Symmetry reduction and exact solutions of nonlinear biwave equations

W.I. FUSHCHYCH, O.V. ROMAN, R.Z. ZHDANOV

Symmetry analysis of a class of the biwave equations $\Box^2 u = F(u)$ and of a system of wave equations which is equivalent to it is performed. Reduction of the nonlinear biwave equations by means of the Ansätze invariant under non-conjugated subalgebras of the extended Poincaré algebra $A\widetilde{P}(1,1)$ and the conformal algebra $AC(1,1)$ is carried out. Some exact solutions of these equations are obtained.

1 Introduction

It was customary for the classical mathematical physics to use as the mathematical models for describing real physical processes linear partial differential equations (PDE) of the order not higher than two. All fundamental equations of mathematical physics such as the Laplace, heat, Klein–Gordon–Fock, Maxwell, Dirac, Schrödinger equations are the first- or the second-order linear partial differential equations. But now there are strong evidences that linear description is not satisfactory (especially it is the case in the quantum field theory [1]). That is why, it was attempted to generalize the classical equations in a non-linear way in order to get more satisfactory models. There exist different principles of the choice of such generalizations but up to our mind the most natural and systematic is the *symmetry selection principle*. A classical illustration is a group classification of nonlinear wave equations

$$\Box u = F(u). \tag{1}$$

Here and further $\Box = \partial^2/\partial x_0 - \partial^2/\partial x_1 - \cdots - \partial^2/\partial x_n$ is the d'Alembertian in the $(n+1)$-dimensional pseudo-Euclidean space $\mathbb{R}(1,n)$ with the metric tensor $g_{\mu\nu} = \operatorname{diag}(1,-1,\dots,-1)$, $\mu,\nu = \overline{0,n}$; $x_\mu = x^\nu g_{\mu\nu}$; $F(u)$ is an arbitrary smooth function; $u = u(x)$ is a real function; the summation over the repeated indices from 0 to n is understood.

With an arbitrary $F(u)$ equation (1) is invariant under the $\frac{(n+1)(n+2)}{2}$-parameter Poincaré group $P(1,n)$ having the following generators:

$$P_\mu = \frac{\partial}{\partial x_\mu}, \qquad J_{\mu\nu} = x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu}, \qquad \mu,\nu = \overline{0,n}. \tag{2}$$

But equation (1) taken with an arbitrary nonlinearity $F(u)$ is too "general" to be a reasonable mathematical model for describing a specific physical phenomena. To specify a form of $F(u)$ symmetry properties of the *linear* wave equation are utilized. It is well-known that PDE (1) with $F(u) = 0$ in addition to the Poincaré group admits a one-parameter scale transformation group and a $(n+1)$-parameter group of special conformal transformations (see, e.g. [2]). Therefore, it is not but natural to

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postulate that those nonlinearities are admissible which preserve a symmetry of the linear equation. It has been proved in [3] that there are only two functions $F(u)$, namely

$$F(u) = \lambda(u + C)^k, \quad F(u) = \lambda \exp Cu, \tag{3}$$

where C, $k \neq 0$ are arbitrary constants, such that Poincaré-invariant equation (1) admits a one-parameter scale transformation group. Furthermore, it was known long ago that the only equation of the form (1) admitting the conformal group $C(1, n)$ is the one with $F(u) = \lambda(u + C)^{\frac{n+3}{n-1}}$. Consequently, choosing from the whole set of PDE (1) equations having the highest symmetry we get the ones with very specific nonlinearities.

A procedure described above is called group or symmetry classification of PDE (1). A method used is the classical infinitesimal Lie's method. Given a representation of a Lie transformation group (which is fixed by a requirement that this group should be admitted by the linear wave equation), the problem of symmetry classification of equations (1) is reduced to solving some linear over-determined system of PDE. This system is called determining equations (for more detail, see [2, 5]).

But what is most important, the Lie's method can be applied not only to classify invariant equations but also to construct their explicit solutions by means of *symmetry reduction procedure*. And one more important remark is that equations having broad Lie symmetry often admit non-trivial *conditional symmetry*, which can be also used to obtain their particular solutions [2].

In [6] the description has been suggested of different physical processes with the help of nonlinear partial differential equations of high order, namely

$$\Box^l u = F\Big(u, \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}\Big). \tag{4}$$

where $\Box^l = \Box(\Box^{l-1})$, $l \in \mathrm{N}$; $F(\cdot\,,\cdot)$ is an arbitrary smooth function.

