DRAFT LECTURE NOTES ON HIGHER-RANK GRAPHS AND THEIR C^*-ALGEBRAS

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ABSTRACT. These are notes for a short lecture course on k-graph C^*-algebras to be delivered at the Summer School on *C^*-algebras and their interplay with dynamics* at the Sophus Lie Conference Centre in Nordfjordeid, Norway in June 2010. They are not even remotely comprehensive of the work that many authors have done on k-graphs, nor are all details even of the material covered included. In addition, there are likely to be plenty of typo's and possibly more serious errors, and I would be grateful if you could pass any you find on to me. These notes are not intended for distribution, but as an aid to those attending the course.

These notes are also not properly — or comprehensively — referenced. Instead, I have just tried to attribute major results and definitions to the people who proved them. There are many people who have been involved in the area who have not been mentioned; I apologise for my oversights.

1. Higher-rank graphs, coloured graphs and skeletons

In these notes, a directed graph is a quadruple (E^0, E^1, r, s) where E^0, E^1 are countable (discrete) sets, and r, s are maps from E^1 to E^0. A *path* in E is a sequence $\alpha_1 \dots \alpha_n$ with each $\alpha_i \in E^1$ and with $s(\alpha_i) = r(\alpha_{i+1})$ for all i. We regard the set E^* of all paths as a category with objects E^0 and composition given by concatenation of paths.

Definition 1.1. Let $k \in \mathbb{N}$. A *graph of rank k* or a k-graph is a countable category Λ equipped with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree functor* satisfying the following factorisation property:

for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$ there are unique elements $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

Lemma 1.2. *Let Λ be a k-graph. Then $d^{-1}(0) = \{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda)\}$.*

Proof. If $o \in \mathrm{Obj}(\Lambda)$, then

$$d(\mathrm{id}_o) = d(\mathrm{id}_o \mathrm{id}_o) = 2d(\mathrm{id}_0),$$

forcing $d(\mathrm{id}_o) = 0$. Thus $\{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda) \subset d^{-1}(0)$.

For the reverse inclusion, fix λ with $d(\lambda) = 0$. We have $d(\lambda) = 0 + 0$, and

$$\mathrm{id}_{\mathrm{cod}(\lambda)} \lambda = \lambda = \lambda \, \mathrm{id}_{\mathrm{dom}(\lambda)}.$$

Uniqueness of factorisations therefore forces $\lambda = \mathrm{id}_{\mathrm{dom}(r(\lambda))}$. □

Notation 1.3. We will adopt the following notation throughout these notes.

- $\Lambda^n := d^{-1}(n)$.

Date: May 27, 2010.

- $r(\lambda) := \mathrm{id}_{\mathrm{cod}(\lambda)} \in \Lambda^0$ and $s(\lambda) := \mathrm{id}_{\mathrm{dom}(\lambda)} \in \Lambda^0$.
- for $E \subset \Lambda$ and $\alpha \in \Lambda$, we write $\alpha E := \{\alpha\lambda : \lambda \in E, r(\lambda) = s(\alpha)\}$, and $E\alpha := \{\lambda\alpha : \lambda \in E, s(\lambda) = r(\alpha)\}$. So in particular, for $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, $v\Lambda^n = \{\lambda \in \Lambda : r(\lambda) = v \text{ and } d(\lambda) = n\}$.

We describe k-graphs in terms of their k-coloured skeletons.

Definition 1.4. Let $k \in \mathbb{N}$. A k-coloured graph is a directed graph (E^0, E^1, r, s) together with a colour map $c : E^1 \to \{1, \dots, k\}$.

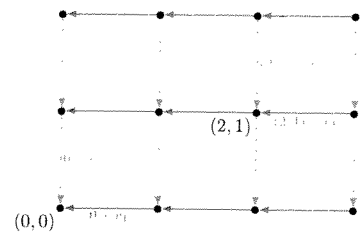
Given a k-coloured graph E, we extend the colour map c to a functor $c : E^* \to \mathbb{F}_k^+$; so $c(\alpha) = c(\alpha_1)c(\alpha_2) \dots c(\alpha_{|\alpha|})$ for $\alpha \in E^*$.

Example 1.5. Fix $k \in \mathbb{N}$ and $m \in \mathbb{N}^k$. The coloured graph $E_{k,m}$ has vertices $E^0_{k,m} = \{n \in \mathbb{N}^k : n \leq m\}$, and edges $E^1_{k,m} = \{n + v_i : n, n + e_i \in E^0_{k,m}\}$ with structure maps

$$r(n + v_i) = n, \quad s(n + v_i) = n + e_i, \quad \text{and} \quad c(n + v_i) = i.$$

For example, $E_{3,(3,2)}$ could be drawn as follows:

$E_{2,(3,2)}$



A graph morphism φ from a graph E to a graph F is a pair of maps $\varphi^0 : E^0 \to F^0$ and $\varphi^1 : E^1 \to F^1$ such that $r(\varphi^1(e)) = \phi^0(r(e))$ and $s(\varphi^1(e)) = \varphi^0(s(e))$ for all $e \in E^1$. We will often simply write ϕ for each of φ^0 and φ^1. A coloured-graph morphism between k-coloured graphs is then a graph morphism which preserves colour.

For distinct $i, j \leq k$, an ij-square in a k-coloured graph E is a coloured-graph morphism $\varphi : E_{k,e_i+e_j} \to E$.

Definition 1.6. A *complete and associative collection of squares* for a k-coloured graph E is a set $\mathcal{C}$ of squares in E such that

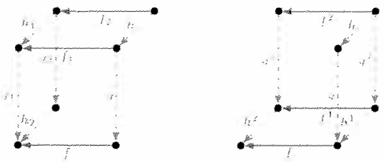
(1) for each ij-coloured path $fg \in E^2$ there is a unique $\varphi \in \mathcal{C}$ such that $\varphi(0+v_i) = f$ and $\varphi(e_i + v_j) = g$; and
(2) if we write $fg \sim g'f'$ whenever there is a square ϕ such that

$$\varphi(0 + v_i) = f, \quad \varphi(e_i + v_j) = g, \quad \varphi(0 + v_j) = g' \quad \text{and} \quad \varphi(e_j + v_i) = f',$$

then if fgh is a tri-coloured path and

$$fg \sim g_1 f_1, \quad f_1 h \sim h_1 f_2, \quad g_1 h_1 \sim h_2 g_2,$$
$$gh \sim h^1 g^1, \quad fh^1 \sim h^2 f^1 \quad \text{and} \quad f^1 g^1 \sim g^2 f^2,$$

then $f_2 = f^2$, $g_2 = g^2$ and $h_2 = h^2$.



Given a k-coloured graph E and a coloured-graph morphism $\varphi : E_{k,m} \to E$, we say that an ij-square ψ in E *occurs in* φ if there exists $n \in \mathbb{N}^k$ such that $n + e_i + e_j \leq m$ and

$$\varphi(n + v_i) = \psi(0 + v_i), \qquad \varphi(n + e_i + v_j) = \psi(e_i + v_j),$$
$$\varphi(n + v_j) = \psi(0 + v_j) \quad \text{and} \quad \varphi(n + e_j + v_i) = \psi(e_j + v_i).$$

If E is a k-coloured graph and $\mathcal{C}$ is a complete and associative collection of squares in E, we say that a coloured-graph morphism $\varphi : E_{k,m} \to E$ is $\mathcal{C}$-compatible if every square which occurs in φ belongs to $\mathcal{C}$.

The next lemma is due to Robbie Hazlewood and is the key step in our construction of a k-graph from a coloured graph.

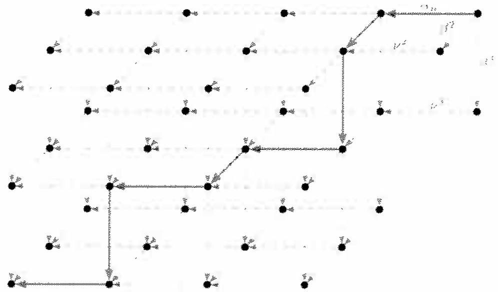
Lemma 1.7. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete and associative collection of squares in E. Let $\pi : \mathbb{F}_k^+ \to \mathbb{N}^k$ be the homomorphism satisfying $\pi(i) = e_i$. Then for each path $\alpha = \alpha_1\alpha_2 \dots \alpha_{|\alpha|} \in E$, there is a unique $\mathcal{C}$-compatible coloured-graph morphism $\varphi : E_{k,\pi(c(\alpha))} \to E$ such that*

$$(1.1) \qquad \varphi(\pi(c(\alpha_1 \dots \alpha_l)) + v_{c(\alpha_{l+1})}) = \alpha_{l+1} \qquad \textit{for all } l < |\alpha|.$$

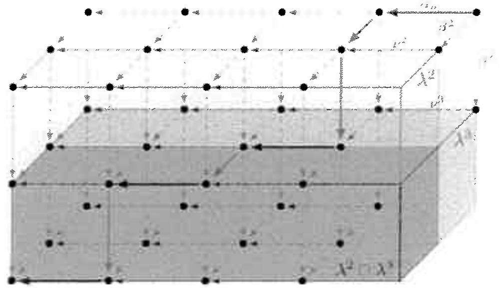
Proof. We proceed by induction on $|\alpha|$. If $|\alpha| = 0$ then the assertion is trivial.

Now fix $n \geq 1$ and suppose that there is a unique φ satisfying (1.1) whenever $|\alpha| < n \in \mathbb{N}$, and fix $\alpha \in E^n$. Let $i := c(\alpha_n)$, and let $m := \pi(c(\alpha))$.

By the inductive hypothesis, there is a unique $\mathcal{C}$-compatible coloured-graph morphism $\psi : E_{k,m-e_i} \to E$ which is traversed by $\alpha_1 \dots \alpha_{n-1}$. For each $j \in \{1, \dots, k\} \setminus \{i\}$ such that $m_j \neq 0$, that $\mathcal{C}$ is a complete collection of squares ensures that there is a unique $\nu^j \in c^{-1}(j)$ and $\beta^j \in c^{-1}(i)$ such that $\psi((m - e_i - e_j) + v_j)\alpha_n \sim \nu^j\beta^j$.



For each j, the inductive hypothesis applied to $\xi\nu^j$ for any traversal ξ of $\psi|_{E_{k,m-e_i-e_j}}$ yields a unique $\mathcal{C}$-compatible morphism λ^j traversed by $\xi\nu^j$.



We claim that for distinct p, q, the morphisms λ^p and λ^q agree on the intersection of their domains, namely $E_{k,m-e_p-e_q}$. To see this, let $\tau := \lambda^q((m - e_p - e_q - e_i) + v_i)$. Then $\tau = \lambda^p((m - e_p - e_q - e_i) + v_i)$ because the two are the paths h^2 and h_2 obtained from Definition 1.6(2) with

$$(1.2) \qquad f = \psi((m - e_p - e_q - e_i) + v_p), \quad g = \psi((m - e_q - e_i) + v_q) \quad \text{and} \quad h = \alpha_n.$$

Hence each of $\lambda^p|_{E_{k,m-e_p-e_q}}$ and $\lambda^q|_{E_{k,m-e_p-e_q}}$ is traversed by $\zeta\tau$ for any traversal ζ of $\psi|_{E_{k,m-e_p-e_q-e_i}}$. The inductive hypothesis therefore gives

$$(1.3) \qquad \lambda^p|_{E_{k,m-e_p-e_q}} = \lambda^q|_{E_{k,m-e_p-e_q}}.$$

Since $E^1_{k,m} = \left(\bigcup_{m_p \neq 0, p \neq i} E^1_{k,m-e_p}\right) \cup \{(m - e_p) + v_p : m_p \neq 0, p \neq i\}$, equation (1.3) implies that there is a well-defined coloured-graph morphism $\varphi : E_{k,m} \to E$ determined by

$$\varphi|_{E_{k,m-e_p}} = \lambda^p \quad \text{whenever } p \neq i \text{ and } m_p \neq 0, \text{ and}$$
$$\varphi((m - e_p) + v_p) = \beta^p \quad \text{for all } p \neq i \text{ with } m_p \neq 0.$$

Every square which occurs in φ either occurs in one of the λ^p or occurs in the cube $\kappa^{p,q}$ traversed by the path fgh of (1.2) for some p, q. Since the λ^p and the $\kappa^{p,q}$ are all $\mathcal{C}$-compatible, it follows that φ is also. That the β^p and λ^p were uniquely determined by requiring that all squares occurring in them belonged to $\mathcal{C}$ implies that φ is the unique $\mathcal{C}$-compatible morphism traversed by α. □

Corollary 1.8. *Let E be a k-coloured graph, and let $\mathcal{C}$ be a complete and associative collection of squares for E. If*

$$\varphi : E_{k,m} \to E \quad \textit{and} \quad \psi : E_{k,n} \to E$$

are $\mathcal{C}$-compatible coloured-graph morphisms such that $\varphi(m) = \psi(0)$, then there is a unique $\mathcal{C}$-compatible morphism $(\phi\psi) : E_{k,m+n} \to E$ such that

$$(1.4) \qquad \begin{aligned} (\phi\psi)(p + v_i) &= \phi(p + v_i) \quad \textit{whenever } p + e_i \leq m, \textit{ and} \\ (\phi\psi)(p + v_i) &= \psi((p - m) + v_i) \quad \textit{whenever } m \leq p \leq m + n - e_i. \end{aligned}$$

Moreover, this defines an associative partial multiplication on the set

$$\Lambda_{E,\mathcal{C}} = \bigcup_{m\in\mathbb{N}^k} \{\varphi : E_{k,m} \to E \mid \varphi \text{ is a } \mathcal{C}\text{-compatible coloured-graph morphism}\}.$$

Proof. Fix paths α^φ and α^ψ in E which traverse φ and ψ. Then Lemma 1.7 implies that there is a unique $\mathcal{C}$-compatible coloured-graph morphism $\varphi\psi$ traversed by $\alpha^\varphi\alpha^\psi$. The uniqueness assertion of Lemma 1.7 implies that $\varphi\psi$ satisfies (1.4). Moreover, any coloured-graph morphism π satisfying (1.4) is traversed by $\alpha^\varphi\alpha^\psi$ and hence another application of uniqueness from Lemma 1.7 implies that $\pi = \varphi\psi$.

