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On the Peculiarities of Stochastic Invariant Solutions of a Hydrodynamic System Accounting for Non-local Effects

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Abstract

A set of travelling wave solutions of a system of PDE describing nonequilibrium processes in relaxing media is investigated. These solutions satisfy a certain dynamic system obtained from the initial one via the group theory reduction. The dynamic system is shown to possess stochastic oscillatory-type solutions that might play role of a non-trivial intermediate asymptotics for the initial system of PDE. Fine structure of a strange attractor arising in the dynamic system is studied by means of Poincaré sections technique.

The problem of description of a multicomponent medium subjected to high-rate high-intense load actually is far from being solved. The classical continual models are not valid for this purpose since high-rate load induces irreversible processes of energy exchange between components and, besides, different structural changes or chemical reactions initiated by shock wave propagation might take place. A route to the equilibrium in such systems is rather difficult to analyze since the detailed mechanism of relaxation in most cases remains unknown. Yet for the processes that are not far from equilibrium, individual features turn out to be unessential and irreversible thermodynamics formalism may be employed in order to construct constitutive equations [1–4]. In the early 90-th, V.A. Danylenko [5, 6] proposed to describe pulse load afteraction in active and relaxing media with the help of the following equations:

$$\begin{gathered}
\rho\frac{du}{dt}+\frac{\partial p}{\partial x}=\Im, \qquad \frac{d\rho}{dt}+\rho\frac{\partial u}{\partial x}=0,\\
\tau\frac{dp}{dt}-\chi\frac{d\rho}{dt}\rho^{n-1}=\kappa\rho^n-p-h\left\{\frac{d^2p}{dt^2}+\rho^{n-1}\left[\frac{2}{\rho}\left(\frac{d\rho}{dt}\right)^2-\frac{d^2\rho}{dt^2}\right]\right\}
\end{gathered} \tag{1}$$

The first two equations of system (1) represent the standard balance equations for mass and momentum, taken in hydrodynamic approximation. The third equation called the dynamic equation of state contains the information about relaxing properties of the medium. Its parameters have following physical meaning: τ is the relaxation time, χ is the volume viscosity coefficient, h is the coefficient of structural relaxation, $n=1+\Gamma_{V\infty}$, $\Gamma_{V\infty}$ is the isochoric Gruneisen coefficient [7, 5], $\sqrt{\kappa}$ is proportional to the sound velocity in equilibrium.

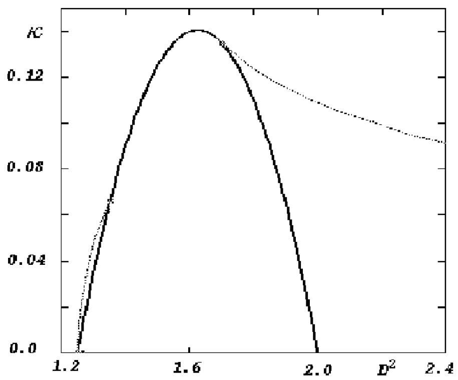


Fig.1. Bifurcation diagram of system (4) in the plane (D^2, κ), obtained for $\beta = -0.8$ and $\xi = -1.25$. Solid line corresponds to a Hopf bifurcation while the dotted lines correspond to a twin cycle bifurcation

System (1) was investigated by means of asymptotic methods and within this approach it was shown the existence of various non-trivial solutions (periodic, quasiperiodic, soliton-like) [6, 8] in the long wave approximation. In this work, we put up the problem of investigation of some class of exact solutions that might play a role of intermediate asymptotics [9, 10] for the Cauchy (boundary-value) problems, connected with strong pulse load afteraction and to study conditions leading to the wave patterns formation. These solutions satisfy an ODE system obtained from the initial PDE system by group-theoretic reduction. The ODE system is studied both by analytical tools and numerical methods enabling to state the existence of domains in the parametric space corresponding to stochastic autowave solutions.