The equations (4) were considered from different points of view in [2, 7, 8], where the pseudo-differential equations of type (4) were also studied (in this case l is fractional or negative).

Assuming $l = 1$ and $F = F(u)$ in (4) we obtain the standard nonlinear wave equation (1), which describes a scalar spin-less uncharged particle in the quantum field theory. Symmetry properties of the equation (1) were studied in [2, 3, 4] and wide classes of its exact solutions with certain concrete values of the function $F(u)$ were obtained in [2, 3, 9, 10, 11].

In this paper we restrict ourselves to symmetry analysis of the biwave equation

$$\Box^2 u = F(u), \tag{5}$$

which is one of the simplest equations of type (4) of the order higher than two ($l = 2$, $F = F(u)$).

2 Symmetry classification of biwave equations

In order to carry out a symmetry classification of the equation (5) we shall establish at first the maximal transformation group admitted by the equation (5), provided $F(u)$

is an arbitrary function. Next, we shall determine all the functions $F(u)$ such that the equation (5) admits a more extended symmetry.

Results of symmetry classification of the equation (5) are presented below.

Lemma 1 *The maximal invariance group of the equation (5) with an arbitrary function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators (2).*

Theorem 1 *Any equation of type (5) admitting a more extended invariance algebra than the Poincaré algebra $AP(1,n)$ is equivalent to one of the following PDE:*

$$1.\ \Box^2 u = \lambda_1 u^k, \qquad \lambda_1 \neq 0,\ k \neq 0, 1; \tag{6}$$

$$2.\ \Box^2 u = \lambda_2 e^u, \qquad \lambda_2 \neq 0; \tag{7}$$

$$3.\ \Box^2 u = \lambda_3 u, \qquad \lambda_3 \neq 0; \tag{8}$$

$$4.\ \Box^2 u = 0. \tag{9}$$

Here λ_1, λ_2, λ_3, k are arbitrary constants.

Theorem 2 *The symmetry of the equations (6)–(9) is described as follows:*

1. (a) The maximal invariance group of the equation (6) when $k \neq (n+5)/(n-3)$, $k \neq 0, 1$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (2) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + \frac{4}{1-k} u \frac{\partial}{\partial u}. \tag{10}$$

(b) The maximal invariance group of the equation (6) when $k = (n+5)/(n-3)$, $n \neq 3$ is the conformal group $C(1,n)$ generated by the operators (2) and operators

$$\begin{aligned} D^{(1)} &= x_\mu \frac{\partial}{\partial x_\mu} + \frac{3-n}{2} u \frac{\partial}{\partial u}, \\ K^{(1)}_\mu &= 2x^\mu D^{(1)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}. \end{aligned} \tag{11}$$

2. (a) The maximal invariance group of the equation (7) when $n \neq 3$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (2) and

$$D^{(2)} = x_\mu \frac{\partial}{\partial x_\mu} - 4 \frac{\partial}{\partial u}. \tag{12}$$

(b) The maximal invariance group of the equation (7) when $n = 3$ is the conformal group $C(1,n)$ generated by the operators (2) and operators

$$K^{(2)}_\mu = 2x^\mu D^{(2)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}. \tag{13}$$

3. The maximal invariance group of the equation (8) is generated by the operators (2) and

$$Q = h(x) \frac{\partial}{\partial u}, \quad I = u \frac{\partial}{\partial u},$$

where $h(x)$ is an arbitrary solution of the equation (8).

4. The maximal invariance group of the equation (9) is generated by the operators (2), (11) and

$$Q = q(x)\frac{\partial}{\partial u}, \qquad I = u\frac{\partial}{\partial u},$$

where $q(x)$ is an arbitrary solution of the equation (9).

The proof of the Lemma 1 and the Theorems 1, 2 is carried out by means of the infinitesimal algorithm of S. Lie [2, 5]. Since it requires very cumbersome computations, we adduce a general scheme of the proof only.

Within the framework of the Lie's approach an infinitesimal operator of the equation (5) invariance group is looked for in the form

$$X = \xi^\mu(x,u)\frac{\partial}{\partial x_\mu} + \eta(x,u)\frac{\partial}{\partial u}. \tag{14}$$

The criterion of invariance of the equation (5) with respect to a group generated by the operator (14) reads

$$\underset{4}{X}(\Box^2 u - F(u))\Big|_{\Box^2 u = F(u)} = 0, \tag{15}$$

where $\underset{4}{X}$ is the 4-th prolongation of the operator X.

