DOI 10.1007/s10958-015-2467-5
*Journal of Mathematical Sciences, Vol. 208, No. 5, August, 2015*

# MODEL OF A DYNAMICAL SYSTEM OF THE "FIRE–WATER" CONFLICT TYPE

**T. V. Karataeva and V. D. Koshmanenko** UDC 517.9

We propose a model of dynamical system generated by the conflict interaction between the alternative elements of the "fire–water" type with systematic external replenishment of the resources and internal mixing. The behavior of the trajectories is investigated. We establish the convergence of trajectories of the system to $\omega$ -limit cyclic orbits.

## 1. Introduction

The mathematical science of conflict appeared about 100 years ago and was extensively developed. As early as in 1944, a comprehensive presentation of this theory was proposed in the fundamental monograph by von Neumann and Morgenstern (see the jubilee edition published in 2006 [1]) with an aim not only to order a great number of problems from the pure game theory but also to use these problems in economic and social models. The game theory, as an abstract theory of various types of conflicts, has been extensively developed in numerous publications (see, e.g., [2, 3] and the references therein). We also especially mention the monograph "The Strategy of Conflict" by Schelling [4] in which the theory of standard game situation known as the "zero-sum game" was significantly improved.

A new stage in the development of conflict theory was connected with the investigations of complex systems in the case where the methods of pure game theory were combined with the methods of dynamical systems [5–7] and nonlinear dynamics [13–16]. A class of models encountered in the conflict theory was significantly enlarged and included not only economic and social problems, but also biological (see [8]), political, religious, and ecological problems, i.e., in fact, almost all actual problems of our civilization formulated in terms of the theory of dynamical systems [10–12, 21]. In the conflict theory of complex systems, a final result of the process of conflict interaction, i.e., of the "game," is not only a fixed equilibrium state but also a cyclic orbit, the choice between the collection of equilibrium states or cyclic orbits, and even chaos or collapse [16].

An important direction in the conflict theory is known as differential games [17–20].

For the last ten years, the methods of simulation of complex conflict systems have been enriched with new approaches: statistical interpretation and regionalization of the space [22–27].

In the present paper, we construct and study the behavior of a maximally simplified model of complex dynamical system generated by the transformations of conflict interaction between the alternative elements $F$ and $W$ of the form of fire and water distributed over the sectors of a closed space. In addition, we assume that the element $F$ is subjected to the systematic external influence (replenishment similar to the solar radiation with simultaneous dissipation) and the distribution of the element $W$ is transformed according to a certain law (mixing). In the considered model, the space of distributions of the elements $F$ and $W$ is a disk split into $n \geq 2$ sectors (regions) $\Delta_k$, $k = 1, \ldots, n$. The presence of the elements $F$ and $W$ in the region $\Delta_k$ is stochastically determined by their weight coefficients. At the initial time $t = 0$, the elements $F$ and $W$ are associated with a pair of stochastic vectors $p, r \in \mathbb{R}^n_+$ whose coordinates $p_k, r_k \geq 0$, $k = 1, \ldots, n$, have the meaning of the probabilities of finding $F$ and $W$ in a region $\Delta_k$ :

$$p_k := \mathbb{P}(F \in \Delta_k), \quad r_k := \mathbb{P}(W \in \Delta_k).$$

Institute of Mathematics, Ukrainian National Academy of Sciences Ukraine, 01601, Kyiv, 3 Tereshchenkivs'ka Str.

Translated from Neliniini Kolyvannya, Vol. 17, No. 2, pp. 228–247, April–June, 2014. Original article submitted December 31, 2013.

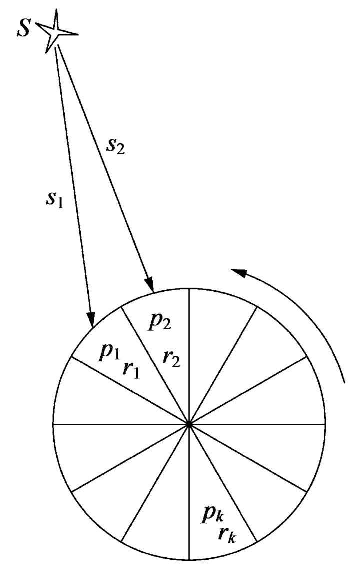


**Fig. 1.** Schematic diagram of the model.

The evolution of the vectors $p(t)$, $r(t)$, $t \geq 0$, is described by the following system of $2n$ ordinary differential equations:

$$\frac{dp_k}{dt} = \tilde{p}_k(\theta - \tilde{r}_k), \quad \frac{dr_k}{dt} = \tilde{r}_k(\theta - \tilde{p}_k), \quad k = 1, \ldots, n, \tag{1}$$

specifying the law of conflict interaction between $F$ and $W$. The influence of external replenishment and internal mixing has the additive character (to within the normalization)

$$\tilde{p}_k = \frac{p_k + s_k}{z_p} \quad \text{and} \quad \tilde{r}_k = \frac{r_k + h_k}{z_r}, \tag{2}$$

where $s_k$, $h_k \geq 0$ are periodic functions. The quantity $\theta(p, r)$ is the *index of conflict* between the elements $F$ and $W$ at time $t$. The normalizing denominators $z_p$ and $z_r$ in (2) guarantee the stochastic nature of the vectors at any time. An equivalent model is given by the system of equations

$$\dot{p}_k = \frac{p_k(\theta - r_k) + s_k}{z_p}, \quad \dot{r}_k = \frac{r_k(\theta - p_k) + h_k}{z_r}, \quad k = 1, \ldots, n. \tag{3}$$

The schematic diagram of the model is shown in Fig. 1.

We split the investigation of the behavior of trajectories of the model given by Eqs. (1), (2) or (3) into several stages. In the first stage, we consider an autonomous system with pure internal conflict interaction. Then we study a model without conflict interaction between the elements in the case where each element is subjected to a periodic external action. Finally, we analyze the complete model given by Eqs. (1)–(3) and simulating the process of conflict interaction between a pair of alternative subsystems (opponents) with systematic external replenishment.

The stability of the system is guaranteed by a constant dissipation of element $F$ (an analog of the thermal energy) and the invariance of the total amount of element $W$ (an analog of water) in the entire space.

We now briefly describe the evolution of investigated model in the discrete time. In the space of pairs of stochastic vectors $p, r \in \mathbb{R}^n_+$, $n \geq 2$, we consider a nonlinear mapping $(p^0 = p, r^0 = r)$

$$\mathbb{R}^n_+ \times \mathbb{R}^n_+ \ni \{p^N, r^N\} \xrightarrow{*} \{p^{N+1}, r^{N+1}\} \in \mathbb{R}^n_+ \times \mathbb{R}^n_+, \quad N = 0, 1, \ldots. \tag{4}$$

In terms of coordinates, it is described by the formulas

$$p_i^{N+1} = \frac{\tilde{p}_i^N (1 - \tilde{r}_i^N)}{c_p^N}, \quad r_i^{N+1} = \frac{\tilde{r}_i^N (1 - \tilde{p}_i^N)}{c_r^N}, \quad i = 1, \ldots, n, \tag{5}$$

where $c_p^N$ and $c_r^N$ are the normalizing denominators guaranteeing the stochasticity of the vectors $p^{N+1}$ and $r^{N+1}$ and

$$\tilde{p}_i^N = \frac{p_i^N + s_i^{N+1}}{z_p^N}, \quad \tilde{r}_i^N = \frac{r_i^N + h_i^N}{z_r^N}, \tag{6}$$

where $s_i^{N+1} = s_{i+1}^N$ for $1 \leq i < n$, $s_n^{N+1} = s_1^N$, and $s^{N=0} = (s_1, \ldots, s_n)$; $h^{N=0} = (h_1 \ldots, h_n)$ are fixed vectors from $\mathbb{R}^n_+$. Their dependence on $N$ obeys a certain law. According to relation (5), the mapping $*$ can be interpreted as an alternative conflict interaction between a pair of physical systems in the states $p^N$ and $r^N$ at discrete times $N = 1, 2, \ldots$. Moreover, every system operates under the conditions of systematic external "replenishment" described by relations (6).

In the present paper, we study the asymptotic behavior of trajectories of the dynamical system (1). In particular, under certain conditions imposed on the vectors $s$, $h$ and $p$, $r$, we establish the existence of $\omega$-limit periodic trajectories. Namely, in the presence of an external source of influence (an analog of solar radiation) for one element and a deterministic or stochastic mixing over the regions for the other element (an analog of wind), we prove the existence of cyclic orbits such that the trajectories of the model asymptotically converge to these orbits. This result is a multidimensional analog of the well-known Poincaré–Bendixson theorem (see, e.g., [10]) on the existence of cycles for two-dimensional dynamical systems and a consequence of the following two facts: the existence of a fixed $\omega$-limit point of the dynamical system given by relations (5) with $\tilde{p}_i^N = p_i^N$ and $\tilde{r}_i^N = r_i^N$ (see [28, 29]) and the oscillating character of uniformly bounded external replenishment and internal mixing.