It is well known that symmetry properties of a given system of PDE can be employed to reduce the number of independent variables [11]. In the case of one spatial variable, this procedure gives rise to an ODE system. By straightforward calculation, one can check that system (1) is invariant under the Galilei algebra $AG(1,1)$ spanned by the following operators:

$$\widehat{P}_0 = \frac{\partial}{\partial t}, \qquad \widehat{P}_1 = \frac{\partial}{\partial x}, \qquad \widehat{G} = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

If $\Im = \gamma\rho^{(n+1)/2}$, then system (1) admits an extra one-parameter group generated by the operator

$$\widehat{\Re} = \frac{n-1}{2}x\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial \rho} + np\frac{\partial}{\partial p}.$$

To the end of this work, we shall analyze the case where $\Gamma_{V\infty}$ is negligible small (and, hence, $n = 1$). Without loss of generality, we may assume that $\tau = \chi = 1$ and the third (constitutive) equation of system (1) contains only two dimensionless parameters.

A passage from system (1) to a subsequent system of ODE will be performed with the help of the following ansatz

$$u = D + U(\omega), \quad \omega = x - Dt, \quad \rho = \exp[\xi t + S(\omega)], \quad p = \rho Z(\omega), \tag{2}$$

built on invariants of the first-order operator

$$\widehat{\mathbf{X}} = \widehat{P}_0 + \widehat{P}_1 + \xi\widehat{\Re}. \tag{3}$$

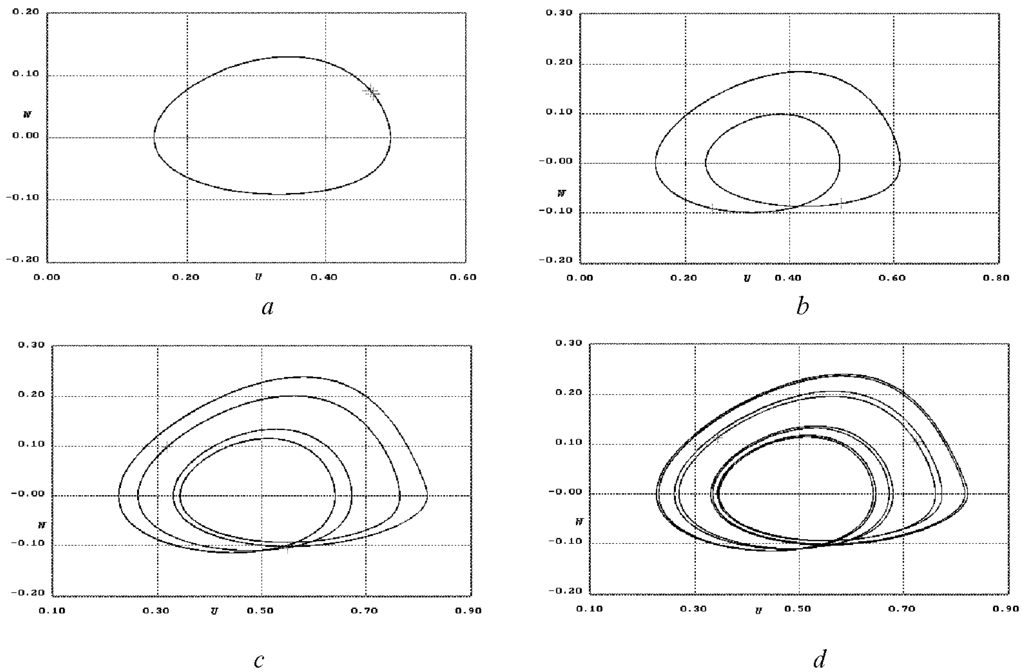


Fig.2. Period doubling bifurcation in system (4) . $\beta=-0.8,\ \xi=-1.25,\ \kappa=0.05,\ \gamma=\kappa\xi/D$. For case *a*: $D=\sqrt{3}$; for case *b*: $D=\sqrt{3.2}$; for case *c*: $D=\sqrt{3.7}$; for case *d*: $D=\sqrt{3.71}$

Inserting (2) into the formula (1), we obtain an ODE system cyclic with respect to the variable S. If one introduces a new variable $W=dU/d\omega\equiv\dot{U}$, then the following dynamic system is obtained:

$$
\begin{aligned}
&U\dot{U}=UW,\\
&U\dot{Z}=\gamma U+\xi Z+W(Z-U^2)\equiv\phi,\\
&U\dot{W}=[\beta(1-U^2)]^{-1}\{M\phi+Z-\kappa+W[1-MZ]\}-W^2,
\end{aligned}
\tag{4}
$$

where $\beta=-h<0$, $M=1-\beta\xi$.