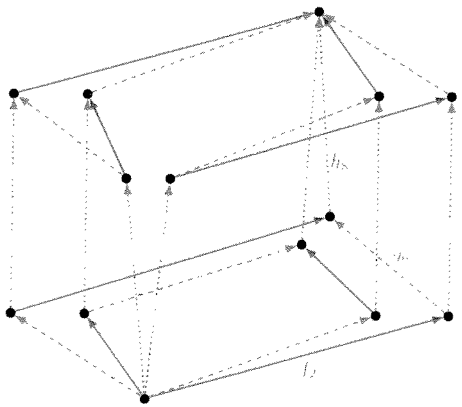
Associativity follows from associativity of concatenation of paths in E. □

Theorem 1.9. *Let E be a k-coloured graph, and let $\mathcal{C}$ be a complete and associative collection of squares for E. Let $\Lambda = \Lambda_{E,\mathcal{C}}$ be as in Corollary 1.8, and define $d : \Lambda \to \mathbb{N}^k$ by $d(\varphi) = m$ if $\mathrm{dom}(\varphi) = E_{k,m}$. Then Λ is the unique k-graph such that $\Lambda^{e_i} = c^{-1}(i)$ for each i and $fg = g'f'$ in Λ if and only if $fg \sim g'f'$ in E.*

Proof. Corollary 1.8 shows that Λ is a category, and it has $\Lambda^{e_i} = c^{-1}(i)$ and $fg = g'f'$ whenever $fg \sim g'f'$ in E by definition. To see that Λ is a k-graph, we must verify the factorisation property. This follows from Lemma 1.7 and uniqueness of factorisations of paths in E.

For uniqueness, observe that if Γ is a k-graph with the given properties, then each $\gamma \in \Gamma$ determines a $\mathcal{C}$-compatible coloured-graph morphism φ_γ by $\varphi_\gamma(n + v_i) = \alpha$ where α is the unique path satisfying $\gamma = \gamma'\alpha\gamma''$ with $d(\gamma') = m$, $d(\alpha) = e_i$ and $d(\gamma'') = d(\gamma) - m - e_i$. □

Example 1.10. The associative condition is necessary in three or more dimensions as is demonstrated by the following three-coloured graph due to Jack Spielberg:

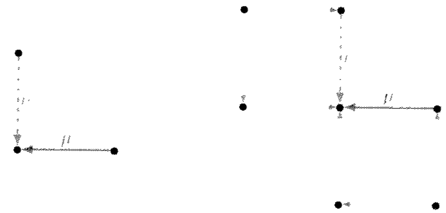


There is a unique complete collection of squares in this graph, but the collection is not associative as can be seen by chasing through the possible factorisations of the path fgh.

2. k-GRAPH C^*-ALGEBRAS AND THE GAUGE-INVARIANT UNIQUENESS THEOREM

A k-graph is *row-finite* if $|v\Lambda^n| < \infty$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. It is *locally convex* if whenever $\mu \in \Lambda^{e_i}$ and $\nu \in \Lambda^{e_j}$ with $i \neq j$ and $r(\mu) = r(\nu)$, we have $s(\mu)\Lambda^{e_j} \neq \emptyset$ and $s(\nu)\Lambda^{e_i} \neq \emptyset$.

Pictorially, the graph on the left is not allowed unless the two edges pictured extend to squares as on the right.



Remark 2.1. If Λ is locally convex, then a straightforward induction shows that if $m \wedge n = 0$ and $\mu \in \Lambda^m$ and $\nu \in \Lambda^n$ with $r(\mu) = r(\nu)$, then $s(\mu)\Lambda^n$ and $s(\nu)\Lambda^m$ are nonempty.

We write $\Lambda^{\leq n}$ for the set

$$\Lambda^{\leq n} = \{\lambda \in \Lambda : d(\lambda) \leq n \text{ and } d(\lambda)_i < n_i \implies s(\lambda)\Lambda^{e_i} = \emptyset\}.$$

Lemma 2.2. *Let Λ be a locally convex k-graph. Fix $m, n \in \mathbb{N}^k$. We have $\Lambda^{\leq(m+n)} = \Lambda^{\leq m}\Lambda^{\leq n}$.*

Proof. If $\mu \in \Lambda^{\leq m}$ and $\nu \in \Lambda^{\leq n}$, then certainly $d(\mu\nu) \leq m + n$. Suppose $d(\mu\nu)_i < (m+n)_i$. There are two cases to consider: $d(\nu)_i < n_i$ or $d(\mu)_i < m_i$. If $d(\nu)_i < n_i$, then $s(\mu\nu)\Lambda^{e_i} = s(\nu)\Lambda^{e_i} = \emptyset$. On the other hand, if $d(\mu)_i < m_i$, then $s(\mu)\Lambda^{e_i} = \emptyset$, and then $s(\mu\nu)\Lambda^{e_i} = s(\nu)\Lambda^{e_i} = \emptyset$ by the factorisation property. So $\Lambda^{\leq m}\Lambda^{\leq n} \subset \Lambda^{\leq(m+n)}$.

Now suppose that $\lambda \in \Lambda^{\leq(m+n)}$. Let $m' := m \wedge d(\lambda)$, and let $n' := n \wedge d(\lambda) - m'$. It is straightforward to check that $m' + n' = (m+n) \wedge d(\lambda)$. Let $\mu = \lambda(0, m')$. Clearly $d(\mu) \leq m$ and $d(\nu) \leq n$. If $d(\nu)_i < n_i$ then $d(\lambda)_i < (m' + n)_i \leq (m+n)_i$, and hence $s(\nu)\Lambda^{e_i} = s(\lambda)\Lambda^{e_i} = \emptyset$, giving $\nu \in \Lambda^{\leq n}$. Now suppose that $d(\mu)_i < m_i$. Then $d(\mu)_i = d(\lambda)_i$, so $d(\nu)_i = 0$. Moreover, $d(\lambda)_i < m_i \leq (m+n)_i$ whence $s(\lambda)\Lambda^{e_i} = \emptyset$. It then follows from Remark 2.1 that $r(\nu)\Lambda^{e_i} = \emptyset$. So $\mu \in \Lambda^{\leq m}$. □

The following definition of a Cuntz-Krieger Λ-family, due originally to Yeend, is the one suitable to locally convex row-finite k-graphs. However, it is very closely modelled on Kumjian and Pask's original definition for row-finite k-graphs with no sources. Likewise, our analysis in this section leading up to the gauge-invariant uniqueness theorem is largely due to Raeburn-S-Yeend but is heavily based on Kumjian and Pask's seminal work.

Definition 2.3. Let Λ be a locally convex row-finite k-graph. A *Cuntz-Krieger Λ-family* in a C^*-algebra B is a function $t : \Lambda \to B$, $\lambda \mapsto t_\lambda$ such that

(CK1) $\{t_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections;

(CK2) $t_\mu t_\nu = t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $t_\mu^* t_\mu = t_{s(\mu)}$ for all $\mu \in \Lambda$; and
(CK4) $t_v = \sum_{\lambda \in v\Lambda^{\leq n}} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

We write $C^*(t)$ for $C^*(\{t_\lambda : \lambda \in \Lambda\})$.

To give an example of a Cuntz-Krieger Λ-family, we introduce filters in k-graphs. The idea of using filters and ultrafilters to construct representations of combinatorial objects such as k-graphs is due to Exel in the context of inverse semigroups, though the procedure is greatly simplified in our setting. To introduce filters, we need the notion of a *minimal common extension* of paths in Λ.

Definition 2.4. Let Λ be a k-graph, and let $\mu, \nu \in \Lambda$ We say that λ is a *minimal common extension* of μ and ν if $d(\lambda) = d(\mu) \vee d(\nu)$ and $\lambda = \mu\mu' = \nu\nu'$ for some $\mu', \nu' \in \Lambda$. We write $\mathrm{MCE}(\mu, \nu)$ for the set of all minimal common extensions of μ and ν.

A *filter* of a k-graph Λ is a set $x \subset \Lambda$ such that
(F1) if $\lambda \in x$ and $\lambda = \mu\mu'$, then $\mu \in x$; and
(F2) if $\mu, \nu \in x$ then $\mathrm{MCE}(\mu, \nu) \cap x \neq \emptyset$.

It follows that if x is a filter of Λ, then $\Lambda^0 \cap x$ contains a unique element $r(x)$, and also that if $\mu, \nu \in x$ then there is a unique element $\mu \vee_x \nu$ of $\mathrm{MCE}(\mu, \nu)$ which belongs to x.

An *ultrafilter* of Λ is a filter which is maximal with respect to containment. A standard Zorn's Lemma argument shows that for each $\lambda \in \Lambda$ there exists an ultrafilter x of Λ such that $\lambda \in x$. We write $\widehat{\Lambda}$ for the set of filters of Λ, and $\widehat{\Lambda}_\infty$ for the set of ultrafilters of Λ.

Lemma 2.5. *Let Λ be a row-finite locally-convex k-graph. Let $x \in \widehat{\Lambda}$ and fix $\lambda \in x$ and $\mu \in \Lambda r(x)$. Then*
(1) *$\lambda^* \cdot x := \{\alpha : \lambda\alpha \in x\}$ and $\mu \cdot x := \{\beta : \beta\Lambda \cap \mu x \neq \emptyset\}$ are filters;*
(2) *$\lambda \cdot (\lambda^* \cdot x) = x = \mu^* \cdot (\mu \cdot x)$.*
(3) *If x belongs to $\widehat{\Lambda}_\infty$, then so do $\lambda^* \cdot x$ and $\mu \cdot x$.*

Proof. (1) If $\alpha \in \lambda^* x$ and $\alpha = \beta\alpha'$ then $\lambda\beta\alpha \in x$ and then (F1) forces $\lambda\beta \in x$ and hence $\beta \in \lambda^* \cdot x$. If $\alpha \in \mu \cdot x$ and $\alpha = \beta\alpha'$, then $\emptyset \neq \alpha\Lambda \cap \mu x \subset \beta\Lambda \cap \mu x$, so $\beta \in \mu \cdot x$. So $\lambda^* \cdot x$ and $\mu \cdot x$ satisfy (F1).

For (F2), suppose that $\alpha, \beta \in \lambda^* \cdot x$. Then $\lambda\alpha \vee_x \lambda\beta$ belongs to $\mathrm{MCE}(\lambda\alpha, \lambda\beta) \cap x = \lambda\,\mathrm{MCE}(\alpha, \beta) \cap x$. Hence $\mathrm{MCE}(\alpha, \beta) \cap \lambda^* \cdot x \neq \emptyset$. If $\alpha, \beta \in \mu \cdot x$, then there exists $\mu\mu' \in x$ such that $\mu\mu' = \alpha\alpha' = \beta\beta'$ for some α', β'. Use the factorisation property to write $\mu\mu' = \tau\eta$ where $d(\tau) = d(\alpha) \vee d(\beta)$. Then $\tau \in \mathrm{MCE}(\alpha, \beta)$ and $\tau\Lambda \cap \mu x \neq \emptyset$, so $\tau \in \mathrm{MCE}(\alpha, \beta) \cap \mu \cdot x$.

(2) We calculate

$$\begin{aligned} \alpha \in \lambda \cdot \lambda^* \cdot x &\iff \alpha\Lambda \cap \lambda(\lambda^* \cdot x) \neq \emptyset \\ &\iff \alpha\Lambda \cap \{\lambda\beta : \lambda\beta \in x\} \neq \emptyset \\ &\iff \alpha\Lambda \cap x \neq \emptyset. \end{aligned}$$

Similarly, $\beta \in \mu^* \cdot \mu \cdot x \iff \mu\beta \in \mu \cdot x \iff \mu\beta\Lambda \cap \mu x \neq \emptyset \iff \beta \in x$.

(3) Suppose that $\lambda^* x \subset y \in \widehat{\Lambda}$. Then $x = \lambda \cdot \lambda^* \cdot x \subset \lambda \cdot y$, so $\lambda \cdot y = x$ and then $y = \lambda^* \cdot \lambda \cdot y = \lambda^* \cdot x$. Similarly for $\mu \cdot x$. □

Lemma 2.6. *Let Λ be a locally convex row-finite k-graph. If $x \in \widehat{\Lambda}_\infty$ and $n \in \mathbb{N}^k$, then $r(x)\Lambda^{\leq n} \cap x \neq \emptyset$.*

Proof. Fix an increasing cofinal subsequence $(\mu_i)_{i=1}^\infty$ of x such that $\mu_0 = r(x)$. For each i, each $\alpha \in s(\mu_i)\Lambda^{\leq n}$, and each $j \leq i$, Lemma 2.2 implies that there is a unique $\beta \in s(\alpha_j)\Lambda^{\leq n}$ such that $\mu_i\alpha \in \mu_j\beta\Lambda$. Since each $s(\mu_i)\Lambda^{\leq n}$ is finite, we may inductively choose $\alpha_i \in s(\mu_i)\Lambda^{\leq n}$ such that $\mu_i\alpha_i \in \mu_j\alpha_j\Lambda$ whenever $j \leq i$, and such that $\mu_i\alpha_i\Lambda \cap \alpha_l\Lambda^{\leq n}$ is nonempty for infinitely many (and hence all) $l \geq i$.

The set $y := \{\beta \in \Lambda : \mu_i\alpha_i \in \beta\Lambda \text{ for some } i\}$ is a filter of Λ which contains x. Since x is an ultrafilter, $y = x$. Since $\alpha_0 \in y$ by definition, and since $\alpha_0 \in r(x)\Lambda^{\leq n}$, the result follows. □

Example 2.7. Let Λ be a locally convex row-finite k-graph, and let $H := \ell^2(\widehat{\Lambda}_\infty)$. Routine calculations show that

$$T_\lambda \xi_x := \chi_x(s(\lambda))\xi_{\lambda \cdot x}$$

yields a Cuntz-Krieger Λ-family T in $\mathcal{B}(H)$ (it satisfies (CK4) by Lemma 2.6). Moreover, for $\lambda \in \Lambda$,

$$T_\lambda^* \xi_x = \chi_x(\lambda)\xi_{\lambda^* \cdot x}, \qquad \text{so} \qquad T_\lambda^* T_\lambda = \mathrm{proj}_{\overline{\mathrm{span}}\{\xi_x : \lambda \in x\}}.$$

Example 2.7 shows in particular that for any k-graph Λ there exist Cuntz-Krieger Λ-families in which every t_λ is nonzero.

