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Nonlinear Conformally Invariant Wave Equations and Their Exact Solutions

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Abstract

New nonlinear representations of the conformal algebra and the extended Poincaré algebra are found and new classes of conformally invariant wave equations are constructed. Exact solutions of the equations in question containing arbitrary functions are obtained.

1 Introduction

The classical and quantum scalar fields are known to be described by the Poincaré-invariant wave equations for the complex function. Therefore, according to the symmetry selection principle, it is interesting to construct classes of nonlinear wave equations admitting wider symmetry, in particular invariant under the different representations of the extended Poincaré algebra and the conformal one, which include the Poincaré algebra as a subalgebra.

It has been stated [1, 2], that the Poincaré-invariant wave equation

$$\Box u = F(|u|)u$$

(F is an arbitrary smooth function, $u = u(x^0 \equiv ct, x^1, \ldots, x^n)$, $\Box \equiv p_\mu p^\mu$ is the d'Alembertian in the $(n+1)$–dimensional pseudo-Euclidean space $R(1,n)$, $|u| = \sqrt{uu^*}$, the asterisk designates the complex conjugation) admits the extended Poincaré algebra $A\widetilde{P}(1,n) = \langle P_\mu, J_{\mu\nu}, D^{(2)}\rangle$ iff it is of the form:

$$\Box u = \lambda_1 |u|^k u, \quad k \neq 0, \tag{1}$$

and admits the conformal algebra $AC(1,n) = \langle P_\mu, J_{\mu\nu}, D^{(1)}, K^{(1)}_\mu\rangle$ iff it is of the form:

$$\Box u = \lambda_2 |u|^{4/(n-1)} u, \quad n \neq 1. \tag{2}$$

Here $\lambda_1,\ \lambda_2,\ k$ are real parameters,

$$P_\mu = p_\mu \equiv i\frac{\partial}{\partial x^\mu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{3}$$

$$D^{(1)} = x^\mu p_\mu + \frac{1-n}{2}[u p_u + u^* p_{u^*}], \qquad K^{(1)}_\mu = 2x_\mu D^{(1)} - (x_\nu x^\nu) p_\mu, \tag{4}$$

$$D^{(2)} = x^\mu p_\mu - \frac{2}{k} u p_u - \frac{2}{k} u^* p_{u^*}; \qquad p_u = i\frac{\partial}{\partial u}, \qquad p_{u^*} = i\frac{\partial}{\partial u^*},$$

sumation under repeated idices from 0 to n is understood, raising or lowering of the vector indices is performed by means of the metric tensor $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, \ldots, -1)$, i.e. $p_\mu = g_{\mu\nu} p^\nu$, $p^\mu = g^{\mu\nu} p_\nu$.

Some equations of the form

$$\Box u = F(u, u^*, p_\mu u, p_\mu u^*),$$

that are invariant under *linear* representations of $A\widetilde{P}(1,n)$ and $AC(1,n)$ for $n \geq 2$ are described in [3]. Some equations of the second order, invariant under linear representations of $A\widetilde{P}(1,n)$ and $AC(1,n)$ are adduced in [4], where for those in $(1+n)$–dimensional space ($n \geq 3$) the functional basis of the differential invariants has been constructed. It should be noted that equations (1) and (2) are invariant under linear representations of $A\widetilde{P}(1,n)$ and $AC(1,n)$, correspondingly.

The following natural question arises: do there exist nonlinear representations of the conformal algebra and the extended Poincaré algebra for a complex scalar field? Our answer to this question is that there exist such representations.

Here we present some classes of nonlinear wave equations invariant under unusual representations of the extended Poincaré algebra and the conformal algebra. In particular, we have found [5], that the equation

$$\Box u = \frac{\Box |u|}{|u|} u + m^2 c^2 u$$

is invariant under nonlinear representation of the conformal algebra $AC(1, n+1)$. It should be noted that this equation is proposed by Gueret and Vigier [6] and by Guerra and Pusterla [7]. It arises in the modelling of the equation for de Broglie's theory of double solution [8].

Also we describe some equations admitting both the standard linear representation of the conformal algebra and a nonlinear representation of the extended Poincaré algebra and find their exact solutions.