Note that various methods for the construction of complex dynamical systems with internal conflict interaction were studied in [30, 31, 34, 35]. Some preliminary results on the behavior of "fire–water" model were reported at the conferences [36, 37].

## 2. Model of Pure Conflict

Consider the case of pure conflict of alternative elements without external replenishment and internal mixing, $s_k = h_k = 0$, $k = 1, \ldots, n$. In this case, the system of equations (1) is simplified and takes the form

$$\dot{p}_k = p_k(\theta - r_k), \quad \dot{r}_k = r_k(\theta - p_k), \quad k = 1, \ldots, n, \tag{7}$$

where the index of conflict

$$\theta(t) = (p, r) = \sum_{k=1}^{n} p_k r_k$$

is the scalar product of the vectors $p(t)$ and $r(t)$. We show that, in this case, each trajectory $\{p(t), r(t)\}$, $t \geq 0$, of the corresponding dynamical system converges to a fixed point (compromise state).

In view of the stochasticity of the vectors $p(t)$ and $r(t)$, for any $t \geq 0$, the equalities

$$\dot{p} = \frac{d}{dt}\left(\sum_{k=1}^{n} p_k\right) = 0, \quad \dot{r} = \frac{d}{dt}\left(\sum_{k=1}^{n} r_k\right) = 0$$

are true. Moreover, the analysis of Eqs. (1) shows that the derivatives of the coordinates $p_k$ and $r_k$ vanish with time:

$$\lim_{t\to\infty} \dot{p}_k(t) = 0 = \lim_{t\to\infty} \dot{r}_k(t), \quad k = 1, 2, \ldots .$$

This implies that all coordinates $p_k(t)$ and $r_k(t)$ converge to fixed limit values:

$$\lim_{t\to\infty} p_k(t) = p_k^{\infty} \quad \text{and} \quad \lim_{t\to\infty} r_k(t) = r_k^{\infty}.$$

Hence, every trajectory of the dynamical system of pure conflict converges to an $\omega$-limit fixed point in the space $\mathbb{R}_+^n \times \mathbb{R}_+^n$:

$$p^{\infty} = \lim_{t\to\infty} p^t \quad \text{and} \quad r^{\infty} = \lim_{t\to\infty} r^t.$$

The values of the limit coordinates can be exactly described. The following theorem is true:

**Theorem 1** (on pure conflict). *Every trajectory of the dynamical system* $\{p(t), r(t)\}_{t\geq 0}$ *given in terms of coordinates by Eq. (7) converges in the space* $\mathbb{R}_+^n \times \mathbb{R}_+^n$ *to a fixed point*

$$\{p^{\infty}, r^{\infty}\} = \lim_{t\to\infty}\{p(t), r(t)\}.$$

*If, at time* $t = 0$, *the initial vectors are different,* $p \neq r$, *then the limit vectors are orthogonal,* $p^{\infty} \perp r^{\infty}$, *and their coordinates take the values*

$$p_i^{\infty} = \begin{cases} \dfrac{d_i}{D}, & i \in \mathbb{N}_+, \\ \\ 0, & i \notin \mathbb{N}_+, \end{cases} \tag{8}$$

$$r_k^{\infty} = \begin{cases} -\dfrac{d_k}{D}, & k \in \mathbb{N}_-, \\ \\ 0, & k \notin \mathbb{N}_-, \end{cases} \tag{9}$$

*where* $d_i = p_i - r_i$, $\mathbb{N}_+ = \{i|p_i > r_i\}$, $\mathbb{N}_- = \{k|p_k < r_k\}$, *and* $D = 1/2\sum_i |d_i|$. *In particular, if, for any coordinate,* $p_l = r_l$, *then* $p_l^\infty = r_l^\infty = 0$. *In the case where* $p = r$, *the limit vectors are also equal,* $p^\infty = r^\infty$, *and their coordinates are uniformly distributed over the regions,* $p_k^\infty = p_k^\infty = \dfrac{1}{m}$, *where* $m \le n$ *is the number of nonzero coordinates of the vectors* $p$ *and* $r$.

***Proof.*** The difference analog of this theorem was proved in [28, 29]. Here, we present the main ideas used to prove the theorem for the model with continuous time. Assume that, at the initial time, the inequality $p_i > r_i$ holds for any pair of coordinates. Then it follows from Eqs. (7) that the function $d_i(t) := p_i(t) - r_i(t)$ monotonically increases. As $t \to \infty$, the following limit exists in view of the boundedness of each coordinate:

$$d_i^\infty = \lim_{t\to\infty} d_i(t) \le 1.$$

Under the same condition, $p_i > r_i$, we consider the ratio

$$R_i(t) := \frac{p_i(t)}{r_i(t)}.$$

It follows from (7) that the ratio $R_i(t)$ infinitely increases:

$$R_i^\infty = \lim_{t\to\infty} R_i(t) = \infty.$$

This implies that the coordinate $r_i(t)$ converges to zero. As a consequence, we obtain

$$p_i^\infty = \lim_{t\to\infty} p_i(t) > 0.$$

Similarly, we establish the existence of the limit

$$r_k^\infty = \lim_{t\to\infty} r_k(t) > 0$$

and the convergence of $p_k(t)$ to zero in the case where $p_k < r_k$. The existence of these limits necessarily implies that the index of conflict converges to zero:

$$\theta^\infty = \lim_{t\to\infty} \theta(t) = 0.$$

This is equivalent to the orthogonality of the limit vectors $p^\infty \perp r^\infty$ under the condition $p \ne r$ for the initial vectors. In particular, this means that the limits of the identical initial coordinates $p_l = r_l$ are equal to zero: $p_l^\infty = r_l^\infty = 0$. The immobility of the limit state follows from the equality $\theta^\infty = 0$. We obtain relations (8) and (9) for the limit values $p_i^\infty$ and $r_k^\infty$ from the equations

$$\frac{p_i^\infty}{p_{i'}^\infty} = \frac{d_i^\infty}{d_{i'}^\infty} = \frac{d_i}{d_{i'}}, \quad i, i' \in \mathbb{N}_+, \quad \frac{r_k^\infty}{r_{k'}^\infty} = \frac{d_k^\infty}{d_{k'}^\infty} = \frac{d_k}{d_{k'}}, \quad k, k' \in \mathbb{N}_-.$$

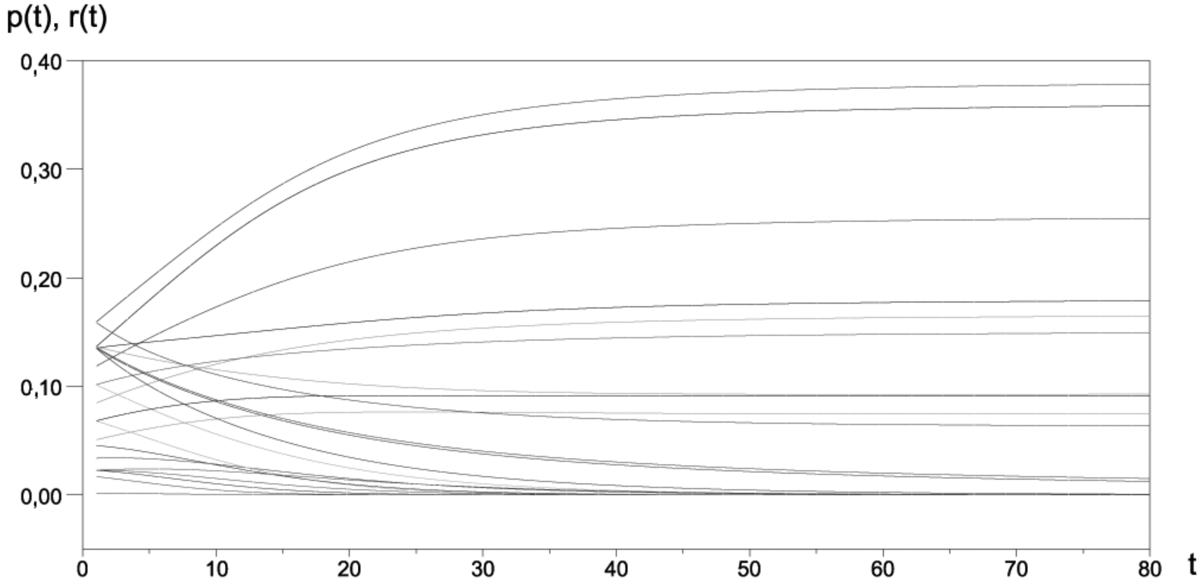


**Fig. 2.** Asymptotic distributions of the limit coordinates.

Finally, the equality $p^{\infty} = r^{\infty}$ and the uniform distribution of coordinates of these vectors over the regions are established in the case of identical initial vectors $p = r$ by ordering the coordinates according to their values for every $t$: $p_1(t) \le p_2(t) \le \ldots \le p_m(t)$ in view of the relation

$$\lim_{t\to\infty}[p_1(t) - p_m(t)] = 0.$$

Hence,

$$\lim_{t\to\infty} p_j(t) = \frac{1}{m}, \quad j = 1, \ldots, m \le n.$$

Theorem 1 is proved.