Lemma 2.8. *Let Λ be a locally convex row-finite k-graph and let t be a Cuntz-Krieger Λ-family. Then for $\mu, \nu \in \Lambda$, we have*

(2.1)
$$\mathrm{MCE}(\mu, \nu) = \mu\Lambda^{\leq(d(\mu)\vee d(\nu))-d(\mu)} \cap \nu\Lambda^{\leq(d(\mu)\vee d(\nu))-d(\nu)} = \mu\Lambda^{(d(\mu)\vee d(\nu))-d(\mu)} \cap \nu\Lambda^{(d(\mu)\vee d(\nu))-d(\nu)}$$

and

$$t_\mu^* t_\nu = \sum_{\mu\mu' = \nu\nu' \in \mathrm{MCE}(\mu,\nu)} t_{\mu'} t_{\nu'}^*. \tag{2.2}$$

Proof. To establish (2.1) first note that

$$\mathrm{MCE}(\mu, \nu) = \mu\Lambda^{(d(\mu)\vee d(\nu))-d(\mu)} \cap \nu\Lambda^{(d(\mu)\vee d(\nu))-d(\nu)} \subset \mu\Lambda^{\leq(d(\mu)\vee d(\nu))-d(\mu)} \cap \nu\Lambda^{\leq(d(\mu)\vee d(\nu))-d(\nu)}$$

by definition. For the reverse inclusion, note that $\lambda \in \mu\Lambda \cap \nu\Lambda \implies d(\lambda) \geq d(\mu) \vee d(\nu)$.

To establish (2.2), let $m := d(\mu)$, $n := d(\nu)$ and use (CK4) to calculate

$$t_\mu^* t_\nu = t_\mu^* t_\mu t_\mu^* t_\nu t_\nu^* t_\nu = t_\mu^* \Big(\sum_{\mu' \in \Lambda^{\leq(m\vee n)-m}} t_{\mu\mu'} t_{\mu\mu'}^* \Big) \Big(\sum_{\nu' \in \Lambda^{\leq(m\vee n)-n}} t_{\nu\nu'} t_{\nu\nu'}^* \Big) t_\nu^*. \tag{2.3}$$

By Lemma 2.2, each $\mu\mu', \nu\nu' \in \Lambda^{\le m\vee n}$, so another application of (CK4) ensures that each $t_{\mu\mu'}t^*_{\mu\mu'}t_{\nu\nu'}t^*_{\nu\nu'} = \delta_{\mu\mu',\nu\nu'}t_{\mu\mu'}t^*_{\mu\mu'}$. Hence

$$\begin{aligned} t^*_\mu t_\nu &= t^*_\mu\Big(\sum_{\lambda\in\mu\Lambda^{\le(m\vee n)-m}\cap\nu\Lambda^{\le(m\vee n)-n}} t_\lambda t^*_\lambda\Big)t^*_\nu \\ &= \sum_{\mu\mu'=\nu\nu'\in\mathrm{MCE}(\mu,\nu)} t^*_\mu t_\mu t_{\mu'}t^*_{\nu'}t_\nu t_\nu \qquad \text{by (2.1)} \\ &= \sum_{\mu\mu'=\nu\nu'\in\mathrm{MCE}(\mu,\nu)} t_{\mu'}t^*_{\nu'} \end{aligned}$$

by (CK3). □

Corollary 2.9. *Let Λ be a locally convex row-finite k-graph and let t be a Cuntz-Krieger Λ-family. Then $C^*(t) = \overline{\operatorname{span}}\{t_\mu t^*_\nu : \mu,\nu\in\Lambda\}$.*

Our proof of the following result is taken more or less directly from Raeburn's notes on graph algebras from the CBMS conference held at the University of Iowa in 2004.

Proposition 2.10. *There is a C^*-algebra $C^*(\Lambda)$ generated by a Cuntz-Krieger Λ-family s which is universal in the sense that each Cuntz-Krieger Λ-family t induces a homomorphism $\pi_t : C^*(\Lambda) \to C^*(t)$ satisfying $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$. Moreover, each s_λ is nonzero.*

Proof. Let $\Lambda * \Lambda := \{(\mu,\nu) \in \Lambda\times\Lambda : s(\mu) = s(\nu)\}$. Let $A_0 := c_c(\Lambda *_s \Lambda)$ and for each μ,ν, let $\delta_{(\mu,\nu)} \in A_0$ denote the indicator function. Define $^* : A_0 \to A_0$ by $f^*(\mu,\nu) := \overline{f(\nu,\mu)}$, and define a multiplication on A_0 by extending the assignment

$$\delta_{(\mu,\nu)}\delta_{(\eta,\zeta)} \mapsto \sum_{\nu\nu'=\eta\eta'\in\mathrm{MCE}(\nu,\eta)} \delta_{(\mu\nu',\zeta\eta')}$$

to a bilinear map. For each Cuntz-Krieger Λ-family t on Hilbert space, the partial isometries $t_\mu t^*_\nu$ satisfy the same relations as the $\delta_{\mu,\nu}$, so each such family determines a representation π_t of A_0 such that $\pi_t(\delta_{(\mu,\nu)}) = t_\mu t^*_\nu$ for all μ,ν.

Each $t_\mu t^*_\nu$ is a partial isometry, so its norm is less than or equal to 1. Hence for $f \in A_0$,

$$\|\pi_t(f)\| = \Big\|\sum_{(\mu,\nu)\in\Lambda*\Lambda} f(\mu,\nu)\pi_t(\delta_{(\mu,\nu)})\Big\| \le \sum_{(\mu,\nu)} |f_{(\mu,\nu)}|\|t_\mu t^*_\nu\| = \sum_{(\mu,\nu)} |f_{(\mu,\nu)}| = \|f\|_1.$$

Hence $\|f\|_0 := \sup_{t \text{ a CK } \Lambda\text{-family}} \|\pi_t(f)\|$ defines a seminorm on A_0. Let $I := \{f : \|f\|_0 = 0\}$, and let $A := A_0/I$. Let $C^*(\Lambda)$ be the completion of A in the norm induced by $\|\cdot\|_0$, and let $s_\lambda := \delta_{(\lambda,s(\lambda))} + I$ for all λ. Then A is a C^*-algebra, and is universal by construction.

Since the Cuntz-Krieger family T of Example 2.7 consists of nonzero partial isometries, the universal property of $C^*(\Lambda)$ ensures that the s_λ are nonzero as well. □

Remark 2.11. Let Λ be a locally convex row-finite k-graph and let t be a Cuntz-Krieger Λ-family. Fix $\mu,\nu\in\Lambda$ with $s(\mu) = s(\nu) = v$, and suppose that $t_v \ne 0$. Then

$$\|t_\mu t^*_\nu\|^2 = \|t_\nu t^*_\mu t_\mu t^*_\nu\| = \|t_\nu t^*_\nu\| = \|t_\nu\|^2 = \|t^*_\nu t_\nu\| = \|t_v\| \ne 0.$$

In particular, each $s_\mu s^*_\nu \ne 0$ in $C^*(\Lambda)$.

For each $z \in \mathbb{T}^k$, the map $\lambda \mapsto z^{d(\lambda)}s_\lambda$ is a Cuntz-Krieger Λ-family, so the universal property of $C^*(\Lambda)$ gives an endomorphism

$$\gamma_z : C^*(\Lambda) \to C^*(\Lambda) \quad \text{such that} \quad \gamma_z(s_\lambda) = z^{d(\lambda)}s_\lambda \text{ for all } \lambda.$$

Since $\gamma_w \circ \gamma_z(s_\lambda) = \gamma_{wz}(s_\lambda)$ and $\gamma_1(s_\lambda) = s_\lambda$ for all λ, each γ_z is an automorphism of $C^*(\Lambda)$ and $z \mapsto \gamma_z$ is an action of $\mathbb{T}^k$. If $z_n \to z$, then $\gamma_{z_n}(s_\mu s^*_\nu) \to \gamma_z(s_\mu s^*_\nu)$ for all μ,ν, and then an $\frac{\varepsilon}{3}$ argument shows that γ is strongly continuous. It is then standard that $\Phi^\gamma(a) := \int_{\mathbb{T}^k}\gamma_z(a)\,dz$ defines a faithful conditional expectation from $C^*(\Lambda)$ to $C^*(\Lambda)^\gamma := \{a \in C^*(\Lambda) : \gamma_z(a) = a \text{ for all } z\}$.

Proposition 2.12. *Let Λ be a locally convex row-finite k-graph. Then*

(1) $C^*(\Lambda)^\gamma = \overline{\operatorname{span}}\{s_\mu s^*_\nu : d(\mu) = d(\nu), s(\mu) = s(\nu)\}$;
(2) $C^*(\Lambda)^\gamma = \varinjlim_{n\in\mathbb{N}^k}\bigoplus_{v\in\Lambda^0, m\le n}\overline{\operatorname{span}}\{s_\mu s^*_\nu : s(\mu) = s(\nu) = v, \mu,\nu\in\Lambda^{\le n}\cap\Lambda^m\}$.
(3) *Each* $\overline{\operatorname{span}}\{s_\mu s^*_\nu : s(\mu) = s(\nu) = v, \mu,\nu\in\Lambda^{\le n}\cap\Lambda^m\} \cong \mathcal{K}(\ell^2((\Lambda^{\le n}\cap\Lambda^m)v))$.

Proof. (1) For $\mu,\nu\in\Lambda$, we have

$$\Phi^\gamma(s_\mu s^*_\nu) = \int_{\mathbb{T}^k} z^{d(\mu)-d(\nu)}s_\mu s^*_\nu = \begin{cases} s_\mu s^*_\nu & \text{if } d(\mu) = d(\nu) \\ 0 & \text{otherwise.}\end{cases}$$

Since $\Phi^\gamma\circ\Phi^\gamma = \Phi^\gamma$ and $\Phi^\gamma(C^*(\Lambda)) = C^*(\Lambda)^\gamma$, this proves (1).

(2) Since $\mu,\nu\in\Lambda^{\le n}$ implies $s^*_\mu s_\nu = \delta_{\mu,\nu}s_{s(\mu)}$, we have

$$\overline{\operatorname{span}}\{s_\mu s^*_\nu : \mu,\nu\in\Lambda^{\le n}\} = \bigoplus_{v\in\Lambda^0, m\le n}\overline{\operatorname{span}}\{s_\mu s^*_\nu : s(\mu) = s(\nu) = v, \mu,\nu\in\Lambda^{\le n}\cap\Lambda^m\}$$

for each n, v, m. If $n \le p$, and $\mu,\nu\in\Lambda^{\le n}$ with $s(\mu) = s(\nu)$, then

$$s_\mu s^*_\nu = \sum_{\lambda\in s(\mu)\Lambda^{\le p-n}} s_{\mu\lambda}s^*_{\nu\lambda} \in \overline{\operatorname{span}}\{s_\eta s^*_\zeta : \eta,\zeta\in\Lambda^{\le p}\}$$

by Lemma 2.2. Hence

$$\overline{\operatorname{span}}\{s_\mu s^*_\nu : \mu,\nu\in\Lambda^{\le n}\} \subset \overline{\operatorname{span}}\{s_\mu s^*_\nu : \mu,\nu\in\Lambda^{\le p}\},$$

and (2) follows.

(3) Fix $n\in\mathbb{N}^k$, $m \le n$ and $v \in \Lambda^0$. For $\mu,\nu,\sigma,\tau\in(\Lambda^{\le n}\cap\Lambda^m)v$, we have $(s_\mu s^*_\nu)^* = s_\nu s^*_\mu$ and $s_\mu s^*_\nu s_\eta s^*_\zeta = \delta_{\nu,\zeta}s_\mu s^*_\zeta$. Remark 2.11 therefore implies that the $s_\mu s^*_\nu$ form a family of nonzero matrix units indexed by $(\Lambda^{\le n}\cap\Lambda^m)v$, and the result then follows from the uniqueness of $\mathcal{K}(\ell^2((\Lambda^{\le n}\cap\Lambda^m)v))$. □

Proposition 2.13. *Let Λ be a locally convex row-finite k-graph. Suppose that t is a Cuntz-Krieger Λ-family such that each t_v is nonzero, and suppose that there is a linear map $\Psi : C^*(t) \to C^*(t)$ such that $\Psi(t_\mu t^*_\nu) = \delta_{d(\mu),d(\nu)}t_\mu t^*_\nu$ for all μ,ν. Then $\pi_t : C^*(\Lambda) \to C^*(t)$ is injective.*

Proof. By Remark 2.11, whenever $s(\mu) = s(\nu)$, we have $t_\mu t^*_\nu \ne 0$. Since $\mathcal{K}(\ell^2((\Lambda^{\le n}\cap\Lambda^m)v))$ is simple, Proposition 2.12(3) implies that π_t is injective, hence isometric, on each $\overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in (\Lambda^{\leq n} \cap \Lambda^m)v\}$. So Proposition 2.12(2) implies that π_t is isometric on $C^*(\Lambda)^\gamma$. Since $\Psi \circ \pi_t = \pi_t \circ \Phi^\gamma$, and since Φ^γ is faithful on positive elements, we have

$$\pi_t(a) = 0 \implies \Psi(\pi_t(a^*a)) = 0 \implies \pi_t(\Phi^\gamma(a^*a)) = 0 \implies \Phi^\gamma(a^*a) = 0 \implies a = 0.$$

□

The following is one of the many generalisations to date of an Huef and Raeburn's gauge-invariant uniqueness theorem for unital Cuntz-Krieger algebras, and its proof like all the others is more or less identical to the one originally given by an Huef and Raeburn. It will be the single most useful result in our repertoire later in the course, and plays a similar role in the theory of k-graph algebras in general.

Corollary 2.14 (The gauge-invariant uniqueness theorem). *Let Λ be a locally convex row-finite k-graph. Suppose that t is a Cuntz-Krieger Λ-family such that each t_v is nonzero and such that there is an action β of $\mathbb{T}^k$ on $C^*(t)$ such that $\beta_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ for all λ. Then π_t is injective.*

Proof. The map $\Psi : a \mapsto \int_{\mathbb{T}} \beta_z(a)\, dz$ from $C^*(t)$ to $C^*(t)$ satisfies the hypotheses of Proposition 2.13. □

3. The Cuntz-Krieger uniqueness theorem and simplicity

The formulations of aperiodicity and cofinality used in this section are due to Lewin. The aperiodicity condition, in particular, is the latest refinement of a condition originally given by Kumjian and Pask which has since been re-cast and sharpened by many authors including Raeburn-S-Yeend, D. Robertson, and Shotwell.