2 Different representations of the extended Poincaré algebra and the conformal algebra

Let us investigate symmetry properties of the more general wave equation with nonlinearities containing the second order derivatives, namely:

$$\Box u = F\left(|u|, (\nabla |u|)^2, \Box |u|\right) u. \tag{5}$$

Here, $F(\cdot,\cdot,\cdot)$ is an arbitrary smooth function, $u = u(x^0 \equiv ct, x^1, \ldots, x^n)$, $\nabla \equiv (p_0, p_1, \ldots, p_n)$, $(\nabla|u|)^2 \equiv (\nabla|u|)(\nabla|u|) \equiv (p_\mu |u|)(p^\mu |u|)$.

Theorem 1. *The maximal invariance algebra (MIA) of equation (5) for an arbitrary function F is the Poincaré algebra $AP(1,n) \oplus Q \equiv \langle P_\mu, J_{\mu\nu}, Q\rangle$ with the basis operators (3) and*

$$Q = i[u p_u - u^* p_{u^*}].$$

Let us introduce the following notations: R designates an arbitrary function, λ, k, l are arbitrary real parameters, and λ is not equal to zero.

Theorem 2. *Equation (5) is invariant under the extended Poincaré algebra $A\widetilde{P}(1,n)$ iff it is of the form:*

$$\Box u = \left\{ \frac{\Box|u|}{|u|} + \left(\frac{\Box|u|}{|u|}\right)^{1-2l} R\left(\frac{|u|\Box|u|}{(\nabla|u|)^2}, |u|^2\left(\frac{\Box|u|}{|u|}\right)^k\right)\right\} u, \tag{6}$$

MIA: $\langle P_\mu, J_{\mu\nu}, D, Q\rangle$, $\quad D = x^\mu p_\mu + l\ln(u/u^*)[up_u - u^* p_{u^*}] + k[up_u + u^* p_{u^*}]$.

We can see that, when $l \neq 0$, equation 6 is invariant under the nonlinear representation of $A\widetilde{P}(1,n)$, because in this case the operator D generates the following nonlinear finite transformations of variables x and u:

$$x'_\mu = x_\mu \exp(\tau), \qquad u' = |u|\exp(k\tau)(u/u^*)^{\exp(2l\tau)/2},$$

where τ is a group parameter.

Theorem 3. *Equation (5) is invariant under the conformal algebra iff it takes one of the following forms:*

1. $$\Box u = |u|^{4/(n-1)} R\left(|u|^{(3+n)/(1-n)}\Box|u|\right) u, \qquad n \neq 1, \tag{7}$$

   MIA: $\langle P_\mu, J_{\mu\nu}, D^{(1)}, K^{(1)}_\mu, Q\rangle$;

2. $$\Box u = \Box|u| R\left(\frac{\Box|u|}{(\nabla|u|)^2}, |u|\right) u, \qquad n = 1, \qquad \textit{MIA:}\ \langle Z^{(1)}, Q\rangle, \tag{8}$$

$$Z^{(1)} = \left[s^{(1)}_1(x^0+x^1) + s^{(1)}_2(x^0-x^1)\right]p_0 + \left[s^{(1)}_1(x^0+x^1) - s^{(1)}_2(x^0-x^1)\right]p_1,$$

$s^{(1)}_1$, $s^{(1)}_2$ *are real smooth functions;*

3. [5] $$\Box u = \frac{\Box|u|}{|u|}u + \lambda u, \tag{9}$$

   $$\textit{MIA:}\quad \langle P_\mu, P_{n+1}, J_{\mu\nu}, J_{\mu n+1}, D^{(3)}, K^{(3)}_\mu, K^{(3)}_{n+1}, Q_3\rangle, \tag{10}$$

$$P_{n+1} \equiv p_{n+1} \equiv i\frac{\partial}{\partial x^{n+1}} = i\sqrt{|\lambda|}[up_u - u^* p_{u^*}], \qquad J_{\mu n+1} = x_\mu p_{n+1} - x_{n+1}p_\mu,$$

$$D^{(3)} = x^\mu p_\mu + x^{n+1}p_{n+1} - \frac{n}{2}[up_u + u^* p_{u^*}], \qquad Q_3 = up_u + u^* p_{u^*},$$