Splitting the equation (15) with respect to the independent variables, we come to the system of partial differential equations for functions $\xi^\mu(x,u)$ and $\eta(x,u)$:

$$\begin{aligned}
&\xi^\mu_u = 0, \quad \eta_{uu} = 0, \quad \mu = \overline{0,n},\\
&\xi^i_0 = \xi^0_i, \quad \xi^i_j = -\xi^j_i, \quad i \neq j, \ i,j = \overline{1,n},\\
&\xi^0_0 = \xi^1_1 = \cdots = \xi^n_n,\\
&2\eta_{\nu u} = (3-n)\xi^\nu_{00}, \quad \nu = \overline{0,n},
\end{aligned} \tag{16}$$

$$\Box^2\eta - \eta F'(u) + F(u)(\eta_u - 4\xi^0_0) = 0. \tag{17}$$

Besides, when $n = 1$, there are additional equations

$$\eta_{00u} = 0, \quad \eta_{01u} = 0, \tag{18}$$

that do not follow from the equations (16) and (17).

In the above formulae we use the notations $\xi^\mu_\nu = \partial\xi^\mu/\partial x_\nu$, $\eta_\mu = \partial\eta/\partial x_\mu$ and so on.

System (16) is one of the Killing equations in the Minkowski space-time. Its general solution is well-known and can be represented in the following form:

$$\begin{aligned}
&\xi^\nu = 2x^\nu x_\mu c^\mu - x_\mu x^\mu c^\nu + b_{\nu\mu}x^\mu + dx_\nu + a_\nu,\\
&\eta = ((3-n)c^\mu x_\mu + p)u + \varkappa(x),
\end{aligned} \tag{19}$$

where c_μ, $b_{\nu\mu} = -b_{\mu\nu}$, d, a_ν, p are arbitrary constants, $\varkappa(x)$ is an arbitrary smooth function.

Substituting the expression (19) into the classifying equation (17) and splitting it with respect to u we arrive at the statements of the Lemma 1 and the Theorems 1, 2 according to the form of $F(u)$.

It follows from the assertions proved that the equation of type (4) is invariant under the extended Poincaré group $\widetilde{P}(1,n)$ if and only if it is equivalent to one of the equations (6), (7) or (9). Let us note that an analogous result was obtained for the wave equation (1) in [3].

The following statement is also a consequence of the Theorems 1, 2 but because of its importance we adduce it as a theorem.

Theorem 3 *Equation (5) admits the conformal group $C(1,n)$ if and only if it is equivalent to the following:*

$$1.\ \Box^2 u = \lambda_1 u^{(n+5)/(n-3)}, \quad n \neq 3; \tag{20}$$

$$2.\Box^2 u = \lambda_2 e^u, \quad n = 3. \tag{21}$$

Let us note that conformal invariance of the equation (20) has been first ascertained in [12] and that of equation (21) – in [2] by means of the Baker–Campbell–Hausdorff formula. It is also worth noting that conformal invariance of the nonlinear polyharmonic equations has been studied in [13], which enables constructing some their exact solutions.

In conclusion of the Section let us emphasize an important property of the linear biwave equation (9) with $n = 3$, which is a consequence of the Theorems 2, 3.

Corollary. *There exist two inequivalent representations of the Lie algebra of the conformal group $C(1,3)$ on the solution set of the equation (9) [2, 6, 8]:*

$$\begin{aligned} 1.\quad & P^{(1)}_\mu = P_\mu, \quad J^{(1)}_{\mu\nu} = J_{\mu\nu}, \\ & D^{(1)} = x_\mu \frac{\partial}{\partial x_\mu}, \quad K^{(1)}_\mu = 2x^\mu D^{(1)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}; \\ 2.\quad & P^{(2)}_\mu = P_\mu, \quad J^{(2)}_{\mu\nu} = J_{\mu\nu}, \\ & D^{(2)} = x_\mu \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial u}, \quad K^{(2)}_\mu = 2x^\mu D^{(2)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}, \end{aligned}$$

where the operators P_μ, $J_{\mu\nu}$ are determined in (2).

