Mean field model of a game for power

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H I G H L I G H T S

- Model equations are designed to describe dynamics of the distribution of power.
- The equations capture the Matthew effect: 'rich gets richer'.
- Unstable fixed points indicate boundaries between basins of attraction.
- The results are interpreted in terms of modeling of coercive power.

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Our aim is to model a game for power (equivalent to total energy) as a dynamical process, where an excess of power possessed by a player allows him to gain even more power. Such a positive feedback is often termed as the Matthew effect. Analytical and numerical methods allow to identify a set of stationary states, i.e. fixed points of the model dynamics. The positions of the unstable fixed points give an insight on the basins of attraction of the stable fixed points. The results are interpreted in terms of modeling of coercive power.

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1. Introduction

The gap between exact sciences and social sciences gets narrow and fuzzy. On the sociological side, authors who refer to physical ideas or apply physical methods are in minority, yet the trend is at least constant [1–3]. On the side of statistical physics, the interest in social phenomena is growing abruptly; for a list of approaches motivated by social sciences see [4,5]. As noted in the Editorial, this volume is devoted to *Econo- and sociophysics in turbulent world*. Having this in mind, here we propose a sociological problem: a formal approach to a competition for power, where power is understood as an ability to get more power. The approach is based on a simple but universal law of the conflict interaction between players (see Eq. (1)), which provides more power for a stronger player on every step of the game. We explore the dynamics of a set of variables, which represent fractions of power of the players. Another set represents their strategies, more or less aggressive. From the mathematical point of view, the system is equivalent to a coupled map lattice.

Dynamics of coupled map lattices is of interest both for their computational flexibility and applications [6–11]. As a rule, analytical considerations are less widespread and often resort to a global coupling or mean-field approximation [12,13], where correlations between different variables are neglected. The latter concept is borrowed directly from statistical physics [14,15].

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We note that while sociology is often claimed as a possible field of application of various theoretical approaches of the coupled map lattices, often it remains merely a declaration [16–18] (see [19] for an exception). To our knowledge, competition for power has not been modeled within this scheme.

In the next section, we refer to the Matthew effect [20] which is at the core of our interpretation of the results. Next, the model is explained in details. Further, analytical results are presented in the form of mathematical theorems and stability analysis of the model equations. These results are backed with numerical calculations, shown in a separate section. Last section is devoted to the interpretation and discussion.

2. The Matthew effect

In his famous essay [20], Robert Merton has introduced the so-called Matthew effect to social sciences, when discussing biased distribution of recognition for scientific achievements. As Merton puts it: '… the Matthew effect consists in the accruing of greater increments of recognition (…) to scientists of considerable repute and the withholding of such recognition from scientists who have not yet made their mark.' Since then, the Matthew effect has been the subject of research in education [21], technology [22], economy [22–24], statistics [25], and science again [26–29], to name only a few [22,30–32]. The effect, commonly cited as 'accumulated advantage' or 'rich gets richer', can be defined as a positive feedback between an amount of possessed goods and an ability of acquiring even more goods. Even when the term 'Matthew effect' is not cited literally, the phenomenon itself is at the center of attention of historians, sociologists, economists and political scientists [33–37].

Our interest is modeling of dynamics of power, one of central concept in sociology [38]. We accept the classical definition by Max Weber: power is 'the ability of an individual or group to achieve their own goals or aims when others are trying to prevent them' [39]. More specifically, we imagine a zero-sum game for coercive power, with the latter not based on a social structure but rather on individual characteristics of social actors. In social simulations, the Matthew effect is often called to interpret the assumption of preferential attachment in growing networks [40,41]. In this sense 'rich gets richer' means that a node (actor) of large degree (number of neighbors) has more chances to get even more neighbors, and therefore her/his position in the network, as measured by centrality, betweenness etc. [42], gets improved. However, this kind of position is not equivalent with a player's individual power, but comes from the structure of the network. In other models which could be used to simulate conflicts [43–47], it is only the amount of actors of given orientation what matters for the final outcome of a model dynamics. Perhaps this limitation is a legacy of statistical physics, where phase transitions are considered of a system of identical objects.

Despite its obvious validity for conflicts, dynamics of power of individual players has been ignored in most computational models. As an exception, we note the Bonabeau model [48] (note however, that the term "Matthew effect" has not been used there). In this model, when two players meet they fight; the winner gets more power and the loser − gets less. These gains and loses are relevant for the outcome of subsequent fights. Main result of the Bonabeau model is a transition between egalitarian and hierarchical phase of a model society, identified by means of simulations and mean field modeling [49–51].

Our aim here is to trace consequences of an individual strategy for the player who selected it. Hence the model dynamics includes individual characteristics of players. Each player is endowed with the willingness to commit himself to conflicts, which stands for his strategy, and with an initial value of the power. The former remains constant in time, while the latter is a subject of model dynamics. Taking into account the principle 'rich gets richer' we can expect a clear difference between winners and losers. As will be demonstrated below, the model outcome is that 'winner takes all'. The problem to solve is, how the distribution of the model parameters allows to appoint the winner. Apart from the random assignment of the parameters among actors, the model is purely deterministic.

3. The model

Let us denote the number of players by m, and the player index by $i = 1, 2, \ldots, m$. The power of ith player at time t is p_i^t, and it is kept nonnegative. The willingness of ith player to commit into conflict is denoted by c_i, kept in the range $[0, 1]$. The equation of motion, equivalent to the system of coupled map lattices, is (cf. with [52,53])

$$p_i^{t+1} = \frac{p_i^t(1 - c_i r_i^t)}{z^t}, \quad t = 0, 1, 2, \ldots. \tag{1}$$

where

$$r_i^t = \frac{\sum_{k \neq i} p_k^t}{m - 1} \equiv \frac{1 - p_i^t}{m - 1} \tag{2}$$

represents a mean player other than ith one. The coupling between players is introduced *via* the normalization constant $1/z^t$, which is taken as to assure that

$$\sum_{i=1}^{m} p_i^{t+1} = 1, \tag{3}$$

what marks that the total amount of power remains constant. In other words, we have a zero-sum game. Due to (3) we find, that

$$z^t = 1 - \theta^t, \quad \theta^t := \sum_{i=1}^{m} p_i^t c_i r_i^t. \tag{4}$$

We will refer on (1) as the formula of conflict interaction.

Note that in general the law of conflict redistribution of power is unknown. Our rather simple version of conflicting fight presented by the system of Eq. (1) in terms of coordinated expresses the natural primitive principle: *each against all*. In fact, these system of equations admit the presentation in a form of non-linear Schrödinger type equation with discrete time,

$$\Psi^{t+1} = (H_0^t - V^t)\Psi^t.$$

Here Ψ^t denotes the vector distribution of energy (power) $p^t = (p_1^t, \ldots, p_m^t)$ along a finite number of player positions, H_0^t stands for the Hamiltonian of free evolution, and non-positive potential V^t corresponds to a perturbation which is produced by the point singular type of repulsive interaction between players (abstract particles). We refer to [54–56] for the complete theory of the point interaction in various physical models. The fixed points in Theorems 3.2, 3.3, 3.5 (see below) present eigenvalues corresponding to the point spectrum of our game model complex system.