The only critical point of system (4) belonging to the physical parameter range (i.e., lying in the half-space $Z>0$ beyond the manifold $U\beta(1-U^2)=0$) is a point $\mathbf{A}$ having the coordinates

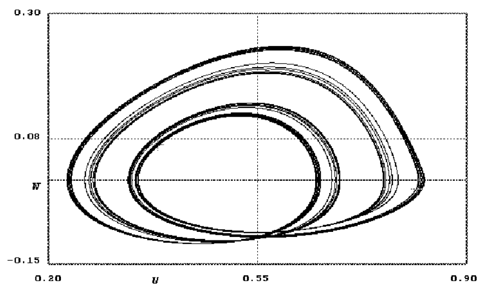
$$U_0=-\kappa\xi/\gamma,\quad Z_0=\kappa,\quad W_0=0.$$

We are going to analyze localized solutions of system (4) in the vicinity of this point.

To begin with, note that the critical point $\mathbf{A}(U_0,\ \kappa,\ 0)$ corresponds, under certain conditions, to an invariant stationary solution of system (1). Indeed, consider a set of time-independent functions belonging to family (2). A simple calculation shows that functions satisfying the above requirements should have the following form

$$u_1=U_0+D,\quad U_0=\text{const},\quad \rho_1=\rho_0\exp(\xi x/D),\quad p_1=Z_0\rho_1.$$

These functions will satisfy system (1) if $\gamma=\kappa\xi/D$ ($U_0=-D$) and $Z_0=\kappa$. Note that the parameter ξ has a clear physical interpretation as defining an inclination of inhomogeneity



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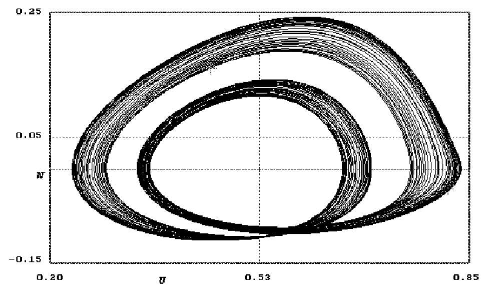

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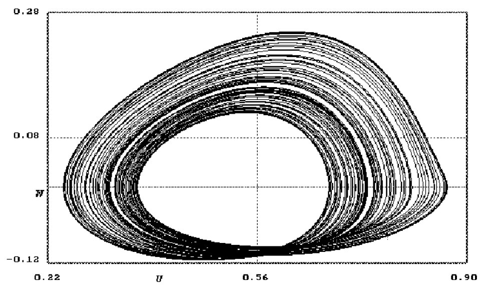

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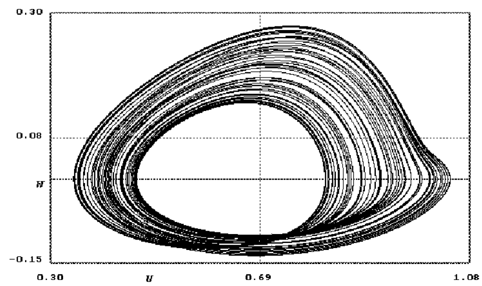

d

Fig.3. Patterns of chaotic trajectories of system (14). $\beta = -0.8$, $\xi = -1.25$, $\kappa = 0.05$, $\gamma = \kappa\xi/D$. For case *a*: $D = \sqrt{3.73}$; for case —it b: $D = \sqrt{3.75}$; for case *c*: $D = \sqrt{3.83}$; for case *d*: $D = \sqrt{4.299}$

of the time-independent invariant solution represented by the critical point $\mathbf{A}(-D,\ \kappa,\ 0)$. In what follows, we assume the validity of the above conditions.