$$K^{(3)}_\mu = 2x_\mu D^{(3)} - (x_\nu x^\nu + x_{n+1}x^{n+1})p_\mu,$$

$$K^{(3)}_{n+1} = 2x_{n+1}D^{(3)} - (x_\nu x^\nu + x_{n+1}x^{n+1})p_{n+1},$$

where the additional variable x^{n+1} *is determined in the following way:*

$$x^{n+1} = \frac{i}{2\sqrt{|\lambda|}} \ln(u^*/u),$$

and a new metric tensor

$$\begin{array}{ll} g_{\mathrm{ij}} = \mathrm{diag}(1, -1, \dots, -1, -1), & \lambda > 0 \\ g_{\mathrm{ij}} = \mathrm{diag}(1, -1, \dots, -1, 1), & \lambda < 0, \qquad \mathrm{i}, \mathrm{j} = \overline{0, n+1} \end{array} \tag{11}$$

is introduced in the space of variables $(x_0, x_1, \dots, x_n, x_{n+1})$.

Direct verification shows that the symmetry operators of equation 9, namely

$$\langle P_\mu, P_{n+1}, J_{\mu\nu}, J_{\mu n+1}, D^{(3)}, K_\mu^{(3)}, K_{n+1}^{(3)} \rangle, \tag{12}$$

satisfy the commutational relations of the conformal algebra $AC(1, n+1)$ when $\lambda > 0$ and of $AC(2, n)$ when $\lambda < 0$.

To give a geometric interpretation of these operators, we rewrite equation 9 in the amplitude-phase terms:

$$A\Box\theta + 2(\nabla A)(\nabla\theta) = 0, \tag{13}$$

$$(\nabla\theta)^2 = -\lambda, \tag{14}$$

where $A = |u| = \sqrt{uu^*}$ and $\theta = (i/2)\ln(u^*/u)$.

The symmetry algebra of 9 is actually obtained by, first, calculating the symmetry algebra of system 13, 14. The maximal invariance algebra of this system is described by operators 10, where

$$\begin{array}{ll} P_{n+1} = \sqrt{|\lambda|} p_\theta, & J_{\mu n+1} = \sqrt{|\lambda|}(x_\mu p_\theta + (\theta/\lambda) p_\mu), \\ D^{(3)} = x^\mu p_\mu + \theta p_\theta - (n/2) A p_A, & K_\mu^{(3)} = 2x_\mu D^{(3)} - (x_\nu x^\nu - \theta^2/\lambda) p_\mu, \\ K_{n+1}^{(3)} = \sqrt{|\lambda|}\left(2(\theta/\lambda) D^{(3)} + (x_\nu x^\nu - \theta^2/\lambda) p_\theta\right), & Q_3 = A p_A. \end{array} \tag{15}$$

Here, we have introduced the following notations: $p_A = i\dfrac{\partial}{\partial A}$, $p_\theta = i\dfrac{\partial}{\partial \theta}$.

From 15 we see that, in the symmetry operators of system 13, 14, the phase variable $x^{n+1} = \theta/\sqrt{|\lambda|}$ is added to the $(n+1)$–dimensional geometric space $(x_0, x_1, \dots, x_n)$. In addition, the metric tensor 11 is introduced. This is the same effect we see for the eikonal equation [1]. The symmetry of equation 9 has the same property because 14 is the eikonal equation for the function θ, and equation 13, which is the continuity one, does not reduce the symmetry of 14.

It should be noted that the Lie algebra 12 realizes the nonlinear representation of the conformal algebra. Solving the corresponding Lie equations, we obtain that the operators $K_\mu^{(3)}, K_{n+1}^{(3)}$ generate the following nonlinear finite transformations of variables x_μ, A, θ:

$$x'^\mu = \frac{x^\mu - b^\mu(x_\delta x^\delta - \theta^2/\lambda)}{1 - 2x_\nu b^\nu - 2b_{n+1}\theta/\sqrt{|\lambda|} + b\cdot b(x_\delta x^\delta - \theta^2/\lambda)},$$

$$A' = A[1 - 2x_\nu b^\nu - 2b_{n+1}\theta/\sqrt{|\lambda|} + b\cdot b(x_\delta x^\delta - \theta^2/\lambda)]^{n/2},$$

$$\theta' = \frac{\theta - \sqrt{|\lambda|} b^{n+1}(x_\delta x^\delta - \theta^2/\lambda)}{1 - 2x_\nu b^\nu - 2b_{n+1}\theta/\sqrt{|\lambda|} + b\cdot b(x_\delta x^\delta - \theta^2/\lambda)},$$

where b is the vector of group parameters in the $(n+2)$–dimensional space with metric the tensor 11.