To clarify the sociological motivation for the form of conflict interaction between players given by formula (1), let us rewrite it as follows:

$$p_i^{t+1} = I_i^t - C_i^t, \quad I_i^t = \frac{p_i^t}{z^t}, \quad C_i^t = c_i \frac{p_i^t r_i^t}{z^t}.$$

The term I_i^t is responsible for increasing of the ith player's power, since always $I_i^{t+1} > I_i^t$ due to $z^t < 1$. In turn, the term $-C_i^t$ brings the losses caused by an interaction with whole society. These losses in general are different and depend of a strength of competition between ith player and his conditional co-player presented by r_i^t (see Eq. (2)). The parameter c_i just regulates this strength. If the players do not enter to the conflict ($c_i = 0$ for each i), they just preserve their power. Acting with $c_i > 0$, an ith player loses. However, other players in the conflict lose as well. The reduction of p_i can be more than compensated (via the normalization z^t) if p_i^t is large enough. Then, each player has to decide to what extent the conflict is fruitful.

This question has a direct counterpart in the sociological literature [29]. There we find a careful analysis of 270 conflicts in two American universities, with several examples, where the decision – to contest or not – was determined by expected gains and losses. To clarify this, let us quote one example from [29]: "… Aeneas had to tolerate seeing Cindy's name go before his on the author list of several papers he had largely written and to which Cindy had contributed virtually nothing. But Cindy was going up for promotion to full professor, and Bruce, her senior full professor spouse and the holder of the grant from which the publication data were generated, insisted she be placed ahead of Aeneas. The sheer injustice galled Aeneas. But he was stuck: He needed Bruce's data, and it was too late to switch to another major professor. He therefore put up with what he considered to be theft of his work". [29]. Apparently, Aeneas decided to reduce his c_A coefficient.

Simplifying this example, suppose that at time t the power of Aeneas is $p_A^t = \varepsilon$, the power of Bruce is $p_B^t = 1-2\varepsilon$, and a third person (also of power ε) does not enter the conflict. Keeping in Eq. (1) only terms linear in ε, we get $p_A^{t+1} \approx \varepsilon(1-c_A/2)$ and $p_B^{t+1} \approx 1 - 2\varepsilon + \varepsilon c_A/2$. In other words, while Aeneas loses with c_A, Bruce gains with c_A for any strategy of himself.

More generally, p_i^{t+1} increases with p_i^t, what activates the Matthew effect. On the other hand, the whole contribution to p_i^{t+1} from the conflict is negative. More precisely, p_i^{t+1} grows, for players with $c_i r_i^t < \theta^t$, and falls, if $c_i r_i^t > \theta^t$. Our computational problem is twofold:
– what is the strategy c_i which drives a player to success?
– how this strategy depends on the actual distribution of power?
To answer, we need a more deep mathematical analysis.

4. Analytical results

4.1. Short analysis of the conflict formula

At first let us put $c_i = 1$ for all $i = 1, \ldots, m$. Then using (2) we can rewrite (1) in terms of coordinates of vector $p^t = (p_1^t, \ldots, p_m^t)$ from $(m-1)$-dimensional simplex $\mathbf{S}_+^{m-1}$ as follows

$$p_i^{t+1} = p_i^t \cdot \frac{m - 2 + p_i^t}{m - 2 + L^t}, \tag{5}$$

where L^t denotes the square norm of p^t, i.e.,

$$L^t \equiv \|p^t\|^2 := \sum_{k=1}^{m} (p_k^t)^2. \tag{6}$$

Thus

$$p_i^{t+1} = p_i^t \cdot k_i^t, \quad k_i^t := \frac{m - 2 + p_i^t}{m - 2 + L^t}. \tag{7}$$

Now we observe that if $p_i^t > L^t$, then $k_i^t > 1$ and therefore p_i^{t+1} increases. p_i^{t+1} will decrease, if $p_i^t < L^t$.

Thus, the value L^t may be considered as a threshold which divides the conflicting society into three classes of players:

$$I_-^t := \{i \ : p_i^t < L^t\}, \quad I_0^t := \{i \ : p_i^t = L^t\}, \quad I_+^t := \{i \ : p_i^t > L^t\}. \tag{8}$$

It is easy to check that in general both subsets I_-^t and I_+^t, $t \geq 0$ are always non-empty. In the excluding case, when $p_i^{t=0} = 1/m, i = 1, \ldots, m, \ L^t = 1/m$ too, and the set $I_0^t = m$ for all $t \geq \infty$. In all other cases I_0^t is non-empty only temporarily.

4.2. A single winner is generic

For further manipulations in situation $c_i = 1$ for all i we rewrite (1) in a form

$$p_i^{t+1} = p_i^t \left(1 + \frac{p_i^t - L^t}{m - 2 + L^t}\right) = (1 + \delta_i^t) \cdot p_i^t, \tag{9}$$

where

$$\delta_i^t := \frac{p_i^t - L^t}{m - 2 + L^t}. \tag{10}$$

Three next propositions follows directly from (5) and (10).

Let us fix some initial distribution of power between players, i.e., we fix $p \equiv p^{t=0} \in \mathbf{S}_+^{m-1}$, a stochastic vector from the positive simplex.

Proposition 3.1. *If some couple of initial coordinates satisfies $p_i^{t=0} = p_k^{t=0}, i \neq k$, then $p_i^t = p_k^t$ for all $t = 1, 2, \ldots$ Moreover, if $p_i^{t=0} < p_k^{t=0}$, then $p_i^t < p_k^t$ for all $t \geq 1$.*

Thus,

$$p_i^t \leq p_k^t \Longrightarrow p_i^{t+1} \leq p_k^{t+1}, \quad t = 0, 1, \ldots \tag{11}$$

It means that the conflict interaction does not change the initial ordering of players on their power:

$$0 \leq p_{i_1}^{t=0} \leq p_{i_2}^{t=0} \leq \cdots p_{i_m}^{t=0} \leq 1 \Longrightarrow 0 \leq p_{i_1}^t \leq p_{i_2}^t \leq \cdots p_{i_m}^t \leq 1, \quad t = 1, 2, \ldots \tag{12}$$

In fact a sign of the difference $p_i^t - L^t$ in (9) defines whether p_i^t grows or falls on $t + 1$-step.

Proposition 3.2. *If $p_i^t < L^t$, then*

$$p_i^{t+1} < p_i^t, \tag{13}$$

and if $p_i^t > L^t$, then

$$p_i^{t+1} > p_i^t. \tag{14}$$

Proposition 3.3. *The sequence L^t converges to a bounded limit:*

$$0 < \lim_{t \to \infty} L^t = b \leq 1. \tag{15}$$

Proof. Obviously $0 < L^t \leq 1$, since $L^t = \|p^t\|^2$ and all vectors p^t are stochastic. We have to show that the sequence L^t is monotonically growing,

$$L^{t+1} - L^t > 0, t \geq 0. \tag{16}$$

With this aim we use the decomposition (8). If $i' \in I_-^t$, then $p_{i'}^t - L^t < 0, \quad \delta_{i'}^t < 0$. Therefore due to (9) the difference $p_{i'}^{t+1} - p_{i'}^t = \delta_i^t \cdot p_i^t$ is negative. Denote it by $-d_{i'}^t$ with $d_{i'}^t > 0$. In the case $i'' \in I_+^t$ the opposite inequality is fulfilled, $\delta_{i''}^t > 0$. Then $p_{i''}^{t+1} - p_{i''}^t =: d_{i''}^t > 0$.