Let us introduce new variables $X = U + D$, $Y = Z - \kappa$ In the coordinates $X,\ Y,\ W$, system (4) may be rewritten as follows:

$$\frac{d}{dT}\begin{pmatrix} X \\ Y \\ W \end{pmatrix} = \begin{pmatrix} 0, & 0, & U_0 \\ \gamma, & \xi, & \Delta \\ L\gamma, & L\xi + G_1, & \sigma \end{pmatrix}\begin{pmatrix} X \\ Y \\ W \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} \tag{5}$$

where $d(\cdot)/dT = Ud(\cdot)/d\omega,\ L = M/K,\ G_1 = K^{-1},\ K = \beta(1 - U_0^2),\ \Delta = \kappa - U_0^2,\ \sigma = (1 - MD^2)/K$,

$$\begin{aligned} &H_1 = WX, \qquad H_2 = W[Y - X(X + 2U_0)], \\ &H_3 = 2U_0X\beta(L\gamma X + L\xi Y + \sigma W)/K - W(2U_0LX + W) - LWX^2(1 + 2U_0\beta/K)+ \\ &\qquad \beta\left(1 + 4U_0^2\beta/K\right)X^2(L\gamma X + L\xi Y + \sigma W)/K + O(|X|^3, |Y|^3, |W|^3). \end{aligned} \tag{6}$$

We are going to analyze the case where the matrix $\widehat{M}$ standing on the RHS of equation (5) has one negative eigenvalue $\lambda_1 = a < 0$ and a pair of pure imaginary eigenvalues $\lambda_{2,3} = \pm i\Omega$. Taking into account that function $U(\omega)$ standing on the LHS of equation (4) is negative in the vicinity of the critical point, we can write down the above conditions as follows:

$$a = -[(\xi\beta + 1) - D^2]/K < 0 \tag{7}$$

$$\Omega^2 = (\xi - \kappa + D^2)/K > 0 \tag{8}$$

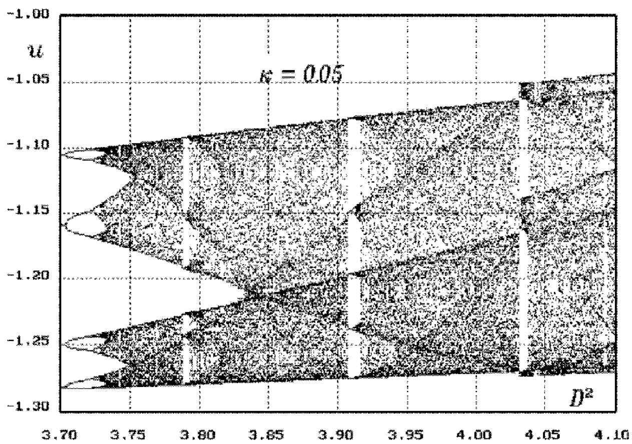

Fig.4. Bifurcation diagram of system (14) in the plane $(D^2,\ U)$. $\beta = -0.8$, $\xi = -1.25$, $\kappa = 0.05$, $\gamma = \kappa\xi/D$.

$$a\Omega^2 = \kappa\xi/K \tag{9}$$

On analyzing relations (7)–(9), one concludes that positiveness of ξ leads to the inequality $\kappa > D^2$. But this relation is unacceptable since it implies instability of the corresponding wave pack in the limiting case where $h = 0$ [12, 13].

So let $\xi < 0$ (and, hence, $K > 0$). We may then rewrite relations (7)–(9) as

$$1 + \xi\beta > D^2 > \kappa - \xi, \tag{10}$$

$$\kappa = \frac{(\xi\beta + 1 - D^2)(\xi + D^2)}{1 + D^2(\xi\beta - 1)}. \tag{11}$$

Equation (11) was solved numerically for $\beta = -0.8$ and $\xi = -1.25$ (see Fig.1, where function $\kappa(D^2)$ is plotted by the solid line). One can easily get convinced that, for the given values of the parameters ξ and β, the segment of curve (11) lying in the positive half-plane $\kappa > 0$ belongs to the open set defined by inequalities (10).