***Example 1.*** The model of evolution of the dynamical system with pure internal conflict interaction in continuous time.

The trajectory of the stochastic vectors $p(t)$ and $r(t)$ whose coordinates satisfy Eq. (7) converges to a fixed point $\{p^{\infty}, r^{\infty}\} \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ with coordinates depicted in Fig. 2.

## 3. A Model of Dynamical System without Conflict Interaction

Consider the case where the conflict interaction between the elements $F$ and $W$ is absent but each of these elements suffers an external periodic action. We show that this yields the existence of limit cyclic orbits.

It follows from (1) that, in the absence of conflict interaction, the system is determined by two collections of independent equations

$$\dot{p}_k(t) = \frac{p_k(t) + s_k(t)}{z_p(t)}, \quad \dot{r}_k(t) = \frac{r_k(t) + h_k(t)}{z_r(t)}, \quad k = 1, \ldots, \quad n \ge 2, \quad t \ge 0, \tag{10}$$

where the periodic functions $s_k(t)$ and $h_k(t)$ describe the external influence and the normalizing denominators $z_p(t)$ and $z_r(t)$ guarantee the stochasticity of the vectors $p(t)$ and $r(t)$ at any time.

Further, we investigate the behavior of one independent vector $(p(t)$ or $r(t))$; the behavior of the other vector is similar. We choose the vector $p(t)$. To distinguish it from the case of presence of the conflict interaction between $p(t)$ and $r(t)$, we denote this vector by $x(t)$. Hence, our aim is to study a system of equations

$$\dot{x}_k(t) = \frac{x_k(t) + s_k(t)}{z(t)}, \quad k = 1, \dots, n, \quad t \geq 0. \tag{11}$$

Actually, we analyze a difference version of system (11) at fixed times divisible by a certain angle $\alpha$ :

$$x_k(t_{N+1}) = \frac{x_k(t_N) + s_k(t_N)}{z(t_N)}, \quad k = 1, \dots, n, \quad t_N = \alpha N, \quad N = 0, 1, \dots, \tag{12}$$

under the assumption that the external action obeys the following law:

$$s_k(t_N) = s(\alpha k + t_N), \quad \alpha > 0. \tag{13}$$

Here, $s(t)$ is a positive periodic continuous function such that $s(t) = s(\alpha n + t)$, e.g., $s(t) = c(1 + \sin t)$, $c > 0$, $\alpha = 2\pi/n$.

It is easy to see that the normalizing denominator (12) guaranteeing the stochasticity of the vector $x(t)$ is independent of time and can be found in the form

$$z = z(t_N) = 1 + s, \quad s = \sum_{k=1}^{n} s_k, \quad s_k = s(\alpha k). \tag{14}$$

Denote

$$x_k^N := x_k(t_N), \quad s_k^N := s_k(t_N) = s(\alpha k + t_N) \equiv s_{k+N}.$$

Then the system of equations (12) takes the form

$$x_k^{N+1} = \frac{x_k^N + s_k^N}{1 + s}, \quad x_k^0 \equiv x_k(t_0) = x_k, \quad k = 1, \dots, n, \quad N = 0, 1, \dots. \tag{15}$$

Iterating this system, we can find the coordinates $x_k(t_N)$ at any time $t_N$. Hence, Eqs. (15) determine a dynamical system in the space $\mathbb{R}_+^n$ with discrete time:

$$x^0 \equiv x(t_0) \xrightarrow{T_1} x^1 \xrightarrow{T_2} x^2 \xrightarrow{T_3} \dots \xrightarrow{T_{N-1}} x^{N-1} \xrightarrow{T_N} x^N \xrightarrow{T_{N+1}} \dots, \tag{16}$$

where $T_1, T_2, \dots$ is a sequence of time-dependent transformations given by relations (15). By virtue of (13), this sequence of transformations is periodically repeated every $n$ steps. Hence, for any $k = 1, \dots, n$, we get $T_k = T_{k+n} = T_{k+Nn}$, $N = 0, 1, \dots$. We are interested in the behavior of the vector $x^N \equiv x(t_N)$ as $N \to \infty$. It is shown that, for any initial values of the coordinates $x_k(t_0)$, the sequence of vectors $x(t_N)$ approaches a fixed cyclic orbit with period $n$.

**Theorem 2.** *Each trajectory $x^N = x(t_N)$ of the dynamical system (16) specified by the system of difference equations (12) converges in the space $\mathbb{R}^n_+$ to the $\omega$-set $\Gamma^\infty$ invariant under an ordered sequence of transformations $T_1, \dots, T_n$. The set $\Gamma^\infty$ consists of $n$ vectors*

$$\Gamma^\infty = \{\Gamma_j^\infty\}_{j=1}^n.$$

*The coordinates of the vector $\Gamma_1^\infty$ are determined by a collection of values $s_1, \dots, s_n$ of the source of external influence according to the formulas*

$$\gamma_1^\infty = \frac{s_1 + s_2 z + \ldots + s_n z^{n-1}}{z^n - 1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\gamma_k^\infty = \frac{s_k + s_{k+1} z + \ldots + s_{k-1} z^{n-1}}{z^n - 1} \equiv \frac{1}{z^n - 1} \sum_{j=0}^{n-1} s_{|k+j|(\operatorname{mod} n)} z^j,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\gamma_n^\infty = \frac{s_n + s_1 z + \ldots + s_{n-1} z^{n-1}}{z^n - 1},$$

*where $z = 1 + s$, $s = s_1 + \ldots + s_n$. Each vector $\Gamma_j^\infty$, $j > 1$, is determined by an $(j-1)$-fold cyclic shift of the coordinates of the first vector $\Gamma_1^\infty = (\gamma_1^\infty, \dots, \gamma_n^\infty)$ :*

$$\gamma_1^\infty \to \gamma_2^\infty, \dots, \quad \gamma_k^\infty \to \gamma_{k+1}^\infty, \dots, \gamma_n^\infty \to \gamma_1^\infty.$$

*The limit $\omega$-set $\Gamma^\infty$ is a cyclic orbit of the dynamical system (16). It does not depend on the initial point $x = x(t_0)$ of the trajectory $x(t_N)$ and is completely determined by the source of external influence.*

***Proof.*** The proof is based on the analysis of the behavior of the coordinates $x_k(t_N)$ as $N \to \infty$. By using (12), we arrive at the following evolution (sequence) of values of the first coordinate:

$$\begin{aligned}
x_1^1 &= \frac{x_1 + s_1}{z} = \frac{x_1}{z} + \frac{s_1}{z}, \\
x_1^2 &= \frac{x_1^1 + s_2}{z} = \frac{x_1}{z^2} + \frac{s_1}{z^2} + \frac{s_2}{z}, \\
x_1^3 &= \frac{x_1^2 + s_3}{z} = \frac{x_1}{z^3} + \frac{s_1}{z^3} + \frac{s_2}{z^2} + \frac{s_3}{z}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
x_1^N &= \frac{x_1^{N-1} + s_N}{z} = \frac{x_1}{z^N} + \frac{s_1}{z^N} + \frac{s_2}{z^{N-1}} + \ldots + \frac{s_N}{z}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{17}$$

The second coordinate has a similar evolution:

$$
\begin{aligned}
&x_2^1 = \frac{x_2 + s_2}{z},\\
&x_2^2 = \frac{x_2}{z^2} + \frac{s_2}{z^2} + \frac{s_3}{z},\\
&x_2^3 = \frac{x_2}{z^3} + \frac{s_2}{z^3} + \frac{s_3}{z^2} + \frac{s_4}{z},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&x_2^{N-1} = \frac{x_2}{z^{N-1}} + \frac{s_2}{z^{N-1}} + \ldots + \frac{s_N}{z},\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\tag{18}
$$

where, both in the first and second cases, the sequence $s_N$ cyclically repeats its values $s_1, \ldots, s_n$ so that $s_{n+l} = s_l$, $l = 1, \ldots, n$. Comparing (17) and (18), we conclude that, for any $N$,

$$x_1^N - x_2^{N-1} = \frac{x_1}{z^N} + \frac{s_1}{z^N} - \frac{x_2}{z^{N-1}}.$$

Since $z > 1$, we conclude that

$$\lim_{N\to\infty} \left(x_1^N - x_2^{N-1}\right) = 0.$$

Similarly, for all $k < n$,

$$\lim_{N\to\infty} \left(x_k^N - x_{k+1}^{N-1}\right) = 0.$$

For $k = n$, in exactly the same way, we get

$$\lim_{N\to\infty} \left(x_n^N - x_1^{N-1}\right) = 0.$$

Hence, for $\gamma_k^\infty := x_k^{N=\infty}$, we obtain the following system of equations as $N \to \infty$:

$$\frac{\gamma_1^\infty + s_1}{z} = \gamma_2^\infty, \ldots, \frac{\gamma_k^\infty + s_k}{z} = \gamma_{k+1}^\infty, \ldots, \frac{\gamma_n^\infty + s_n}{z} = \gamma_1^\infty.$$