Since both p^t and p^{t+1} are stochastic there exist $0 < s < m$ such that

$$0 = \sum_{k=1}^m p_k^{t+1} - \sum_{k=1}^m p_k^t = \sum_{i'=1}^s (p_{i'}^{s+1} - p_{i'}^s) + \sum_{i''=1}^{m-s} (p_{i''}^{s+1} - p_{i''}^s) = -\sum_{i'=1}^s d_{i'}^t + \sum_{i''=1}^{m-s} d_{i''}^t.$$

By this

$$\sum_{i''=1}^{m-s} d_{i''}^t - \sum_{i'=1}^{s} d_{i'}^t = 0. \tag{17}$$

Consider now the difference $L^{t+1} - L^t \equiv \|p^{t+1}\|^2 - \|p^t\|^2$. Since $p_{i'}^{t+1} = p_{i'}^t - d_{i'}^t$ and $p_{i''}^{t+1} = p_{i''}^t + d_{i''}^t$, using the inequalities $p_{i''}^t > \|p^t\|^2$ ($i'' \in I_+^t$) and $p_{i'}^t < \|p^t\|^2$ ($i' \in I_-^t$) we obtain

$$\|p^{t+1}\|^2 - \|p^t\|^2 > 2L^t \left(\sum_{i''=1}^{m-s} d_{i''}^t - \sum_{i'=1}^{s} d_{i'}^t \right) + \sum_{i'=1}^{s} d_{i'}^{t\,2} + \sum_{i''=1}^{m-s} d_{i''}^{t\,2} > 0.$$

Finally due to (17) we have:

$$L^{t+1} - L^t = \sum_{i'=1}^{s} d_{i'}^{t\,2} + \sum_{i''=1}^{m-s} d_{i''}^{t\,2} > 0.$$

Thus (16) is proved. This shows that L^t is a growing bounded sequence. Therefore (15) is true.

Let us denote

$$p_{\max}^t := \max_{1 \le i \le m} \{p_i^t\}.$$

Now we will prove one of the main results of the paper.

Theorem 3.1. *Assume for a vector* $p \equiv p^{t=0} \in \mathbf{S}_+^{m-1}$, $m > 2$ *all its coordinates are non-zero and mutually different,*

$$p_i \ne p_j, \quad i \ne j. \tag{18}$$

Then

$$\lim_{t \to \infty} p_i^t = 0, \quad p_i \ne p_{max}. \tag{19}$$

and

$$p_{max}^\infty := \lim_{t \to \infty} p_{max}^t = 1, \tag{20}$$

Proof. From obvious inequalities

$$\min_k \{p_k^t\} \le \|p^t\|^2 \equiv L^t \le \max_k \{p_k^t\} \tag{21}$$

and Propositions 3.2, 3.3 it follows that the sequence $p_{\max}^t$ grows with $t \to \infty$. Since it is bounded, there exists a limit $a = \lim_{N \to \infty} p_{\max}^t \le 1$. Due to condition (18) without loss of generality we can assert that coordinates of vectors p^t are ordered in such a manner that

$$0 < p_1^t < p_2^t < \cdots < p_m^t < 1. \tag{22}$$

This order does not depend from t (see Proposition 3.1). By this the latter coordinate is maximal for all times $p_{\max}^t = p_m^t$. Thus, the following estimate holds:

$$0 < a := \lim_{t \to \infty} p_m^t = p_m^\infty \le 1.$$

Let us prove that $a = 1$. At first we show that $a = b$, where $b = L^\infty := \lim_{t \to \infty} L^t$. Indeed, from existence of the limits for p_m^t and L^t we have:

$$a = p_m^\infty = k_m^\infty \cdot p_m^\infty = \frac{m - 2 + p_m^\infty}{m - 2 + L^\infty} \cdot p_m^\infty = \frac{m - 2 + a}{m - 2 + b} \cdot p_m^\infty. \tag{23}$$

By this $k_m^\infty = 1$. It means that $a = b$. In fact both, a and b are equal to one. This consequence one can draw from analysis of behavior of the lasting coordinate p_{m-1}^t. Indeed, due to (22) and Proposition 3.1, the inequality $p_{m-1}^t < p_m^t$ holds for all t. Therefore the ratio p_m^t/p_{m-1}^t exceeds one and grows. In particular, $p_{m-1}^t < a$ always. It means that this ratio goes to infinity and therefore $\lim p_{m-1}^t = 0$. If we assume the contrary, then by (7) we have the equality $\lim p_{m-1}^t = L^\infty = b = a$, that is a contradiction. Similarly one can assert that all other coordinates converge to zero, $\lim p_i^t = 0$, $i \ne m$. Thus, $a = 1$. The theorem is proved.

Fig. 1 illustrates the above result.

By this theorem if all initial coordinates $p_i^{t=0}$ are different, then

$$|I_-^t| \longrightarrow m - 1, \quad |I_+^t| \longrightarrow 1, \quad t \longrightarrow \infty,$$

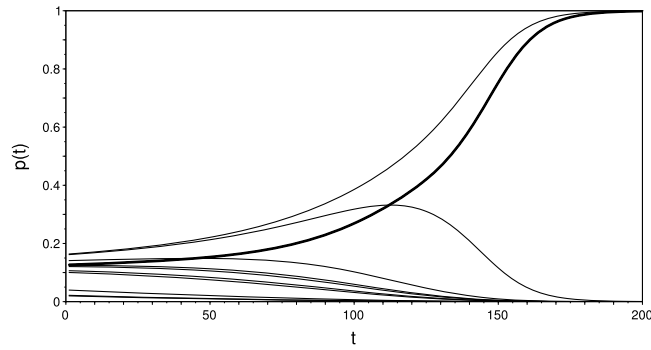where $|I_{\text{sign}}^t|$ denotes a cardinality of a set.

Fig. 1. The winner is determined by the maximal initial value of social power. $m = 10, c_i = 1$ for all $i = \overline{1, 10}$. The initial values are : $p_1 = 0.05, p_2 = 0.01, p_3 = 0.01, p_4 = 0.02, p_5 = 0.053, p_6 = 0.08, p_7 = 0.06, p_8 = 0.07, p_9 = 0.062, p_{10} = 0.011$. Bold line shows the growth of L^t.

Thus we proved that if the society joints players with nonzero and different values of power, there exist a single winner which is determined by the maximal initial power. In other terms this means that the richest player becomes richer and captures with time the whole wealth, all other agents do not get anything.

Let $\mathcal{P}^*$ denotes the set of fixed points for the nonlinear map generated by formula (1). By construction, all limit points in Theorem 3.1 are fixed points, write $p^\infty = p^* \in \mathcal{P}_1^*$, where index 1 means that p^* has only one nonzero coordinate equals to 1.

Theorem 3.2. *Under condition* (18) *there exist m fixed points*

$$p_j^* = (0, 0, \ldots, 0, p_{ji}, 0, \ldots, 0), \quad p_{ji} = \delta_{ji}, \quad j, i = 1, \ldots, m, \tag{24}$$

where δ_{ji} stands for the Kronecker symbol. All these points are stable.

Proof. We have only to show the stability of the fixed points $p_j^\infty = p_j^*, j = 1, \ldots, m$. It follows from the fact that any ε-perturbation of the vector $p_j^* = (0, 0, \ldots, 0, \delta_{ji}, 0, \ldots, 0)$ preserves for its j-coordinate to have the maximal value. And by Theorem 3.1 the limits on t for all other coordinates are zero.