Definition 3.1. We say that a k-graph Λ is *aperiodic* if, for all $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, there exists $\tau \in s(\mu)\Lambda$ such that $\text{MCE}(\mu\tau, \nu\tau) = \emptyset$.

Lemma 3.2. *Let Λ be a k-graph, and fix $v \in \Lambda^0$ and a finite subset H of Λv. Then there exists $\tau \in v\Lambda$ such that $\text{MCE}(\mu\tau, \nu\tau) = \emptyset$ for all $\tau \in H$.*

Proof. We proceed by induction on H. If $|H| = 1$ there is nothing to do.

Suppose there exists τ with the desired property whenever $|H| < n$, and fix $H \subset \Lambda v$ with $|H| = n$. Let λ be any element of H and let $G := H \setminus \{\lambda\}$. By the inductive hypothesis there exists τ_0 such that $\text{MCE}(\mu\tau_0, \nu\tau_0) = \emptyset$ for all $\mu, \nu \in G$. Enumerate $G = \{\mu_1, \dots, \mu_{n-1}\}$ and iteratively choose paths $\tau_i \in \Lambda$ such that for each i, $\text{MCE}((\mu_{i+1}\tau_0 \dots \tau_i)\tau_{i+1}, (\lambda\tau_0\tau_i)\tau_{i+1}) = \emptyset$. Then $\tau := \tau_0\tau_1 \dots \tau_{n-1}$ has the desired property: if $\mu, \nu \in G$, then

$$\text{MCE}(\mu\tau, \nu\tau) \subset \mu\tau_0\Lambda \cap \nu\tau_0\Lambda = \emptyset,$$

and if $\mu = \mu_i \in G$, then

$$\text{MCE}(\mu\tau, \lambda_\tau) =\subset \mu_i\tau_0 \dots \tau_i\Lambda \cap \lambda\tau_0 \dots \tau_i = \emptyset.$$

□

The following theorem is a generalisation of Cuntz and Krieger's original uniqueness result theorem for their C^*-algebras associated to $\{0, 1\}$-matrices. Indeed, the proof is, modulo the details, very much like the one given by Cuntz in his analysis of $\mathcal{O}_n$.

Theorem 3.3 (The Cuntz-Krieger uniqueness theorem). *Let Λ be a locally convex row-finite k-graph. Suppose that Λ is aperiodic. Let t be a Cuntz-Krieger Λ-family such that $t_v \neq 0$ for all $v \in \Lambda^0$. Then π_t is injective.*

Proof. We aim to apply Proposition 2.13. It suffices to show that for any finite $F \subset \Lambda$ and any collection of scalars $\{a_{\mu,\nu} : \mu, \nu \in F\}$,

$$\Big\| \sum_{\substack{\mu,\nu \in F \\ d(\mu)=d(\nu)}} a_{\mu,\nu} t_\mu t_\nu^* \Big\| \leq \Big\| \sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^* \Big\|,$$

for this implies that there is a well-defined linear map Ψ satisfying $\Psi(t_\mu t_\nu^*) = \delta_{d(\mu),d(\nu)} t_\mu t_\nu^*$ for all μ, ν.

Let $n := \bigvee_{\lambda \in F} d(\lambda)$. Then $\sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^* \in \overline{\text{span}}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda^{\leq n}\}$ by Proposition 2.12(2). Hence, by the same result there exist $v \in \Lambda^0$ and $m \leq n$ such that the strict-topology limit

$$P_{v,m} := \sum_{\lambda \in \Lambda^{\leq n} \cap \Lambda^m v} t_\mu t_\mu^*$$

satisfies

$$\Big\| \sum_{\substack{\mu,\nu \in F \\ d(\mu)=d(\nu)}} a_{\mu,\nu} t_\mu t_\nu^* \Big\| = \Big\| P_{v,m} \sum_{\substack{\mu,\nu \in F \\ d(\mu)=d(\nu)}} a_{\mu,\nu} t_\mu t_\nu^* P_{v,m} \Big\|.$$

Since Λ is row finite, $F\Lambda \cap \Lambda^{\leq n} \cap \Lambda^m$ is finite, so Lemma 3.2 implies that there exists $\tau \in v\Lambda$ such that whenever $\mu, \nu \in F$ with $d(\mu) \neq d(\nu)$, and $\mu\mu', \nu\nu' \in \Lambda^{\leq n} \cap \Lambda^m v$, we have $\text{MCE}(\mu'\tau, \nu'\tau) = \emptyset$. Let P be the strict-topology limit

$$P := \sum_{\lambda \in \Lambda^{\leq n} \cap \Lambda^m v} t_{\mu\tau} t_{\mu\tau}^* \leq P_{v,m}.$$

We have $P_{v,m}\overline{\text{span}}\{t_\mu t_\nu^* : \mu, \nu \in F\}P_{v,m} \subset \overline{\text{span}}\{t_\eta t_\zeta^* : \eta, \zeta \in \Lambda^{\leq n} \cap \Lambda^m v\}$. By Proposition 2.12(3), $t_\eta t_\zeta^* \mapsto \Theta_{\eta,\zeta}$ determined an isomorphism

$$\overline{\text{span}}\{t_\eta t_\zeta^* : \eta, \zeta \in \Lambda^{\leq n} \cap \Lambda^m v\} \cong \mathcal{K}(\ell^2(\Lambda^{\leq n} \cap \Lambda^m v)).$$

Since $P t_\eta t_\zeta^* P = t_{\eta\tau} t_{\zeta\tau}^*$ for all $\eta, \zeta \in \Lambda^{\leq n} \cap \Lambda^m v$ and since the $t_{\eta\tau} t_{\zeta\tau}^*$ also form a family of nonzero matrix units, compression by P is an injective C^*-homomorphism, and hence isometric on $\overline{\text{span}}\{t_\eta t_\zeta^* : \eta, \zeta \in \Lambda^{\leq n} \cap \Lambda^m v\}$. In particular,

$$\Big\| P \sum_{\substack{\mu,\nu \in F \\ d(\mu)=d(\nu)}} a_{\mu,\nu} t_\mu t_\nu^* P \Big\| = \Big\| P_{v,m} \sum_{\substack{\mu,\nu \in F \\ d(\mu)=d(\nu)}} a_{\mu,\nu} t_\mu t_\nu^* P_{v,m} \Big\| = \Big\| \sum_{\substack{\mu,\nu \in F \\ d(\mu)=d(\nu)}} a_{\mu,\nu} t_\mu t_\nu^* \Big\|.$$

Moreover, if $\mu, \nu \in F$ with $d(\mu) \neq d(\nu)$, then

$$P t_\mu t_\nu^* P = \sum_{\mu\mu', \nu\nu' \in \Lambda^{\leq n} \cap \Lambda^m v} t_{\mu\mu'\tau} t_{\mu\mu'\tau}^* t_\mu t_\nu^* t_{\nu\nu'\tau} t_{\nu\nu'\tau}^* = \sum_{\mu\mu', \nu\nu' \in \Lambda^{\leq n} \cap \Lambda^m v} t_{\mu\mu'\tau} t_{\mu'\tau}^* t_{\nu'\tau} t_{\nu\nu'\tau}^* = 0$$

by choice of τ. Hence

$$\Big\| \sum_{\substack{\mu,\nu \in F \\ d(\mu)=d(\nu)}} a_{\mu,\nu} t_\mu t_\nu^* \Big\| = \Big\| P \sum_{\substack{\mu,\nu \in F \\ d(\mu)=d(\nu)}} a_{\mu,\nu} t_\mu t_\nu^* P \Big\| = \Big\| P \sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^* P \Big\| \leq \Big\| \sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^* \Big\|$$

as required. □

Corollary 3.4. *Let Λ be a locally convex row-finite k-graph. The following are equivalent*

(1) *Λ is aperiodic;*

(2) *each nontrivial ideal of* $C^*(\Lambda)$ *contains* s_v *for some* $v \in \Lambda^0$.

Proof. (1) $\implies$ (2) is the Cuntz-Krieger uniqueness theorem.

For (2) $\implies$ (1) we prove the converse. Suppose that there exist distinct $\mu, \nu \in \Lambda$ such that $\mathrm{MCE}(\mu\tau, \nu\tau) \neq \emptyset$ for all $\tau \in s(\mu)\Lambda$.

We claim that an ultrafilter x of Λ contains μ if and only if it contains ν. To see this, it suffices by symmetry to show that $\mu \in x$ implies $\nu \in x$. Fix an ultrafilter x such that $\mu \in x$. Fix a cofinal sequence $(\mu_i)_{i=0}^\infty$ of x such that $\mu_0 = \mu$. For each i, $\mu_i = \mu\tau_i$ for some $\tau_i \in \Lambda$. By assumption, $\mathrm{MCE}(\mu\tau_i, \nu\tau_i) \neq \emptyset$ for all i. For $j \leq i$, we have $\mathrm{MCE}(\mu\tau_i, \nu\tau_i) \subset \mathrm{MCE}(\mu\tau_j, \nu\tau_j)\Lambda$. Since each $\mathrm{MCE}(\mu\tau_i, \nu\tau_i)$ is finite, we may inductively choose $\gamma_i \in \mathrm{MCE}(\mu\tau_i, \nu\tau_i)$ such that $\gamma_i \in \gamma_j\Lambda$ for all $j \leq i$, and such that $\gamma_i\Lambda \cap \mathrm{MCE}(\mu\tau_l, \nu\tau_l) \neq \emptyset$ for infinitely many, and hence all, $l > i$. Let $y = \{\alpha \in \Lambda : \gamma_j \in \alpha\Lambda$ for some $j\}$. Then y is a filter. We have $x \subset y$ because the μ_i were cofinal. Hence $y = x$. Since $\nu \in y$ by definition, we conclude that $\nu \in x$. This proves the claim.

By the preceding paragraph, the Cuntz-Krieger Λ-family T of Example 2.7 satisfies $T_\mu T_\nu^* = T_\mu T_\mu^*$. Moreover, $T_v \notin \ker(\pi_T)$ for all T. So it suffices to show that $s_\mu s_\nu^* \neq s_\mu s_\mu^*$. We have $s_\mu s_\nu^* \neq 0$ by Remark 2.11. Since $\mathrm{MCE}(\mu, \nu) \neq 0$ and $\mu \neq \nu$, we have $d(\mu) \neq d(\nu)$. Hence there exists $z \in \mathbb{T}^k$ such that $z^{d(\mu)-d(\nu)} = -1$. Now

$$(1 - \gamma_z)(s_\mu s_\mu^* - s_\mu s^*\nu) = 2s_\mu s_\mu^* \neq 0$$

and hence $s_\mu s_\mu^* \neq s_\mu s_\nu^*$ as required. □

Definition 3.5. We say that a locally convex row-finite k-graph Λ is *cofinal* if, for all $v, w \in \Lambda^0$, there exists $n \in \mathbb{N}^k$ such that $w\Lambda s(\lambda) \neq \emptyset$ for all $\lambda \in v\Lambda^{\leq n}$.

Proposition 3.6. *Let* Λ *be a locally convex row-finite* k*-graph. The following are equivalent*

(1) Λ *is cofinal;*
(2) *each ideal* I *of* $C^*(\Lambda)$ *such that* $s_v \in I$ *for some* $v \in \Lambda^0$ *satisfies* $I = C^*(\Lambda)$.

Proof. (1) $\implies$ (2). Fix an ideal I and a vertex w such that $s_w \in I$. Fix $v \in \Lambda^0$. Since Λ is cofinal, there exists $n \in \mathbb{N}^k$ and paths $\{\mu_\lambda : \lambda \in v\Lambda^{\leq n}\}$ such that $\mu_\lambda \in w\Lambda s(\lambda)$ for each $\lambda \in v\Lambda^{\leq n}$. Hence

$$s_v = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^* = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_{\mu_\lambda}^* s_w s_{\mu\lambda} s_\lambda^* \in I.$$

(2) $\implies$ (1). We prove the contrapositive. Fix $v, w \in \Lambda^0$ and suppose that for each $n \in \mathbb{N}^k$ there exits $\lambda \in v\Lambda^{\leq n}$ such that $w\Lambda s(\lambda) = \emptyset$. As before, we may inductively choose paths $\mu_n \in v\Lambda^{\leq n}$ such that each $\mu_n \in \mu_m\Lambda$ for all $m \leq n$ and such that for infinitely many (and hence all) $p > n$ there exists $\eta \in \mu_n\Lambda \cap \Lambda^{\leq p}$ such that $w\Lambda s(\eta) = \emptyset$. The set $x = \{\alpha : \mu_n \in \eta\Lambda$ for some $n\}$ is a filter. It is an ultrafilter because if $\beta \notin x$, then $\mu_{d(\beta)} \neq \beta$, and then since $\beta, \mu_{d(\beta)} \in \Lambda^{\leq} d(\beta)$ we have $\mathrm{MCE}(\mu_{d(\beta)}, \beta) = \emptyset$, and so there is no filter containing x which also contains β. Let

$$X_w := \{x \in \widehat{\Lambda}_\infty : w\Lambda s(\lambda) = \emptyset \text{ for all } \lambda \in x\}.$$

Then $\ell^2(X_w) \subset \ell^2(\widehat{\Lambda}_\infty)$ is invariant for the Cuntz-Krieger Λ-family T of Example 2.7, so $S_\lambda := T_\lambda|_{\ell^2(X_w)}$ determines a Cuntz-Krieger Λ-family S with $S_w = 0$ and $S_v \neq 0$. Hence $\ker(\pi_S)$ is a proper ideal containing a vertex projection. □

The following corollary, as stated, is due to D. Robertson. In its most general form, for finitely aligned k-graphs, it was first proved by Shotwell.

Corollary 3.7. *Let* Λ *be a locally convex row-finite* k*-graph. Then* $C^*(\Lambda)$ *is simple if and only if* Λ *is aperiodic and cofinal.*

Proof. Suppose that Λ is aperiodic and cofinal, and fix a nontrivial ideal I of $C^*(\Lambda)$. Then Corollary 3.4 implies that $s_v \in I$ for some $v \in \Lambda^0$, and then Proposition 3.6 implies that $I = C^*(\Lambda)$. For the converse observe that if Λ is not cofinal, then Proposition 3.6 yields a nontrivial proper ideal of $C^*(\Lambda)$, and if Λ is not cofinal, then Corollary 3.4 does the same job. □

4. Constructions of k-graphs

Constructions of $(k+l)$-graphs from k-graphs have appeared in many contexts beginning with the cartesian product construction of Kumjian and Pask, and including many authors since — we shan't list them here, but we shall see a number of specific examples later in these notes. The notion of a k-morph was introduced by Kumjian-Pask-S as a unifying framework for these constructions.

