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On Symmetries of a Generalized Diffusion Equation

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Abstract

The non-linear diffusion eqaution, describing the vertical transfer of both heat and moisture in the absence of solutes are considered. Lie symmetries of the equation are obtained for some specific form of diffusion coefficient.

The group properties of diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial \vec{x}}\left[D(u)\frac{\partial u}{\partial \vec{x}}\right] = 0, \tag{1}$$

(where $u = u(\vec{x}, t)$, $D(u)$ are real functions) describing nonlinear processes of heat conductivity are investigated by many authors [1–3].

In the present paper we investigate a particular case of eq.(1), where u is vector function and $D(u)$ is a non-singular matrix, defining the diffusive properties of the soil. Recently, such systems have been extensively studied, from both mathematical and biological viewpoints [4, 5]. It is motivated by a successful application of these models to a wide range of developmental and ecological systems. We will analyze symmetry properties of given models.

Let us consider non-linear diffusion equation with sources

$$\frac{\partial u^a}{\partial t} - \frac{\partial}{\partial x}\left[K^{ab}(\vec{u})\frac{\partial u^b}{\partial \vec{x}}\right] = M^a(\vec{u}), \qquad a, b = 1, 2, \ldots, n, \tag{2}$$

where K^{ab} is a n-dimensional matrix.

Classical symmetry groups for coupled non-linear diffusion equation with $M^a = 0$ were found in [6].

Let us rewrite eq.(2) in the form:

$$\frac{\partial u^a}{\partial t} - \frac{K^{ab}}{\partial u^c}\frac{\partial u^c}{\partial x}\frac{\partial u^b}{\partial x} - K^{ab}\frac{\partial^2 u^b}{\partial x^2} - M^a = 0. \tag{3}$$

The symmetry operator Q is defined by

$$Q = \xi^1(x, t, \vec{u})\frac{\partial}{\partial t} + \xi^2(x, t, \vec{u})\frac{\partial}{\partial x} + \eta^a(x, t, \vec{u})\frac{\partial}{\partial u^a}. \tag{4}$$

The prolongation operator of eq.(3) has the form

$$\underset{2}{Q} = Q + \eta^a_1\frac{\partial}{\partial u^a_t} + \eta^a_2\frac{\partial}{\partial u^a_x} + \eta^a_{11}\frac{\partial}{\partial u^a_{tt}} + \eta^a_{12}\frac{\partial}{\partial u^a_{xt}} + \eta^a_{22}\frac{\partial}{\partial u^a_{xx}}, \tag{5}$$

where

$$\eta_1^a = \eta_t^a + u_t\eta_u - u_t\xi_t^1 - u_x\xi_t^2 - (u_t)^2\xi_u^1 - u_tu_x\xi_u^2,$$
$$\eta_2^a = \eta_x^a + u_x\eta_u - u_t\xi_x^1 - u_x\xi_x^2 - u_xu_t\xi_u^1 - (u_x)^2\xi_u^2,$$
$$\eta_{22}^a = \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + (u_x)^2\eta_{uu} - 2u_{xt}\xi_x^1 - 2u_{xx}\xi_x^2 - 2u_{xt}u_x\xi_u^1 - 2u_{xx}u_x\xi_u^2 - u_{xx}u_t\xi_u^1 - u_{xx}u_x\xi_u^2 - 2u_tu_x\xi_{xu}^1 - 2(u_x)^2\xi_{xu}^2 - u_x\xi_{xx}^2 - u_t\xi_{xx}^1 - (u_x)^2u_t\xi_{uu}^1 - (u_x)^3\xi_{uu}^2. \tag{6}$$

To find the invariance condition we act by $\underset{2}{Q}$ on the eq.(3).

As a result we obtain:

$$-\eta^aK_{u^cu^a}^{ab}u_x^cu_x^b - \eta^aK_{u^a}^{db}u_{xx}^b - M_{u^a}^d\eta^a + \eta_1^a - \eta_2^aK_{u^c}^{da}u_x^c - \eta_2^aK_{u^a}^{db}u_x^b + \eta_{22}^aK^{da}\Big|_{u_t^a=K^{ab}u_{xx}^b+K_{u^c}^{ab}u_x^cu_x^b+M^a} = 0. \tag{7}$$