3 Symmetry classification of system of wave equations

Introducing a new variable $v = \Box u$ in (5) we get a system of partial differential equations

$$\begin{aligned} &\Box u = v, \\ &\Box v = F(u), \end{aligned} \tag{22}$$

which is equivalent to the biwave equation (5).

Symmetry properties of the system (22) are investigated by analogy with the previous Section. That is why, we restrict ourselves to formulating the corresponding assertions omitting their proofs.

Lemma 2 *The maximal invariance group of the system (22) with an arbitrary function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators (2).*

Theorem 4 *Any system of type (22) admitting a more extended invariance algebra than the Poincaré algebra $AP(1,n)$ is equivalent to one of the following:*

$$1.\quad \begin{aligned} &\Box u = v, \\ &\Box v = \lambda_1 u^k, \quad \lambda_1 \neq 0,\ k \neq 0, 1; \end{aligned} \tag{23}$$

$$2.\quad \begin{aligned} &\Box u = v, \\ &\Box v = \lambda_2 u, \quad \lambda_2 \neq 0; \end{aligned} \tag{24}$$

$$3.\quad \begin{aligned} &\Box u = v, \\ &\Box v = 0. \end{aligned} \tag{25}$$

Theorem 5 *The symmetry of the systems (23)–(25) is described in the following way:*

1. The maximal invariance group of the system (23) is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (2) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + \frac{4}{1-k} u \frac{\partial}{\partial u} + \frac{2(1+k)}{1-k} v \frac{\partial}{\partial v}.$$

2. The maximal invariance group of the system (24) is generated by the operators (2) and

$$Q_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \qquad Q_2 = v\frac{\partial}{\partial u} + \lambda_2 u \frac{\partial}{\partial v},$$
$$Q_3 = h_1(x)\frac{\partial}{\partial u} + h_2(x)\frac{\partial}{\partial v},$$

where $(h_1(x), h_2(x))$ is an arbitrary solution of the system (24).

3. The maximal invariance group of the system (25) is generated by the operators (2) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + 2u\frac{\partial}{\partial u}, \qquad Q_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$
$$Q_2 = v\frac{\partial}{\partial u}, \quad Q_3 = q_1(x)\frac{\partial}{\partial u} + q_2(x)\frac{\partial}{\partial v},$$

where $(q_1(x), q_2(x))$ is an arbitrary solution of the system (25).

It follows from the statements above that, unlike the biwave equations, the extended Poincaré group $\widetilde{P}(1,n)$ is the invariance group of the system (22) only in two cases, namely, when the system (22) is equivalent to (23) or (25). Moreover, there are no systems of the form (22) which are invariant under the conformal group. Therefore, in the class of Lie operators, the invariance algebras of the biwave equations and the corresponding systems of the wave equations are essentially different.

4 Reduction and exact solutions of the equation $\Box^2 u = \lambda e^u$

As follows from the Theorem 2 the maximal invariance group of the equation (7) with $n = 1$ is the extended Poincaré group $\widetilde{P}(1,1)$ with the generators

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = \frac{\partial}{\partial x_1}, \quad J_{01} = x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_0}, \tag{26}$$

$$D^{(2)} = x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} - 4 \frac{\partial}{\partial u}. \tag{27}$$

To construct exact solutions of the above equation we shall make use of the symmetry reduction procedure. A principal idea of the said procedure is a special choice of a solution to be found. This choice is motivated by a representation of symmetry group admitted. It is known that if an equation admits a Lie transformation group having a symmetry operator

$$X = \xi^\mu(x) \frac{\partial}{\partial x_\mu} + \eta(x) \frac{\partial}{\partial u}, \tag{28}$$

then its solutions can be looked for in the form [2]:

$$u(x) = \varphi(\omega) + g(x), \tag{29}$$

where $\varphi(\omega)$ is an arbitrary smooth function, and what is more, functions $\omega(x)$ and $g(x)$ are to satisfy the following conditions:

$$\xi^\mu(x) \frac{\partial \omega}{\partial x_\mu} = 0, \qquad \xi^\mu(x) \frac{\partial g(x)}{\partial x_\mu} = \eta(x).$$

To obtain all the $\widetilde{P}(1,1)$ non-conjugated Ansätze (29) we have to describe all the inequivalent one-dimensional subalgebras of the Lie algebra $\widetilde{AP}(1,1)$ spanned by the operators (26) and (27) (see [2, 11]). In the paper we make use of a classification adduced in [11]. Omitting cumbersome intermediate computations we give $\widetilde{P}(1,1)$ non-conjugated Ansätze in the Table 1.