Definition 4.1. A k*-morph* between k-graphs Λ and Γ (or a Λ–Γ-morph for short) is a countable set X equipped with maps $r : X \to \Lambda^0$ and $s : X \to \Lambda^0$ and a bijection $\theta : X_s{*}_r\Gamma \to \Lambda_s{*}_rX$ such that whenever $\theta(x, \gamma) = (\lambda, y)$ we have

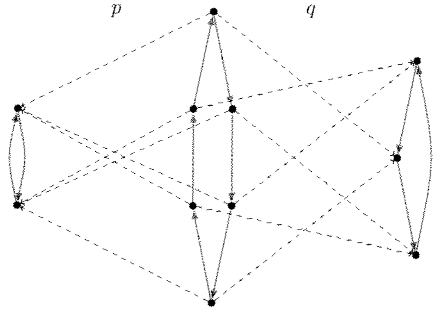
(M1) $d(\lambda) = d(\gamma)$;
(M2) $r(\lambda) = r(x)$;
(M3) $s(\gamma) = s(y)$;

and whenever, in addition, $\theta(y, \eta) = (\mu, z)$, we have

(M4) $\theta(x, \gamma\eta) = (\lambda\mu, z)$.

if $\Lambda = \Gamma$, we call X a Λ-*endomorph*.

Examples 4.2. (1) Fix k-graphs Λ, Σ, Γ and coverings $p : \Sigma \to \Lambda$ and $q : \Sigma \to \Gamma$; that is, degree-preserving functors which restrict to bijections on each $v\Sigma$ and Σv. Let $X =_p X_q := \{x_w : w \in \Sigma^0\}$, and define $r(x_w) := p(x_w)$ and $s(x_w) = q(x_w)$. Define $\theta : X * \Gamma \to \Lambda * X$ by $\theta(x_{r(\sigma)}, q(\sigma)) = (p(\sigma), x_{s(\sigma)})$. (To see that this makes sense observe that since q is a covering, σ can be recovered from $r(\sigma)$ and $q(\sigma)$.) In the picture below, Λ and Γ are cycles of length 2 and 3 and Σ is the common covering cycle of length 6.



(2) Fix a k-graph Λ and an automorphism α of Λ. Let $X_\alpha := \{x_v : v \in \Lambda^0\}$ with $r = \alpha$, $s = \mathrm{id}$, $\theta(x_{r(\lambda)}, \lambda) = (\alpha(\lambda), x_{s(\lambda)})$. Then X_α is a Λ-endomorph. In fact, X is precisely ${}_\alpha X_{\mathrm{id}}$ from (1).

Theorem 4.3. *Let Λ and Γ be k-graphs.*

(1) *Let X be a Λ–Γ-morph. There is a unique $(k+1)$-graph Σ, called the* linking graph *for X admitting an isomorphism $i = i_\Lambda, i_\Gamma : \Lambda \sqcup \Gamma \to \{\sigma \in \Sigma : d(\sigma)_{k+1} = 0\}$ and a bijection $i_X : X \to \Sigma^{e_{k+1}}$ such that $r(i_X(x)) = i_\Lambda(r(x))$ and $s(i_X(x)) = i_\Gamma(s(x))$ for all $x \in X$ and $i_X(x) i_\Gamma(\gamma) = i_\Lambda(\lambda) i_X(y)$ whenever $\theta(x, \gamma) = (\lambda, y)$.*

(2) *Let Y be a Λ-endomorph. There is a unique $(k+1)$-graph $\Lambda \times_Y \mathbb{N}$ admitting an isomorphism $i_\Lambda : \Lambda \to \{\gamma \in \Lambda \times_Y \mathbb{N} : d(\gamma)_{k+1} = 0\}$ and a bijection $i_Y : Y \to (\Lambda \times_Y \mathbb{N})^{e_{k+1}}$ such that $r(i_Y(y)) = i_\Lambda(r(y))$ and $s(i_Y(y)) = i_\Lambda(s(y))$ for all $y \in Y$, and such that $i_Y(y) i_\Lambda(\mu) = i_\Lambda(\nu) i_Y(z)$ whenever $\theta(y, \mu) = (\nu, z)$.*

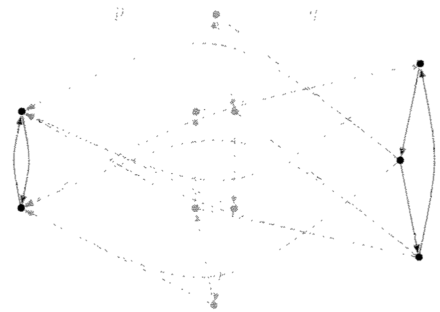
Proof. (1) Let $E = E_{\Lambda \sqcup \Gamma}$ be the k-coloured graph with colour map c associated to $\Lambda \sqcup \Gamma$. Define F by $F^0 := E^0$ and $F^1 := E^1 \sqcup \{i_X(x) : x \in X\}$, $r, s : F^1 \to F^0$ inherited from Λ, Γ and X, and colour map agreeing with c on E^1 and with $c(i_X(x)) = k+1$ for all $x \in X$. Define a collection $\mathcal{C}$ of squares to consists of those occurring in $\Lambda \sqcup \Gamma$ together with those of the form

$$\varphi(0 + v_{k+1}) = i_X(x), \quad \varphi(e_{k+1} + v_i) = \gamma, \quad \varphi(0 + v_i) = \lambda, \text{ and } \quad \varphi(e_i + v_{k+1}) = i_X(y)$$

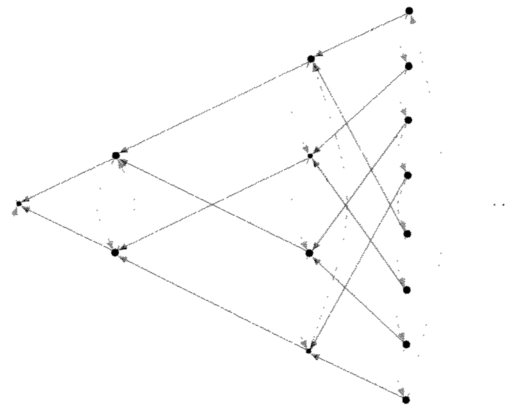
whenever $\theta(x, \gamma) = (\lambda, y)$. Then $\mathcal{C}$ is a complete collection of squares because θ is a bijection, and is associative by (M4). Let Σ be the $(k+1)$-graph obtained from F, c and $\mathcal{C}$ as in Theorem 1.9. In particular, $\Lambda^0 \Sigma \Lambda^0$ is a k-graph with the same coloured graph as Λ, so the uniqueness assertion of Theorem 1.9 gives an isomorphism $i_\Lambda : \Lambda \to \Lambda^0 \Sigma \Lambda^0$, and similarly for Γ. The $(k+1)$-graph Σ satisfies the desired factorisation regime by definition. Uniqueness of Σ follows from another application of the uniqueness assertion of Theorem 1.9.

(2) The proof is basically the same as that of (1), except that $i_\Lambda : \Lambda \to \Sigma$ maps onto $\{\sigma \in \Sigma : d(\sigma)_{k+1} = 0\}$ rather than $\Lambda^0 \Sigma \Lambda^0$. □

Examples 4.4. (1) The common covering of the 2-cycle and the 3-cycle by the 6-cycle above gives the following linking graph:



(2) Let Λ be the complete directed binary tree described with the vertices at level n indexed by $\mathbb{Z}/2^n\mathbb{Z}$ and an edge from the vertex i at level n to the vertex j at level $n-1$ if i is congruent to j mod 2^{n-1}. There is a unique automorphism α which acts on the vertices at level n by addition of 1 modulo 2^n. The resulting endomorph crossed product $\Lambda \times_{X_\alpha} \mathbb{N}$ has the following coloured graph.



Theorem 4.5. *Let Λ and Γ be locally convex row-finite k-graphs, and let X be a Λ–Γ-morph in which both r and s are surjective, and r is finite-to-one. Let Σ be the linking graph. Then $\sum_{v \in \Lambda^0} s_{i_\Lambda(v)}$ and $\sum_{w \in \Gamma^0} s_{i_\Gamma(w)}$ converge to full projections P_Λ and P_Γ in the multiplier algebra $\mathcal{M}C^*(\Sigma)$. The map $i^*_\Lambda : s_\lambda \mapsto s_{i_\Lambda(\lambda)}$ determines an injective homomorphisms $i^*_\Lambda : C^*(\Lambda) \to P_\Lambda C^*(\Sigma) P_\Lambda$, and the map $i^*_\Gamma : s_\gamma \mapsto s_{i_\Gamma(\gamma)}$ determines an isomorphism $i^*_\Gamma : C*(\Gamma) \cong P_\Gamma C^*(\Sigma) P_\Gamma$.*

Proof. Recall that $C^*(\Sigma) = \overline{\operatorname{span}}\{s_\sigma s_\tau^* : \sigma, \tau \in \Sigma\}$. For any finite linear combination $a = \sum_{\sigma,\tau\in F} a_{\sigma,\tau} s_\sigma s_\tau^*$, the projection $P_{r(F)} := \sum_{v\in r(F)} s_v$ satisfies $P_{r(F)} a P_{r(F)} = a$. It is therefore straightforward that $\sum_{v\in\Lambda^0} s_{i_\Lambda(v)}$ and $\sum_{w\in\Gamma^0} s_{i_\Gamma(w)}$ converge in the strict topology.

To see that they are full, fix $\sigma \in \Sigma$. Then

$$s_\sigma = \sum_{\alpha\in s(\sigma)\Sigma^{\le e_{k+1}}} s_\sigma s_\alpha P_\Gamma s_\alpha^* \in C^*(\Sigma) P_\Gamma C^*(\Sigma),$$

so P_Γ is full. Moreover since $s : X \to \Gamma^0$ is surjective, we may choose a surjective section $x : \Gamma^0 \to X$ for s, and then $P_\Gamma = \sum_{v\in\Gamma^0} s^*_{i_X(x(v))} P_\Lambda s_{i_X(x(v))}$, and it follows that P_Λ is full also.

The map $\lambda \mapsto s_{i_\Lambda(\lambda)}$ is a Cuntz-Krieger Λ-family in $C^*(\Sigma)$ because the relations in $C^*(\Sigma)$ include all those from $C^*(\Lambda)$; so the universal property of $C^*(\Lambda)$ gives a homomorphism $i_\Lambda^* : s_\lambda \mapsto s_{i_\Lambda(\lambda)}$. The gauge action on $C^*(\Sigma)$ restricts to an action β of $\mathbb{T}^k$ on $P_\Lambda C^*(\Sigma) P_\Lambda$ such that $\beta_z(i_\Lambda^*(s_\lambda)) = z^{d(\lambda)} i_\Lambda^*(s_\lambda)$ for all λ. Hence the gauge-invariant uniqueness theorem implies that i_Λ^* is injective. The same argument applies to i_Γ^*.

We clearly have $i_\Lambda^* : C^*(\Lambda) \subset P_\Lambda C^*(\Sigma) P_\Lambda$ and $i_\Gamma^* : C*(\Gamma) \subset P_\Gamma C^*(\Sigma) P_\Gamma$. Moreover, for each $\sigma, \tau \in \Sigma$, we have

$$P_\Gamma s_\sigma s_\tau^* P_\gamma = \begin{cases} s_\sigma s_\tau^* & \text{if } r(\sigma), r(\tau) \in i_\Gamma(\Gamma^0) \\ 0 & \text{otherwise.} \end{cases}$$

Since $r(\sigma) \in i_\Gamma(\Gamma^0)$ implies $\sigma \in i_\Gamma(\Gamma)$, it follows that $P_\Gamma C^*(\Sigma) P_\Gamma \subset i_\Gamma^* : C*(\Gamma)$. □

Corollary 4.6. *Under the hypotheses of Theorem 4.5 the vector space $H_X := P_\Lambda C^*(\Sigma) P_\Gamma$ is a $C^*(\Lambda)$–$C^*(\Gamma)$-correspondence with linking algebra $C^*(\Sigma)$. If $\Lambda = \Gamma$ so that X is a Λ-endomorph, then H_X is a $C^*(\Lambda)$–$C^*(\Lambda)$-correspondence, and $C^*(\Lambda \times_X \mathbb{N}) \cong \mathcal{O}_{H_X}$.*

Proof sketch. The first statement follows from general nonsense - the C^*-identity ensures that the norm on the linking algebra is the same as the norm on H_X which is the same as the restriction of the norm on $C^*(\Sigma)$.

For the second statement, recall that $\mathcal{O}_{H_X}$ is generated by a copy $j_\Lambda(C^*(\Lambda))$ of $C^*(\Lambda)$ and a copy $j_H(H_X)$ of H_X. Setting

$$t_{i_\Lambda(\lambda)} := j_\Lambda(s_\lambda) \text{ for } \lambda \in \Lambda \quad \text{and} \quad t_{i_X(x)} := j_H(s_x) \text{ for } x \in X,$$

and extending this to a map $t : \Lambda \times_X \mathbb{N} \to \mathcal{O}_{H_X}$ by (CK2) gives a Cuntz-Krieger $(\Lambda \times_X \mathbb{N})$-family which generates $\mathcal{O}_X$. The universal property of $\mathcal{O}_{H_X}$ implies that it carries an action β of $\mathbb{T}^{k+1}$ which matches up with the gauge action on $j_\Lambda(C^*(\Lambda))$ and satisfies $\beta_z(t_{i_X(x)}) = z_{k+1} t_x$ for all $x \in X$, so the gauge-invariant uniqueness theorem implies that $\pi_t : C^*(\Lambda \times_X \mathbb{N}) \to \mathcal{O}_{H_X}$ is injective. □

Remark 4.7. It follows from the above construction that if Λ is a locally convex row-finite k-graph, then $C^*(\Lambda)$ is an iterated Cuntz-Pimsner algebra in the sense of Deaconu: given a k-graph Γ, the set $X_k := \Gamma^{e_k}$ is an endomorph of the $(k-1)$-graph $\Lambda := \{\lambda \in \Gamma : d(\lambda_k) = 0\}$ whose endomorph crossed-product is Γ. Hence $C^*(\Gamma) \cong \mathcal{O}_{H_{X_k}}$. Iterating this construction k-times gives an iterated Cuntz-Pimsner algebra construction of $C^*(\Gamma)$ with initial coefficient algebra $c_0(\Gamma^0)$.