Substituting into (7) explicit forms of η_1^a, η_2^a, η_{22}^a and equating the coefficients of the various partial derivatives we obtain the following defining equations:

$$\xi_u^1 = \xi_x^1 = 0, \tag{8}$$

$$K^{da}\xi_{u^c}^2 + 2K^{dc}\xi_{u^a}^2 - \xi_{u^f}^2K^{fc}\delta_{da} = 0, \tag{9}$$

$$-2K^{da}\eta_{u^bu^c}^a - K_{u^a}^{dc}\eta_{u^b}^a - K_{u^a}^{db}\eta_{u^c}^a + 2K_{u^b}^{dc}\xi_x^2 + 2K_{u^c}^{db}\xi_x^2 - K_{u^c}^{da}\eta_{u^b}^a - K_{u^b}^{da}\eta_{u^c}^a + \eta_{u^f}^dK_{u^c}^{fb} + \eta_{u^f}^dK_{u^b}^{fc} - \xi_t^1K_{u^c}^{db} - \xi_t^1K_{u^b}^{dc} - \eta^aK_{u^cu^a}^{db} - \eta^aK_{u^au^b}^{dc} = 0, \tag{10}$$

$$-K^{da}\eta_{u^b}^a + 2K^{db}\xi_x^2 + \eta_{u^c}^dK^{cb} - \xi_t^1K^{db} - \eta^aK_{u^a}^{db} = 0, \tag{11}$$

$$K^{da}\xi_{xx}^2 - 2K^{db}\eta_{xu^a}^b - K_{u^b}^{da}\eta_x^b - \xi_t^2\delta_{ad} = 0, \tag{12}$$

$$-M_{u^a}^d\eta^a + \eta_t^d - \xi_t^1M^d + \eta_{u^b}^dM^b - \eta_{xx}^aK^{da} = 0. \tag{13}$$

It follows from (8), (9) that

$$\xi^1 = \xi^1(t), \qquad \xi^2 = \xi^2(x,t).$$

Solving of eqs.(8)–(13) is a complicated problem. We consider some particular cases which can find applications in mathematical biology.

Choosing

$$K^{11} = D, \qquad K^{12} = -B(u^1)u^1, \qquad K^{21} = 0, \qquad K^{22} = D, \qquad M^a = 0, \tag{14}$$

we reduce eq.(3) to the following system:

$$\begin{cases} u_t^1 = Du_{xx}^1 - B_{u^1}u_x^1u^1u_x^2 - Bu_x^1u_x^2 - Bu_1u_{xx}^2, \\ u_t^2 = Du_{xx}^2. \end{cases} \tag{15}$$

Theorem 1.* *The invariance algebra of system (15) is a 5-parameter algebra Lie whose basis elements have a following form:*

$$\begin{aligned} &Q_1 = \partial_t, \qquad Q_2 = \partial_x, \qquad Q_3 = \partial_{u^2}, \\ &Q_4 = B(u_1)\partial_{u^1}, \qquad Q_5 = x\partial_x + 2t\partial_t. \end{aligned} \tag{16}$$

Proof. Substituting (14) into determing eqs.(8)–(13) we find the general solutions of η^a, ξ^a $(a = 1, 2)$:

$$\begin{aligned} &\eta^1 = C^1 B(u^1)u^1, \qquad \eta^2 = C_2, \\ &\xi^1 = 2C_3 t + C_4, \qquad \xi^2 = C_3 x + C_5. \end{aligned} \tag{17}$$

It is easily to verify using (17) and (4) that basis elements have the form (16).
In the case

$$M^a = 0, \qquad K = K(x, t) \tag{18}$$

we come to

Theorem 2. *Eq.(3), (18) invariant under 5-parameter Lie algebra, whose operator are:*

$$\begin{aligned} &Q_1 = \partial_t, \qquad Q_2 = \partial_x, \qquad Q_3 = x\partial_x + 2t\partial_t, \\ &Q_4 = (K^{-1})^{ab}u^b x\partial_{u^a} - 2t\partial_x, \qquad Q_5 = (K^{-1})^{ab}u^b\partial_{u^a}, \end{aligned} \tag{19}$$

iff a diffusion matrix K has a specific dependence on t, x:

$$K = K\left[\frac{\gamma t + \sigma}{\left(\frac{\gamma}{2}x + \alpha\right)^2}\right], \tag{20}$$

where γ, α, δ are constants.

In the conclusion we consider a cell equation of the following general form:

$$\begin{aligned} &n_t + (nu_t)_x = 0, \\ &\rho_t + (\rho u_t)_x = 0, \\ &\mu u_{xxt} + u_{xx} + \Big[\tau n(\rho + \gamma\rho_{xx})\Big]_x = s\rho u, \end{aligned} \tag{21}$$

where μ, τ, s are constants, $n = n(t, x)$, $u = u(t, x)$, $\rho = \rho(t, x)$.

Using classical Lie methods [7] we obtained that system (21) is invariant under Heisenberg algebra $H(1)$.

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