We employed the curve given by equation (11) (a Hopf bifurcation curve [14]) as a starting point for our numerical study. On the opposite sides of this curve stability types of the critical point are different. It is a stable focus beneath the curve and an unstable focus above the curve. The local change of stability can be induced by different global processes: by the unstable limit cycle disappearance (subcritical Hopf bifurcation) or by the stable limit cycle creation (supercritical Hopf bifurcation).

A local stability analysis based on the central manifold theorem and Poincaré normal forms technique [14, 15] shows that the neutral stability curve (11) may be divided into three parts. The domain lying between the values ${D_1}^2 = 1.354076$ and ${D_2}^2 = 1.693168$ corresponds to the subcritical Hopf bifurcation while two other segments correspond to the supercritical one. Thus, the dynamic system's solutions change in a different way with growth of the parameter κ when D^2 is fixed. If $1.354076 < D^2 < 1.693168$, then solutions of the system become globally unstable and tend to infinity after the Hopf bifurcation takes place. In the case where D^2 is beyond this interval, the growth of the parameter κ leads to the self-oscillating solutions appearance. The amplitudes of oscillations grow with parameter's κ growth until one of the twin cycle bifurcation curves (shown on Fig.1 as dotted lines) is attained. Above these curves, system (4) again becomes globally unstable. Numerical experiments show that twin cycle bifurcation curves are attached with one end

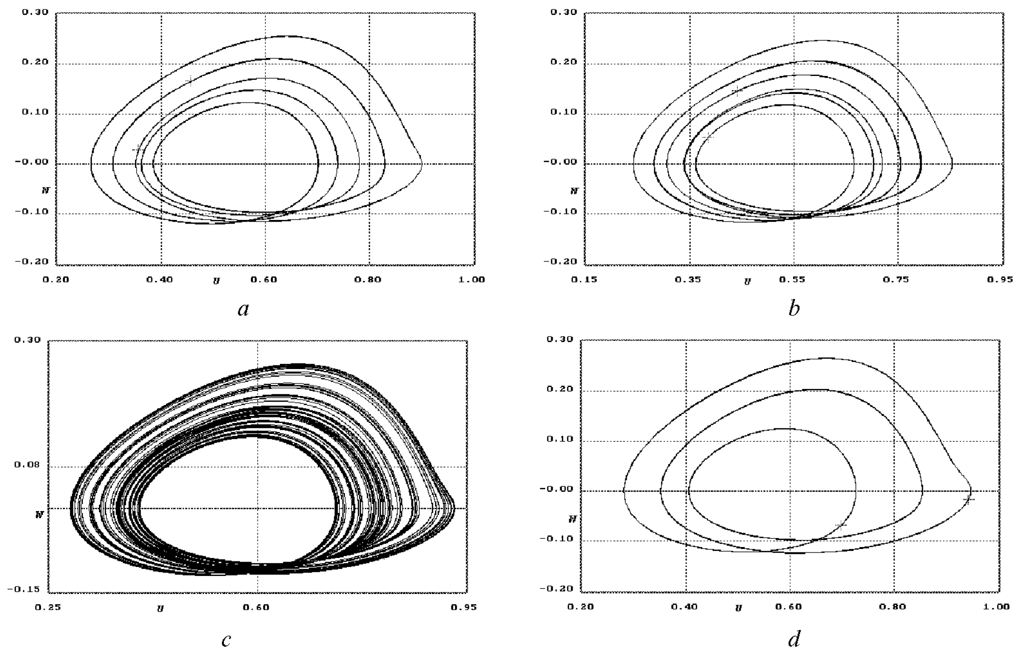

Fig.5. Phase portraits of system (14) obtained *a*: for $D^2 = \sqrt{3.91}$; *b*: for $D^2 = \sqrt{3.789}$; *c*, *d*: for $D^2 = \sqrt{4.005}$

