On some new exact solutions of nonlinear d'Allembert and Hamilton equations

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Some new exact solutions of d'Allembert–Hamilton and d'Allembert equations are obtained. The necessary conditions of integrability of over-determined d'Allembert–Hamilton system of nonlinear differential equations are established.

1. It was the Euler's idea (1734–1740 y.) that problem of integrating partial differential equations (PDE) could be solved by reducing them to ordinary equations (ODE). But one can not apply this idea to arbitrary PDE. Therefore it was suggested by Fushchych [4, 5] to restrict oneself by PDE possessing wide symmetry groups. This program was realized for some nonlinear wave equations by Fushchych and Serov [7], Fushchych and Shtelen [8] and Fushchych and Zhdanov [9] (see also [1, 11, 12]). The vast list of references on this point can be found in Fushchych and Nikitin [6].

When reducing PDE to ODE one has always to deal with the problem of investigating compatibility of some systems of PDE. For example, nonlinear d'Allembert equation

$$\Box u = F_1(u), \quad \Box = \partial^2_{x_0} - \partial^2_{x_1} - \partial^2_{x_2} - \partial^2_{x_3} \tag{1}$$

with the aid of ansatz [4]

$$u = \varphi(\omega), \quad \omega = \omega(x_0, x_1, x_2, x_3), \tag{2}$$

is reduced to the ODE having variable coefficients [7]

$$\omega_\mu \omega^\mu \ddot{\varphi} + \Box\omega\dot{\varphi} = F_1(\varphi), \tag{3}$$

where $\omega_\mu \equiv \frac{\partial\omega}{\partial x_\mu}$, $\mu = \overline{0,3}$, $\dot{\varphi} \equiv \frac{d\varphi}{d\omega}$. Hereafter the summation over repeated indices in Minkowsky space having the metric $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$ is supposed, i.e. $\omega_\mu\omega^\mu \equiv g^{\mu\nu}\omega_\mu\omega_\nu = \omega_0^2 - \omega_1^2 - \omega_2^2 - \omega_3^2$.

We demand new variable ω to satisfy d'Allembert and Hamilton equations simultaneously

$$\Box\omega = F_2(\omega), \tag{4}$$

$$\omega_\mu\omega^\mu = F_3(\omega). \tag{5}$$

As a result equation (3) takes the form

$$F_3(\omega)\ddot{\varphi} + F_2(\omega)\dot{\varphi} = F_1(\varphi). \tag{6}$$

Winternitz and collaborators (see [1, 11]) construct new variables ω by using subgroup structure of the Poincaré group $P(1,3)$. One can be easily convinced that invariants obtained in this way satisfy system (4), (5).

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So to obtain set of variables ω making possible to reduce multi-dimensional PDE (1) to ODE one has to consider the problem of compatibility of system (4), (5) and then to integrate it.

In the present paper compatibility of equations (4), (5) is investigated, i.e. all smooth functions ensuring the compatibility of d'Allembert–Hamilton system are described.

The direct application of Cartan's method of investigation of compatibility of overdetermined PDE [2] is rather difficult. To avoid arising difficulties we essentially use symmetry properties of system (4), (5) [7, 8].

System (4), (5) via the change of dependent variable $z = z(\omega)$ can be reduced to the following system

$$\Box\omega = F(\omega), \tag{7}$$

$$\omega_\mu\omega^\mu = \lambda, \quad \lambda = \text{const}, \tag{8}$$

ODE (6) taking the form

$$\lambda\ddot{\varphi} + F(\omega)\dot{\varphi} = F_1(\varphi). \tag{9}$$

Before formulating the principal result of the paper we adduce without proof some auxiliary statements.

Lemma 1. *Solutions of system (7), (8) satisfy the identities*

$$\begin{aligned} &\omega_{\mu\nu_1}\omega^{\mu\nu_1} = -\lambda\dot{F}(\omega),\\ &\omega_{\mu\nu_1}\omega^{\nu_1\nu_2}\omega^\mu_{\nu_2} = \frac{1}{2!}\lambda^2\dot{F}(\omega),\\ &\omega_{\mu\nu_1}\omega^{\nu_1\nu_2}\ddot{\omega}^{\nu_n\mu} = \frac{1}{n!}(-\lambda)^n\frac{d^nF(\omega)}{d\omega^n}, \quad n\geq 0, \end{aligned} \tag{10}$$

where $\omega_{\alpha\beta} \equiv \frac{\partial^2\omega}{\partial x_\alpha\partial x_\beta}$, $\alpha,\beta = \overline{0,3}$.