One can easily get a solution of this system of equations

$$\gamma_1^\infty = \frac{s_1 + s_2 z + \ldots + s_n z^{n-1}}{z^n - 1},$$

$$\gamma_2^\infty = \frac{s_2 + s_3 z + \ldots + s_n z^{n-2} + s_1 z^{n-1}}{z^n - 1},$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$$\gamma_k^\infty = \frac{s_k + s_{k+1}z + \ldots + s_{k-1}z^{n-1}}{z^n - 1},$$

$$\gamma_n^\infty = \frac{s_n + s_1 z + \ldots + s_{n-1}z^{n-1}}{z^n - 1}.$$

We introduce a vector $\Gamma_1^\infty = (\gamma_1^\infty, \ldots, \gamma_n^\infty)$. By using the explicit form of transformations $T_1, \ldots, T_n$ given by Eqs. (12), we directly show that

$$(T^1 \Gamma_1^\infty)_k = \frac{\gamma_k^\infty + s_k}{z} = \gamma_{k+1}^\infty, \quad k < n.$$

Thus, for $k = 1$, we get

$$(T_1 \Gamma_1^\infty)_1 = \frac{\gamma_1^\infty + s_1}{z} = \frac{s_1 + s_2 z + \ldots + s_n z^{n-1}}{(z^n - 1)z} + \frac{s_1}{z}$$

$$= \frac{s_1 + s_2 z + \ldots + s_n z^{n-1} + s_1(z^n - 1)}{(z^n - 1)z} = \frac{s_2 z + \ldots + s_n z^{n-1} + s_1 z^n}{(z^n - 1)z} = \gamma_2^\infty.$$

In particular, for $k = n$, we find

$$(T_1 \Gamma_1^\infty)_n = \frac{\gamma_n^\infty + s_n}{z} = \gamma_{n+1}^\infty = \gamma_1^\infty.$$

Hence, we have shown that $T_1 \Gamma_1^\infty = \Gamma_2^\infty$, where

$$\Gamma_2^\infty := (\gamma_2^\infty, \ldots, \gamma_n^\infty, \gamma_1^\infty).$$

Similarly, $T_2 \Gamma_2^\infty = \Gamma_3^\infty$, where the vector $\Gamma_3^\infty$ is formed by a single cyclic shift of the coordinates of the vector $\Gamma_2^\infty$ to the left. Further, we apply the transformation $T_3$ to the vector $\Gamma_3^\infty$ and successively continue these actions. By construction, we return to the vector $\Gamma_1^\infty = T^n \Gamma_n^\infty$ at the $n$ th step.

Theorem 2 is proved.

The established result can be reformulated in the form of a different theorem with simpler proof by introducing another dynamical system generated by a time-independent transformation. To this end, in the space $\mathbb{R}_+^n$, we construct a time-independent transformation $T$ generating a dynamical system equivalent to (16) but in the other coordinate system. For $N = 0$, we denote $y^0 = x^0$, $y_k^0 = x_k^0$, $k = 1, 2, \ldots, n$, and, for $N = 1$, we define a transformation $T$ by the formulas

$$(Ty)_k \equiv y_k^1 = \frac{y_{k+1}^0 + s_k}{1 + s}, \quad s_k = s(\alpha\, k), \quad k = 1, \ldots, n, \tag{19}$$

where we set

$$y_{n+1}^0 = y_1^0 \quad \text{for} \quad k = n.$$

By analogy with (19), for any $N \geq 1$, the coordinates of the vector $y^N$ are given by the formulas

$$(T^N y)_k \equiv y_k^N = \frac{y_{k+1}^{N-1} + s_k}{1+s}, \quad y_{n+1}^{N-1} = y_1^{N-1}, \quad k = 1, \dots, n. \tag{20}$$

It turns out that, for any initial values of the coordinates $y_k^0 = x_k^0$, a sequence of vectors $y^N$ converges to a unique fixed point in the space $\mathbb{R}_+^n$.

**Theorem 3.** *Every trajectory* $\{y^N\}_{N=0}^{\infty}$ *of the dynamical system generated by the transformation* $T$ *according to relations (19) and (20) converges in the space* $\mathbb{R}_+^n$ *to a fixed point specified by the vector*

$$y^\infty = \lim_{N\to\infty} T^N y \quad \forall y \in \mathbb{R}^n$$

*whose coordinates are determined solely by the collection of values* $s_1, \dots, s_n$ *of the source of external influence:*

$$y_k^\infty = \frac{s_k + s_{k+1}z + \ldots + s_{k-1}z^{n-1}}{z^n - 1} \equiv \frac{1}{z^n - 1} \sum_{j=0}^{n-1} s_{(k+j)(\mathrm{mod}\, n)} z^j, \quad k = 1, \dots, n, \tag{21}$$

*where* $z = 1 + s$ *[see (14)]. In particular,*

$$y_1^\infty = \frac{s_1 + s_2 z + \ldots + s_n z^{n-1}}{z^n - 1}, \quad y_n^\infty = \frac{s_n + s_1 z + \ldots + s_{n-1} z^{n-1}}{z^n - 1}.$$

***Proof.*** By using relations (20), we conclude that, for large $N$, each coordinate $y_k^N$ can be rewritten in the form of the sum of two terms. The first term has the form $y_l^0 z^{-N}$, $l = 1, \dots, n$, and depends only on $k$ and $N$. It is clear that this term converges to zero because all $|y_l^0| \leq 1$ and $z > 1$. The second term is the sum of $N$ terms of the geometric progression with common ratio $q = \dfrac{1}{z^n}$ and the first term

$$a_k = s_n z^{-n} + s_{n+1} z^{-n+1} + \ldots + s_{k-1} z{-1},$$

where $s_0 = s_n$ and $s_{n+1} = s_1$. As $N \to \infty$, for $y_k^N$, we get a value from collection (21).

Theorem 3 is proved.

***Example 2.*** Model of evolution of a dynamical system with cyclic replenishment and mixing (without conflict interaction).

The trajectory of each vector $p(t_N)$, $r(t_N)$ of the dynamical system (16) for any $p(0)$, $r(0) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ converges to the cyclic orbit $\{\Gamma^\infty, \Psi^\infty\}$ with period $n$ (see Fig 3). Here, $\Psi^\infty$ denotes the limit orbit for $\{r(t_N)\}_{N=0}^{\infty}$, which exists by virtue of Theorem 2.

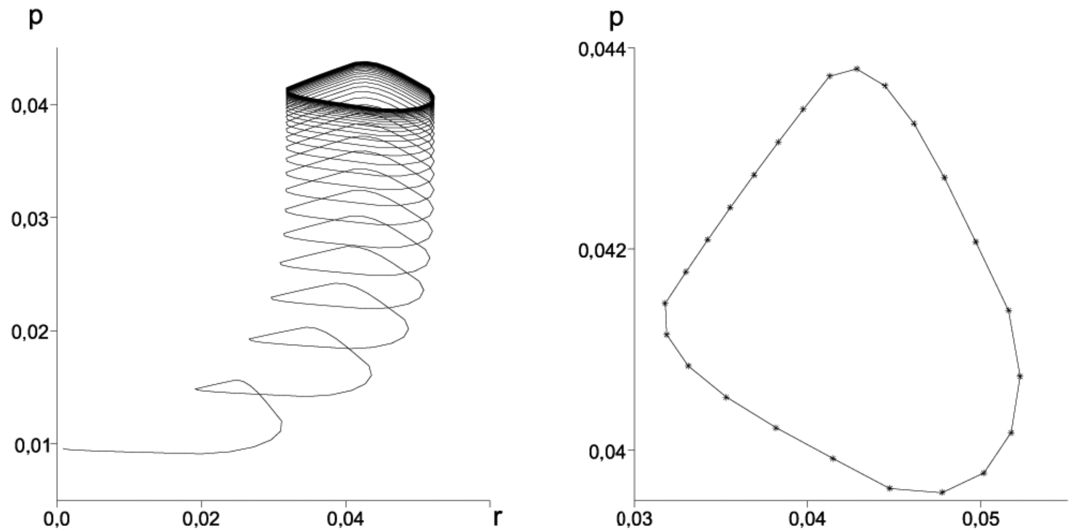


**Fig. 3.** Evolution and the limit cycle for any pair of coordinates in the phase space.

## 4. Existence of Cyclic Orbits in the General Case

In the general case, the evolution of the dynamical system (4) occurs under the influence of simultaneous interactions of various nature: internal conflict interaction between the elements $F$ and $W$, interaction with an external source of influence upon the element $F$ in the form of periodic replenishment, and the process of migration described in terms of internal mixing of the distributions of element $W$ over the regions. It turns out that, in this case, almost all trajectories of the dynamical system asymptotically approach fixed cyclic orbits.

