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PROJECTION–ITERATION METHOD OF SOLUTION OF THE EQUATIONS OF QUANTUM FIELD THEORY AND ITS CONNECTION WITH THE THEORY OF RENORMALIZATION.

THE EQUATIONS OF QUANTUM FIELD THEORY AND IMPROPERLY POSED PROBLEMS OF MATHEMATICAL PHYSICS

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The equations for the coefficient functions of the S matrix are considered in Euclidean space-time with dimension $d = 4$. It is shown that they can be defined in a pair of fundamental spaces of translationally invariant functions. The problem of finding the solutions to such equations is improperly posed, and therefore the iterative method of solution leads to the appearance of ultraviolet divergences. A projection–iteration method of solution is constructed that leads to a renormalized perturbation series. It is shown that the renormalized perturbation series is a quasisolution to the original equations.

1. Introduction

Historically, the equations of quantum field theory have been solved by perturbation theory. Divergences have been encountered in the higher orders of the theory, and these have been eliminated by means of Bogolyubov's subtractional procedure. Because of the occurrence of the divergences, the opinion has arisen (see, for example, [1]) that the original equations have no meaning, that they are internally contradictory. Their perturbative solutions are also meaningless, although the perturbation series renormalized by means of the subtractional procedure leads in individual cases (for example, in quantum electrodynamics) to good agreement with the experiments.

Another view has been put forward, namely, that the equations of field theory are consistent and that the divergences are due to the use of perturbation theory and would not arise in a nonperturbative approach. There now exist methods of investigation that do not use perturbation theory [2, 3, 4], but as yet they have been applied only to models in two-dimensional space-time; it is not known what happens to the models in four-dimensional space-time.

In the pioneering papers of Bogolyubov and Parasyuk [5-7] it was shown that the divergences of the contributions from the Feynman diagrams arise from the circumstance that these are expressed by integrals of products of generalized functions, and the products are not defined for coincident arguments (in the configuration space). If the definition of the product is modified appropriately, the integral of it converges, and this gives the renormalized contribution of the Feynman diagram. In the momentum space and in the Euclidean region, the contributions from the Feynman diagrams are expressed by means of integrals of a product of ordinary functions, which decrease weakly at infinity, so that the integrals diverge (ultraviolet divergences).

A great success was the proof that the contribution from an arbitrary diagram can be renormalized by means of the subtractional procedure (R operation) [5-7], which was developed further in [8-11].

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Later, it was found that one can write down renormalized equations for the Green's functions or coefficient functions [12-14] which have the property that their perturbation series no longer contain divergences in their individual terms, i.e., in the renormalized contributions from the Feynman diagrams.

On the basis of these facts there has developed in quantum field theory a definite view as to how the equations encountered in the investigation of the models are to be solved. Briefly, the system of rules can be formulated as follows:

1) the original equations (for the Green's functions, coefficient functions, etc.) deduced from the unrenormalized Lagrangian are meaningless, since they lead to divergences when solved by perturbation theory;

2) the perturbation series must be renormalized by means of the subtractional procedure;

3) the true equations are the renormalized equations, since their solution by perturbation theory is the renormalized series, which does not contain divergences in its individual terms. This point of view has been made into a principle. It developed historically together with quantum field theory.

What can one say about the above principles? They contain a weak point – there is no rigorous analysis of the original equations whose solution is sought by perturbation theory. The perturbation series were written down already at the end of the fifties, and the necessary mathematical analysis of the original equations was not made by anyone. Note that the contributions from the Feynman diagrams were initially written down by Feynman without any equation [15], and it was shown later by Dyson [16] that they arise on the solution of the equation for the S matrix. The subtractional procedure was developed for the already existing expressions corresponding to the contributions from the Feynman diagrams.

Thus, the initial equations were solved perturbatively without the necessary investigation of their mathematical structure.