Lemma 2. *Solutions of the system (7), (8) satisfy the following equality:*

$$\det(\omega_{\mu\nu}) = 0. \tag{10$'$}$$

Let us now formulate the principal statement.

Theorem 1. *The necessary condition of compatibility of overdetermined system (7), (8) is as follows*

$$F(\omega) = \begin{cases} 0,\\ \lambda(\omega+C_1)^{-1},\\ 2\lambda(\omega+C_1)[(\omega+C_1)^2+C_2]^{-1},\\ 3\lambda((\omega+C_1)^2+C_2)[(\omega+C_1)^3+3C_2(\omega+C_1)+C_3]^{-1}, \end{cases} \tag{11}$$

where C_1, C_2, C_3 *are arbitrary constants.*

Proof. By direct (and rather tiresome) verification one can be convinced that the following identity holds

$$\begin{aligned} &6(\omega_{\mu\nu_1}\omega^{\nu_1\nu_2}\omega_{\nu_2\nu_3}\omega^{\nu_3\mu}) - 8(\omega^\mu_\mu)(\omega_{\mu\nu_1}\omega^{\nu_1\nu_2}\omega^\mu_{\nu_2}) -\\ &\quad - 3(\omega_{\mu\nu_1}\omega^{\mu\nu_1})^2 + 6(\omega^\mu_\mu)^2(\omega_{\mu\nu_1}\omega^{\mu\nu_1}) - (\omega_\mu\omega^\mu)^4 = 24\det(\omega_{\mu\nu}). \end{aligned} \tag{12}$$

Substituting (10), (10′) into (12) one obtains nonlinear ODE for $F(\omega)$

$$\lambda^3\dddot{F} + 4\lambda^2 F\ddot{F} + 3\lambda^2\dot{F}^2 + 6\lambda\dot{F}F^2 + F^4 = 0, \tag{13}$$

where $\dot{F} \equiv \frac{dF}{d\omega}$.

General solution of equation (13) is given by formulae (11). Theorem is proved.

Note 1. Compatibility of three-dimensional d'Allembert–Hamilton system has been investigated in detail by Collins [3]. Collins essentially used geometrical methods which could not be generalized to higher dimensions.

Using Lie's method (see e.g. [10]) one can prove the following statement.

Theorem 2. *The sytem of PDE (7), (8) is invariant under the 15-parameter conformal group $C(1,3)$ iff*

$$F(\omega) = 3\lambda(\omega + C)^{-1}, \quad \lambda > 0, \quad C = \text{const}. \tag{14}$$

Note 2. Formula (14) can be obtained from (11) by putting $C_2 = C_3 = 0$. So Theorem 2 demonstrates close connection between compatibility of a system of PDE and its symmetry.

Note 3. It is common knowledge that PDE (7) is invariant under the group $C(1,3)$ iff $F(\omega) = \lambda\omega^3$ [7]. Consequently, an additional constraint (8) changes essentially symmetry properties of d'Allembert equation (choosing $F_3(\omega)$ in a proper way one can obtain conformally-invariant system of the form (4), (5) under arbitrary $F_2(\omega)$).

2. Let us list explicit form of some exact solutions of d'Allembert–Hamilton system and reduced ODE for function $\varphi(\omega)$.