This result can be qualitatively explained as follows: According to Theorem 1, the presence of pure conflict interaction between $F$ and $W$ guarantees the existence of the fixed limit values $p_k^\infty$ and $r_k^\infty$ for each pair of coordinates $p_k^N$, $r_k^N$, $k = 1, \ldots, n$. The periodic external replenishment for $F$ and the process of mixing for $W$, which is also periodic, lead to fixed periodic shifts of the prelimiting values of these coordinates. In this case, it is worth noting that the increments of these coordinates, in view of the repulsive conflict interaction between $F$ and $W$, are negligible as compared with the external influence and internal mixing. Hence, as a rule, all pairs of coordinates $p_k^N$, $r_k^N$, $k = 1, \ldots, n$ approach, as $N \to \infty$, the same cyclic orbit obtained as a perturbation of the $\omega$-limit set $\{\Gamma^\infty, \Psi^\infty\}$ in Theorem 2.

For the exact formulation of this result, we consider a dynamical system

$$\{p^N, r^N\} \xrightarrow{*} \{p^{N+1}, r^{N+1}\}, \quad p^0, r^0 \in \mathbb{R}_+^n, \quad N = 0, 1, \ldots, \tag{22}$$

where the mapping $*$ is given by a collection of difference equations formulated in terms of the coordinates

$$\begin{aligned} p_k^{N+1} &= \frac{p_k^N(\theta^N + 1 - r_k^N) + s_{k+N}}{z_p}, \\ r_k^{N+1} &= \frac{r_k^N(\theta^N + 1 - p_k^N) + h_{k+N}}{z_r}, \quad k = 1, \ldots, n, \end{aligned} \tag{23}$$

with $z_p = 1 + s$ and $z_r = 1 + h$, where the sequences $s_{k+N}$ and $h_{k+N}$ are periodic functions of discrete time: $s_{N+n} = s_N$ and $h_{N+n} = h_N$.

The computer simulation of numerous specific examples shows that every trajectory of the dynamical system (22) given by mapping (23) approaches a certain limit cyclic orbit. Moreover, the projections of this orbit onto the phase spaces of each pair of coordinates $\{p_k, r_k, k = 1, \ldots, n\}$ have the same form. The theorem presented below seems to be plausible but, most likely, hypothetical because, at present, its proof is, in fact, schematic.

**Theorem 4.** *If all pairs of coordinates* $\{p_k^N, r_k^N\}$, $k = 1, \ldots, n$, *converge to the same cyclic orbit in a plane, then the limit* $\omega$*-set* $\{\Gamma, \Psi\}$ *of the dynamical system (22) consists of a collection of* $2n$ *vectors:* $\Gamma = \{\Gamma_j\}_{j=1}^n$, $\Psi = \{\Psi_j\}_{j=1}^n$, $\Gamma_j, \Psi_j \in \mathbb{R}_+^n$. *The coordinates of the first pair of vectors* $\Gamma_1 = (\gamma_1, \ldots, \gamma_n)$, $\Psi_1 = (\psi_1, \ldots, \psi_n)$ *are solutions of the system of* $2n$ *algebraic equations*

$$\gamma_{k+1} = \frac{\gamma_k(\theta + 1 - \psi_k) + s_k}{1 + s},$$
$$\psi_{k+1} = \frac{\psi_k(\theta + 1 - \gamma_k) + h_k}{1 + h}, \quad k = 1, \ldots, n, \tag{24}$$

*where* $\gamma_{n+1} = \gamma_1$, $\psi_{n+1} = \psi_n$, *and* $\theta = \Sigma_{k=1}^n \gamma_k \psi_k$. *Each pair of vectors* $\Gamma_j$, $\Psi_j$, $1 < j \le n$, *is formed by a cyclic shift of the coordinates of the first pair of vectors* $\Gamma_1$ *and* $\Psi_1$ *by* $j - 1$ *steps to the left. Moreover, a one-time shift of the coordinates of the vectors* $\Gamma_n$ *and* $\Psi_n$ *gives* $\Gamma_1$ *and* $\Psi_1$, *respectively.*

***Proof.*** The direct iteration of the coordinates $p_k^N$ and $r_k^N$ according to relations (23) shows that, for $N = Mn + l$, $l = 1, \ldots, n$, $M = 1, 2, \ldots$, each of these coordinates (in what follows, only $p_k^N$ is investigated in detail) can be represented in the form of two groups of terms with different types of behavior:

$$p_k^N = \text{I}_k^N + \Pi_k^N,$$

where

$$\text{I}_k^N = \frac{F_k^N(p, r)}{z_p^N}, \quad \Pi_k^N = \sum_{j=1}^n S_{kj}^N,$$

$F_k^N(p, r)$ is a polynomial of coordinates of the vectors $p$ and $r$ and, by virtue of the periodic properties of the sequences $s_N$ and $h_N$, the quantity $\Pi_k^N$ is decomposed in a combination of partial sums $S_{kj}^N$ of geometric progressions with the same common ratio (that does not exceed one) and different first terms expressed via $s_i$ and $h_i$, $i = 1, \ldots, n$. It is easy to see that the values of $F_k^N$ are uniformly bounded because all coordinates $p_k$, $r_k \le 1$. Hence,

$$\lim_{N\to\infty} \text{I}_k^N = 0, \quad k = 1, \ldots, n,$$

because $z_p = 1 + s > 1$. In view of the fact that all geometric progressions used to form $S_{kj}^N$ are convergent, we conclude that the following limits exist:

$$\gamma_k = \lim_{M\to\infty} \Pi_k^{Mn}, \quad k = 1, \ldots, n.$$

A comprehensive analysis of the structure of the terms $S_{kj}^N$ performed with regard for relations (23) gives the following formulas for the terms $\Pi_k^N$:

$$\Pi_1^N = \Pi_n^{N+1}, \quad \Pi_k^N = \Pi_{k-1}^{N+1}, \quad k > 1. \tag{25}$$

Hence, as $Mn = N \to \infty$, by the assumption of the theorem, the limit values $\gamma_k$ differ from the limit values $\gamma_k^\infty$ in Theorem 2 by relatively small quantities called shifts and denoted by $d_k^\gamma$. They are also sums of convergent geometric progressions whose first terms are given by the products $s_k \cdot h_k$ and the products of initial coordinates $p_k \cdot r_k$. Similarly, we show that the limits

$$\gamma_k^l = \lim_{M\to\infty} \Pi_k^{Mn+l}, \quad l = 1, \ldots, n-1,$$

exist. Moreover, in view of (25), these limits cyclically depend on $l$. Hence, we conclude that the dependences of the shifts $d_k^\gamma$ on the index $l$ are also cyclic and

$$d_k^\gamma(l) = d_{k+l}^\gamma, \quad d_{n+1}^\gamma = d_1^\gamma. \tag{26}$$

This means that the limit values $\gamma_k^l$, $k = 1, \ldots, n$, $l = 1, \ldots, n-1$, can be obtained by cyclic shifts of the coordinates $\gamma_k$ by $l$ steps to the left

$$(\gamma_1, \ldots, \gamma_n) \to (\gamma_l, \ldots, \gamma_n, \ldots, \gamma_1, \ldots, \gamma_{l-1}).$$

Note that relations (26), together with similar relations for the shifts $d_k^\psi$, form the exact mathematical formulation of the assumption of the theorem on the unique form of orbit for each pair of coordinates. Thus, under condition (26), the sequence $p_1^N$ converges, as $N \to \infty$, to the limit $\omega$-set, which forms a vector $\Gamma_1$ with coordinates $(\gamma_1, \ldots, \gamma_n)$, and the sequence $p_l^N$, $1 < l \le n$, converges to a similar limit $\omega$-set, which forms a vector $\Gamma_l$ with coordinates $(\gamma_l, \ldots, \gamma_n, \ldots, \gamma_1, \ldots, \gamma_{l-1})$.

By using similar reasoning and statements for the iteration of the coordinates $r_l^N$, we arrive at a system of vectors $\Psi_l$, which can be obtained, for $1 < l \le n$, by successive shifts of the coordinates of the first vector $\Psi_1 = (\psi_1, \ldots, \psi_n)$.