Corollary 4.8. *Let $\alpha : \Lambda \to \Lambda$ be an automorphism. Then there is an automorphism $\tilde\alpha$ of $C^*(\Lambda)$ satisfying $\tilde\alpha(s_\lambda) = s_{\alpha(\lambda)}$ for all λ, and $C^*(\Lambda \times_{X_\alpha} \mathbb{N}) \cong C^*(\Lambda) \times_{\tilde\alpha} \mathbb{Z}$.*

Proof sketch. Let $X = X_\alpha$. Since α is bijective, H_X is isomorphic as a vector-space to $C^*(\Lambda)$. This isomorphism carries the inner product on H_X to the standard right inner-product $\langle a, b\rangle := a^* b$ on $C^*(\Lambda)$, so $H_X \cong C^*(\Lambda)_{C^*(\Lambda)}$ as a right Hilbert module. By definition of $\theta : X * \Lambda \to \Lambda * X$, the left action on H_X is given by $t_\lambda \cdot t_\mu = t_{\alpha(\lambda)\mu}$, so H_X is isomorphic as a right-Hilbert bimodule to ${}_{\tilde\alpha}C^*(\Lambda)$. Pimsner's theorem therefore shows that $\mathcal{O}_{H_X} \cong C^*(\Lambda) \times_{\tilde\alpha} \mathbb{Z}$; combined with Corollary 4.6, this proves the result. □

Hooptedoodle. In fact, the assignments $X \mapsto H_X$ and $\Lambda \mapsto C^*(\Lambda)$ determine a contravariant functor from a category M_k whose objects are k-graphs and whose morphisms are isomorphism classes of k-morlips (the fibred product of k-morphs determines a composition) to the category C with C^*-algebras as objects and isomorphism classes of C^*-correspondences as morphisms. We can then construct graphs of k-morphs: these are functors from 1-graphs to M_k. Indeed, this can be made to work for l-graphs of k-morphs, though in that instance more information is required than just the functor. We won't go into all this as we are only interested in a special case.

Proposition 4.9. *Let $\Lambda_0, \Lambda_1, \dots, \Lambda_n$ be locally convex row-finite k-graphs, and let X_i be a Λ_{i-1}–Λ_i-morph with r, s surjective and r finite-to-one for each $1 \le i \le n$. There is a unique $(k+1)$-graph Σ admitting an isomorphism $i : \bigsqcup_{i=0}^n \Lambda_i \to \{\sigma \in \Sigma : d(\sigma)_{k+1} = 0\}$, bijections $i_{X_i} : i_{\Lambda_{i-1}}(\Lambda_{i-1}^0)\Sigma^{e_{k+1}} i_{\Lambda_i}(\Lambda_i^0)$ such that the factorisation property in Σ is inherited from the Λ_i and the bijections $\theta_i : X_i * \Lambda_i \to \Lambda_{i-1} * X_i$. The maps i_{Λ_i} determine injective homomorphisms $i_{\Lambda_i}^* : C^*(\Lambda_i) \to C^*(\Sigma)$, each $P_{\Lambda_i} = \sum_{v\in\Lambda_i^0} i_{\Lambda_i}^*(s_v)$ is full, and $P_{\Lambda_n} C^*(\Sigma) P_{\Lambda_n} = i_{\Lambda_n}^*(C^*(\Lambda_n))$.*

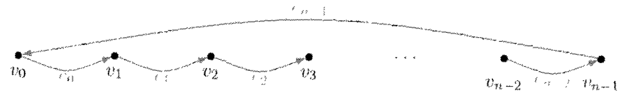
Proof. The proof is almost identical to those of Theorems 4.3 and 4.5. □

Remark 4.10. If $n = \infty$ in Proposition 4.9, it is still straightforward to establish the existence of the enveloping $(k+1)$-graph Σ and that that P_{Λ_i} are full, but in general $P_{\Lambda_n} C^*(\Sigma) P_{\Lambda_n} \neq i_{\Lambda_n}^*(C^*(\Lambda_n))$ for any n.

5. Rank-2 Bratteli diagrams and A$\mathbb{T}$-algebras.

The results in this section are due to Pask-Raeburn-Rørdam-S, though we have proved them in a very different manner to streamline arguments and highlight demonstrate how the k-morph construction can be used.

We will write c_n for the 1-graph with vertices $\{v_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ and edges $\{e_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ with $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$.



Fix, for the section, a sequence $(\Lambda_n)_{n=1}^\infty$ of 1-graphs such that each $\Lambda_n = \bigsqcup_{i=1}^{m_n} \Lambda_{n,i}$ where each $\Lambda_{n,i} \cong C_{|\Lambda_{n,i}^0|}$. For each $n \ge 1$, and each pair i, j with $i \le m_{n-1}$ and $j \le m_n$, fix $c^n_{i,j} \in \mathbb{N}$; we assume that for each i there exits j such that $c^n_{i,j} \neq 0$ and that for each j there exists i such that $c^n_{i,j} \neq 0$.

Whenever $c^n_{i,j} \neq 0$, let $X^n_{i,j} := {}_pX_q$ be the k-morph of Example 4.2(1) for the canonical coverings

$$p : C_{c^n_{i,j} \gcd(|\Lambda^0_{n-1,i}|,|\Lambda^0_{n,j}|)} \to \Lambda_{n-1,i} \quad \text{and } q : C_{c^n_{i,j} \gcd(|\Lambda^0_{n-1,i}|,|\Lambda^0_{n,j}|)} \to \Lambda_{n,j}.$$

For each n, $X^n := \bigsqcup_{c^n_{i,j} \neq 0} X^n_{i,j}$ is a Λ_{n-1}–Λ_n-morph.

As in Remark 4.10, there is a unique 2-graph Γ such that $\Gamma^0 = \bigsqcup \Lambda^0_n$, $\Gamma^{e_1} = \bigsqcup \Lambda^1_n$ and $\Gamma^{e_2} = \bigsqcup X^n$, and where the factorisation rules are determined by the bijections $\theta^n_{i,j} : X^n_{i,j} * \Lambda_{n,j} \to \Lambda_{n-1,i} * X^n_{i,j}$. We call Γ a *rank-2 Bratteli diagram.*

To analyse $C^*(\Gamma)$, we first study its building blocks.

Lemma 5.1. *For $n \geq 1$, $C^*(C_n) \cong M_n \otimes C(\mathbb{T})$.*

Proof. For $i \in \mathbb{Z}/n\mathbb{Z} \setminus \{0\}$, define $t_{e_i} := \theta_{i+1,i} \otimes 1 \in M_n \otimes C(\mathbb{T})$, and define $t_{e_0} := \theta_{1,0} \otimes z$. Now define $t_{v_i} := t^*_{e_i} t_{e_i} = \theta_{i,i} \otimes 1$ for each $v_i \in C^0_n$, and for $\alpha = \alpha_1 \dots \alpha_{|\alpha|}$ define $t_\alpha = t_{\alpha_1} \dots t_{\alpha_{|\alpha|}}$. Then t is a Cuntz-Krieger C_n-family which clearly generates $M_n \otimes C(\mathbb{T})$. There is an action β of $\mathbb{T}$ on $M_n \otimes C(\mathbb{T})$ determined by $\beta_z(\theta_{i,j} \otimes f) := \theta_{i,j} \otimes f(z^{i-j} \cdot)$, and we have $\beta_z \circ \pi_t = \pi_t \circ \gamma_z$ for all z. Hence the gauge-invariant uniqueness theorem implies that π_t is an isomorphism. □

Corollary 5.2. *For each $N \in \mathbb{N}$, let $\Gamma[0, N] := \left(\bigcup_{i=1}^n \Lambda^0_n\right)\Gamma\left(\bigcup_{i=1}^n \Lambda^0_n\right)$. Then $C^*(\Gamma) \cong \bigoplus_{i=1}^{m_n} M_{l_i} \otimes C(\mathbb{T})$ for some collection $l_1, \dots, l_{m_N} \in \mathbb{N} \setminus \{0\}$.*

Proof. Proposition 4.9 and shows that

$$C^*(\Gamma_{[0,N]}) \sim_{\text{Me}} P_{\Lambda_N} C^*(\Gamma_{[0,N]}) P_{\Lambda_N} \cong C^*(\Lambda_N) = \bigoplus_{i=1}^{m_N} C^*(\Lambda_{N,i}).$$

Lemma 5.1 then implies that $C^*(\Gamma_{[0,N]}) \sim_{\text{Me}} \bigoplus_{i=1}^{m_N} C(\mathbb{T})$. Since $C^*(\Gamma_{[0,N]})$ is unital (with identity $\sum_{v \in \Gamma^0} s_v$), the result follows because of the deep result due to amongst separable C^*-algebras, Morita equivalence is the same as stable isomorphism. □

Proposition 5.3. *For each $N \in \mathbb{N}$, let $P_N := \sum_{n=0}^N \sum_{v \in \Lambda^0_n} s_v \in C^*(\Gamma)$. Then each $P_N C^*(\Gamma) P_N \cong C^*(\Gamma_{[0,N]})$, and*

$$C^*(\Gamma) = \overline{\bigcup_{N=0}^{\infty} P_N C^*(\Gamma) P_N}. \tag{5.1}$$

In particular, $C^*(\Gamma)$ is an $A\mathbb{T}$-algebra.

Proof. Yet another application of the gauge-invariant uniqueness theorem gives the isomorphism of $C^*(\Gamma_{[0,N]})$ onto $P_N C^*(\Gamma) P_N$. For $\gamma \in \Gamma$, we have $s(\gamma) \in \Lambda^0_n$ for some n, and then $s_\gamma \in P_N C^*(\Gamma) P_N$ for all $N \geq n$, establishing (5.1). The last statement is by definition of $A\mathbb{T}$-algebras. □

Theorem 5.4. *Suppose that*

(1) *for all $w \in \Lambda^0$ there exists $n \in \mathbb{N}$ such that $w\Lambda v \neq \emptyset$ for all $v \in \Lambda^0_n$; and*
(2) *for all $l \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $|X^n_{i,j}| \geq l$ whenever $c^n_{i,j} \neq 0$.*

Then $C^*(\Gamma)$ is simple and has real rank 0.

Proof. For simplicity, we just need to show that Γ is cofinal and aperiodic — Corollary 3.7 does the rest.

For cofinality, fix $v, w \in \Gamma^0$. By (1), there exists $n_0 \in \mathbb{N}$ such that $w\Gamma u \neq \emptyset$ for all $u \in \Lambda^0_{n_0}$. Since the X_n are k-morphs, so their source maps are surjective, we then have $w\Gamma u \neq \emptyset$ for all $u \in \bigcup_{n=n_0}^\infty \Lambda^0_n$. Let $m \in \mathbb{N}$ be the integer such that $v \in \Lambda^0_m$, and let $N := \max\{n_0, m\}$. Then $s(v\Gamma^{\leq (N-m)e_2}) \subset \Lambda^0_N$, and hence $w\Gamma s(\lambda) \neq \emptyset$ for all $\lambda \in v\Gamma^{\leq (N-m)e_2}$. For aperiodicity, fix distinct paths $\mu, \nu \in \Gamma$. If $d(\mu) = d(\nu)$, or $s(\mu) \neq s(\nu)$ or $r(\mu) \neq r(\nu)$, then $\tau := s(\mu)$ satisfies $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$. Moreover, if $d(\mu)_2 \neq d(\nu)_2$, then either $r(\mu) \neq r(\nu)$ or $s(\mu) \neq s(\nu)$ since one pair or the other must be in different levels of Γ^0. So suppose that $r(\mu) = r(\nu)$, $s(\mu) = s(\nu) \in \Lambda^0_n$ (and hence $d(\mu)_2 = d(\nu)_2$) and that $d(\mu) \neq d(\nu)$; so $d(\mu)_1 \neq d(\nu)_1$. Factorise $\mu = \mu'\mu''$ and $\nu = \nu'\nu''$ where $d(\mu')_1 = d(\nu')_1 = 0 = d(\mu'')_2 = d(\nu'')_2$. Using (2), fix $m \geq n+1$ such that $|X^m_{i,j}| > |d(\mu)_1 - d(\nu)_1|$ for all i, j such that $c^m_{i,j} \neq 0$. Let τ be any element of $s(\mu)\Gamma^{(m-n)e_1}$. By definition of the $X^l_{i,j}$ we may identify each with $\mathbb{Z}/|X^l_{i,j}|\mathbb{Z}$, so we can identify τ with a sequence $[p_{n+1}][p_{n+2}] \dots [p_m]$ where each $[p_l] \in \mathbb{Z}/|X^l_{i,j}|\mathbb{Z}$ for some i, j. In particular, $[p_m] \in \mathbb{Z}/|X^m_{i,j}|\mathbb{Z}$ for some i, j and by choice of m it follows that $[p_m + d(\mu)_1] \neq [p_m + d(\nu)_1]$. By definition of the $X^l_{i,j}$, we have

$$\mu\tau = \mu'\mu''\tau = \mu'[p_{n+1} + d(\mu)_1] \dots [p_m + d(\mu)_1]\mu''' \quad \text{and}$$
$$\nu\tau = \nu'\nu''\tau = \mu'[p_{n+1} + d(\nu)_1] \dots [p_m + d(\nu)_1]\nu'''$$

For some μ''', ν'''. In particular,

$$\mu\tau(d(\mu'\tau) - e_1, d(\mu'\tau)) = [p_m + d(\mu)_1] \neq [p_m + d(\nu)_1] = \nu\tau(d(\mu'\tau) - e_1, d(\mu'\tau)),$$

so $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$.

