(y, R)$ be a ball in X of radius R centered at a point y, i.e.,
  $$B_X(y, R) := \{x \in X \colon \ \|x - y\| \le R\}.$$

  For a compact set A and $\varepsilon > 0$, we introduce the entropy numbers $\varepsilon_k(A, X)$ (see [Höllig 1980]):
  $$\varepsilon_k(A, X) := \inf \left\{ \varepsilon \colon \quad \exists y^1, \dots, y^{2^k} \in X \colon \quad A \subseteq \bigcup_{j=1}^{2^k} B_X(y^j, \varepsilon) \right\}.$$

Main results

Theorem 1 [1]. Let $r_1 > 0$ and $1 \le \theta \le \infty$. Then it holds

$$\varepsilon_M(B_{\infty,\theta}^{\boldsymbol{r}}, B_{1,1}) \gg M^{-r_1} (\log^{\nu-1} M)^{r_1 + 1 - \frac{1}{\theta}}.$$

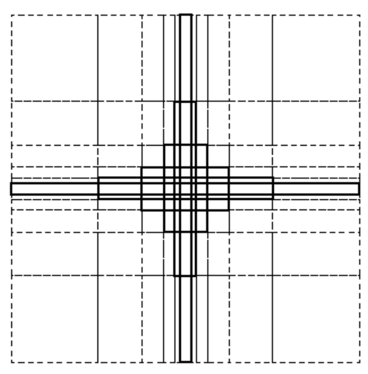
Theorem 2 [1]. Let $d \ge 2$, $r_1 > 0$, $1 \le q \le p \le \infty$, $1 \le \theta \le \infty$. Then it holds

$$\varepsilon_M(B_{p,\theta}^{\boldsymbol{r}}, B_{q,1}) \asymp d_M(B_{p,\theta}^{\boldsymbol{r}}, B_{q,1}) \asymp M^{-r_1} (\log^{\nu-1} M)^{r_1 + 1 - \frac{1}{\theta}}.$$

Note, that the orders of the Kolmogorov widths in Theorem 2 are realized by the subspace of trigonometric polynomials with spectrum form the step hyperbolic crosses

$$Q_n^\gamma := \bigcup_{(\boldsymbol{s}, \gamma) < n} \rho(\boldsymbol{s}),$$

where $n \in \mathbb{N}$, $M \asymp 2^n n^{\nu-1}$, $\gamma \in \mathbb{R}^d$, $\gamma_j = r_j / r_1$, $j = 1, \dots, d$, $\boldsymbol{s} \in \mathbb{N}^d$, $\rho(\boldsymbol{s}) := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \ 2^{s_j - 1} \le |k_j| < 2^{s_j}, \ j = 1, \dots, d\}$.



Step hyperbolic cross $Q_n^{\mathbf{1}}$, $\mathbf{1} = (1, \dots, 1)$, in the case $d = 2$

The respective estimates for the Sobolev classes $W_{p,\alpha}^{\boldsymbol{r}}$ were obtained in [2].

Comparison to the estimates in L_q

1. **Case $p = q$.**

- $p = q = 1$ — open problem for $d \ge 2$ for both $d_M(B_{1,\theta}^{\boldsymbol{r}}, L_1)$ and $\varepsilon_M(B_{1,\theta}^{\boldsymbol{r}}, L_1)$.
  In particular, for the Kolmogorov width $d_M(H_1^{\boldsymbol{r}}, L_1)$.

- $p = q = \infty$.

  - $d = 2$, $\boldsymbol{r} = (r_1, r_1)$, $r_1 > 0$, $1 \le \theta \le \infty$
    $$\varepsilon_M(B_{\infty,\theta}^{\boldsymbol{r}}, L_\infty) \asymp d_M(B_{\infty,\theta}^{\boldsymbol{r}}, L_\infty) \asymp M^{-r_1} (\log M)^{r_1 + 1 - \frac{1}{\theta}}.$$
    For $1 \le \theta < \infty$, the estimate was obtained in [Romanyuk 2019].
    For $\theta = \infty$, we combine by the Carl's lemma the upper estimate for $d_M(H_\infty^{\boldsymbol{r}}, L_\infty)$ from [Temlyakov 1989] and the lower estimate for $\varepsilon_M(H_\infty^{\boldsymbol{r}}, L_\infty)$ from [Temlyakov 1995], see [1] for details.

    The respective orders in $B_{\infty,1}$ and L_∞ coincide.

  - $d \ge 2$, $r_1 > 0$ — only for $\theta = 1$ and the Kolmogorov widths [Romanyuk 2005]
    $$d_M(B_{\infty,1}^{\boldsymbol{r}}, L_\infty) \asymp M^{-r_1} (\log^{\nu-1} M)^{r_1}.$$
    The respective orders in $B_{\infty,1}$ and L_∞ coincide.

  - $d \ge 3$, $1 < \theta \le \infty$ — open problem for both $d_M(B_{\infty,\theta}^{\boldsymbol{r}}, L_\infty)$ and $\varepsilon_M(B_{\infty,\theta}^{\boldsymbol{r}}, L_\infty)$.

- **$2 \le p = q < \infty$, $2 \le \theta \le \infty$, $r_1 > 0$.**
  For $d \ge 2$ it holds [1]
  $$\varepsilon_M(B_{p,\theta}^{\boldsymbol{r}}, L_p) \asymp d_M(B_{p,\theta}^{\boldsymbol{r}}, L_p) \asymp M^{-r_1} (\log^{\nu-1} M)^{r_1 + \frac{1}{2} - \frac{1}{\theta}}.$$
  The respective orders in $B_{\infty,1}$ and L_∞ differ.

2. **Case $p \ne q$.**

   For $d \ge 2$, $r_1 > 0$, $2 \le p \le \infty$, $1 \le q < p$, $2 \le \theta \le \infty$ it holds [Romanyuk 2015]
   $$\varepsilon_M(B_{p,\theta}^{\boldsymbol{r}}, L_q) \asymp d_M(B_{p,\theta}^{\boldsymbol{r}}, L_q) \asymp M^{-r_1} (\log^{\nu-1} M)^{r_1 + \frac{1}{2} - \frac{1}{\theta}}.$$

   The respective orders in $B_{\infty,1}$ and L_∞ differ.

For the remaining values of the parameters p, q and θ, that fulfil the conditions of Theorem 2, the orders of the quantities $d_M(B_{p,\theta}^{\boldsymbol{r}}, L_q)$ and $\varepsilon_M(B_{p,\theta}^{\boldsymbol{r}}, L_q)$, to the best of our knowledge, **remain unknown.**

References

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