Thus, in terms of the limit vectors, one step of the transformation specified by Eqs. (23) yields the equality

$$T_1\Gamma_1 = \Gamma_2,$$

where

$$(T_1\Gamma_1)_k = (\Gamma_2)_k = \frac{\gamma_k(\theta + 1 - \chi_k) + s_k}{1+s}, \quad k = 1, \ldots, n.$$

Moreover, the coordinates of the vector $\Gamma_2$ are formed by shifting the coordinates of the vector $\Gamma_1$ to the left:

$$(\Gamma_2)_k = \gamma_{k+1},$$

where $\gamma_{n+1} = \gamma_1$. The coordinates $(T_1\Psi_1)_k$ satisfy a similar equality

$$(T_1\Psi_1)_k = (\Psi_2)_k = \psi_{k+1}.$$

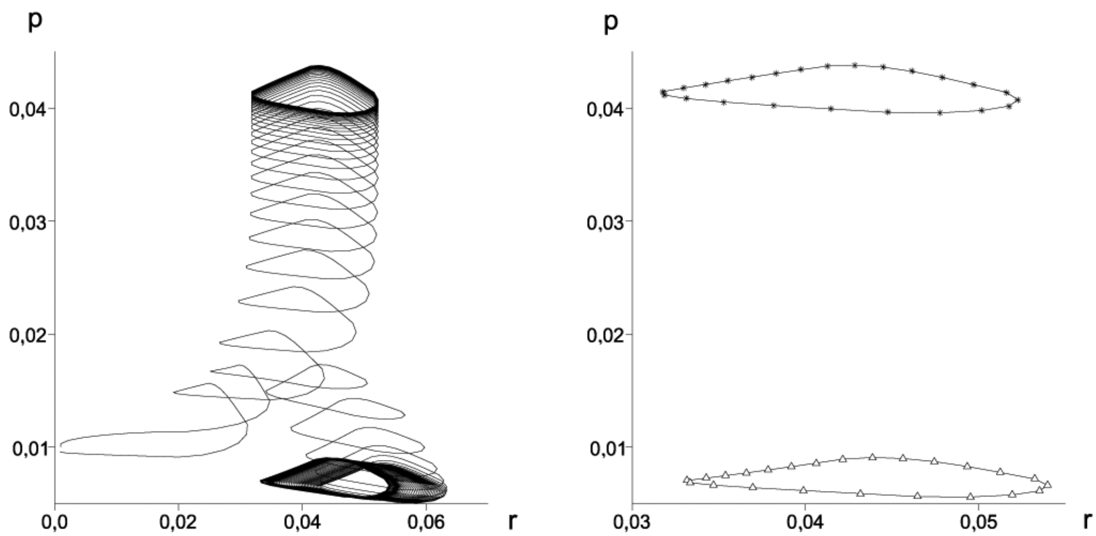


**Fig. 4.** Comparison of the evolutions and limit orbits.

In the second step, we get the vectors $\Gamma_3$ and $\Psi_3$. In the $j$ th step, we find $\Gamma_j = T_j \circ \ldots \circ T_1 \Gamma_1$ or, in terms of the coordinates,

$$(T_j \circ \ldots \circ T_1 \Gamma_1)_k = \gamma_{k+j}, \quad (T_j \circ \ldots \circ T_1 \Psi_1)_k = \psi_{k+j}, \quad j < n.$$

For $j = n,$ we obtain

$$T_n \circ \ldots \circ T_1 \Gamma_1 = \Gamma_1$$

or, in terms of coordinates,

$$(T_n \circ \ldots \circ T_1 \Gamma_1)_k = \gamma_k, \quad (T_n \circ \ldots \circ T_1 \Psi_1)_k = \psi_k.$$

Thus, we have proved that, as $N \to \infty,$ the vectors $p^N$ and $r^N$ approach a cyclic orbit with period equal to the dimension of the space $\mathbb{R}^n_+$ :

$$\{\Gamma_1, \Psi_1\} \overset{T_1}{\to} \{\Gamma_2, \Psi_2\} \overset{T_2}{\to} \ldots \overset{T_{n-1}}{\to} \{\Gamma_n, \Psi_n\} \overset{T_n}{\to} \{\Gamma_1, \Psi_1\}.$$

It is worth noting that the scalar product $(\Gamma_j, \Psi_j)$ is independent of the index $j = 1, \ldots, n.$ Hence, the index of conflict $\theta$ is a constant depending, generally speaking, on the initial pair of vectors $\{p, r\}.$

The fact that the coordinates of the limit vectors satisfy Eq. (24) directly follows from the explicit form of the conflict mapping $*.$

Theorem 4 is proved.

Thus, in the general case where the dynamical system is defined by the mappings given by relations (23) and condition (26) is satisfied, every trajectory $\{p^N, r^N\}$ converges in the space $\mathbb{R}^n_+$ to an $\omega$ -set formed by the cyclic orbit $\{\Gamma, \Psi\}.$ The invariance of this set under the generator $*$ of the dynamical system follows from relations (24).

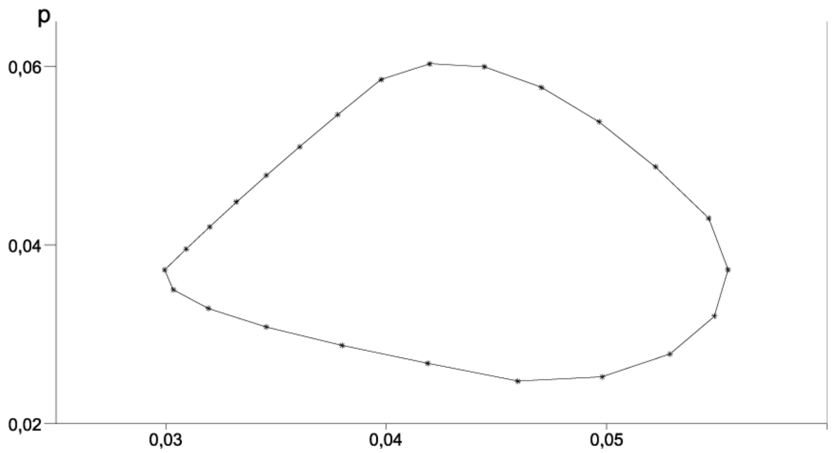


**Fig. 5.** Typical form of the limit orbit.

***Example 3.*** A model of dynamical system whose generator includes conflict interaction, cyclic replenishment, and mixing.

For dynamical systems of conflict with external replenishment and internal mixing, fixed equilibrium states do not exist. Every trajectory of this system approaches a cyclic orbit.

The influence of conflict interaction leads to the deformation of cyclic orbit in Example 2. In Fig. 4, we show the trajectory of evolution and the limit orbit of the dynamical system with conflict interaction, cyclic replenishment, and mixing (at the bottom) and compare it with the trajectory and the limit orbit of the dynamical system without conflict interaction (at the top).

## 5. Computer Analysis

If the directions of replenishment and mixing are identical, then each pair of coordinates $p_k^N$, $r_k^N$, $k = 1, \ldots, n$, approaches the same fixed cyclic orbit (see Fig. 5). According to relations (24), the exact form of the orbit depends on the quantities $s_k$ and $h_k$.

A decrease in $h_k$ is accompanied by the deformation of orbits in some regions (see Fig. 6). Significant deformation of orbits appears as a consequence of predominance of the conflict interaction.

In the case of the opposite directions of oscillations of replenishment and mixing, every pair of coordinates $p_k^N$ and $r_k^N$ approaches its individual cyclic orbit (see Fig. 7). Moreover, the form of the orbit in each specific region depends on the nonlocal conflict interaction.

A similar effect occurs in the case of constant (nonperiodic) mixing. Moreover, we can encounter the coordinates $r_j^\infty = 0$ (see Fig. 8). The coordinate $r_j^N$ approaches zero as a result of the minimization of mixing.

## 6. Open Questions

The aim of the present paper is to show that, in the elementary statement, the behavior of the trajectories of a complex model used to describe the conflict interaction between abstract alternative elements $F$ and $W$ subjected to the action of an external source of replenishment for $F$ and a fixed law of internal mixing (migration) for $W$ is similar to the physical picture of redistribution of the thermal energy and moisture over the surface of a planet.