It remains to show that $C^*(\Gamma)$ has real rank 0. To do this we apply a powerful result of Blackadar-Bratteli-Elliott-Kimjian which says that a simple $A\mathbb{T}$ algebra has real rank 0 if and only if projections separate tracial states. For this, fix a trace τ on $C^*(\Gamma)$. Fix paths $\alpha, \beta \in \Gamma$ with $d(\alpha)_1 = d(\beta)_1 = 0$ and a path μ such that $d(\mu)_2 = 0$, and suppose that $\tau(s_\alpha s_\mu s^*_\beta) \neq 0$. Then $\tau(s^*_\beta s_\alpha s_\mu) \neq 0$, forcing $s^*_\beta s_\alpha \neq 0$, so $r(\alpha) = r(\beta)$. Since $d(\mu)_2 = 0$, each of $r(\mu)$ and $s(\mu)$ belong to the same Λ^0_n, and it follows that $d(\alpha) = d(\beta)$, and then $s^*_\beta s_\alpha \neq 0$ forces $\alpha = \beta$, and μ is a cycle. Choose $m > n$ such that $|X^m_{i,j}| > |\mu|$ whenever $c^m_{i,j} \neq 0$. Then

$$0 \neq \tau(s_\alpha s_\mu s^*_\beta) = \tau(s^*_\alpha s_\alpha s_\mu) = \sum_{\eta \in s(\mu)\Lambda^{(m-n)e_2}} \tau(s_\mu s_\eta s^*_\eta) = \sum_{\eta \in s(\mu)\Lambda^{(m-n)e_2}} \tau(s_{\eta'} s_{\mu'} s^*_\eta)$$

where each $\mu\eta = \eta'\mu'$ with $d(\mu') = d(\mu)$; and this forces $\eta' = \eta$ for all η. By choice of m, this forces $d(\mu) = 0$. A similar argument applies to show that $\tau(s_\alpha s^*_\mu s_\beta) not = 0$ forces $\alpha = \beta$ and $d(\mu) = 0$. The factorisation property and the Cuntz-Krieger relations show that $C^(\Gamma)$ is spanned by elements of the form $t_\alpha t_\mu t^*_\beta$ and $t_\alpha t^*_\mu t^*_\beta$, and it follows that if traces τ_1, τ_2 agree on all elements of the form $t_\alpha t^*_\alpha$, then they are equal. In particular, projections separate tracial states as required. □

It now follows from Elliott's classification theorem that $C^*(\Gamma)$ is classified by its K-theory.

Theorem 5.5. *Let E_0 be the directed graph with one vertex $u_{n,j}$ for each $\Lambda_{n,j}$ and $|u_{n,i}E_0^1u_{n+1,j}| = |X_{i,j}^n|/|\Lambda_{n,i}^0|$, and let E_1 be the directed graph with one vertex $w_{n,j}$ for each $\Lambda_{n,j}$ and $|w_{n,i}E_0^1w_{n+1,j}| = |X_{i,j}^n|/|\Lambda_{n+1,j}^0|$. Then $K_0(C^*(\Gamma))$ is the dimension group associated to the Bratteli diagram E_0, and $K_1(C^*(\Gamma))$ is group-isomorphic to the dimension group associated to the Bratteli diagram E_1.*

Proof sketch. Let $A := \{\lambda \in \Gamma : d(\lambda)_1 = 0\}$ regarded as a 1-graph. So $A^1 = \bigsqcup_{n=1}^\infty X^n$. The map $v_i \mapsto v_{i+1}$ is a bijection of each $X_{i,j}^n$ and determines an automorphism of A. It is straightforward to see that $\Gamma \cong A \times_{\alpha X} \mathbb{N}$, so Corollary 4.8 implies that $C^*(\Gamma) \cong C*(A) \times_{\tilde\alpha} \mathbb{Z}$.

A theorem of Drinen shows that $C^*(A)$ is Morita equivalent to the AF algebra with Bratteli diagram A. The Pimsner-Voiculescu exact sequence in K-theory then implies that $K_0(C^*(\Gamma)) = \operatorname{coker}(1 - \tilde\alpha_*)$ and $K_1(C^*(\Gamma)) = \ker(1 - \tilde\alpha_*)$.

To describe the K-theory of $C^*(A)$, recall that $K_*(M_n) = (\mathbb{Z}, \{0\})$ with generator $[p]$ for any minimal projection p. Hence

$$K_1(C^*(A)) = 0 \quad \text{and} \quad K_0(C^*(A)) = \varinjlim \bigoplus_{v\in\Lambda_n^0} \mathbb{Z}[s_v],$$

with linking maps determined by

$$[s_v] = \sum_{\alpha\in vA^1} [s_\alpha s_\alpha^*] = \sum_{\alpha\in vA^1} [s_{s(\alpha)}] = \sum_{w\in\Lambda_{n+1}^0} |vA^1w|[s_w].$$

The automorphism $\tilde\alpha$ permutes the s_w for w in a given $\Lambda_{n,i}^0$. So $\ker(1 - \tilde\alpha_*)$ consists of functions which are constant on cycles. That is

$$\ker(1 - al\tilde{p}ha_*) \cap \bigoplus_{v\in\Lambda_n^0} \mathbb{Z}[s_v] = \bigoplus_{i=1}^{m_n} \mathbb{Z}[P_{n,i}]$$

where $P_{n,i} = \sum_{v\in\Lambda_{n,i}^0} s_v$. Relation (CK4) gives

$$\begin{aligned}[P_{n,i}] &= \sum_{j=1}^{m_{n+1}} \sum_{\alpha\in\Lambda_{n,i}^0 A^1 \Lambda_{n+1,j}^0} [s_{s(\alpha)}] \\ &= \sum_{j=1}^{m_{n+1}} \frac{|\Lambda_{n,i}^0 A^1 \Lambda_{n+1,j}^0|}{|\Lambda_{n+1,j}^0|}[P_{n+1,j}] \\ &= \sum_{j=1}^{m_{n+1}} \frac{|X_{i,j}^{n+1}|}{|\Lambda_{n+1,j}^0|}[P_{n+1,j}].\end{aligned}$$

Continuity of K-theory then establishes the formula for $K_1(C^*(\Gamma))$.

Similarly, for each n, i, the classes $[s_v] \in C^*(A)$ where $v \in \Lambda_{n,i}^0$ are all equivalent modulo the image of $(1 - \tilde\alpha_*)$. Hence $\operatorname{coker}(1 - \tilde\alpha_*) \cap \bigoplus_{v\in\Lambda_n^0} \mathbb{Z}[s_v] = \bigoplus_{i=1}^{m_n} \mathbb{Z}[s_{w_{n,i}}]$ where $(n, i) \mapsto w_{n,i}$ is a fixed choice of representative for each $\Lambda_{n,i}^0$. The Cuntz-Krieger relations for A show that in $\operatorname{coker}(1 - \tilde\alpha_*)$,

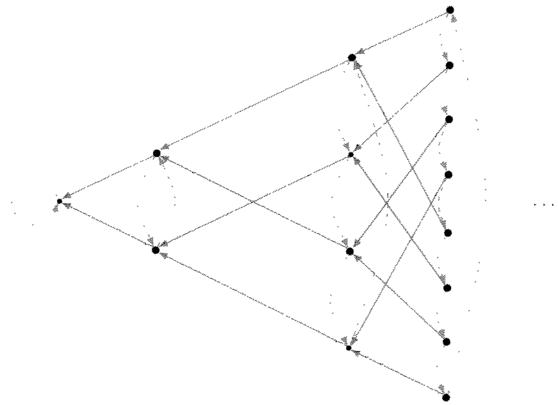
$$|\Lambda_{n,i}^0|[s_{w_{n,i}}] = [P_{n,i}] = \sum_j |\Lambda_{n,i}^0 A^1 \Lambda_{n+1,j}^0|[s_{w_{n+1,j}}].$$

so

$$[s_{w_{n,i}}] = \sum_j |X_{i,j}^{n+1}|/|\Lambda_{n,i}^0|[s_{w_{n+1,j}}].$$

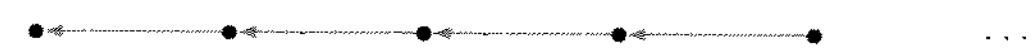
Continuity of K-theory once again establishes the formula for $K_0(C^*(\Gamma))$. □

Examples 5.6. (1) For the rank-2 Bratteli diagram Γ with coloured graph



the graphs E_1 and E_2 are

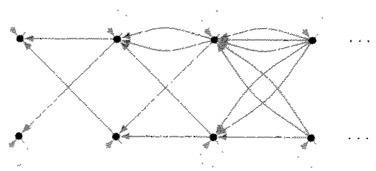
E_1:



E_0:

so we have $K_*(C^*(\Gamma)) = (\mathbb{Z}[\frac{1}{2}], \mathbb{Z})$ and hence $C^*(\Gamma)$ is stably isomorphic to the 2^∞ Bunce-Deddens algebra.

(2) For the rank-2 Bratteli diagram Γ with coloured graph



both E_0 and E_1 are isomorphic to the 1-graph obtained by deleting the loops. Hence results of Effros and Shen show that the K_0-group associated to this diagram is $\mathbb{Z} + \theta\mathbb{Z}$ where θ is the irrational number $\frac{1+\sqrt{5}}{2}$. Hence $K_*(C^*(\Gamma)) =$

$(\mathbb{Z}+\theta\mathbb{Z}, \mathbb{Z}^2)$, and it follows that $C^*(\Gamma)$ is Morita equivalent to the irrational rotation algebra for rotation θ.

6. Coactions, crossed-products and coverings

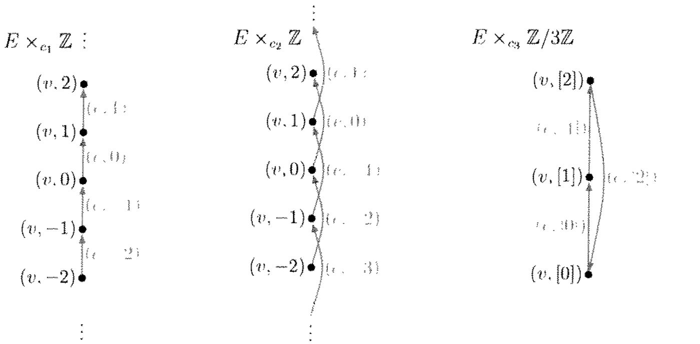
The connection between skew products and coaction crossed-products was first established for graph C^*-algebras by Kaliszewski-Quigg-Raeburn and was extended to k-graphs by Pask-Quigg-Raeburn.

Definition 6.1. Let Λ be a k-graph, and let $c : \Lambda \to G$ be a functor into a discrete group G. The skew-product k-graph $\Lambda \times_c G$ is given by $(\Lambda \times_c G)^n := \Lambda^n \times G$ with

$$r(\lambda, g) = (r(\lambda), c(\lambda)g), \quad s(\lambda, g) = (s(\lambda), g) \quad \text{and} \quad (\lambda, c(\mu)g)(\mu, g) = (\lambda\mu, g).$$

It is straightforward to see that $(\Lambda \times_c G)^{\leq n} = \Lambda^{\leq n} \times G$.

Example 6.2. Let E be the 1-graph with a single vertex v and a single edge e. The following are, from left to right, the skew-product graphs for the functors determined by $c_1(e) = 1 \in \mathbb{Z}$, $c_2(e) = 2 \in \mathbb{Z}$ and $c_3(e) = [1] \in \mathbb{Z}/3\mathbb{Z}$.



Theorem 6.3. *Let Λ be a locally convex row-finite k-graph and let $c : \Lambda \to G$ be a functor into a discrete group. Then there is a coaction δ of G on $C^*(\Lambda)$ determined by $\delta(s_\lambda) = s_\lambda \otimes c(\lambda)$ for each $\lambda \in \Lambda$. Moreover, $C^*(\Lambda \times_c G) \cong C^*(\Lambda) \times_\delta G$ via an isomorphism which carries $s_{(\lambda,g)}$ to $\iota(s_\lambda)q_g$ where $\iota : C^*(\Lambda) \to C^*(\Lambda)\times_\delta G$ is the canonical inclusion, and the q_g are the images of the indicator functions $\chi_{\{g\}} \in c_0(G)$.*

Proof. Define $t : \Lambda \to C^*(\Lambda) \otimes C^*(G)$ by $t_\lambda := s_\lambda \otimes c(\lambda)$. Since c is a cocycle, we have $t_v = s_v \otimes 1$ for each $v \in \Lambda^0$ and it follows that t satisfies (CK1). If $s(\mu, g) = r(\nu, h)$, then $g = c(\nu)h$, and

$$t_\mu t_\nu = (s_\mu \otimes c(\mu))(s_\nu \otimes c(\nu)) = s_\mu s_\nu c(\mu)c(\nu) = t_\mu$$

since s satisfies (CK2) and c is a cocycle. So t satisfies (CK2). For $\lambda \in \Lambda$, we have

$$t_\lambda^* t_\lambda = s_\lambda^* s_\lambda \otimes c(\lambda)^* c(\lambda) = s_{s(\lambda)} \otimes 1 = t_{s(\lambda)}$$

because s satisfies (CK3) and the $c(g)$ are unitaries. Similarly, for $v \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$\sum_{\lambda \in v\Lambda^{\leq n}} t_\lambda t_\lambda^* = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^* \otimes c(\lambda)c(\lambda)^* = \Big(\sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^*\Big) \otimes 1 = s_v \otimes 1 = t_v.$$

The universal property of $C^*(\Lambda)$ yields a homomorphism $\delta : C^*(\Lambda) \to C^*(\Lambda) \otimes C^*(G)$ such that $\delta(s_\lambda) = t_\lambda = s_\lambda \otimes c(\lambda)$. This δ is nondegenerate because increasing finite sums of the form $P_F = \sum_{v\in F} s_v$ form an approximate identity for $C^*(\Lambda)$ with $\delta(P_F) = P_F \otimes 1$ for all F. For the coaction identity, we calculate

$$(\delta \otimes 1) \circ \delta(s_\lambda) = \delta(s_\lambda) \otimes c(\lambda) = (s_\lambda \otimes c(\lambda)) \otimes c(\lambda) = s_\lambda \otimes \delta^G(c(\lambda)) = (1 \otimes \delta^G) \circ \delta(s_\lambda)$$

for all λ.

