On the Galilean-invariant equations for particles with arbitrary spin

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In our preceding paper [1] the equations of motion which are invariant under the Galilei group G have been obtained starting from the assumption that the Hamiltonian of a nonrelativistic particle has positive eigenvalues and negative ones. These nonrelativistic equations as well as the relativistic Dirac equation lead to the spin-orbit and to the Darwin interactions by the standard replacement $p_\mu \to \pi_\mu = p_\mu - eA_\mu$. Previously it was generally accepted the hypothesis that the spin-orbit and the Darwin interactions are truly relativistic effects [2].

In [1] only the equations for the particles with the lowest spins $s = \frac{1}{2}, 1, \frac{3}{2}$ have been obtained. What puts the equations [1] in a class by themselves is that the transformation properties of a wave function are rather complicated (nonlocal) and it is difficult to establish their invariance under the Galilei transformations after the replacement $p_\mu \to \pi_\mu$.

In the present note equations for arbitrary-spin particles are obtained which possess as good physical properties as the equations [1].

Moreover the wave function has simple transformation properties in the case of the equation describing the interaction with an external field as well as in the case of the absence of interaction.

We shall start from the assumption that under the Galilei transformation

$$\begin{aligned} &\boldsymbol{x} \to \boldsymbol{x}' = R\boldsymbol{x} + \boldsymbol{V}t + \boldsymbol{a}, \\ &t \to t' = t + b, \end{aligned} \tag{1}$$

the $2(2s+1)$-component wave function $\Psi(t, \boldsymbol{x})$ transforms as

$$\Psi(t, \boldsymbol{x}) \to \Psi'(t', \boldsymbol{x}') = \exp[if(t, \boldsymbol{x})] D^s(R, \boldsymbol{V}) \Psi(t, \boldsymbol{x}), \tag{2}$$

where $D^s(R, \boldsymbol{V})$ is some numerical matrix, depending on the parameters of the transformation (1), $\exp[if(t, \boldsymbol{x})]$ is the phase factor [3]

$$f(t, \boldsymbol{x}) = m\boldsymbol{V} \cdot R\boldsymbol{x} + \frac{1}{2} m v^2 t. \tag{3}$$

The generators of the group G, which corresponds to the transformation (2), have the form

$$\begin{aligned} &P_0 = i\frac{\partial}{\partial t}, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a p_b - x_b p_a + S_{ab}, \\ &G_a = t p_a - m x_a + \lambda_a, \qquad S_{ab} = \begin{pmatrix} s_{ab} & 0 \\ 0 & s_{ab} \end{pmatrix}, \end{aligned} \tag{4}$$

where s_{ab} are the generators of the irreducible representation $D(s)$ of the group O_3, λ_a are some numerical matrices, which have to be such that the operators (4) satisfy

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the commutation relations of the algebra G. It can be shown that the most general (up to equivalence) form of the matrices λ_a satisfying this requirement is

$$\lambda_a = k(\sigma_3 + i\sigma_2)S_a, \qquad S_a = \frac{1}{2}\varepsilon_{abc}S_{bc}, \tag{5}$$

where σ_2, σ_3 are the $2(2s+1)$-dimensional Pauli matrices which commute with S_{ab}, k is an arbitrary constant.

To find the motion equations for arbitrary-spin particles

$$i\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x}) = H_s(\boldsymbol{p},\boldsymbol{s})\Psi(t,\boldsymbol{x}) \tag{6}$$

it is sufficient to construct such an operator (Hamiltonian) $H_s(\boldsymbol{p},\boldsymbol{s})$ that eq. (6) be invariant under the Galilei group G. Equation (6) will be invariant relative to G, if the following conditions are satisfied:

$$[H_s(\boldsymbol{p},\boldsymbol{s}), P_a]_- = 0, \qquad [H_s(\boldsymbol{p},\boldsymbol{s}), J_{ab}]_- = 0, \qquad [H_s(\boldsymbol{p},\boldsymbol{s}), G_a]_- = -iP_a. \tag{7}$$

Thus our problem has been reduced to the solution of the commutation relations (7)[1].