All these equations are operator equations, and the unknowns in them are sequences of functions of an increasing number of variables; the generating operator which determines them acts on the space of such sequences. Clearly, a rigorous mathematical investigation of these equations on the basis of functional analysis is required. It is necessary to select a space on which the generating operator acts, find its domain of definition and range, and also define the space in which the initial data lie, if one is considering evolution equations, or the space in which the free term lies, if one is considering equations of resolvent type.

So far as we know, such an analysis has been made only for the models of a scalar field in two-dimensional space-time and nonpolynomial nonlocal models, i.e., models in which ultraviolet divergences do not arise (volume divergences occur in any translationally invariant model and have a completely different origin).

In the present paper, we shall make the necessary analysis of the equations for the coefficient functions of the S matrix for the example of the model of a real scalar field in Euclidean four-dimensional space-time with interaction Lagrangian

$$\mathscr{L}(x) = -\lambda{:}\varphi(x)^4{:}, \quad x = (x^0, \ldots, x^3), \tag{1.1}$$

where λ is the coupling constant.

We shall show that the equations for the coefficient functions can be given a well-defined mathematical meaning in a pair of function spaces of sequences of functions of an increasing number of arguments. The generating operator of these equations is unbounded, but has an everywhere dense domain of definition in the space of sequences of integrable functions, and its range lies in a different space of sequences of bounded functions. The free term does not belong to the domain of definition of the generating operator but belongs to its range. In addition, the powers of the generating operator are not defined on any element in the domain of definition or the operator itself is not defined on its range.

It becomes clear why divergences arise in the individual terms of the perturbation series; the generating operator and all its powers are not defined on the free term, but each term of the perturbation series involves powers of the generating operator on the free term. Because the powers of the generating operator are not defined, it is not possible to solve the equations for the coefficient functions perturbatively in the form of a series of iterations. For equations for which the domain of definition of the generating operator does not contain the free term, one cannot even construct a formal solution by perturbation theory. Equations of such type must be solved by direct methods, i.e., one must find an element in the domain of definition of the generating operator that satisfies the equation.

The problem of determining solutions of the equations for the coefficient functions and, in general, the equations of quantum field theory can be included in the class of problems that are improperly posed in the sense of Tikhonov [17]. It is possible that the equations for the coefficient functions do not have solutions in the classical sense at all but only quasisolutions in the sense of Ivanov [18].

When one compares the methods of quantum field theory and the methods of solving improperly posed problems, one is struck by the circumstance that many approaches in these fields of science have developed independently in parallel and that even the terminology is frequently the same. It is obvious that the use of the methods of solution to improperly posed problems in quantum field theory presents a problem ripe for solution.

In the present paper, we show that the renormalized perturbation series is a solution, not to the original equation, but to some modified equation, in which the generating operator is replaced by a product of the generating operator and the operator of projection onto the domain of definition. Evidently, the renormalized series is a formal quasisolution.

2. Equations for the Coefficient Functions

2.1. Equations of Resolvent Type. Equations of resolvent type for the coefficient functions of the S matrix of the model (1.1) are described in [19]. For convenience, we write them in the momentum space in the Euclidean domain. We denote by $F_N(p_1,\ldots,p_N)$ the N-th coefficient function, and $p_i = (p_i^1, \ldots, p_i^4)$, $i = 1, 2, \ldots, N$, $N = 1, 2, \ldots$. The equations have the form

$$F_N(p_1,\ldots,p_N) = -\frac{4\lambda}{N}\sum_{s=-1}^{2}\binom{3}{1+s}\frac{(N-2s)!}{N!}\sum_{i_1\neq\ldots\neq i_{s+1}}\int dq_1\ldots dq_{2-s}\times$$

$$\frac{(2\pi)^4\delta(p_{i_1}+\ldots+p_{i_{2+s}}-q_1-\ldots-q_{2-s})}{(2\pi)^4(q_1^2+\mu^2)\ldots(2\pi)^4(q_{2-s}^2+\mu^2)}F_{N-2s}(q_1,\ldots,q_{2-s},p_1,\ldots,\hat{p}_{i_1},\ldots,\hat{p}_{i_{2+s}},\ldots,p_N). \tag{2.1}$$

Here, $F_0=1$, $F_{-1}=F_{-2}=F_{-3}=0$, $q^2=(q^1)^2+\ldots+(q^4)^2$, $dq=dq^1\ldots dq^4$, and the symbol $\hat{p}_i$ means that the argument p_i is omitted.