To see that $C^*(\Lambda \times_c G) \cong C^*(\Lambda) \times_\delta G$, define $u : \Lambda \times_c G \to C^*(\Lambda) \times_\delta G$ by $u_{(\lambda,g)} := \iota(s_\lambda)q_g$. Since the q_g are mutually orthogonal, and since the $\iota(s_v)$ are also, and since the q_g commute with the $\iota(s_v)$, the map u satisfies (CK1). For (CK2) we calculate

$$u_{(\mu, c(\nu)g)} u_{(\nu,g)} = \iota(s_\mu) q_{c(\nu)g} \iota(s_\nu) q_g = \iota(s_\mu s_\nu) q_{c(\nu)^{-1}c(\nu)g} q_g = \iota(s_\mu s_\nu) q_g = u_{(\mu\nu, g)},$$

so u satisfies (CK2). Also,

$$u_{(\lambda,g)}^* u_{(\lambda,g)} = q_g \iota(s_\lambda^* s_\lambda) q_g = q_g \iota(s_{s(\lambda)}) q_g = \iota(s_{s(\lambda)}) q_g = u_{s(\lambda,g)}$$

because $\delta(s_{s(\lambda)}) = s_\lambda \otimes e$. Hence u satisfies (CK3). Finally, for $(v, g) \in (\Lambda \times_c G)^0$ and $n \in \mathbb{N}^k$,

$$\sum_{(\lambda,h)\in(v,g)(\Lambda\times_c G)^{\leq n}} u_{(\lambda,h)} u_{(\lambda,h)}^* = \sum_{\lambda\in v\Lambda^{\leq n}} \iota(s_\lambda) q_{c(\lambda)^{-1}g} \iota(s_\lambda^*) = \sum_{\lambda \in v\Lambda^{\leq n}} \iota(s_\lambda)\iota(s_\lambda^*) q_g = u_{(v,g)},$$

so u satisfies (CK4). Thus there is a homomorphism $\pi_u : C^*(\Lambda \times_c G) \to C^*(\Lambda) \times_\delta G$. The universal property of $C^*(\Lambda) \times_\delta G$ ensures that the gauge action γ on $C^*(\Lambda)$ induces an action β of $\mathbb{T}^k$ on $C^*(\Lambda) \times_\delta G$ such that $\beta_z \circ \iota = \iota \circ \gamma_z$ and $\beta_z(q_g) = q_g$ for all z, g. In particular, $\beta_z(u_{(\lambda,g)}) = z^{d(\lambda,g)} u_{(\lambda,g)}$ for all g. Since the s_v go to $s_v \otimes 1$ under δ, the $u_{(v,g)}$ are all nonzero, and the gauge-invariant uniqueness theorem implies that π_u is an isomorphism. □

Corollary 6.4. *Let Λ be a locally convex row-finite k-graph. Then $C^*(\Lambda) \times_\gamma \mathbb{T}^k \cong C^*(\Lambda \times_d \mathbb{Z}^k)$.*

Proof. We have $\mathbb{T}^k = \widehat{\mathbb{Z}^k}$, and the action γ of $\mathbb{T}^k$ on $C^*(\Lambda)$ corresponds to the coaction ε of $\mathbb{Z}^k$ given by $\varepsilon(s_\lambda) = s_\lambda \otimes d(\lambda)$. The result therefore follows from Theorem 6.3 □

Lemma 6.5. *Let Λ be a locally convex row-finite k-graph. Then $C^*(\Lambda \times_d \mathbb{Z}^k)$ is AF.*

Proof sketch. Fix a finite subset F of $\Lambda \times_d \mathbb{Z}^k$. Let $D_F := \{m \in \mathbb{N}^k : (\lambda, m) \in F$ for some $\lambda \in \Lambda\}$. Let $N := \bigvee D_F \in \mathbb{N}^k$, and let $\overline{F} := \bigcup_{(\lambda,m)\in F}\{(\lambda\lambda', p) : m \leq p \leq N, \lambda' \in s(\lambda)\Lambda^{p-m}\}$. It is straightforward to check that for $(\mu, m), (\nu, n) \in \Lambda \times_d \mathbb{Z}^k$, if $(\lambda, p) \in \mathrm{MCE}((\mu, m), (\nu, n))$ then $p = m \vee n$. Using this and (2.2) it is straightforward to check that $\operatorname{span}\{s_\eta s_\zeta^* : \eta, \zeta \in \overline{F}\}$ is closed under multiplication, and hence a finite-dimensional subalgebra of $C^*(\Lambda \times_d \mathbb{Z}^k)$ which contains $C^*(\{s_\mu s_\nu^* : \mu, \nu \in F\})$. Since $C^*(\Lambda \times_d \mathbb{Z}^k)$ is the increasing union of the subalgebras $\operatorname{span}\{s_\eta s_\zeta^* : \eta, \zeta \in \overline{F}\}$, the result follows. □

Corollary 6.6. *Let Λ be a locally convex row-finite k-graph. Then $C^*(\Lambda)$ is stably isomorphic to a crossed product of an AF algebra by $\mathbb{Z}^k$.*

Proof. Let $\hat{\varepsilon}$ be the dual action of $\mathbb{Z}^k$ on $C^*(\Lambda) \times_\varepsilon \mathbb{Z}^k$ given by $\hat{\varepsilon}_n(\iota(s_\lambda)q_m) = \iota(s_\lambda)q_{m+n}$. By Takai duality, $C^*(\Lambda) \sim_{\text{Me}} C^*(\Lambda) \times_\varepsilon \mathbb{Z}^k \times_{\hat{\varepsilon}} \mathbb{Z}^k$. Corollary 6.4 and Lemma 6.5 combine to show that $C^*(\Lambda) \times_\varepsilon \mathbb{Z}^k \cong C^*(\Lambda \times_d \mathbb{Z}^k)$ is AF, and the result follows. □

We finish with a third take on the Bunce-Deddens algebra of type 2^∞. We have seen it as an $A\mathbb{T}$-algebra and as a crossed product of an AF-algebra by $\mathbb{Z}$. Now we will see it as a coaction crossed product by the profinite group of 2-adic numbers.

Fix a discrete group G and a sequence $G = H_0 \rhd H_1 \rhd H_2 \rhd \cdots$ of finite-index normal subgroups of G. For each n, let $G_n := G/H_n$. We obtain a projective system

$$\{e\} = G_0 \overset{q_1}{\longleftarrow} G_1 \overset{q_2}{\longleftarrow} G_2 \cdots$$

of finite groups. Fix a locally convex row-finite k-graph Λ, and a sequence of cocycles $c_n : \Lambda \to G_n$ such that $q_n(c_n(\lambda)) = c_{n-1}(\lambda)$ for all λ, n. We for $g \in G$, we will write $[g]_n$ for the class of g in G_n.

Each $\Gamma_n := \Lambda \times_{c_n} G_n$ is a k-graph, and the map $\varphi_n : \Gamma_n \to \Gamma_{n-1}$ given by $\varphi_n(\lambda, [g]_n) = (\lambda, q_n([g_n])) = (\lambda, [g]_{n-1})$ is a covering. Let Σ be the infinite $(k+1)$-graph of Remark 4.10 obtained from the tower of k-morphs X_{φ_n}. For each n, let $P_n := \sum_{v \in \Gamma_n^0} s_v = \sum_{v \in \Lambda^0, [g]_n \in G_n} s_{(v,[g]_n)} \in C^*(\Sigma)$.

Lemma 6.7. *We have $P_0 C^*(\Sigma) P_0 \cong \varinjlim C^*(\Gamma_n)$ under inclusions satisfying $s_{(\lambda,[g_n])} \mapsto \sum_{q_{n+1}([h]_{n+1}) = [g]_n} s_{(\lambda,[h]_{n+1})}$.*

Proof. For each $n \in \mathbb{N}^k$, let $V_n := \sum_{\alpha \in X_1 * \cdots X_n} s_\alpha$. Since the source map on each X is a bijection, the V_n are all partial isometries with $V_n^* v_n = P_n$ and $V_n V_n^* = P_0$. The map $a \mapsto V_n a V_n^*$ is an injective homomorphism from $C^*(\{s_\alpha : \alpha \in \iota(\Gamma_n)\})$ to $P_0 C^*(\Sigma) P_0$ (gauge-invariant uniqueness theorem again). Every spanning element of $P_0 C^*(\Sigma) P_0$ belongs to $V_n C^*(\{s_\alpha : \alpha \in \iota(\Gamma_n)\}) V_n^*$ for large enough n, so it follows that $P_0 C^*(\Sigma) P_0 \cong \varinjlim C^*(\Gamma_n)$.

To calculate the connecting maps, note that

$$\begin{aligned} V_{n+1}^* V_n s_{(\lambda,[g]_n)} V_n^* V_{n+1} &= \sum_{x,y \in X_{n+1}} s_x^* s_{(\lambda,[g]_n)} s_y \\ &= \sum_{q([h]_{n+1}) = q([h']_{n+1}) = [g_n]} s_{(\lambda,[h']_{n+1})} s_{[h']_{n+1}}^* s_{[h]_{n+1}} \\ &= \sum_{q([h]_{n+1}) = [g_n]} s_{(\lambda,[h]_{n+1})} \end{aligned}$$

as required. □

Corollary 6.8. *For each n, let δ_n be the coaction of G_n on $C^*(\Lambda)$ determined by $\delta_n(s_\lambda) = s_\lambda \otimes c_n(\lambda)$. Then*

$$P_0 C^*(\Sigma) P_0 \cong \varinjlim (C^*(\Lambda) \times_{\delta_n} G_n)$$

under inclusions satisfying $s_\lambda q_{[g]_n} \mapsto \sum_{q_{n+1}([h]_{n+1}) = [g]_n} s_\lambda g_{[h]_{n+1}}$.

Proof. Combine Lemma 6.7 and Theorem 6.3. □

For the following theorem, let $G_\infty := \varprojlim G_n$, and let $q_n^\infty : G_\infty \to G_n$ be the canonical surjection for each n. We G_∞ identify with the set of sequences $([g_n]_n)_{n=1}^\infty \in \prod_{n=1}^\infty G_n$ such that $q_{n+1}([g_{n+1}]_{n+1}) = [g_n]_n$ for all n. Observe that the $c_n : \Lambda \to G_n$ determine a cocycle $c_\infty : \Lambda \to G_\infty$ by $c_\infty(\lambda) := (c_n(\lambda))_{n=1}^\infty$.

Theorem 6.9. *There is a coaction δ_∞ of G_∞ on $C^*(\Lambda)$ satisfying $\delta_\infty(s_\lambda) = s_\lambda \otimes c_\infty(\lambda)$ for all λ. Moreover,*

$$C^*(\Lambda) \times_{\delta_\infty} G_\infty \cong \varinjlim C^*(\Lambda) \times_{\delta_n} G_n,$$

and in particular is Morita equivalent to $C^*(\Sigma)$.

Proof. More or less the same argument as in Theorem 6.3 shows that there is a coaction $\delta_\infty : C^*(\Lambda) \to C^*(\Lambda) \otimes C^*(G_\infty)$[1] satisfying $\delta_\infty(s_\lambda) = s_\lambda \otimes c_\infty(\lambda)$ — nondegeneracy as a coaction follows from nondegeneracy as a homomorphism by a result of Landstad because G_∞ is amenable. Since $C_0(G_\infty) = \varinjlim c_0(G_n)$, for each n the map $\delta_{[g]_n} \mapsto \chi_{h \in G_\infty : q_n^\infty(h) = [g]_n}$ determines a homomorphism of $c_0(G_n)$ into $C_0(G_\infty)$. Thus Theorem 6.3 implies that $t_{(\lambda,[g]_n)} := \iota(s_\lambda)\iota_G(\chi_{h \in G_\infty : q_n^\infty(h) = [g]_n})$ determines a Cuntz-Krieger Γ_n-family in $C^*(\Lambda) \times_{\delta_\infty} G_\infty$, and hence a homomorphism $\pi_n : C^*(\Gamma_n) \to C^*(\Lambda) \times_{\delta_\infty} G_\infty$, for each $n \in \mathbb{N}$. The universal property of $C^*(\Lambda) \times_{\delta_\infty} G_\infty$ implies that there is an action β of $\mathbb{T}^k$ on $C^*(\Lambda) \times_{\delta_\infty} G_\infty$ which fixes the copy of $C_0(G_\infty)$ and satisfies $\beta_z \circ \iota = \iota \circ \gamma_z$, and it follows that $\beta_z \circ \pi_n = \pi_n \circ \gamma_z$ for each n. The gauge-invariant uniqueness theorem therefore implies that the π_n are injective.

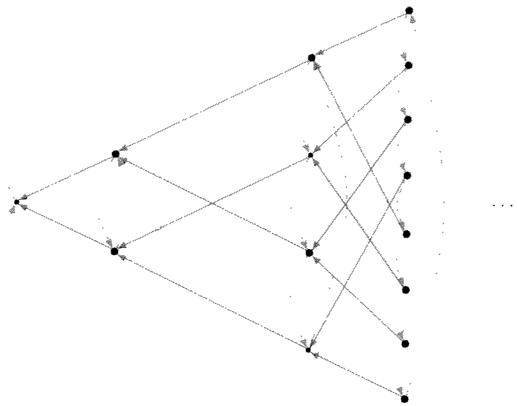
The universal property of $\varinjlim C^*(\Lambda) \times_{\delta_n} G_n$ then gives $\pi_\infty : \varinjlim C^*(\Lambda) \times_{\delta_n} G_n \to C^*(\Lambda) \times_{\delta_\infty} G_\infty$, and π_∞ is injective because the π_n are all injective. It is surjective because the $\chi_{h \in G_\infty : q_n^\infty(h) = [g]_n}$ span a dense subalgebra of $C_0(G_\infty)$ so the image of π_∞ contains all the generators of $C^*(\Lambda) \times_{\delta_\infty} G_\infty$. Remark 4.10 implies that P_0 is full, so the Morita equivalence of $C^*(\Lambda) \times_{\delta_\infty} G_\infty$ with $C^*(\Sigma)$ follows from Corollary 6.8. □

Remark 6.10. In fact the continuity of coaction crossed products by projective systems of finite discrete groups is a general phenomenon, but the proof is more involved.

Example 6.11. Let Λ be the 1-graph with one edge e and one vertex v. Let $G := \mathbb{Z}$ and $H_n = 2^n\mathbb{Z}$ for all n, so $G_n = G/H_n$ is the finite cyclic group of order 2^n for all n.

[1]Not $\mathcal{M}C^*(G)$ because the projective limit is compact

Hence $G_\infty = \mathbb{Z}_2$ the group of 2-adic numbers. Then the 2-graph Σ is



which is precisely the rank-2 Bratteli diagram corresponding to the 2^∞ Bunce-Deddens algebra as described in Example 5.6(1). By Theorem 6.9 and Lemma 5.1, we have $P_0C^*(\Sigma)P_0 \cong C(\mathbb{T}) \times_{\delta_\infty} \mathbb{Z}_2$.

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