In order to solve relation (7) we expand H_s in a complete system of the orthoprojectors and Pauli matrices

$$H_s(\boldsymbol{p},\boldsymbol{s}) = \sum_{\mu,r}\sigma_\mu a_r^\mu \Lambda_r, \qquad \mu = 0,1,2,3, \tag{8}$$

where

$$\Lambda_r = \prod_{r\neq r'}\frac{\boldsymbol{s}\cdot\boldsymbol{p}/p - r'}{r-r'}, \qquad r,r' = -s,-s+1,\ldots,s,$$

and σ_0 is the $2(2s+1)$-dimensional unit matrix, $a_r^\mu(p)$ are unknown coefficient functions. Substituting (8) into (7), using the relations [4]

$$\begin{gathered}[\Lambda_r, x_a] = \frac{S_{ab}p_b}{2p^2}(2\Lambda_r - \Lambda_{r+1} - \Lambda_{r-1}) + \frac{i}{2p}\left(S_a - \frac{p_a}{p}\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\right)(\Lambda_{r+1} - \Lambda_{r-1}),\\ [\Lambda_r, S_{ab}] = p_a[\Lambda_r, x_b] - p_b[\Lambda_r, x_a],\end{gathered} \tag{9}$$

and taking into account the completeness and the orthogonality of the orthoprojeotors, we have found that, up to equivalence, the general form of the Hamiltonian $H_s(\boldsymbol{p},\boldsymbol{s})$, satisfying (7), is given by the formula

$$H_s = m_0 + \sigma_3\eta m + \frac{p^2}{2m} - \sigma_1 2i\eta h\boldsymbol{S}\cdot\boldsymbol{p} - (\sigma_3 + i\sigma_2)\eta k^2\frac{(\boldsymbol{S}\cdot\boldsymbol{p})^2}{m}, \tag{10}$$

where η is an arbitrary constant.

Formula (10) gives the free nonrelativistic Hamiltonian for a particle with an arbitrary spin. Equation (6) with the Hamiltonian (10) is invariant under the group G. For the spin $\frac{1}{2}$ particle (when $s=\frac{1}{2}$, $k=-i$, $\eta=1$) equation (6) may be written in the compact form

$$(\gamma_\mu p^\mu + m)\Psi = (1 + \gamma_4 - \gamma_0)\frac{p^2}{2m}\Psi, \tag{11}$$

where γ_μ are the Dirac matrices.

[1]The analogous problem has been eolved in the relativistic case in [4]. Lately the method of the work [4] has been further developed in works of R.F. Guertin [5].

The Hamiltonian (10) and the generators (4) are non-Hermitian under the usual scalar product. They are, however, Herinitian under

$$(\Psi_1, \Psi_2) = \int d^3x \; \Psi_1^\dagger M \Psi_2, \tag{12}$$

where M is the positive-definite metric operator

$$M = 1 + [i(k - k^*)\sigma_3 - (k + k^*)\sigma_2]\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{m} + 2|k|^2(1+\sigma_1)\left(\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{m}\right)^2. \tag{13}$$

Besides, if η, k satisfy the condition $\eta k = (\eta k)^*$, the Hamiltonians are Hermitian also in the indefinite metric

$$(\Psi_1, \Psi_2) = \int d^3x \; \Psi_1^\dagger \xi \Psi_2, \tag{14}$$

where

$$\xi = \begin{cases} \sigma_3, & \text{if} \quad \eta^* = \eta, \;\; k^* = k, \\ \sigma_2, & \text{if} \quad \eta^* = -\eta, \;\; k^* = -k. \end{cases}$$