to the Hopf bifurcation curve (11) just at the point where the type of the Hopf bifurcation is changed. In addition, it was shown that system (4) has in these points stable limit cycles surrounded by unstable ones, These unstable attractors cannot be associated with the self-oscillating regimes – in contrast to them, they are not manifested explicitly. On the other hand, the unstable cycles restrict the development of self-oscillations that grow with growth of the parameter κ. The domain of the unstable limit cycle existence in the parameter space (κ, D^2) is restricted by the subcritical branch of the Hopf bifurcation curve from one side and the bifurcation curves of twin cycles (having one of the multipliers equal to unity) from another. On the intersection of the system's parameters of both of these curves, the unstable limiting cycle disappears (note that when the parameters approach the Hopf bifurcation curve, a radius of the periodic trajectory tends to zero, while the cycle disappears with non-zero amplitude on the dotted curves).

The above analysis suggests that another direction of movement in the parametric space should be tried. Below we describe scenario obtained in the case where the parameter D^2 is varied, while the rest of parameters are fixed as follows: $\beta = -0.8$, $\xi = -1.25$, $\kappa = 0.05$.

So, at $D^2 = 1.927178$, a stable limit cycle appears. The amplitude of the limit cycle grows within some interval when D^2 grows (for $D^2 = 3$, it is shown in Fig.2*a*). But finally, the limit cycle loses stability giving rise to another cycle with approximately two times greater period ($2T$-cycle). The $2T$ cycle obtained for $D^2 = 3.2$ is shown in Fig.2*b*. This cycle also loses stability when D^2 approaches a certain critical value at which the $4T$-period cycle arises (Fig.2*c* shows $4T$ cycle obtained for $D^2 = 3.7$). Further growth of the parameter D^2 is accompanied by creation of the series of cycles having periods $8T$ (Fig.2*d* at $D^2 = 3.71$), $16T$, …, 2^nT. The period-doubling cascade of bifurcation is well known in the theory of non-linear oscillation [15–17]. Very often it leads to the appearance stochastic regimes (strange attractors).

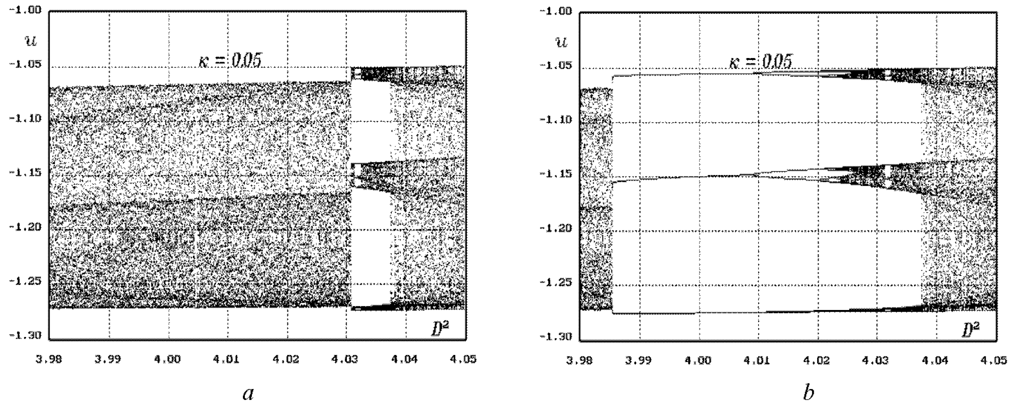

Fig.6. Bifurcation diagrams of system (14) in the vicinity of the period $3T$ window obtained (a) when the parameter D^2 increases and (b) when the parameter D^2 decreases

In fact, system (4) has a strange attractor. Within the interval $3.7288 < D^2 < 3.8328$, it has a band structure. Four bands may be seen in Fig.3a corresponding to $D^2 = 3.73$, while in Fig.3b, corresponding to $D^2 = 3.75$, trajectories form only two bands. On a further growth of the parameter D^2, narrow chaotic bands join each other (this process is known as a "reverse" period-doubling cascade) and finally give rise to a Rössler-type attractor, covering a certain domain of the phase space (it is seen in Fig.3c obtained for $D^2 = 3.833$). An outlook of the strange attractor just before the destruction taking place at $D^2 = 4.3$ is shown in Fig.3d. Specific hump on the right edge of the attractor suggests that its destruction results from the interaction with a homoclinic loop.