Consider the exotic situation when initial vector $p \in \mathbf{S}_+^{m-1}$ has $1 < k < m$ equal coordinates with the maximal value. Obviously the set of such vectors has zero $(m-1)$-dimensional Lebesgue measure. By slightly modified argumentations as above one can prove that all non-maximal coordinates $p_{i\neq\max}^t$ converge to zero, as $t \to \infty$, and coordinates with maximal value come to $1/k$. Thus, the limiting set of fixed points, denote it by $\mathcal{P}_k^*$, contains the family of C_k^m vectors $\{p^* \equiv p^\infty\}$ whose k nonzero coordinates are equal to $1/k$.

Theorem 3.3. *Every fixed point from family $\mathcal{P}_k^*$, $1 < k \leq m$ is unstable.*

Proof. Obviously, a general ε-perturbation of a vector $p^* \in \mathcal{P}_k^*$ does not preserve the condition that $k \geq 2$ coordinates are equal and have the maximal value. Therefore by Theorem 3.1 the limiting vector will not belong to $p^* \in \mathcal{P}_k^*$.

4.3. An arbitrary conflict activity

Consider the general situation when $0 \leq c_i \leq 1$ are arbitrary.
In this case the conflict formula (1) after using (2) has a view

$$p_i^{t+1} = p_i^t \cdot \frac{m - 1 - c_i(1 - p_i^t)}{m - 1 - L_c^t} = p_i^t \cdot k_{i,c}^t \tag{25}$$

where

$$k_{i,c}^t := \frac{m - 1 - c_i(1 - p_i^t)}{m - 1 - L_c^t} \tag{26}$$

and

$$L_c^t := \sum_{k=1}^m c_k p_k^t (1 - p_k^t). \tag{27}$$

Obviously now the value $0 \leq L_c^t \leq 1$ has more complex non-linear dependence from p_i^t, in particular, it changes non-monotonically with time.

In a slightly other form the conflict formula views as follows:

$$p_i^{t+1} = p_i^t \left(1 + \frac{L_c^t - c_i(1 - p_i^t)}{m - 1 - L_c^t} \right) = p_i^t(1 + \delta_{i,c}^t), \tag{28}$$

where

$$\delta_{i,c}^t := \frac{L_c^t - c_i(1 - p_i^t)}{m - 1 - L_c^t}. \tag{29}$$

From (25) and (28) we see that p_i^t increases under the following condition

$$L_c^t > c_i(1 - p_i^t). \tag{30}$$

Let

$$L_{i,c}^t := \sum_{k \neq i} c_k p_k^t(1 - p_k^t), \tag{31}$$

then (30) has a form

$$\frac{L_{i,c}^t}{c_i} > (1 - p_i^t)^2. \tag{32}$$

Unfortunately, in general, no one of both conditions (30), (32) guarantee the global increasing for p_i^t, but only the local behavior. Nevertheless, we are able to get some strategic characteristic of the relative behaviors for players in terms of their ratios

$$R_{ik}^t := \frac{p_i^t}{p_k^t}, \quad i, k = 1, \ldots, m.$$

Since due to (25)

$$R_{ik}^{t+1} = R_{ik}^t \cdot \frac{m - 1 - c_i(1 - p_i^t)}{m - 1 - c_k(1 - p_k^t)}, \tag{33}$$

we obtain

Proposition 3.4. *The ratio R_{ik}^t grows with $t \to \infty$, iff*

$$c_i(1 - p_i^t) < c_k(1 - p_k^t). \tag{34}$$

Theorem 3.4. *Assume*

$$c_i(1 - p_i^t) < L_c^t < c_k(1 - p_k^t). \tag{35}$$

hold for a single $i = i_1$ and all $k \neq i_1$. Then

$$p_{i_1}^\infty = \lim_{t \to \infty} p_{i_1}^t = 1, \quad p_{k \neq i_1}^\infty = \lim_{t \to \infty} p_k^t = 0. \tag{36}$$

All these limit points are stable.

Proof. By the left part of (35), $p_{i_1}^{t+1}$ grows (see (28)). Obviously $R_{i_1 k}^{t+1} > 1$ since due to (35), we have

$$c_{i_1}(1 - p_{i_1}^t) < c_k(1 - p_k^t). \tag{37}$$

and therefore the inequalities

$$c_{i_1}(1 - p_{i_1}^{t+1}) < c_k(1 - p_k^{t+1}) \tag{38}$$

are also true. They, in general, do not guarantee that $p_{i_1}^{t+1}$ grows quicker than each p_k^{t+1}. But thanks to the right part of (35), all p_k^{t+1} in fact decrease. Since (35) are fulfilled for each t we get (36). Clearly, the limit points are fixed.

To prove its stability we will consider without of loss generality the case $p^\infty \equiv p^* = (1, 0, 0, \ldots, 0)$ and show this vector attracts all vectors of type $p^{*,\varepsilon} = (1 - \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$, $\varepsilon_1 = \sum_{k \neq 1} \varepsilon_k$ with ε_1 small enough. In fact, we have to check the inequality $(p_1^{*,\varepsilon})^{t=1} > p_1^{*,\varepsilon} = 1 - \varepsilon_1$ only for the first coordinate. It is equivalent (see (35)) to

$$c_1(1 - p_1^{*,\varepsilon}) < L_{c,\varepsilon}, \quad L_{c,\varepsilon} = \sum_k c_k \varepsilon_k(1 - \varepsilon_k).$$

In turn, the equivalent inequality has a form $c_1 \varepsilon_1^2 < \sum_{k \neq 1} c_k \varepsilon_k(1 - \varepsilon_k)$, or $\sum_k c_k \varepsilon_k^2 < \sum_{k \neq 1} c_k \varepsilon_k$. Clearly, the last inequality is fulfilled for all ε_k small enough, since the left side is constituted with square of ε_k.
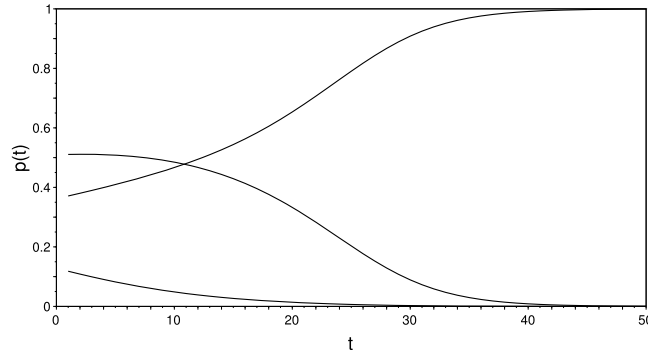
Fig. 2. The second player becomes winner due to conditions (39) from Proposition 3.5. $m = 3, p_1 = 0.4065, p_2 = 0.2513, p_3 = 0.3421$, $c_1 = 0.4588, c_2 = 0.41967, c_3 = 0.2896$.

Proposition 3.5. *Assume $c_{i_1} < c_k$ and for some t the inequalities*

$$c_i(1 - p_i^t) < L_c^t < c_k(1 - p_k^t), \quad p_k^t < 1/2 \tag{39}$$

hold for a single $i = i_1$ and all $k \neq i_1$. Then these inequalities are true for all $t + n, n \geq 1$.