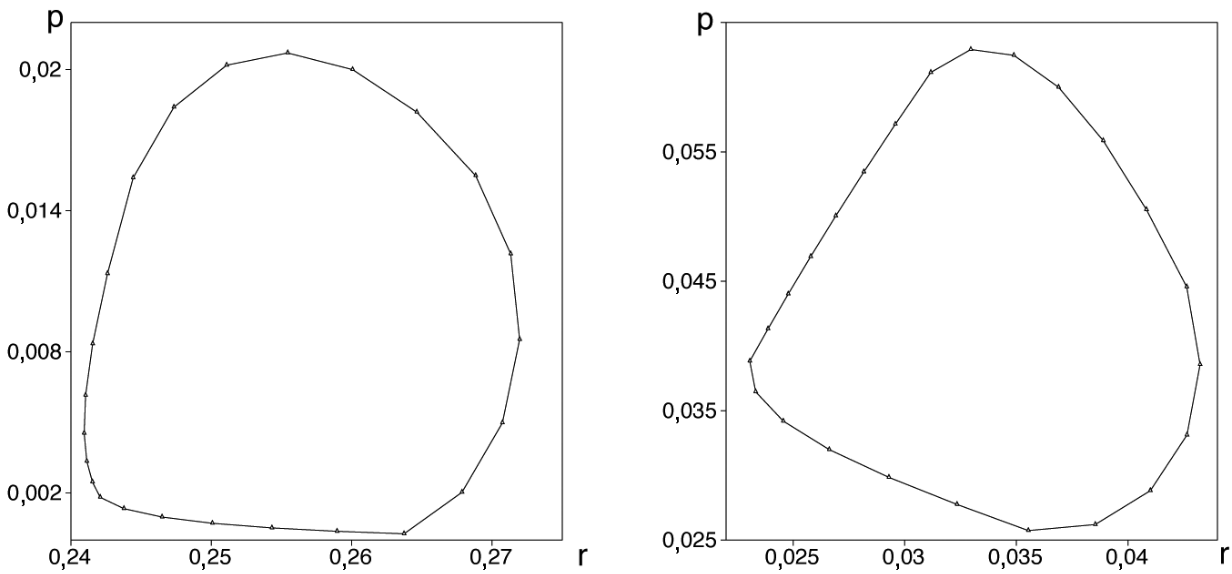


**Fig. 6.** Effect of deformation of limit orbits.

To propose the law of conflict interaction, we assume that the heat (element $F$) and moisture (element $W$) have, in a certain sense, the opposite alternative properties. This leads to a confrontational redistribution of their initial weight factors in different regions of the common space. Nevertheless, this is only one of the causes of formation of the dynamical picture. In the analyzed mathematical model, we also take into account the action of an external energy source, which systematically replenishes the amount of heat in each conflict region according to a certain law. Moreover, the total amount of thermal energy in the entire space remains constant in the mean due to natural dissipation. Mathematically, this is guaranteed by the normalization in each step of conflict transformation.

It is worth noting that the source of external action disturbs possible compromise equilibrium between the alternative elements and, in particular, leads to the moisture transfer from high-temperature regions into colder regions. In the model, this is maintained by the procedure of internal mixing.

In the constructed model, the general mathematical mechanism of conflict interaction [see (5)] between the analogs of the thermal energy (element $F$) and water ( element $W$) is, in fact, described in terms of conditional probabilities of the presence of these elements in each region of the closed space (disk). Moreover, we take into account only some most important (in our opinion) causes of redistribution of the alternative elements over different regions. Clearly, the main "engine" of the evolution of dynamical picture is the external energy source acting upon the element $F$. However, without additional mechanism of internal mixing (migration) for the element $W$, e.g., due to the wind, the evolution of the system becomes fairly primitive, e.g., it is possible that the entire amount of $W$ is concentrated in a single region.

The above-mentioned causes of redistribution are taken into account in three stages. Thus, in Sec. 2, we study a simple abstract version of dynamical system with pure conflict interaction of the "minus–minus" type. We prove the theorem on the evolution behavior of the system and the existence of limit equilibrium state for each trajectory. The exact values of compromise distributions are described in terms of initial states.

In Sec. 3, we investigate the model of two independent subsystems without conflict interaction but with periodic external action. It is shown that every trajectory of this system asymptotically approaches a fixed cyclic orbit independent of the initial point. The limit orbit is completely determined by the source of external action, and only the presence of conflict interaction between the subsystems affects its limit behavior.

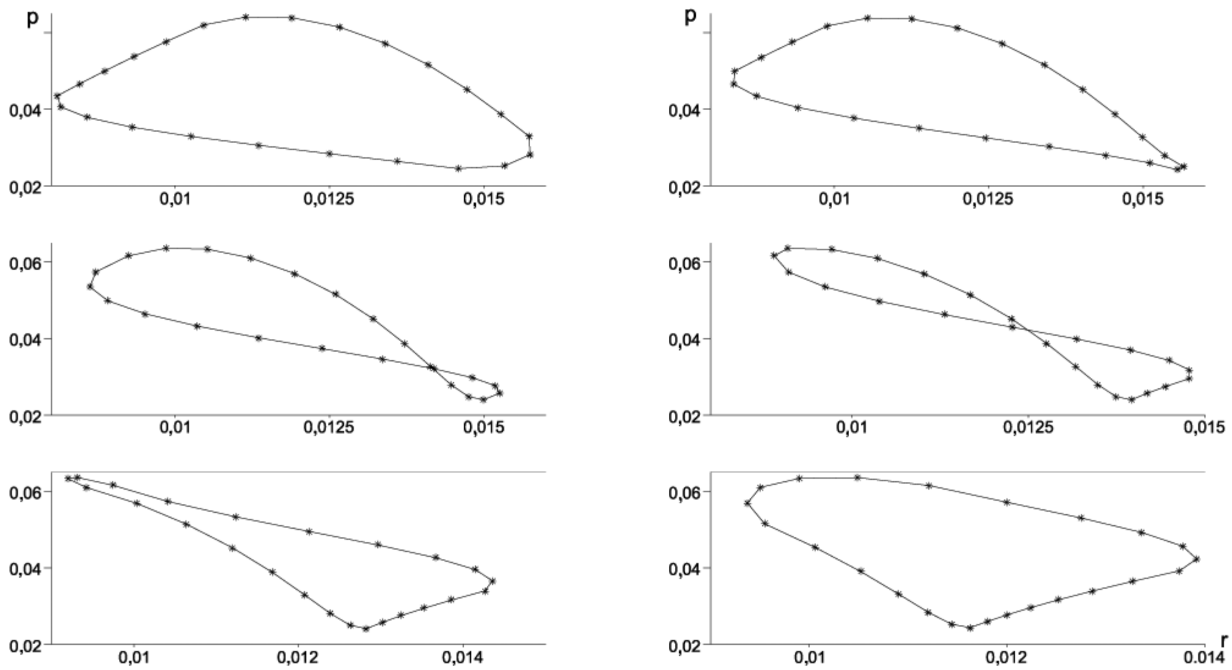


**Fig. 7.** Different forms of the limit orbits.

In Sec. 4, we show that the existence of cyclic orbits and the fact that the trajectories of a dynamical system approach these orbits remains true in the general case, where all three causes of redistribution of the elements $F$ and $W$ over different regions are taken into account. Clearly, in this case, the limit orbits depend on the initial point (they are not attractors) and are sensitive to the ratio of intensities of the external actions upon $F$ and $W$. In particular, we observe the appearance of regions with specific behaviors untypical for the most regions. In the case where the replenishment of the element $F$ acts in the direction opposite to the migration mixing of the element $W$, in almost all regions, we observe their specific limit cyclic orbits. To establish the exact characteristics of these effects, it is necessary to perform additional investigations.

It is possible that the existence of cyclic orbits in Theorems 3 and 4 is a consequence of an abstract statement of the theory of dynamical systems presented in our paper without proof. If the trajectories of dynamical systems generated by the generators $T_1$ and $T_2$ in the same space converge to fixed limit states or cyclic orbits, then the trajectories of the dynamical system generated by the generator $T_3 = T_1 \circ T_2$ also converge to a fixed state or to a cyclic orbit.

We now recall the history of the problem of existence of cyclic orbits for dynamical systems in the plane

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y).$$

It is connected with the 16th Hilbert problem, which has not been completely solved yet. In this direction, the Poincaré–Bendixson theorem is of especial interest. This theorem guarantees the existence of cyclic orbits for continuously differentiable functions $f$ and $g$ if the coordinates $x(t)$ and $y(t)$ are bounded and the equilibrium point is absent.