With the help of the transformation

$$H_s \to H_s' = V H_s V^{-1}, \qquad V = \exp\left[i\frac{\boldsymbol{\lambda}\cdot\boldsymbol{p}}{m}\right], \tag{15}$$

the Hamiltonian (10) may be reduced to the diagonal form

$$H_s' = m_0 + \sigma_3 \eta m + \frac{p^2}{2m}. \tag{16}$$

It is interesting to note that the condition of Galilei invariance admits the possibility to introduce two masses: the rest mass, or the rest energy ($\varepsilon_1 = m_0 + \eta m$, $\varepsilon_2 = m_0 - \eta m$) and the kinetic mass (the coefficient of p^2). Below we consider the case when $m_0 = 0$, $\eta = 1$, i.e. the rest mass is equal to the kinetic mass.

To describe the motion of the charged particle in external electromagnetic fields we make in (6) and (10) the replacement $p_\mu \to \pi_\mu$ (symmetrizing preliminarily the Hamiltonian in p_a [1]). This leads to the equation

$$i\frac{\partial}{\partial t}\Psi(t, \boldsymbol{x}) = H_s(\boldsymbol{\pi})\Psi(t, \boldsymbol{x}), \tag{17}$$

$$H_s(\boldsymbol{\pi}) = \sigma_3 m + \frac{\boldsymbol{\pi}^2}{2m} + \sigma_1 2ik(\boldsymbol{S}\cdot\boldsymbol{p}) + \frac{2k^2}{m}(\sigma_3 + i\sigma_2)\left[(\boldsymbol{S}\cdot\boldsymbol{\pi})^2 + \frac{1}{2}(\boldsymbol{S}\cdot\boldsymbol{M})\right], \tag{18}$$

where $H_a = i\varepsilon_{abc}[\pi_b, \pi_c]_-$ are components of the magnitude of the magnetic field.

It is important to note that eq. (17) as before is invariant with respect to the Galilei transformations (1) and (2), if the vector potential is transformed according to [2]

$$\boldsymbol{A} \to \boldsymbol{A}' = R\boldsymbol{A}, \qquad A_0 \to A_0' = A_0 + \boldsymbol{V}R\boldsymbol{A}. \tag{19}$$

To prove this statement it is sufficient to use the exact form of the matrix $D^s(R,\boldsymbol{V})$ in (2)

$$D^s(R,\boldsymbol{V}) = (1+i\boldsymbol{\lambda}\cdot\boldsymbol{V})\cdot\begin{pmatrix} D^s(R) & 0 \\ 0 & D^s(R)\end{pmatrix}, \tag{20}$$

where $D^s(R)$ the matrices from the representation $D(s)$ of the group O_3.

As in the case of the Dirac equation [6] the Hamiltonian (18) cannot be diagonalized exactly. We shall make the approximate diagonalization of the operator (18) up to the terms of the power $1/m^2$ with the help of the operator

$$V(\boldsymbol{\pi}) = \exp[iB_3^s]\exp[iB_2^s]\exp[iB_1^s], \tag{21}$$

where

$$\begin{gathered}
B_1^s = i\sigma_2 k\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{m}, \qquad E_a = -\frac{\partial A_a}{\partial x_a} - \frac{\partial A_a}{\partial t}, \\
B_2^s = -\sigma_1 k\frac{[\boldsymbol{S}\cdot\boldsymbol{\pi},\boldsymbol{\pi}^2]_-}{4m^2} - i\sigma_1 k^2\frac{(\boldsymbol{S}\cdot\boldsymbol{\pi})^2 - \frac{1}{2}\boldsymbol{S}\cdot\boldsymbol{H}}{m^2} - i\sigma_1 k\frac{\boldsymbol{S}\cdot\boldsymbol{E}}{2m^2}, \\
B_3^s = -\frac{2}{3}ik^3\sigma_2\left(\frac{\boldsymbol{S}\cdot\boldsymbol{\pi}}{m}\right)^3 + ik^3\frac{[\boldsymbol{S}\cdot\boldsymbol{\pi},\boldsymbol{S}\cdot\boldsymbol{H}]_+}{m^3}\sigma_2 + \sigma_2\frac{k^2[(\boldsymbol{S}\cdot\boldsymbol{\pi})^2, eA_0]}{m^3}.
\end{gathered} \tag{22}$$