The chain of equations (2.1) can be written in the form of a single operator equation for the complete sequence of coefficient functions $F=\{F_N\}_{N=1}^{\infty}$:

$$F=-\lambda AF+F^0, \tag{2.2}$$

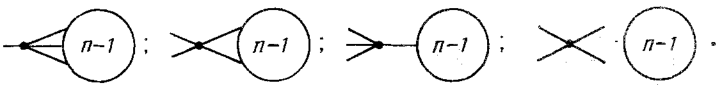
where the generating operator A is defined by the right-hand side of Eq. (2.1), and

$$F_N^0=-\lambda(2\pi)^4\delta(p_1+\ldots+p_4)\delta_{N4}. \tag{2.3}$$

2.2. Structure of the Operator A. To elucidate the structure of A, we consider the formal solution of Eq. (2.2) in the form of the iteration series

$$F=\sum_{n=0}^{\infty}(-\lambda)^n A^n F^0. \tag{2.4}$$

It follows [19] from the derivation of Eqs. (2.1) that the series (2.4) coincides with the perturbation series for the S matrix without the vacuum contributions. For example, the vector F^0 corresponds to the contribution from the diagram of first order (vertex) in the theory with the Lagrangian (1.1). The vector AF^0 is equal to the sum of the contributions from all diagrams of second order. Thus, the (n − 1)-th power of the operator A on the vector F^0 generates the Feynman diagrams of n-th order, since F^0 is of first order in λ. In diagram language, each application of the operator A to the contribution from the diagram of (n − 1)-th order joins to the existing diagram a vertex in one of the following four possible ways:



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Depending on how the new vertex is joined to the diagram of order n − 1, the operator A can be split into a sum of four operators, each of which joins the new vertex in its own characteristic way. With case a), we associate the operator $A_{1,\widetilde{3}}$. The action of this operator on the vector F is determined by the term of the expression (2.1) with $s = -1$. With case b) we associate the operator $A_{2,\widetilde{2}}$, whose action on the vector F is determined by the term of the expression (2.1) with $s = 0$. Similarly, in case c) the operator $A_{3,1}$ corresponds to $s = 1$ and in the case d) the operator $A_{4,0}$ corresponds to $s = 2$. The application of these operators to the contribution from the diagram of order n depends on the topological structure of the diagram itself. In the most general case, the diagram of order n need not be connected, i.e., it can, for example, have k independent parts, each of which is a connected diagram. Then the operator $A_{4,0}$ acts in such a way as to increase the connectedness of the diagrams by unity, and the number of external lines by $4 - 0 = 4$ (since it joins the vertex in a nonconnected manner). The operator $A_{3,1}$ conserves the connectedness of the diagrams, but increases the number of external lines by $3 - 1 = 2$ (since it connects the vertex to one external line, i.e., in a weakly connected manner). The powers of the operator $A_{3,1}$ on F^0 generate skeleton diagrams or so-called trees.

The operator $A_{2,\widetilde{2}}$ can generate different topologically inequivalent diagrams. It cannot change the connectedness of the diagrams. In this case, the vertex is joined to two external lines belonging to one connected component of the diagram. We denote the corresponding operator by $A_{2,2}$.

In the other case, the vertex is joined to two external lines belonging to two connected components. We denote the corresponding operator by $A_{2,11}$. The operator $A_{2,11}$ reduces the connectedness of the diagram by unity. The operator $A_{2,\widetilde{2}}$ acts either as $A_{2,2}$ or as $A_{2,11}$ depending on whether the variables of integration q_1 and q_2 (see the term in (2.1) for $s = 0$) belong to one connected component or different connected components.