Theorem 4 can be regarded as a generalization of the Poincaré–Bendixson theorem in the space $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, $n \geq 2$. Indeed, under additional conditions, Theorem 4 is reduced to the Poincaré–Bendixson theorem. Assume that Eq. (23) has a unique solution for each pair of initial vectors $p, r \in \mathbb{R}^n_+$. Then, according to the implicit-

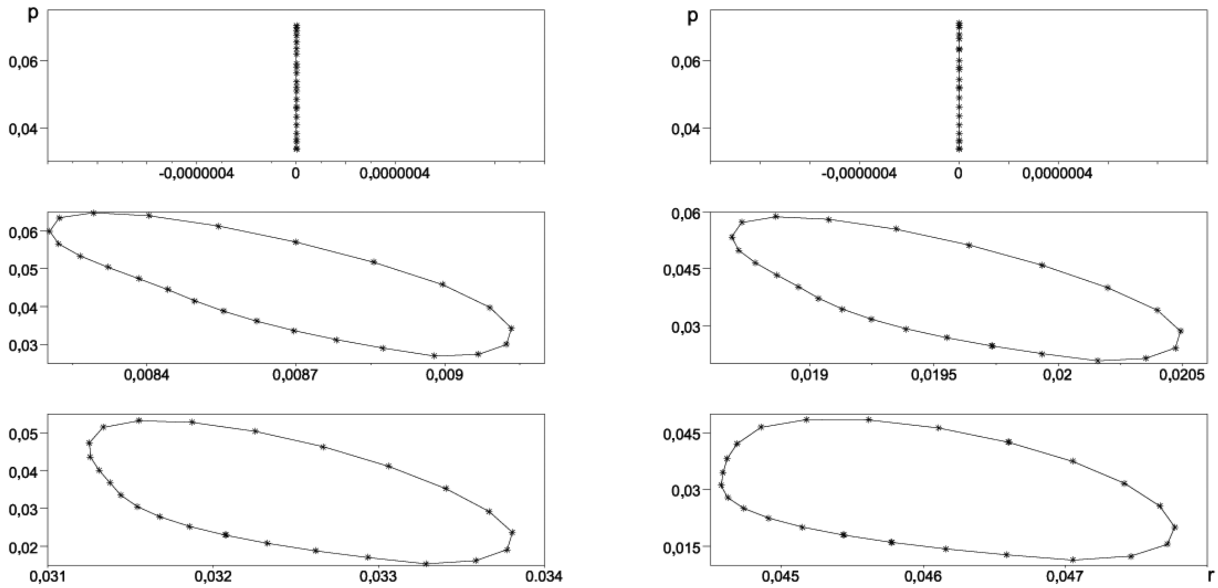


**Fig. 8.** Forms of the orbits in the case of constant mixing.

function theorem, for each pair $p_i(t)$, $r_i(t)$, there exist functions $f_i(p,r)$ and $g_i(p,r)$ such that Eq. (23) can be rewritten in the form

$$\dot{p}_i = f_i(p,r), \quad \dot{r}_i = g_i(p,r).$$

Under an additional assumption that these functions are differentiable, we conclude that closed cyclic orbits exist.

## REFERENCES

1. J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior,* Princeton University Press, Princeton (1960).
2. A. J. Jones, *Game Theory: Mathematical Models of Conflict,* Wiley, New York (1980).
3. G. Owen, *Game Theory,* Academic Press, San Diego (1995).
4. T. C. Schelling, *The Strategy of Conflict,* Harvard University Press, Cambridge (1981).
5. A. N. Sharkovskii, S. F. Kolyada, A. G. Sivak, and V. V. Fedorenko, *Dynamics of One-Dimensional Mappings* [in Russian], Naukova Dumka, Kiev (1989).
6. A. N. Sharkovskii, Yu. L. Maistrenko, and E. Yu. Romanenko, *Difference Equations and Their Applications* [in Russian], Naukova Dumka, Kiev (1986).
7. W. de Melo and S. van Strien, *One-Dimensional Dynamics,* Springer, Berlin (1993).
8. J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics,* Cambridge Univ. Press, Cambridge (1998).
9. J. D. Murray, *Mathematical Biology I: An Introduction,* Springer, New York (2002).
10. J. M. Epstein, *Nonlinear Dynamics, Mathematical Biology, and Social Science,* Addison-Wesley Publishing Company (1997).
11. Y. Lonzonn, S. Solomon, J. Goldenberg, and D. Mazarsky, "World-size global markets lead to economic instability," *Acr. Life,* 357–370 (2003).
12. M. Maron, "Modelling populations: from Malthus to the threshold of artificial life, evolutionary and adaptive systems," *Univ. Sussex* (2003), pp. 1–17.
13. B. Hasseblatt and A. Katok, *A First Course in Dynamics: with a Panorama of Recent Developments,* Cambridge Univ. Press, Cambridge (2003).
14. A. N. Kolmogorov, "General theory of dynamical systems and classical mechanics," in: *Proceedings of the International Congress of Mathematics,* **1** (1954), pp. 315–333.

15. A. N. Kolmogorov, “Qualitative study of the mathematical models of dynamics of populations,” *Probl. Kibern.,* **25**, Issue 2, 101–106 (1972).
16. V. I. Arnol’d, *Theory of Catastrophes* [in Russian], Nauka, Moscow (1986).
17. L. S. Pontryagin, *Ordinary Differential Equations* [in Ukrainian], Kyiv (2000).
18. A. A. Chikrii, *Conflict-Controlled Processes* [in Russian], Naukova Dumka, Kiev (1992).
19. A. Chikrii, I. Matychyn, and K. Chikrii, “Differential game with impulse control,” *Adv. Dynam. Game Theory, Ann. Int. Soc. Dynam. Games,* **9**, 37–57 (2007).
20. Yu. G. Krivonos, I. I. Matichin, and A. A. Chikrii, *Dynamical Games with Discontinuous Trajectories* [in Russian], Naukova Dumka, Kiev (2005).
21. K. I. Takahashi and K. MD. M. Salam, “Mathematical model of conflict with non-annihilating multi-opponent,” *J. Interdiscipl. Math.,* **9**, No. 3, 459–473 (2006).
22. G.-I. Bischi and U. Merlone, “Global dynamics in binary choice models with social influence,” *J. Math. Sociology,* **33**, 1–26 (2009).
23. G.-I. Bischi and E. Tramontana, “Three-dimensional discrete-time Lotka–Volterra models with an application to industrial clusters,” *Comm. Nonlin. Sci. Numer. Simulat.,* **15**, No. 10, 3000–3014 (2009).
24. N. Bellomo, M. Herrero, and A. Tosin, “On the dynamics of social conflicts: looking for the black swan,” *Kinetic Relat. Models,* **6**, 459–479 (2013).
25. G.-I. Bischi, M. Gallegati, L. Gardini, R. Leombruni, and A. Palestrini, “Herd behavior and nonfundamental asset price fluctuations in financial markets,” *MacDynamics,* **10**, 502–528 (2006).
26. T. K. Kar, “Modelling and analysis of a harvested prey-predator system incorporating a prey refuge,” *J. Comput. Appl. Math.,* **185**, 19–33 (2006).
27. E. P. Belan, M. V. Mikhalevich, and I. V. Sergienko, “System analysis, cycles in economic systems with open labor markets,” *Cybern. Syst. Anal.,* **44**, No. 4 (2008).
28. V. D. Koshmanenko, “Theorem on conflicts for a pair of stochastic vectors,” *Ukr. Mat. Zh.,* **55**, No. 4, 555–560 (2003); ***English translation:*** *Ukr. Math. J.,* **55**, No. 4, 671–678 (2003).
29. V. Koshmanenko, “Theorem of conflicts for a pair of probability measures,” *Math. Meth. Oper. Res.,* **59**, No. 2, 303–313 (2004).
30. V. Koshmanenko and I. Samoilenko, “The conflict triad dynamical system,” *Comm. Nonlin. Sci. Numer Simulat.,* **16**, 2917–2935 (2011).
31. V. D. Koshmanenko and I. V. Samoilenko, “Model of a dynamical system of a conflict triad,” *Nelin. Kolyvannya,* **14**, No. 1, 55–75 (2011); ***English translation:*** *Nonlin. Oscillations,* **14**, No. 1, 56–76 (2011).
32. V. D. Koshmanenko and N. V. Kharchenko, “Invariant points of a dynamical system of conflict in the space of piecewise-uniformly distributed measures,” *Ukr. Math. J.,* **56**, No. 7, 927–938 (2004).
33. M. V. Bondarchuk, V. D. Koshmanenko, and N. V. Kharchenko, “Properties of the limit states of a dynamical conflict system,” *Nelin. Kolyvannya,* **7**, No. 4, 446–461 (2004); ***English translation:*** *Nonlin. Oscillations,* **7**, No. 4, 432–447 (2004).
34. M. V. Bondarchuk, V. D. Koshmanenko, and I. V. Samoilenko, “Dynamics of conflict interaction between systems with internal structure,” *Nelin. Kolyvannya,* **9**, No. 4, 435–450 (2006); ***English translation:*** *Nonlin. Oscillations,* **9**, No. 4, 423–437 (2006).
35. S. Albeverio, V. Koshmanenko, and I. Samoilenko, “The conflict interaction between two complex systems: cyclic migration,” *J. Interdiscipl. Math.,* **11**, No. 2, 163–185 (2008).
36. V. Koshmanenko and I. Samoilenko, “The cyclic conflict fire-water model,” in: *Proc. of the Internat. Conf. “System Analysis and Information Technology,”* **43** (2008).
37. V. Koshmanenko and N. Kharchenko, “The conflict fire-water cyclic model,” in: *Proceedings of the International Conference “Problems of Decision Making under Uncertainties,”* Kyiv, **24** (2008).