After this diagonalization one obtains

$$\begin{gathered}
V(\boldsymbol{\pi})H^s(\boldsymbol{\pi})V^{-1}(\boldsymbol{\pi}) = \sigma_3 m + \frac{\boldsymbol{\pi}^2}{2m} + eA_0 + k^2\sigma_3\frac{\boldsymbol{S}\cdot\boldsymbol{H}}{m} - \\
-\frac{k^2}{4m^2}\boldsymbol{S}\cdot(\boldsymbol{\pi}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{\pi}) + \frac{k^2}{6m^2}s(s+1)\,\mathrm{div}\,\boldsymbol{E} + \frac{k^2}{12m^2}Q_{ab}\frac{\partial E_b}{\partial x_a} + \\
+\frac{k^3}{m^2}\boldsymbol{S}\cdot(\boldsymbol{\pi}\times\boldsymbol{H} - \boldsymbol{H}\times\boldsymbol{\pi}) - \frac{1}{3}\frac{k^3}{m^2}Q_{ab}\frac{\partial H_a}{\partial x_b} + o\left(\frac{1}{m^3}\right),
\end{gathered} \tag{23}$$

where Q_{ab} is the tensor of the quadrupole interaction

$$Q_{ab} = 3[S_a,S_b]_+ - 2\delta_{ab}s(s+1). \tag{24}$$

It is readily seen from (23) that $-k^2$ can be interpreted as the dipole magnetic moment of the particle. If $s=\frac{1}{2}$, $-k^2=1$ (it corresponds to the "normal" dipole moment), the first seven constituents of the approximate Hamiltonian coincide on the set $\Phi^+ = \frac{1}{2}(1+\sigma_3)\Phi$ with the Foldy–Wouthuysen Hamiltonian, which had been obtained from the relativistic Dirac equation. The last two terms in (23) may be interpreted as the magnetic spin-orbit and the magnetic quadrupole interactions of the particle with the field.

In conclusion we note that we have not required the invariance with respect to the time reflection for eq. (6). This invariance has been ensured if one doubles (brings to $4(2s+1)$) the number of the components of the wave function and assumes that the particle energy can take both positive and negative values. An analogous situation takes place in the relativistic theory [7].

As in the relativistic theory, it is possible to construct for the particle with spin s the nonrelativistic wave equations with another (different from $2(2s+1)$ or $4(2s+1)$)

number of components. For instance, the spin-one and spin-zero particles may bo described by the Galilean-invariant equations

$$(\beta_\mu p^\mu - m)\Psi = \left[\beta_0 \frac{p^2}{2m} + \beta_0^2 \frac{(\boldsymbol{\beta}\cdot\boldsymbol{p})^2}{m}\right]\Psi, \tag{25}$$

where β_μ are the 10×10- or 5×5-dimensional Kemmer–Duffin–Peteu matrices. These equations will be considered in another work.

Note. The equations obtained in [1] and in the present paper may be considered as those with the broken Lorentz symmetry. Actually, equations (12) from [1] and (11) from the present work have the form of the Dirac equation with the additional term which is noninvariant under the Poincaré group, but is Galilean invariant. The second-order equations with this broken symmetry have the form

$$(p_\mu p^\mu - m^2)\Psi = B\Psi, \tag{26}$$

where $B = p^4/4m^2$ for the equations of ref. [1] and $B = m(1+2\sigma_3) + p^2\sigma_3 + p^4/4m^2$ for the equations from the present paper (if $m_0 = m$, $\eta = 1$).

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