Proof. Obviously $R_{i_1k}^{t+1} > 1$ since due to (39) we have (37) for $i = i_1$ and all $k \neq i_1$ Therefore the inequalities (38) hold too. To show $c_{i_1}(1 - p_{i_1}^{t+1}) < L_c^{t+1}$ one can consider the ratio

$$\frac{L_c^{t+1}}{c_{i_1}(1 - p_{i_1}^{t+1})} = \frac{c_{i_1}p_{i_1}^{t+1}(1 - p_{i_1}^{t+1})}{c_{i_1}(1 - p_{i_1}^{t+1})} + \frac{\sum_{k \neq i_1} c_k p_k^{t+1}(1 - p_k^{t+1})}{c_{i_1}(1 - p_{i_1}^{t+1})}.$$

Using (38) we find that $L_c^{t+1}/c_{i_1}(1 - p_{i_1}^{t+1}) > \sum_{i=1}^m p_i^{t+1} = 1$. Therefore $p_{i_1}^{t+1}$ grows. The proof of validity $L_c^{t+1} < c_k(1 - p_k^{t+1})$ requires more deep observations. At $(t + 1)$th step the value of L_c^t changes due to two reasons. At first, it falls since all p_k^{t+1} falls due to assumption $p_k^t < 1/2$ and by inequalities $L_c^t < c_k(1 - p_k^t)$ (see (28) with $i = k$). At second, it grows since $c_{i_1}(1 - p_{i_1}^t) < L_c^t$ (see also (28) with $i = i_1$). We assert that inequalities $L_c^{t+1} > c_k(1 - p_k^{t+1})$ could not fulfilled if $c_{i_1} < c_k$ for all $k \neq i_1$. The proof is purely geometrical. To show this fact one need to compare the graphics of functions $c_{i_1}p_{i_1}^t(1 - p_{i_1}^t)$ and $c_k p_k^t(1 - p_k^t)$ for k with maximal value of p_k^t. By induction we continue our argumentations for any $n > 1$.

Thus, (36) is also true under conditions of Proposition 3.5 (see Fig. 2).

Denote by $\mathcal{P}_{k,c}^*$ the set of fixed points for the general case $0 \leq c_i \leq 1$, where $1 \leq k \leq m$ means a number of nonzero coordinates.

Theorem 3.5. *Each fixed point $p^* \in \mathcal{P}_{k,c}^*, k > 1$ is unstable.*

Proof. Consider any $p^* \in \mathcal{P}_{k,c}^*, k > 2$ and a couple of it nonzero coordinates $p_{i_1}^*, p_{i_2}^*$. They have to satisfy the equality

$$c_{i_1}(1 - p_{i_1}^*) = L_c^* = c_{i_2}(1 - p_{i_2}^*).$$

Assume $p_{i_1}^* \geq p_{i_2}^*$ and replace $p_{i_1}^*$ on $p_{i_1,\varepsilon}^* = p_{i_1}^* + \varepsilon$ and $p_{i_2}^*$ on $p_{i_2,\varepsilon}^* = p_{i_1}^* - \varepsilon, \varepsilon > 0$. Then we easily check that for any small ε, the above equalities transform into inequalities

$$c_{i_1}(1 - p_{i_1,\varepsilon}^*) < L_{c,\varepsilon}^* < L_c < c_{i_2}(1 - p_{i_2,\varepsilon}^*).$$

Since now $p_{i_2,\varepsilon}^* < 1/2$ due to $k > 2$, we can use Proposition 3.5. Thus, $(p_{i_1,\varepsilon}^*)^t$ increases and $(p_{i_2,\varepsilon}^*)^t$ falls. For the case $k = 2$ see next subsection.

4.4. Structure of fixed points

For clarity our assertion in more details, first we consider the case $m = 2$; although very simple, it provides a good starting point. The set of Eqs. (1) reduce to only one

$$p^{t+1} = \frac{p^t(1 - c_1 + c_1 p^t)}{1 - (c_1 + c_2)p^t(1 - p^t)}. \tag{40}$$

with three fixed points: $p^* = 0, p^* = 1$ and $p^* = c_1/(c_1 + c_2)$. To state their stability, we need to compare $|\partial p^{t+1}/\partial p^t|$, calculated at the fixed point, with one [57]. This expression is equal to $1 - c_1$ and $1 - c_2$ for $p^* = 0$ and $p^* = 1$, respectively.

Hence, both these fixed points are stable except the cases $c_i = 0$, where the stability is marginal. At the third fixed point the derivative is $|\partial p^{t+1}/\partial p^t| = (c_1 + c_2)/(c_1 + c_2 - c_1c_2) > 1$, hence this fixed point is unstable. This is an illustration of the above-given general theorem, that all fixed points different than $\{p_i^*\} = \{0, 0, \ldots, 0, 1, 0, \ldots, 0\}$ are unstable.

It is easy to see that the attraction basin for the fixed point $p* = 1$ ($p* = 0$) is interval $(c_1/(c_1+c_2), 1]$ ($[0, c_1/(c_1+c_2))$). Indeed, since now $L_c = (c_1+c_2)p(1-p)$ from (28) it follows that p^t grows to 1 only if $c_1(1-p) < L_c$, i.e., if $p > c_1/(c_1+c_2)$. Otherwise, i.e., if $p < c_1/(c_1 + c_2)$, that is equivalent to $c_1(1 - p) > L_c$, the value p^t goes to zero. The unstable fixed point $c_1/(c_1 + c_2)$ has its basin empty.

One can put the inverse question (some kind of the two players problem). Given $0 < p < 1$ and $0 < c_2 \le 1$ what c_1 guarantees $p* = 1$? From (28) we find solution $c_1 < c_2p/(1 - p)$. In particular, if $c_2 = 1$ the first player with any initial $p > 0$ wins if he take $c_1 < p/(1 - p)$.

We note, in the case of three players the similar question (see below) requires essentially more effort.

For $m = 3$, the normalization condition reduces the number of equations to two. For simplicity, let us use variables $x, y, 1-x-y$ instead of p_1, p_2, p_3, and primes instead of time index $t + 1$; the time index t will be omitted. Then we have

$$x' = \frac{x(1 - c_1 + c_1 x)}{1 - c_1 x(1 - x) - c_2 y(1 - y) - c_3(x + y)(1 - x - y)} \tag{41}$$

$$y' = \frac{y(1 - c_2 + c_2 y)}{1 - c_1 x(1 - x) - c_2 y(1 - y) - c_3(x + y)(1 - x - y)}$$

Basically, there are seven fixed points: $(x^*, y^*) =$ (i) $(1, 0)$, (ii) $(0, 1)$, (iii) $(0, 0)$, (iv) $(c_1/(c_1 + c_2), c_2/(c_1 + c_2))$, (v) $(c_1/(c_1 + c_3), 0)$, (vii) $(0, c_2/(c_2 + c_3))$, and (vii)

$$\left(\frac{c_1(c_2 + c_3) - c_2c_3}{c_1c_2 + c_2c_3 + c_3c_1}, \frac{c_2(c_3 + c_1) - c_1c_3}{c_1c_2 + c_2c_3 + c_3c_1} \right) \tag{42}$$

However, the coordinates of the last fixed point are not necessarily positive. To keep all coordinates $(x^*, y^*, 1 - x^* - y^*)$ nonnegative, three conditions should be fulfilled:

$$
\begin{aligned}
c_1 &> c_2c_3/(c_2 + c_3) \\
c_2 &> c_3c_1/(c_3 + c_1) \\
c_3 &> c_1c_2/(c_1 + c_2)
\end{aligned}
\tag{43}
$$

Now suppose that with the coefficient c_3 we are at the limit case, *i.e.* $c_3(c_1 + c_2) = c_1c_2$. After some simple algebra we get $x + y = 1$, hence for the seventh fixed point given by Eq. (42) we get $p_3^* = 0$. Also, its first coordinate $x^* = c_1/(c_1 + c_2)$, what means that the two fixed points (*iv*) and (*vii*) collide. When c_3 decreases further, the seventh fixed point leaves the simplex where $\{p_i > 0\}$.