The expression for these transformations differs from the standard one because the variable θ is considered as a geometric variable on the same footing as the variables x_μ and the amplitude A transforms as a dependent variable [5].

Thus, we see that the wave equation 9 which has a nonlinear quantum–potential term $(\Box|u|)/|u|$ has an unusually wide symmetry, namely it is invariant under a nonlinear representation of the conformal algebra.

It should be noted that the maximal invariance algebra of equation 9 for $\lambda \neq 0$ is a infinite-dimensional algebra with the following operators:

$$\langle Z^{(2)}, Q_1, Q_2\rangle, \quad n \neq 1; \qquad \langle Z^{(3)}, Q_1, Q_2\rangle, \quad n = 1,$$

where

$$Z^{(2)} = a^\mu\left(i\ln(u/u^*)\right)P_\mu + b^{\mu\nu}\left(i\ln(u/u^*)\right)J_{\mu\nu} + d\left(i\ln(u/u^*)\right)D^{(1)} + f^\mu\left(i\ln(u/u^*)\right)K_\mu^{(1)},$$

$$Q_1 = q_1\left(i\ln(u/u^*)\right)[up_u + u^*p_{u^*}], \qquad Q_2 = q_2\left(i\ln(u/u^*)\right)[up_u - u^*p_{u^*}],$$

$$Z^{(3)} = \left\{s_1^{(3)}\left(x^0+x^1, i\ln(u/u^*)\right) + s_2^{(3)}\left(x^0-x^1, i\ln(u/u^*)\right)\right\}p_0 + \left\{s_1^{(3)}\left(x^0+x^1, i\ln(u/u^*)\right) - s_2^{(3)}\left(x^0-x^1, i\ln(u/u^*)\right)\right\}p_1.$$

Here, a^μ, $b^{\mu\nu}$, d, f^μ, q_1, q_2, $s_1^{(3)}$, $s_2^{(3)}$ are real smooth functions.

As stated in Theorem 3, the standard representation of the conformal algebra 3, 4 is realized on the set of solutions of equations 7, 8. Moreover, equation 8 admits a wider symmetry, namely an infinite–dimensional algebra. The invariance under the infinite–dimensional algebra $\langle Z^{(1)}\rangle$ dictates the invariance under the conformal algebra $AC(1,1) = \langle P_\mu, J_{\mu\nu}, D^{(1)}, K_\mu^{(1)}\rangle$, $n=1$, as long as the latter is a subalgebra of the former.

It is interesting to note that the class of equations 7, 8 contains ones invariant under both the standard representation of the conformal algebra and under a nonlinear representation of the extended Poincaré algebra.

It follows from Theorems 2 and 3 that the equation

$$\Box u = \left\{\frac{\Box|u|}{|u|} + \lambda|u|^{l(n+3)-1}(\Box|u|)^{l(1-n)+1}\right\}u, \qquad n \neq 1,$$

admitting the conformal algebra 3, 4 is invariant under a nonlinear representation of the extended Poincaré algebra $A\widetilde{P}(1,n) = \langle P_\mu, J_{\mu\nu}, D^{(4)}\rangle$, and the equation

$$\Box u = \left\{\frac{\Box|u|}{|u|} + |u|^{4l-1}\Box|u| R\left(\frac{|u|\Box|u|}{(\nabla|u|)^2}\right)\right\}u, \qquad n = 1,$$

admitting the infinite–dimensional algebra $\langle Z^{(1)}\rangle$ which includes the conformal algebra as a subalgebra, is invariant under a nonlinear representation of the extended Poincaré algebra $A\widetilde{P}(1,n) = \langle P_\mu, J_{\mu\nu}, D^{(5)}\rangle$.