№	λ	$F(\omega)$	$\omega = \omega(x)$	ODE for $\varphi(\omega)$
1.	1	0	$a_\mu x^\mu$	$\ddot{\varphi} = F_1(\varphi)$
2.	1	ω^{-1}	$\left[(a_\mu x^\mu)^2 - (b_\mu x^\mu)^2\right]^{1/2}$	$\ddot{\varphi} + \omega^{-1}\dot{\varphi} = F_1(\varphi)$
3.	1	$2\omega^{-1}$	$\left[(a_\mu x^\mu)^2 - (b_\mu x^\mu)^2 - (c_\mu x^\mu)^2\right]^{1/2}$	$\ddot{\varphi} + 2\omega^{-1}\dot{\varphi} = F_1(\varphi)$
4.	1	$3\omega^{-1}$	$(x_\mu x^\mu)^{1/2}$	$\ddot{\varphi} + 3\omega^{-1}\dot{\varphi} = F_1(\varphi)$
5.	-1	0	$(b_\mu x^\mu)\cos h_1 + (c_\mu x^\mu)\sin h_1 + g_1$ $a_\mu x^\mu - (b_\mu x^\mu)\cos h_2 -$ $- (c_\mu x^\mu)\sin h_2 - g_2 = 0$	$\ddot{\varphi} = -F_1(\varphi)$
6.	-1	$-\omega^{-1}$	$\left[(b_\mu x^\mu + h_1)^2 + (c_\mu x^\mu + h_2)^2\right]^{1/2}$	$\ddot{\varphi} + \omega^{-1}\dot{\varphi} = -F_1(\varphi)$
7.	-1	$-2\omega^{-1}$	$\left[(b_\mu x^\mu)^2 + (c_\mu x^\mu)^2 + (d_\mu x^\mu)^2\right]^{1/2}$	$\ddot{\varphi} + 2\omega^{-1}\dot{\varphi} = -F_1(\varphi)$
8.	0	0	h_1	$0 = F_1(\varphi)$

Here h_1, g_1 are arbitrary smooth functions on $a_\mu x^\mu + d_\mu x^\mu$, h_2, g_2 — on $\omega + d_\mu x^\mu$; a_μ, b_μ, c_μ, d_μ are arbitrary real parameters satisfying conditions of the form

$$-a_\mu a^\mu = b_\mu b^\mu = c_\mu c^\mu = d_\mu d^\mu = -1,$$
$$a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.$$

3. Natural generalization of the formula (2) is given by ansatz of the form [4]

$$u(x) = f(x)\varphi(\omega). \tag{15}$$

Some multi-parameter families of exact solutions of nonlinear d'Allembert equation with nonlinearity $F_1 = \tau u^k$, $\tau, k = \mathrm{const}$, were constructed with the help of ansatz (15) by Fushchych and Serov [7].

Omitting intermediate calculations we write down new family of solutions of equation (1) under $F_1 = \tau u^k$ obtained via ansatz (15)

$$\begin{aligned} u(x) = R^{-1} \left[C_6 + \frac{1}{2}\tau(1-k)^2 \int R^{1-k}(\omega) d\omega \right]^{1/(1-k)} \times \\ \times \left\{ \frac{1}{2}(a_\mu x^\mu - d_\mu x^\mu) - \frac{1}{2}\dot{R}R^{-1}\left[(b_\mu x^\mu)^2 + (c_\mu x^\mu)^2 \right] + \right. \\ \left. + f_0 \left[(b_\mu x^\mu)^2 - (c_\mu x^\mu)^2 \right] + f_1 b_\mu x^\mu + f_2 \right\}^{1/(1-k)}, \end{aligned}$$

where

$$\begin{aligned} f_0(\omega) &= \frac{1}{2} C_1 R^{-2}(\omega), \\ f_1(\omega) &= \frac{1}{2} C_4 R(\omega) \exp\left\{ 4C_1 \int R^{-2}(\omega) d\omega \right\}, \\ f_2(\omega) &= C_4 \int R(\omega) \exp\left\{ 4C_1 \int R^{-2}(\omega) d\omega \right\} d\omega + C_5, \end{aligned}$$

and $\omega = a_\mu x^\mu + d_\mu x^\mu$, $C_1, \ldots, C_6 = \mathrm{const}$.

Function $R = R(\omega)$ is determined by formulae

$$R(\omega) = \begin{cases} \dfrac{\varepsilon}{2}\left[(C_2\omega + C_3)^2 - 16C_1^2 \right]^{1/2}, \\ (8\varepsilon C_1 \omega + C_2)^{1/2}, \quad \varepsilon = \pm 1. \end{cases}$$

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