The detailed investigation of the zone of chaotic oscillations enables us to see (Fig.4) that chaotic patterns are inhomogeneous, including domains with periodic movements where the system has the cyclic solutions with periods that are different from 2^nT. To investigate a fine structure of the chaotic zone, the bifurcation diagrams technique [17, 18] is employed.

A plane $W = 0$ was chosen as a Poincaré section plane in numerical experiments. Only those points were taken into consideration that correspond to the trajectories moving from the half–space $W > 0$ towards its supplement.

Employment of the Poincaré sections technique enables us to conclude that there exist well-known period $5T$ and $6T$ domains (Fig.5a and 5b obtained for $D^2 = 3.91$ and $D^2 = 3.789$, respectively) predicted by the Sharkovskij theorem [19, 15]. In addition, it was stated another features of the chaotic domain inherent to the system under consideration. The most characteristic property of this attractor is as follows. The period $3T$ domain of system (4) manifests hysteresis features. There exist simultaneously two attractors: a chaotic attractor and a period $3T$ attractor (seen in Fig.5c and 5d, both obtained for $D^2 = 4.005$). Scenario of the patterns development on this interval depends on the direction of movement along the parameter D^2 values. Parameter's D^2 increasing causes in the period $3T$ a supercritical bifurcation (Fig.6a). The attractor created in this way has a band structure. When parameter D^2 decreases, the 3-band chaotic attractor undergoes to the inverse cascade of period doubling bifurcations $3 \cdot 2^n \cdot T$, $3 \cdot 2^{n-1} \cdot T$, …, $6T$, $3T$, finally giving rise to the twin cycle bifurcation resulting in the stable and unstable period $3T$ attractor creations followed by the passage to the advanced chaotic regime (Fig.6b).

Thus, the domain of the coexistence of two attractors is restricted by the twin cycle bifurcation curve from one side and crisis of the strange attractor [20, 21] from another.

For $\kappa = 0.03$ and $\kappa = 0.02$, a similar scenario was observed, this time for significantly greater values of the parameter D^2. It seems that a chaotic attractor does not exist when $\kappa < 0.01$.

Concluding remarks

The goal of this work was to study a certain class of solutions of system (1) describing nonequilibrium processes in multicomponent relaxing media, namely the travelling wave solutions satisfying the ODE system (4). From both qualitative analysis and numerical simulations, it has been observed that a wide variety of regimes can be exhibited by this dynamic system – from multiperiodic to chaotic. The existence of these complicated regimes occurs to be possible due to the complex interaction of non-linear terms with terms describing relaxing properties of the medium.

To realize an influence of relaxing features on the patterns formation, it is useful to compare the results of this work with the investigations undertaken in [22], where processes described by the first-order governing equation were considered and merely periodic solutions were shown to exist. Thus, the existence of stochastic invariant solutions of system (1) is directly linked with the presence of higher-derivative terms in the governing (constitutive) equation.

Of special interest is the fact that system (1) possesses complicated travelling wave solutions provided that an external force of special form is present. Let us note that thus far oscillating invariant solutions in hydrodynamic-type systems have been obtained merely in the presence of mass forces [23, 24] that seem to play the key role in the invariant wave patterns formation.

From the numerical study of dynamic system (4), it is also seen that complex oscillating regimes exist over a wide range of parameter's D values, so the bifurcation phenomena as well as the patterns formation can occur practically at arbitrarily large values of the Mach number. Note, that this conclusion is in agreement with the well-known results obtained by Erpenbeck, Fickett and Wood and several other authors, studying stability conditions for overpressurized detonation waves (for comprehensive survey, see, e.g., [25]).

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