As we know from the preceding subsection, the only stable fixed points appear at the corners of the m-cube, where one player got the whole power ($p_i^* = 1$). We can check the stability of such fixed points, taking $x = 1$ as an example. There, the eigenvalues of the Jacobian are $(1 - c_3/2, 1 - c_2/2)$, what is nicely consistent with the case $m = 2$.

The question about the attraction basins for stable fixed points is more complex and here we present only particular numerical results.

Consider the case (i), two next cases, (ii), (iii) are analogical.

Let $c_1 < c_2, c_3$. If $p_1 > p_2, p_3$, then, due to (33), the both inequalities $c_1(1 - p_1) < c_2(1 - p_2), c_3(1 - p_3)$ become with time stricter. This means that p_2', p_3' fall and hence $x^* = 1$. It is only a part of the attractive basin for point $x^* = 1$. Conditions $p_1 = p_2 > p_3$, $p_1 = p_3 > p_2$ give, by same argumentation, else two parts. Moreover, for enough small c_1 the attractive basin of x^* contains points p_1, p_2, p_3 with $p_1 < p_2, p_3$. It follows from the fact that if $c_1 = 0$, then x^* attracts all points with $p_1 \ne 0$ since in this case both ratios $R_{1,2}^t, R_{1,3}^t$ grows (see (33)). Fig. 3 demonstrates above phenomenon for p_1 with minimal value in a general case ($m = 10$).

The fixed points (iv), (v), (vi) are highly unstable. Any small perturbation of points x^*, y^* violates the balance $c_1(1 - x*)/c_2(1 - y*) = 1$ which, due to (33), goes far from 1 with time.

In the general case of unlimited number m of players and $c_i > 0$ $\forall i$, we have m 'corner' fixed points and $m(m - 1)/2$ 'edge' fixed points where $p_i^* = 0$ for all but two players. As we have seen for $m = 3$, more fixed points are possible if the coefficients c_i fulfill appropriate conditions. Accordingly, the maximal number of the fixed points is

$$\sum_{k=1}^{m} \binom{m}{k} = 2^m - 1. \tag{44}$$

The actual number of fixed points can be less, if some of them fall out of the area where $\forall i, p_i > 0$.

The positions of the unstable fixed points can give hints on the boundaries of the basins of attraction of the stable fixed points. This advantage is demonstrated numerically in the next section.
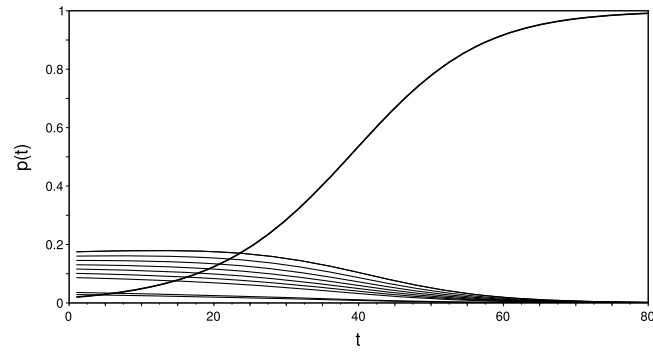
Fig. 3. The player with smallest initial social energy (power) becomes the winner. $m = 10, c_k = 0, p_k = \min_i\{p_i\}, c_i = 1$ for all $i = \overline{1, 10}, i \neq k$. The initial values are: $p_1 = 0.0182, p_2 = 0.0364, p_3 = 0.1309, p_4 = 0.0290, p_5 = 0.1164, p_6 = 0.1018, p_7 = 0.1455, p_8 = 0.0873, p_9 = 0.16, p_10 = 0.1745.$.
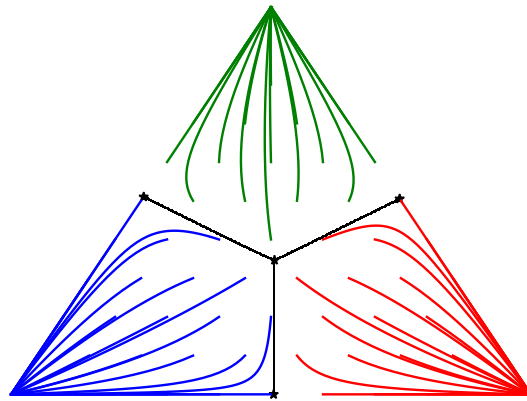

Fig. 4. For close values of conflict activities, $c_1 = 0.5, c_2 = 0.49, c_3 = 0.51$, basins of attraction divide the 2-dimensional simplex into three parts of similar size. The unstable fixed points, marked here by stars, lie at the boundaries between the basins, marked by continuous black lines. Red trajectories tend to $(1, 0, 0)$ (bottom right), blue trajectories tend to $(0, 1, 0)$ (bottom left), and green trajectories tend to $(0, 0, 1)$ (top).
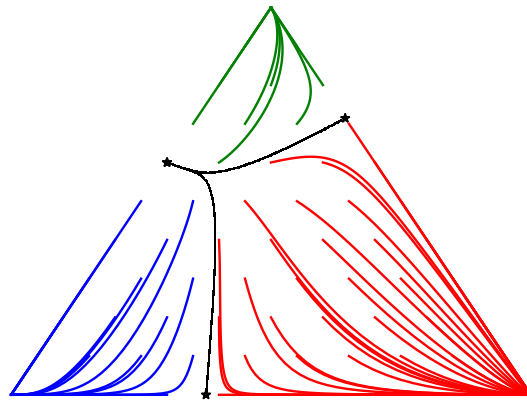

Fig. 5. The fixed point $p^* = (1, 0, 0)$ (bottom right) which represents the player with the smallest conflict activity ($c_1 = 0.24, c_2 = 0.4, c_3 = 0.6,$) has the largest basin of attraction. Here the central (seventh) fixed point collides with one of the edge fixed points..