Table 1

N	Algebra	Invariant variable ω	Ansatz
1°	$D - J_{01}$	$x_0 + x_1$	$u = \varphi(\omega) - 2\ln(x_0 - x_1)$
2°	$D + \alpha J_{01},\ \alpha \neq -1$	$(1+\alpha)\ln(x_1 - x_0) - (1-\alpha)\ln(x_0 + x_1)$	$u = \varphi(\omega) - \dfrac{4}{\alpha+1}\ln(x_0 + x_1)$
3°	$D - J_{01} + P_0$	$\ln(x_0 - x_1 + 1/2) - 2(x_0 + x_1)$	$u = \varphi(\omega) - 2\ln(x_0 - x_1 + 1/2)$
4°	J_{01}	$x_0^2 - x_1^2$	$u = \varphi(\omega)$
5°	P_0	x_1	$u = \varphi(\omega)$
6°	$P_0 + P_1$	$x_0 - x_1$	$u = \varphi(\omega)$

Remark. Inequivalent subalgebras adduced in the Table 1 are constructed by taking into account an obvious fact that equation (7) is invariant under transformations of the form:

$$\begin{array}{lcl} x_0' \to x_0, & \quad\text{and}\quad & x_0' \to x_1, \\ x_1' \to -x_1; & & x_1' \to x_0. \end{array} \tag{30}$$

Substituting the Ansätze obtained into the equation (7) we get the following ordinary differential equations (ODE) for a function $\varphi(\omega)$:

1° $0 = \lambda e^{\varphi}$,

2° $\varphi^{(4)}(\alpha^2-1)^2 + 2\varphi^{(3)}\alpha(1-\alpha^2) - \varphi^{(2)}(1-\alpha^2) = \dfrac{\lambda}{16}\exp\left(\varphi + \dfrac{2\omega}{\alpha+1}\right)$,

3° $\varphi^{(4)} - \varphi^{(3)} = \dfrac{\lambda}{64}e^{\varphi}$,

4° $\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \dfrac{\lambda}{16}e^{\varphi}$,

5° $\varphi^{(4)} = \lambda e^{\varphi}$,

6° $0 = \lambda e^{\varphi}$.

Equation 5° has a particular solution

$$\varphi = \ln\left(\frac{24}{\lambda}(\omega+c)^{-4}\right), \quad \lambda > 0,$$

that leads to the following exact solutions of the equation (7):

$$\begin{aligned} u &= \ln\left(\frac{24}{\lambda}(x_0+c_1)^{-4}\right), \quad \lambda > 0, \\ u &= \ln\left(\frac{24}{\lambda}(x_1+c_2)^{-4}\right), \quad \lambda > 0. \end{aligned} \tag{31}$$

Here c, c_1, c_2 are arbitrary constants. This solutions are invariant under the operators P_0 and P_1 accordingly.

In conclusion of the section let us note that the solutions (31) can be also obtained by making use of the Ansatz in a Liouville form [2]:

$$u = \ln\left\{\frac{24}{\lambda}\frac{\left(\dot\varphi_1(\omega_1)\dot\varphi_2(\omega_2)\right)^2}{(\varphi_1(\omega_1)+\varphi_2(\omega_2))^4}\right\}, \quad \omega_1 = x_0 + x_1, \quad \omega_2 = x_0 - x_1,$$

that reduces the equation (7) to one of the following systems:

1. $\ddot\varphi_1 = 0, \quad \ddot\varphi_2 = 0;$
2. $\ddot\varphi_1 = \dfrac{2\dot\varphi_1^2}{\varphi_1}, \quad \ddot\varphi_2 = \dfrac{2\dot\varphi_2^2}{\varphi_2}.$

Here $\dot\varphi$ and $\ddot\varphi$ stand for the first and the second derivatives with respect to a corresponding argument.

Integrating the above systems we get the following exact solutions of the equation (7):

$$u = \ln\left(\frac{24}{\lambda}\frac{(a^2-b^2)^2}{(ax_0+bx_1+c)^4}\right), \tag{32}$$

where a, b, c are arbitrary constants.

The solution (32) can be obtained from (31) by means of the final transformations of the extended Poincaré group with generators (26) and (27).