Here,

$$D^{(4)} = x^\mu p_\mu + l\frac{n-1}{2}\ln(u/u^*)[up_u - u^*p_{u^*}],$$

$$D^{(5)} = x^\mu p_\mu + l\ln(u/u^*)[up_u - u^*p_{u^*}] + [up_u + u^*p_{u^*}].$$

3 Exact solutions of conformally invariant equations

Given an equation, its symmetry algebra can be exploited to construct ansatzes (see, for example [1]) for the equation, which reduce the problem of solving the equation to one of solving the equation of lower order, even ordinary differential equations. We examine this question for the three-dimensional conformally invariant equation

$$\Box u = \left\{ \frac{\Box |u|}{|u|} + \lambda |u|^{6l-1} (\Box |u|)^{1-2l} \right\} u. \tag{16}$$

As follows from Theorems 2 and 3, when $n = 3$, equation 16 is invariant under the conformal algebra $AC(1,3) = \langle P_\mu, J_{\mu\nu}, D^{(1)}, K_\mu^{(1)} \rangle$ and the operators Q, $D^{(4)}$. Making use of this symmetry, we construct exact solutions of equation 16. To this end, we rewrite equation 16 in the amplitude–phase representation:

$$\begin{cases} A\Box\theta + 2(\nabla A)(\nabla\theta) = 0, \\ (\nabla\theta)^2 + \lambda A^{6l-1}(\Box A)^{1-2l} = 0. \end{cases} \tag{17}$$

To construct solutions containing arbitrary functions, we consider the following ansatz:

$$\begin{aligned} A &= \varphi(\omega_1, \omega_2, \omega_3), \\ \theta &= \psi(\omega_1, \omega_2, \omega_3), \end{aligned} \qquad \omega_1 = \beta x, \ \omega_2 = \gamma x, \ \omega_3 = \alpha x, \tag{18}$$

where α, β, γ are $(n+1)$–dimensional constant vectors in $R(1,n)$, satisfying the following conditions:

$$\alpha^2 = \alpha\beta = \alpha\gamma = \beta\gamma = 0, \quad \beta^2 = \gamma^2 = -1.$$

Substituting 18 into 17, we get the system

$$\begin{cases} \varphi(\psi_{11} + \psi_{22}) + 2(\varphi_1\psi_1 + \varphi_2\psi_2) = 0, \\ -\varphi_1^2 - \varphi_2^2 + \lambda\varphi^{6l-1}(-\varphi_{11} - \varphi_{22})^{1-2l} = 0, \end{cases} \tag{19}$$

containing the variable ω_3 as a parameter. System 19 admits the infinite–dimensional symmetry operator:

$$\widehat{X} = \tau_1\widehat{P}_1 + \tau_2\widehat{P}_2 + \tau_3\widehat{J}_{12} + \tau_4\widehat{D}_1 + \tau_5\widehat{D}_2 + \tau_6\widehat{Q}, \tag{20}$$

where $\tau_1, \ldots, \tau_6$ are arbitrary functions of ω_3, and

$$\widehat{P}_1 = \frac{\partial}{\partial\omega_1}, \qquad \widehat{P}_2 = \frac{\partial}{\partial\omega_2}, \qquad \widehat{J}_{12} = \omega_1\frac{\partial}{\partial\omega_2} - \omega_2\frac{\partial}{\partial\omega_1}, \qquad \widehat{Q} = \frac{\partial}{\partial\psi},$$
$$\widehat{D}_1 = \omega_1\frac{\partial}{\partial\omega_1} + \omega_2\frac{\partial}{\partial\omega_2} - \varphi\frac{\partial}{\partial\varphi}, \qquad \widehat{D}_2 = \omega_1\frac{\partial}{\partial\omega_1} + \omega_2\frac{\partial}{\partial\omega_2} + 2r\psi\frac{\partial}{\partial\psi}.$$

Making use of ansatzes constructed via nonequivalent one-dimensional subalgebras of algebra 20, one can reduce system 19 to ordinary differential equations and find their exact solutions. Let us consider the following subalgebras:

$$A_1 = \langle \widehat{P}_1 \rangle, \quad A_2 = \langle \widehat{D}_1 \rangle, A_4 = \langle \widehat{J}_{12} + a(\alpha x)\widehat{D}_1 \rangle,$$

where a is an arbitrary real function of αx.