A similar classification can be given for the operator $A_{1,\widetilde{3}}$. The operator $A_{1,3}$ joins the vertex to three external lines belonging to one connected component; the operator $A_{1,21}$ joins the vertex to three external lines, two of which belong to one connected component and the third to another; the operator $A_{1,111}$ joins the vertex to three external lines, each belonging to a corresponding connected component. The operator $A_{1,21}$ reduces the connectedness of the diagram by unity and $A_{1,111}$ by two.

A more detailed analysis is given in [20].

The operators $A_{2,11}$, $A_{1,111}$, and $A_{3,1}$ generate weakly connected diagrams. The operators $A_{1,3}$, $A_{2,2}$, and $A_{1,21}$ join the new n-th vertex to a connected component of the diagram of order $n - 1$ by at least two internal lines. Now the contributions from such diagrams contain ultraviolet divergences [20]. Thus, the contributions of the strongly connected diagrams shown above diverge.

As will be shown below, the divergences of the contributions from the diagrams of order n arise, in the language of functional analysis, because the contributions from the diagrams of order $n - 1$ do not lie in the domain of definition of the operators $A_{1,3}$, $A_{2,2}$, and $A_{1,21}$ (the contributions from the diagrams of order $n - 1$ may be finite).

Thus, the operator A is the sum of the operators $A_{i,j}$ introduced above:

$$A=\sum_{i,j} A_{i,j}, \tag{2.5}$$

where the indices (i, j) take the values (1,3), (1,21), (1,111), (2,2), (2,11), (3,1), (4,0).

3. The Space on which the Operator A Is Defined

3.1. The spaces $\mathscr{B}^T$, $\mathscr{E}^T$, and $\mathscr{E}^T_{\mathrm{loc}}$. Thus, the $(n - 1)$-th power of the operator A on the vector F^0 is equal to the sum of the contributions from the Feynman diagrams of order n. The contributions from the diagrams are translationally invariant functions, and therefore the space on which the operator A is defined must consist of translationally invariant functions. Such spaces were introduced in the papers of the first author in connection with the investigation of models of quantum statistical physics [22] and were used to study the equations for the coefficient functions of the model (1.1) in two-dimensional space-time [3, 23]. We shall need a modification of such a space, to the description of which we now turn.

We consider functions of N variables of the form

$$f_N(x_1,\ldots,x_N)=\sum_{k=1}^{N}\sum_{\sigma_k(1,\ldots,N)} f_{N;n_1,\ldots,n_k}(x_{j_1},\ldots,x_{j_{n_1}};\ldots;x_{l_1},\ldots,x_{l_{n_k}}). \tag{3.1}$$

Here, $\sigma_k(1, \ldots, N)$ is a partitioning of the set of N points into k subsets $\{n_1\}=\{x_{j_1},\ldots,x_{j_{n_1}}\},\ldots,\{n_k\}=\{x_{l_1},\ldots,x_{l_{n_k}}\}$, $n_1+\ldots+n_k=N$, and the summation is over all partitionings σ_k of the set $\{N\}$. The function $f_{N;n_1,\ldots,n_k}(x_{j_1},\ldots,x_{j_{n_1}};\ldots;x_{l_1},\ldots,x_{l_{n_k}})$ is translationally invariant with respect to each group of variables $\{x_{j_1},\ldots,x_{j_{n_1}}\},\ldots,\{x_{l_1},\ldots,x_{l_{n_k}}\}$, and in a special case can be thought of as a linear combination of products of the translationally invariant functions $f_{n_1}(x_{j_1},\ldots,x_{j_{n_1}}),\ldots,f_{n_k}(x_{l_1},\ldots,x_{l_{n_k}})$. In the momentum space, the function $f_N(\ldots)$ takes the form