5 Reduction and exact solutions of the equation $\Box^2 u = \lambda u^k$

It follows from the Theorem 2 that the equation (6) with $n=1$ is invariant under the extended Poincaré group $\widetilde{P}(1,1)$ with generators (26) and

$$D = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + \frac{4}{1-k}u\frac{\partial}{\partial u}. \tag{33}$$

If some equation admits a symmetry operator

$$X = \xi^\mu(x)\frac{\partial}{\partial x_\mu} + \eta(x)u\frac{\partial}{\partial u}, \tag{34}$$

then its solutions can be looked for in the form [2]:

$$u(x) = f(x)\varphi(\omega), \tag{35}$$

provided functions $\omega(x)$ and $f(x)$ satisfy the following system:

$$\xi^\mu(x)\frac{\partial\omega}{\partial x_\mu} = 0, \quad \xi^\mu(x)\frac{\partial f(x)}{\partial x_\mu} = \eta(x)f(x). \tag{36}$$

A complete list of $\widetilde{P}(1,1)$ non-conjugated Ansätze invariant under the inequivalent one-dimensional subalgebras of the algebra $\widetilde{P}(1,1)$ is given in the Table 2.

Table 2

N	Algebra	Invariant variable ω	Ansatz
1°	$D - J_{01}$	$x_0 + x_1$	$u = (x_0 - x_1)^{\frac{2}{1-k}}\varphi(\omega)$
2°	$D + \alpha J_{01}, \quad \alpha \neq -1$	$(x_0 - x_1)(x_0 + x_1)^{\frac{\alpha-1}{\alpha+1}}$	$u = (x_0 + x_1)^{\frac{4}{(1-k)(\alpha+1)}}\varphi(\omega)$
3°	$D + J_{01} + P_0$	$(x_0 + x_1 + \frac{1}{2})\times$ $\times\exp\left(2(x_1 - x_0)\right)$	$u = \exp\left(\frac{4}{k-1}(x_1 - x_0)\right)\varphi(\omega)$
4°	J_{01}	$x_0^2 - x_1^2$	$u = \varphi(\omega)$
5°	P_0	x_1	$u = \varphi(\omega)$
6°	$P_0 + P_1$	$x_0 + x_1$	$u = \varphi(\omega)$

Let us note that similar Ansätze for the nonlinear wave equation

$$\Box u = \lambda u^k, \tag{37}$$

were obtained in [3].

Substituting the Ansätze obtained into the equation (6) we get the following ODE for a function $\varphi(\omega)$:

$$1^\circ \quad \frac{1+k}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{32}\varphi^k,$$

$$2^\circ \quad (\alpha-1)^2\varphi^{(4)}\omega^2 + 2(\alpha-1)(\alpha+1)^2\left(\frac{3k+1}{1-k}+2\alpha\right)\omega\varphi^{(3)} + $$
$$+2\left(\alpha^2-4\alpha+3+\frac{6\alpha-10}{1-k}+\frac{8}{(1-k)^2}\right)\varphi^{(2)} = \frac{\lambda}{16}(\alpha+1)^2\varphi^k,$$

$$3^\circ \quad \varphi^{(4)}\omega^2 + \frac{5k-1}{k-1}\varphi^{(3)}\omega + \frac{4k^2}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{64}\varphi^k,$$

$$4^\circ \quad \varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \frac{\lambda}{16}\varphi^k,$$

$$5^\circ \quad \varphi^{(4)} = \lambda\varphi^k,$$

$$6^\circ \quad \lambda\varphi^k = 0.$$

Equations 1°, 2°, 4° have particular solutions of the form:

$$\varphi = \left(\frac{64}{\lambda}\frac{(k+1)^2}{(k-1)^4}\right)^{\frac{1}{k-1}}\omega^{-\frac{2}{k-1}}, \quad k \neq -1$$

and equation 5° has a particular solution of the form

$$\varphi = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}}\omega^{-\frac{4}{k-1}}, \quad k \neq -1, -3, -\frac{1}{3},$$

which lead to the following solutions of the equation (6):

$$u = \left(\frac{64}{\lambda}\frac{(k+1)^2}{(k-1)^4}\right)^{\frac{1}{k-1}}\Big((x_0+x_1+c_1)(x_0-x_1+c_2)\Big)^{-\frac{2}{k-1}}, \quad k \neq -1,$$

$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}}(x_0+c_3)^{\frac{4}{1-k}}, \quad k \neq -1, -3, -\frac{1}{3},$$

$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}}(x_1+c_4)^{\frac{4}{1-k}}, \quad k \neq -1, -3, -\frac{1}{3},$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Note that equation (37) has analogous solutions (see e.g. [2]).