Ansatzes corresponding to these subalgebras are of the form:

$$1.\quad \begin{cases}\varphi = v(\omega),\\ \psi = w(\omega),\end{cases} \qquad \omega = \gamma x;$$

$$2.\quad \begin{cases}\varphi = [(\beta x)^2 + (\gamma x)^2]^{-1/2} v(\omega),\\ \psi = w(\omega),\end{cases} \qquad \omega = \arctan\tfrac{\beta x}{\gamma x}; \tag{21}$$

$$3.\quad \begin{cases}\varphi = [(\beta x)^2 + (\gamma x)^2]^{-1/2} v(\omega),\\ \psi = w(\omega),\end{cases} \qquad \omega = 2a(\alpha x)\arctan\frac{\beta x}{\gamma x} - \ln[(\beta x)^2 + (\gamma x)^2].$$

Group parameters of the group with generator 20 are arbitrary real functions of ω_3. Therefore, acting 21 by the finite transformations of this group, we obtain 3 classes of ansatzes containing arbitrary functions of $\omega_3 = \alpha x$:

$$1.\quad \begin{cases}A = \varphi = \rho_1 v(\omega),\\ \theta = \psi = \rho_4^{-2l} w(\omega) + \rho_5,\end{cases}$$
$$\omega = \rho_1\rho_4(\gamma x\cos\rho_2 - \beta x\sin\rho_2) + \rho_3;$$

$$2.\quad \begin{cases}A = \varphi = \rho_4[(\beta x+\rho_1)^2 + (\gamma x+\rho_2)^2]^{-1/2} v(\omega),\\ \theta = \psi = \rho_4^{2l} w(\omega) + \rho_5,\end{cases}$$
$$\omega = \arctan\frac{(\beta x+\rho_1)\cos\rho_3 + (\gamma x+\rho_2)\sin\rho_3}{(\gamma x+\rho_2)\cos\rho_3 - (\beta x+\rho_1)\sin\rho_3};$$

$$3.\quad \begin{cases}A = \varphi = \rho_4[(\beta x+\rho_1)^2 + (\gamma x+\rho_2)^2]^{-1/2} v(\omega),\\ \theta = \psi = \rho_4^{2l} w(\omega) + \rho_6,\end{cases}$$
$$\omega = 2a(\alpha x)\arctan\frac{(\beta x+\rho_1)\cos\rho_3 + (\gamma x+\rho_2)\sin\rho_3}{(\gamma x+\rho_2)\cos\rho_3 - (\beta x+\rho_1)\sin\rho_3} -$$
$$\ln[(\beta x+\rho_1)^2 + (\gamma x+\rho_2)^2] + \rho_5.$$

Here, $\rho_1, \ldots, \rho_6$ are arbitrary real functions of αx.

The ansatzes constructed reduce system 19 as well as system 17 to the systems of ordinary differential equations. Finding their partial solutions, we obtained the following exact solutions of equation 16 when $l = 1/2$:

$$1.\ u = \pm\sigma\exp\left\{i\sqrt{\lambda}|\sigma|(\gamma x\cos\rho_2 - \beta x\sin\rho_2)\right\};$$

$$2.\ u = \pm\sigma[(\beta x+\rho_1)^2 + (\gamma x+\rho_2)^2]^{-1/2}\times$$
$$\exp\left\{i\sqrt{\lambda}|\sigma|\arctan\frac{(\beta x+\rho_1)\cos\rho_3 + (\gamma x+\rho_2)\sin\rho_3}{(\gamma x+\rho_2)\cos\rho_3 - (\beta x+\rho_1)\sin\rho_3}\right\};$$

$$3.\ u = \pm 2\sigma[(\beta x+\rho_1)^2 + (\gamma x+\rho_2)^2]^{(-1-3a^2)/(6a^2+6)}\times$$

$$\exp\left\{-\frac{2a}{3(a^2+1)}\arctan\frac{(\beta x+\rho_1)\cos\rho_3+(\gamma x+\rho_2)\sin\rho_3}{(\gamma x+\rho_2)\cos\rho_3-(\beta x+\rho_1)\sin\rho_3}+\right.$$

$$3i\sqrt{\lambda}\sqrt{a^2+1}|\sigma|[(\beta x+\rho_1)^2+(\gamma x+\rho_2)^2]^{1/(3a^2+3)}\times$$

$$\left.\exp\left\{-\frac{2a}{3(a^2+1)}\arctan\frac{(\beta x+\rho_1)\cos\rho_3+(\gamma x+\rho_2)\sin\rho_3}{(\gamma x+\rho_2)\cos\rho_3-(\beta x+\rho_1)\sin\rho_3}\right\}\right\}.$$

Here, $\rho_1, \rho_2, \rho_3, a, q$ are arbitrary real functions of αx, and σ is an arbitrary complex function of αx. It should be noted that all the solutions obtained contain arbitrary functions.

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