$$\tilde{f}_N(p_1,\ldots,p_N)=\sum_{k=1}^{N}\sum_{\sigma_k}\delta(p_{j_1}+\ldots+p_{j_{n_1}})\ldots\delta(p_{l_1}+\ldots+p_{l_{n_k}})f_{N;n_1,\ldots,n_k}(p_{j_1},\ldots,p_{j_{n_1}};\ldots;p_{l_1},\ldots,p_{l_{n_k}}), \tag{3.2}$$

where $f_{N;n_1,\ldots,n_k}(p_{j_1},\ldots,p_{j_{n_1}};\ldots;p_{l_1},\ldots,p_{l_{n_k}})$ is the Fourier transform of the function $f_{N;n_1,\ldots,n_k}(x_{j_1},\ldots,x_{j_{n_1}};\ldots;x_{l_1},\ldots,x_{l_{n_k}})$ with respect to the $n_1-1,\ldots,n_k-1$ difference variables $x_{j_1}-x_{j_{n_1}},\ldots,x_{j_{n_1-1}}-x_{j_{n_1}};\ldots;\ x_{l_1}-x_{l_{n_k}},\ldots,x_{l_{n_k-1}}-x_{l_{n_k}}$, the Fourier transform being defined on the intersection of the hypersurfaces $\Omega_{n_1}=(p\,|\,p_{j_1}+\ldots+p_{j_{n_1}}=0),\ldots,\Omega_{n_k}=(p\,|\,p_{l_1}+\ldots+p_{l_{n_k}}=0)$. If $f_N(\ldots)$ is symmetric with respect to $x_1,\ldots,x_N\,(p_1,\ldots,p_N)$, the sum over σ_k does not include permutation of the indices within each group.

Finally, we consider the space $\mathscr{B}^T$, whose elements are sequences of symmetric translationally invariant functions $f=\{\tilde{f}_N\}_{N=1}^{\infty}$. Each function $\tilde{f}_N(\ldots)$ in the momentum space is such that

$$f(N;\sigma_k)=\int\limits_{\Omega_{n_1}\cap\ldots\cap\Omega_{n_k}}\frac{dp_1}{p_1^2+\mu^2}\ldots\frac{dp_N}{p_N^2+\mu^2}\,|f_{N;n_1,\ldots,n_k}(p_{j_1},\ldots,p_{j_{n_1}};\ldots;p_{l_1},\ldots,p_{l_{n_k}})|<\infty. \tag{3.3}$$

We introduce a topology in the space of such sequences by means of the norm

$$\|f\|_{\mathscr{B}^T}=\sum_{N=1}^{\infty}\sum_{k=1}^{N}\sum_{\sigma_k}f(N;\sigma_k), \tag{3.4}$$

where $f(N;\sigma_k)$ is defined by the expression (3.3).

In addition, we consider one further Banach space $\mathscr{E}^T$. The elements of this space are also sequences $f=\{\tilde{f}_N\}_{N=1}^{\infty}$ of translationally invariant functions $f_N(\ldots)$ of the form (3.1) and (3.2). We define the norm in $\mathscr{E}^T$ by

$$\|f\|_{\mathscr{E}^T}=\sup_{N,k,\sigma_k}\ \sup_{p_1,\ldots,p_N\in\Omega_{n_1}\cap\ldots\cap\Omega_{n_k}}|f_{N;n_1,\ldots,n_k}(p_{j_1},\ldots,p_{j_{n_1}};\ldots;p_{l_1},\ldots,p_{l_{n_k}})|. \tag{3.5}$$

The norm (3.5) defines a space of functions of the form (3.2) that are bounded on the intersection of the hypersurfaces $\Omega_{n_1},\ldots,\Omega_{n_k}$.