6 Reduction and exact solutions of the equation $\Box^2 u = \lambda u^{-3}$

It follows from the Theorems 2, 3 that the equation

$$\Box^2 u = \lambda u^{-3} \tag{38}$$

with $n = 1$ is invariant under the conformal group $C(1,1)$ with generators (26) and

$$D^{(1)} = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + u\frac{\partial}{\partial u},$$
$$K_\mu^{(1)} = 2x^\mu D^{(1)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}, \quad \mu,\nu = 0,1. \tag{39}$$

By analogy with the preceding Section solutions of the equation (38) are looked for in the form (35), where functions $\omega(x)$ and $f(x)$ are the solutions of the system (36), and what is more, the operator (34) belongs to the invariance algebra of the equation (38).

To obtain all the $C(1,1)$ non-conjugated Ansätze we use the one-dimensional inequivalent subalgebras of the conformal algebra $AC(1,1)$ adduced in [11].

Solving for each subalgebra equations (36) we arrive at the collection of $C(1,1)$-invariant Ansätze which are presented in the Table 3.

Table 3.

N	Algebra	Invariant variable ω	Ansatz
1°	$P_0 + K_0^{(1)}$	$\mathrm{arctg}(x_1 - x_0) + \mathrm{arctg}(x_1 + x_0)$	$u = \Big((x_0-x_1)^2+1\Big)^{1/2} \times \Big((x_0+x_1)^2+1\Big)^{1/2}\varphi(\omega)$
2°	$P_0 + K_0^{(1)} + \alpha(K_1^{(1)} - P_1)$ $0<\alpha<1$	$(\alpha-1)\mathrm{arctg}(x_0 - x_1) + (\alpha+1)\mathrm{arctg}(x_1 + x_0)$	$u = \Big((x_0-x_1)^2+1\Big)^{1/2} \times \Big((x_0+x_1)^2+1\Big)^{1/2}\varphi(\omega)$
3°	$P_0 + K_0^{(1)} + K_1^{(1)} - P_1$	$x_0 + x_1$	$u = \Big((x_0-x_1)^2+1\Big)^{1/2} \times \varphi(\omega)$
4°	$2P_1 + K_0^{(1)} + K_1^{(1)}$	$x_0 + x_1 + \frac{1}{2}\ln\frac{1+x_0-x_1}{1-x_0+x_1}$	$u = \Big((x_0-x_1)^2+1\Big)^{1/2} \times \varphi(\omega)$
5°	$2P_1 - K_0^{(1)} - K_1^{(1)}$	$x_0 + x_1 + \mathrm{arctg}(x_0 - x_1)$	$u = \Big((x_0-x_1)^2+1\Big)^{1/2} \times \varphi(\omega)$
6°	$P_0 + K_0^{(1)} + K_1^{(1)} - P_1 - \beta(J_{01} + D^{(1)})$, $\beta>0$	$\ln(x_0 + x_1) - \beta\mathrm{arctg}(x_1 - x_0)$	$u = \Big((x_0-x_1)^2+1\Big)^{1/2} \times \Big(x_0+x_1\Big)^{1/2}\varphi(\omega)$

We omit subalgebras not containing the conformal operator (39) since they were considered in the preceding Section.

Substituting Ansätze obtained in the equation (38) we get the following reduced ODE for a function $\varphi(\omega)$:

1° $\quad \varphi^{(4)} + 2\varphi^{(2)} + \varphi = \dfrac{\lambda}{16}\varphi^{-3};$

2° $\quad (\alpha^2-1)^2\varphi^{(4)} + 2(\alpha^2+1)\varphi^{(2)} + \varphi = \dfrac{\lambda}{16}\varphi^{-3};$

3° $\quad \varphi^{(2)} = \dfrac{\lambda}{16}\varphi^{-3};$

4° $\quad \varphi^{(4)} - \varphi^{(2)} = \dfrac{\lambda}{16}\varphi^{-3};$

$5^\circ \quad \varphi^{(4)} + \varphi^{(2)} = \dfrac{\lambda}{16}\varphi^{-3};$

$6^\circ \quad 4\beta^2\varphi^{(4)} + (4-\beta^2)\varphi^{(2)} - \varphi = \dfrac{\lambda}{4}\varphi^{-3}.$

The general solution of the equation 3° is of the form

$$\varphi = \pm\sqrt{\frac{(c_1\omega + c_2)^2}{c_1} + \frac{\lambda}{16c_1}}, \quad \varphi = \pm\sqrt{\frac{1}{2}\sqrt{-\lambda}\omega + c},$$

where c, c_1, c_2 are arbitrary constants, $c_1 \neq 0$.

