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Hill's Equation and Its Instability Intervals

On the whole real axis $\mathbb{R}$ we consider the Hill's equation

$$-y'' + q(x)y = \lambda y, \qquad x \in \mathbb{R}, \tag{1}$$

where $q(x)$ is 1-periodic, real-valued L^2-function:

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k) e^{ik2\pi x} \in L^2(\mathbb{T}, \mathbb{R}), \quad \mathbb{T} := \mathbb{R}/\mathbb{Z},$$

that is

$$\sum_{k\in\mathbb{N}} |\widehat{q}(k)|^2 < \infty, \quad \text{and} \quad \operatorname{Im} q(x) = 0 \Leftrightarrow \widehat{q}(k) = \overline{\widehat{q}(-k)}, \quad k \in \mathbb{Z}.$$

Denote by $\{\lambda_0(q), \lambda_{2n}^{\pm}(q)\}_{n=1}^{\infty}$ and $\{\lambda_{2n+1}^{\pm}(q)\}_{n=0}^{\infty}$ the eigenvalues in periodic and semi-periodic problems associated with (1) and the x-interval (0,1). It is known, see for example [1], that the $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$ occur in the order

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \le \lambda_1^+(q) < \lambda_2^-(q) \le \lambda_2^+(q) < \cdots .$$

Further, if λ lies in any of the open intervals $(-\infty, \lambda_0)$ and $(\lambda_n^-, \lambda_n^+)$, $n \in \mathbb{N}$, then all non-trivial solutions of (1) are unbounded in $\mathbb{R}$. These intervals are called the *instability intervals* of (1). Apart from $(-\infty, \lambda_0)$, some or all of the instability intervals will be absent in the case of double eigenvalues. If λ lies in any of the complementary open intervals (λ_0, λ_1^-) and $(\lambda_n^+, \lambda_{n+1}^-)$, $n \in \mathbb{N}$, then all solutions of (1) are bounded in $\mathbb{R}$, and these intervals are called the *stability intervals* of (1).

Main goal of the talk is to characterize the behaviour of the lengths of instability intervals

$$\gamma_q(n) := \lambda_n^+(q) - \lambda_n^-(q), \quad n \in \mathbb{N}$$

of the Hill's equation (1) in terms of the behaviour of the Fourier coefficients $\{\widehat{q}(n)\}_{n\in\mathbb{Z}}$ of the potential $q(x)$ with respect to the appropriate weight sequence spaces.

Theorem ([2, 3]). *Let the weight sequence $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$ satisfy conditions:*

$$k^s \ll w(k) \ll k^{1+s}, \quad s \in [0, \infty).$$

Then: $\quad q \in H^{\omega}(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^{\omega}(\mathbb{N}),$ *i.e.,* $\displaystyle\sum_{k\in\mathbb{N}} \omega^2(k) |\widehat{q}(k)|^2 < \infty \Leftrightarrow \sum_{k\in\mathbb{N}} \omega^2(k) \gamma_q^2(k).$

The case $\omega(k) = k^s$, $s \in \mathbb{Z}_+$, is due to Marchenko and Ostrovskii (1975).

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[1] Eastham M. The spectral theory of periodic differential equations. — Edinburg: SAP, 1973.

[2] Mikhailets V., Molyboga V. // Methods Funct. Anal. Topology. —2009. —**15**, N 1.

[3] Mikhailets V., Molyboga V. // arXiv: math.SP/0905.4655.