In what follows, to construct solutions of Eqs. (2.2) iteratively we need a larger class of functions, which we denote by $\mathscr{E}^T_{\mathrm{loc}}$. By definition, $\mathscr{E}^T_{\mathrm{loc}}$ is the set of all sequences of functions of the form (3.2) that are locally bounded on the intersection of the hypersurfaces $\Omega_{n_1},\ldots,\Omega_{n_k}$. We do not introduce a topology in $\mathscr{E}^T_{\mathrm{loc}}$, since it will not be used in what follows. We also denote by $D_0(\mathscr{E}^T_{\mathrm{loc}})$ the set of all finite sequences whose components are locally bounded functions on the intersection of the hypersurfaces Ω_{n_i}.

3.2. Domain of Definition and Range of the Operator A. We now characterize the domain of definition and range of A.

THEOREM 3.1. The domain of definition D(A) of the operator A contains the set $D_0(A)$ of finite sequences f ($\tilde{f}_N=0$, if $N > N_0$) of continuous functions $f_{N;n_1,\ldots,n_k}(p_{j_1},\ldots,p_{j_{n_1}};\ldots;p_{l_1},\ldots,p_{l_{n_k}})$ concentrated on compacta in the intersection of the hypersurfaces $\Omega_{n_1},\ldots,\Omega_{n_k}$. It maps sequences $f\in D_0(A)$ to finite sequences in $\mathscr{E}^T$.

Proof. We note first that in what follows we shall consider functions $f_{N;n_1,\ldots,n_k}$ on the intersection of the hypersurfaces $\Omega_{n_1},\ldots,\Omega_{n_k}$ without stating this specifically.

We give the proof for each term of A. We begin with $A_{4,0}$. In accordance with the definitions (2.1) and (3.2),

$$(\widetilde{A_{4,0}f})_N(p_1,\ldots,p_N)=\frac{4}{N^2(N-1)(N-2)(N-3)}\sum_{i_1\neq\ldots\neq i_4=1}^{N}(2\pi)^4\delta(p_{i_1}+\ldots+p_{i_4})\tilde{f}_{N-4}(p_1,$$

$$\ldots,\hat{p}_{i_1},\ldots,\hat{p}_{i_4},\ldots,p_N)=\frac{4(2\pi)^4}{N^2(N-1)(N-2)(N-3)}\sum_{i_1\neq\ldots\neq i_4=1}^{N}\sum_{k=1}^{N-4}\sum_{\sigma_k(1,\ldots,\hat{i}_1,\ldots,\hat{i}_4,\ldots,N)}\delta(p_{i_1}+\ldots+p_{i_4})\times$$

$$f_{N-4;\, n_1,\dots,\, n_k}(p_{j_1},\dots,p_{j_{n_1}};\dots;p_{l_1},\dots,p_{l_{n_k}}),\quad j_1,\dots,j_{n_1};\dots;\,l_1,\dots,\,l_{n_k}\neq i_1,\,i_2,\,i_3,\,i_4;\; n_1+\dots+n_k=N-4. \tag{3.6}$$

It follows from (3.6) that the sequence $A_{4,0}f$ is finite together with f and, in addition, $(A_{4,0}f)_N=0$ for N = 1, 2, 3. Then in accordance with the definition of the norm (3.5) in the space $\mathscr{E}^T$,

$$\| A_{4,0}f \|_{\mathscr{E}^T} = \sup_{N>3,\, k,\, \sigma_k}\ \sup_{p_1,\dots,\, p_{N-4}} \frac{4(2\pi)^4}{N^2(N-1)(N-2)(N-3)} \left| f_{N-4;\, n_1,\dots,\, n_k}(p_{j_1},\dots;\dots;\dots,p_{l_{n_k}})\right|. \tag{3.7}$$

Since the function $f_{N-4;\, n_1,\dots,\, n_k}(\dots)$ is continuous and concentrated on a compact set on the hypersurface $\Omega_{n_1}\cap\dots\cap\Omega_{n}$, the norm (3.7) is finite.

The estimates of the norms for the operators $A_{3,1}$, $A_{2,2}$, and $A_{1,3}$ are similar and differ only in technical details. We shall therefore omit the further calculations, referring the reader to [20].