5. Basins of attraction: $m = 3$ and beyond

In Figs. 4 , 5 , 6, three simplexes are shown for $m = 3$ and various sets of the coefficients c_i. In Fig. 4, the coefficients c_i are approximately equal: $c_1 = 0.5, c_2 = 0.49$, and $c_3 = 0.51$. The seventh internal unstable fixed point is placed almost in the middle of the simplex, and the basins of attraction are almost of the same size. In Fig. 5, the coefficients c_i ($c_1 = 0.24$, $c_2 = 0.4, c_3 = 0.6$) are set as to assure the internal fixed point at the same position as the edge fixed point; hence these
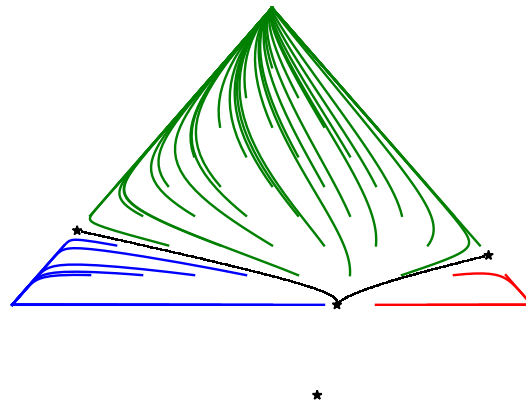
Fig. 6. For activity values $c_1 = 0.5, c_2 = 0.3, c_3 = 0.1, y = p_3$ the central (seventh) fixed point is arranged out of the simplex and the basins of attraction are subjected to a strong deformation..

two fixed points, both unstable, collide. In Fig. 6, the coefficients c_i ($c_1 = 0.5$, $c_2 = 0.3$, $c_3 = 0.1$) are chosen as to make the seventh fixed point out of the simplex. As we see, the pictures in Figs. 5 and 6 are qualitatively the same, except the order of the coefficients c_i. All the unstable fixed points are placed at the boundaries between the basins of attraction.

We conjecture that the same rules apply for higher dimensions of the system. Consider the case of a given m. Having fixed $m - 3$ coordinates of a fixed point equal zero, we are left with a three-dimensional system described above in this subsection. The same rule should apply to any dimensionality m and k. This is a consequence of the model equations (Eq. (1)): each subspace W where $p_i = 0$ for some subset of actors $i \in W$ is invariant, and the mere existence of these actors does not influence the system behavior.

6. Discussion

The structure of the fixed points, described above, allows to summarize the results as follows. Generic trajectories end up at one of the fixed points where $p_i = 1$ for one player i, $p_j = 0$ for all other j-s. Which one of such points is selected, depends on the set of the coefficients c_i and on the initial values of p_i's. The latter dependence can be expressed in the form of basins of attraction of the stable fixed points. As a rule, the unstable fixed points are placed at the boundaries of the basins, hence they provide valuable information on these boundaries. Accordingly, for $m = 2$ there are three fixed points, two stable $(0, 1)$ and $(1, 0)$, and third unstable at the edge between the stable ones. For $m = 3$ there are three stable fixed points at the corners of the simplex, and three unstable fixed points at the edges of the triangle. Out of the coordinates of the latter, one is equal to zero. It is also possible that there is a seventh fixed point, either within the triangle surface or at the edge; in the latter case it coincides with one of fixed points at the edge. This seventh point is also unstable. For higher m, a classification is possible along the same rules. For $m = 4$ there are four stable fixed points (three coordinates of each equal to zero), and 6 unstable 'edge' fixed points with two coordinates equal to zero. Four further 'surface' fixed points can also appear, in the analogy to the case $m = 3$. Finally, one unstable fixed point can appear within the volume of the simplex. If the latter happens to be at the surface, it coincides with the existing one at the same surface. In this way, the structure of all but the last of the fixed points for $m + 1$-dimensional simplex can be reconstructed from the structure for m-dimensional one by adding coordinates equal zero to the existing fixed points.

There are some interesting analogies of these model results and the social reality. First is that basically, the winner is this player who engages minimally in the conflict. The winning strategy is to withdraw from the conflict, what can be carried out by setting $c_i = 0$. If this strategy is accepted by all players, i.e. $c_i = 0$ for each i, there is no conflict at all, and everybody stays with her/his initial power p_i. This is a kind of the Nash equilibrium [58]; whoever enters into conflict, loses. However, we know that people enter into conflict for various reasons, which are out of scope of the paradigm of rational players. Our results indicate that even the most aggressive player (with the largest value of c_i) can win, if his initial power p_i is large enough. In this sense, the Matthew effect is reproduced here. We note that a similar problem has been considered recently [59] from the perspective of voting dynamics and competition between political campaigns. A series of interesting examples of numerical solutions of model differential equations, presented there, is a convincing demonstration that a lot of questions in the field remain open.

We can add that our main point — the condensation of power in hands of one agent (cf Theorem 3.1) has some analogies also in physical systems. When our variable p_i is interpreted as the probability of state i, Eq. (1) is equivalent to a nonlinear master equation. Analysis of master equations for nonlinear kinetics [60] has been shown to lead to the regime of abnormal diffusion that exhibits the Matthew effect [61]. We note that this formulation allows to observe a regime dependent on initial mass distribution [62], similarly to our case, as shown in Figs. 4–6. On the other hand, the time variation of incomes, where the Matthew effect has been identified, has been shown to be describable in terms of

Boltzmann thermodynamics [63]. Last but not least, the network science entered to statistical physics with the idea of preferential attachment [64], which is also an example of the Matthew effect. A casual reference to the Matthew effect is found even in a literature on quantum dots, where the effect is a consequence of asymmetric RKKY interaction [65]. This remote voice nicely confirms our statement in the first section.

Summarizing, a model of conflict is proposed and explored, which takes into account the dependence of strategy on the actual power of a player. The structure of state space of model variables, controlled by nonlinear difference equations, reveals interesting phenomena as collisions of unstable fixed points. Further extensions of the model will include coalitions and state dependent strategies.