Hence we obtain the following exact solutions of the equation (38):

$$1. \quad u = \pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{a_1}\right)^{1/4}\left|(x_0 + x_1 + a_2)^2 - a_1\right|^{1/2}\left|x_0 - x_1 + a_3\right|^{1/2},$$

$$2. \quad u = \pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{b_1}\right)^{1/4}\left|(x_0 - x_1 + b_2)^2 - b_1\right|^{1/2}\left|x_0 + x_1 + b_3\right|^{1/2},$$

$$3. \quad u = \pm\frac{1}{2}\left(\frac{\lambda}{c_1c_2}\right)^{1/4}\left|(x_0 - x_1 + c_3)^2 + c_1\right|^{1/2}\left|(x_0 + x_1 + c_4)^2 + c_2\right|^{1/2},$$

where a_i, b_i, c_j, $i = \overline{1,3}$, $j = \overline{1,4}$ are arbitrary constants.

Besides, the expression

$$u = \pm\lambda^{1/4}\left|(x_0 - x_1 + c_1)(x_0 + x_1 + c_2)\right|^{1/2}$$

(c_1, c_2 are arbitrary constants) was proved in the Section 4 to be the exact solution of the equation (38).

Conclusion

Thus, we have shown that the symmetry selection principle is a natural way of classification of physically admissible nonlinear biwave equations. Requiring an invariance with respect to the extended Poincaré group picks out very specific nonlinearities (3). And the demand of a conformal invariance yields, in fact, a unique nonlinear PDE (20), (21).

As equations obtained in this way admit broad Lie symmetry, one can apply the symmetry reduction procedure to find their exact solutions. An important part of the said procedure is a construction of special substitutions which reduce the equation under study to PDE with less number of independent variables. Given a subgroup classification of the equation under study, a procedure of construction of such substitutions is entirely algorithmic. Of course, there is no guarantee that the reduced equations can be solved explicitly. But our experience as well as a rich experience of other groups engaged in the field of group-theoretical, symmetry analysis of nonlinear partial differential equations evidence that it is almost always possible [2, 5, 14, 16]. The reason is that PDE obtained by means of reduction of some initial PDE admitting broad Lie symmetry also possess a hereditary symmetry. Moreover, in some exceptional cases this symmetry can be much more extensive than the one of the initial equation. An example is given in [15], where it is established that some equations obtained by means of reduction of the nonlinear Poincaré-invariant Dirac equation admit infinite-parameter symmetry groups. Since a maximal symmetry group of the initial equation is the ten-parameter Poincaré group, this symmetry is essentially new. The source of it is the *conditional symmetry* of the nonlinear Dirac equation [2, 15].

In the present paper we have applied the symmetry reduction procedure to reduce to ODE the fourth-order nonlinear biwave equations of the form (5) having two independent variables x_0, x_1 and to construct its explicit solutions. A problem of symmetry reduction of these equations has been completely solved in a sense that any solution of PDE (5) invariant under a subgroup of the conformal group $C(1,1)$ (which is a most extensive group that can be admitted by equation of the form (5)) is equivalent to one of the Ansätze given in the Tables 1–3. And what is more, these Ansätze can be applied to reduce any two-dimensional PDE, provided it is invariant under the Poincaré, extended Poincaré and conformal groups having the generators (2), (10)–(13). But it does not mean that *all* possibilities to reduce PDE (5) to ODE are exhausted. New reductions can be obtained by utilizing conditional symmetry of the biwave equation in the way as it has been done for a number of nonlinear mathematical physics equations in [2]. This problem is under investigation now.

An another interesting problem is to carry out symmetry reduction of the biwave equation in the four-dimensional Minkowski space-time. This work is now in progress and will be reported elsewhere.

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