Thus, Theorem 3.1 establishes that all terms of the operator A, and, hence, the operator A itself are defined on a domain that is everywhere dense in $\mathscr{B}^T$, namely, the domain $D_0(A)$ of finite sequences with components concentrated on compact supports. However, the set $D_0(A)$ does not exhaust the complete domain of definition of A. It is readily noted that the terms $A_{2,2}$, $A_{1,2,1}$, and $A_{1,3}$, whose definitions include integrations over the variables q_1 (for $A_{2,2}$ and $A_{1,21}$) and q_1 and q_2 (for $A_{1,3}$), can be defined on finite sequences whose components are not necessarily concentrated on compacta; all that is needed is sufficiently rapid decrease at infinity with respect to the momentum variables. And the remaining terms of the operator A are defined on all finite sequences $D_0(\mathscr{B}^T)$ in $\mathscr{B}^T$ and map vectors in $D_0(\mathscr{B}^T)$ to finite sequences $D_0(\mathscr{E}^T)$ of the space $\mathscr{E}^T$. In addition, the domain of definition D(A) of A may include nonfinite sequences $f=\{f_N\}_{N=1}^{\infty}$ whose components decrease sufficiently rapidly with respect to the number N, and vectors $f^{(N)}=(0,\dots,0,f_N,0,\dots)\in D(A)$, $N=1,2,\dots$. Then also the vector

$$f=\sum_{N=1}^{\infty} f^{(N)}\in D(A).$$

<u>3.3. The Powers of the Operator A.</u> Irrespective of the topology of the space on which the operator A is defined and the space in which its range lies, the powers of the operator A are defined nowhere. To see this, we show that for any sequence f the sequence A^2f contains divergences.

Since the operator A can be represented in the form (2.5), A^2 is a sum of products of the form $A_{i,j}A_{k,l}$. We consider, for example, the products $A_{1,3}A_{4,0}$ and $A_{2,2}A_{4,0}$. These operators contain terms that are operators of multiplication by functions corresponding to loop diagrams (§2.2). These functions can be expressed by the divergent integrals (for d = 4)

$$\int \frac{dq_1}{(2\pi)^4(q_1^2+\mu^2)}\,\frac{dq_2}{(2\pi)^4(q_2^2+\mu^2)}\,\frac{1}{(2\pi)^4[(p_i-q_1-q_2)^2+\mu^2]}, \tag{3.8}$$

$$\int \frac{dq_1}{(2\pi)^4(q_1^2+\mu^2)}\,\frac{1}{(2\pi)^4[(q-p_{i_1}-p_{i_2})^2+\mu^2]} \tag{3.9}$$

Therefore, the operator A^2 is defined nowhere. The divergence of the integrals in (3.8) and (3.9) also shows that the vector F^0 does not lie in the domain of definition of A, since the expressions (3.8) and (3.9) are equal to the values of the operators $A_{1,3}$ and $A_{2,2}$ on F^0.

Besides the above products, the operator A^2 contains terms of the type $A_{4,0}A_{4,0}$, $A_{3,1}A_{4,0}$, $A_{1,111}A_{3,1}$, etc., which, essentially, are operators of multiplication by bounded functions. These operators are well defined on the set of all finite sequences $D_0(\mathscr{B}^T)$ and map vectors in $D_0(\mathscr{B}^T)$ to vectors in $D_0(\mathscr{E}^T)$. It then follows from the definitions of the operators $A_{4,0}$, $A_{3,1}$, $A_{2,11}$, and $A_{1,111}$ that they are defined not only on $\mathscr{B}^T$, but also on the set of finite sequences of the space $\mathscr{E}^T$, and

$$A_{i,j}D_0(\mathscr{E}^T)\subset D_0(\mathscr{E}^T)\quad \text{for}\quad i,j=4{,}0;\ 3{,}1;\ 2{,}11;\ 1{,}111.$$

Moreover, for the same reason

$$A_{i,j}D_0(\mathscr{E}^T_{\text{loc}})\subset D_0(\mathscr{E}^T_{\text{loc}}).$$