Acknowledgments

References

[1] J.S. Coleman, Introduction to Mathematical Sociology, The Free Press of Glencoe, 1964.
[2] L.C. Freeman, The Development of Social Network Analysis, Book Surge, LLC, North Charleston, 2004.
[3] P. Bonacich, P. Lu, Introduction To Mathematical Sociology, Princeton University Press, Princeton, 2012.
[4] S. Fortunato, C. Castellano, V. Loreto, Statistical physics of social dynamics, Rev. Modern Phys. 81 (2009) 591.
[5] S. Fortunato, M. Macy, S. Redner, Statistical mechanics and social sciences I, J. Stat. Phys. 151 (2013) 1.
[6] K. Kaneko, Overview of coupled map lattices, Chaos 2 (1992) 279.
[7] L.C. Martins, L.G. Brunnet, Multi-state coupled map lattices, Physica A 296 (2001) 119.
[8] T. Nagatani, Phase diagrams in unidirectionally coupled map lattice for open traffic flow, Physica A 289 (2001) 267.
[9] C.A.C. Jousseph, S.E. de S. Pinto, L.C. Martins, M.W. Beims, Influence of impurities on the dynamics and synchronization of coupled map lattices, Physica A 317 (2003) 401.
[10] H. Atmanspacher, H. Scheingraber, Stabilization of causally and non-causally coupled map lattices, Physica A 345 (2005) 435.
[11] J.C.A. de Pontes, A.M. Batista, R.L. Viana, S.R. Lopes, Self-organized memories in coupled map lattices, Physica A 368 (2006) 387.
[12] S. Lepri, W. Just, Mean-field theory of critical coupled map lattices, J. Phys. A: Math. Gen. 31 (1998) 6175.
[13] Ming-Chung Ho, Yao-Chen Hung, I-Min Jiang, Phase synchronization in inhomogeneous globally coupled map lattices, Phys. Lett. A 324 (2004) 450.
[14] H.E. Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford University Press, New York, 1971.
[15] K. Huang, Statistical Mechanics, J. Wiley and Sons, 1963.
[16] K. Kaneko, Chaos as a source of complexity and diversity in evolution, Artif. Life 1 (1993) 163.
[17] N.J. Balmforth, A hierarchy of coupled maps, Chaos 12 (3) (2002) 719.
[18] M.D. Sotelo Herrera, J. San Martín, M.A. Porter, Heterogeneous, weakly coupled map lattices, Commun. Nonlinear Sci. Numer. Simul. 36 (2016) 549.
[19] G. Weisbuch, Environment and institutions: a complex dynamical systems approach, Ecol. Econom. 34 (2000) 381.
[20] R.K. Merton, The Matthew effect in science, Science 159 (3810) (1968) 56–63.
[21] K.E. Stanovich, Matthew effects in reading: Some consequences of individual differences in the acquisition of literacy, Read. Res. Quart. 21 (1986) 360–406.
[22] D. Rigney, The Matthew Effect. How Advantage Begets Further Advantage, Columbia University Press, 2010.
[23] R.H. Wade, The causes of increasing world poverty and inequality; or, why the Matthew effect prevails, Int. J. Health Serv. 35 (4) (2005) 631–653.
[24] D. Stauffer, Income inequality in the 21st century. A biased summary of Piketty's capital in the twenty-first century, Internat. J. Modern Phys. 27 (2) (2016) 163000.
[25] Xubin Pan, Shuifang Zhu, Matthew effect in counting the number of species, Biodivers. Conserv. 24 (2015) 2865.
[26] R. Merton, The Thomas theorem and the Matthew effect, Soc. Forces 74 (2) (1995) 379–424.
[27] M. Davis, A. Stark (Eds.), Conflict of Interest in the Professions, Oxford University Press, 2001, p. 243.
[28] G.Z. Jin, B.F. Jones, S.F. Lu, B. Uzzi, The reverse Matthew Effect: Catastrophe and consequence in scientific teams, NBER Working Paper No. w19489, October 2013. Available at SSRN: https://ssrn.com/abstract=2336355.
[29] M. Cooney, S. Phillips, When will academics contest intellectual conflict? Socius Sociol. Res. Dynam. World 3 (2017). http://dx.doi.org/10.1177/2378023117713099.
[30] M. Perc, The Matthew effect in empirical data, J.R. Soc. Interface 11 (98) (2014) 20140378.
[31] A.M. Petersen, Woo-Sung. Jung, Jae-Suk Yang, H.E. Stanley, Quantitative and empirical demonstration of the Matthew effect in a study of career longevity, Proc. Natl. Acad. Sci. USA 108 (1) (2011).
[32] Th. A. DiPrete, G.M. Eirich, Cumulative advantage as a mechanism for inequality: A review of theoretical and empirical developments, Annu. Rev. Sociol. 32 (2006) 271–297.
[33] Th. Piketty, Capital in the XXI Century, The Belknap Press of Harvard University Press, Cambridge, 2014.
[34] B.J. Moore-Gilbert, Postcolonial Theory: Contexts, Practices, Politics, Verso Books, London, 1997.
[35] M. Mann, The Sources of Social Power: Vol. 1, A History of Power from the Beginning to AD 1760, Cambridge University Press, 1986.
[36] P. Heather, Empires and Barbarians: Migration, Development and the Birth of Europe, Pan Macmillan, London, 2010.
[37] B. Simms, Europe: The Struggle for Supremacy, from 1453 to the Present, Basic Books, New York, 2013.
[38] A. Stewart, Theories of Power and Domination, Sage Publ., London, 2001.
[39] M. Weber, Economy and Society. An Outline of Interpretive Sociology, Univ. of California, 1978.
[40] M.P.H. Stumpf, C. Wiuf (Eds.), Statistical and Evolutionary Analysis of Biological Networks, World Scientific, 2010.
[41] K. Orton-Johnson, N. Prior (Eds.), Digital Sociology: Critical Perspectives, Palgrave Macmillan, London, 2013.
[42] S. Wasserman, K. Faust, Social Network Analysis. Methods and Applications, Cambridge University Press, Cambridge, UK, 1994.
[43] R. Axelrod, The dissemination of culture: A model with local convergence and global polarization, J. Confl. Resol. 41 (1997) 203.

[44] K. Sznajd-Weron, J. Sznajd, Opinion evolution in closed community, Internat. J. Modern Phys. C 11 (2000) 1157.
[45] G. Deffuant, D. Neau, F. Amblard, G. Weisbuch, Mixing beliefs among interacting players, Adv. Compl. Sys. 3 (2000) 87.
[46] R. Hegselmann, U. Krause, Opinion dynamics and bounded confidence: models, analysis and simulation, J. Artif. Soc. Soc. Simul. 5 (3) (2002).
[47] J.M. Epstein, Modeling civil violence: An player-based computational approach, Proc. Natl. Acad. Sci. USA 99 (3) (2002) 7243.
[48] E. Bonabeau, G. Theraulaz, J.-L. Deneubourg, Phase diagram of a model of self-organizing hierarchies, Physica A 217 (1995) 373.
[49] E. Ben-Naim, S. Redner, Dynamics of social diversity, J. Stat. Mech. (2005) L11002.
[50] K. Malarz, D. Stauffer, K. Kułakowski, Bonabeau model on a fully connected graph, Eur. Phys. J. B 50 (2006) 195.
[51] L. Lacasa, B. Luque, Bonabeau hierarchy models revisited, Physica A 366 (2006) 472.
[52] V. Koshmanenko, On the conflict theorem for a pair of stochastic vectors, Ukrainian Math. J. 55 (2003) 4.
[53] V. Koshmanenko, Theorem of conflicts for a pair of probability measures, Math. Methods Oper. Res. 59 (2) (2004) 303.
[54] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, Solvable Models in Quantum Mechanics, Springer-Verlag, Berlin, Heidelberg, 1988.
[55] V. Koshmanenko, M. Dudkin, The Method of Rigged Spaces in Singular Perturbation Theory of Self-Adjoint Operators, in: Operator Theory: Advance and Applications, vol. 253, Birkhäuser, 2016.
[56] S. Albeverio, M. Dudkin, A. Konstantinov, V. Koshmanenko, On the point spectrum of $\mathcal{H}_{-2}$-class singular perturbations, Math. Nachr. 208 (1–2) (2007) 20–27.
[57] P. Glendinning, Stability, Instability and Chaos: An Introduction to the Theory of Nonlinear Differential Equations, Cambridge University Press, Cambridge, 1994.
[58] Ph.D. Straffin, Game Theory and Strategy, Math. Assoc. of America, Washington, DC, 1993.
[59] L. Hébert-Dufresne, A. Allard, P.-A. Noël, J.-G. Young, E. Libby, Strategic tradeoffs in competitor dynamics on adaptive networks, Sci. Rep. 7 (2017) 7576.
[60] G. Kaniadakis, Non-linear kinetics underlying generalized statistics, Physica A 296 (2001) 405.
[61] G. Kaniadakis, D.T. Hristopoulos, Nonlinear kinetics on lattices based on the kinetic interaction principle, Entropy 20 (2018) 426.
[62] D.T. Hristopoulos, A. Muradova, Kinetic model of mass exchange with dynamic Arrhenius transition rates, Physica A 444 (2016) 95.
[63] Y. Tao, Swarm intelligence in humans: a perspective of emergent evolution, Physica A 502 (2018) 436.
[64] A.-L. Barabási, R. Albert, Emergence of scaling in random networks, Science 286 (1999) 509.
[65] A. Nejati, Quantum phase transitions in multi-impurity and lattice Kondo systems (Ph.D. thesis), Bonn University, 2016.