This enables us to extend somewhat the domains of definition of the individual terms of A. We define the set D^* as follows:

$$D^*=D(A)\cup D_0(\mathscr{E}^T)\cup D_0(\mathscr{E}^T_{\text{loc}}), \tag{3.10}$$

where $D(A)\subseteq\mathscr{B}^T$ is the set of all sequences on which the operator A is defined; the sets $D_0(\mathscr{E}^T)$ and $D_0(\mathscr{E}^T_{\text{loc}})$ are defined above. Then the operators $A_{1,3}$, $A_{1,21}$, $A_{2,2}$ are defined on D^* only on D(A), and the operators

$A_{1,111}$, $A_{2,11}$, $A_{3,1}$, and $A_{4,0}$ are defined on the whole of D^*. In this connection, we introduce further notation. Let $D^*(A_\alpha)$ be the set of vectors in D^* on which the operator A_α is defined, i.e., such that $A_\alpha f$ for $f \in D^*$ belongs to one of the spaces $\mathscr{B}^T, \mathscr{E}^T$, or $\mathscr{E}^T_{loc}$.

Thus, in this section we have shown that Eq. (2.2) is defined as an operator equation of resolvent type considered for the pair of function spaces $\mathscr{B}^T$ and $\mathscr{E}^T$. We have also established the reasons that preclude application of the method of successive approximation (the method of iteration) to the finding of a solution to this equation.

In the following sections, we shall discuss possible ways of solving Eq. (2.2) correctly.

3.4. The Improperly Posed Nature of the Problem of Solving Eq. (2.2). Thus, Eq. (2.2) can be considered in the pair of Banach spaces $\mathscr{B}^T$ and $\mathscr{E}^T$. The operator A is unbounded in $\mathscr{B}^T$, and $D(A)\neq\mathscr{B}^T$, and its range R(A), like the range $R(I + \lambda A)$ of the operator $I + \lambda A$, lies in $\mathscr{E}^T$, and it is not clear whether R(A) and $R(I + \lambda A)$ coincide with $\mathscr{E}^T$. It may happen that $F^0 \notin R(I+\lambda A)$, and therefore the problem (2.2) fails to satisfy not only the criteria for a properly posed problem of Hadamard [25] but also the criteria of Tikhonov [26].

As Tikhonov has shown [17], "improperly posed" problems arise in the description of many real physical phenomena. It will be seen from the following analysis that the methods of their solution have much in common with the methods developed in quantum field theory independently of the theory of improperly posed problems. It will be shown in the following section that the well-known method of solving the equations of quantum field theory by means of perturbation theory and renormalizations is equivalent to a certain projection–iteration method, and that the resulting series (the renormalized perturbation series) is, generally speaking, a quasisolution of Eq. (2.2). In addition, we shall trace the analogies between problems of quantum field theory and some improperly posed problems.

4. Construction of an Iterative Series

As we have already noted, the method of iteration is inapplicable to finding a solution to Eq. (2.2), since the first iteration already leads to contributions from the loop diagrams (see §2.2) that diverge. However, a solution to Eq. (2.2) free of ultraviolet divergences may exist. Such a situation can be illustrated by an elementary example.

Consider the integral equation

$$f(x)=-\lambda\int_{-\infty}^{\infty} dy[-\pi(e^{-|x-y|}+|x-y|)]f(y)-\pi|x|. \tag{4.1}$$

It is readily seen that already the first iteration of Eq. (4.1) diverges:

$$f_1(x)=-\pi^2\lambda\int_{-\infty}^{\infty}(e^{-|x-y|}+|x-y|)\,|y|dy=-\infty.$$

However, an exact solution to Eq. (4.1) can be obtained by applying the theory of Fourier transformations of generalized functions. The solution is

$$f(x)=\int dp e^{ipx}\frac{p^2+1}{p^4+p^2+2\pi\lambda},\quad p\in R^1,$$

and if $\lambda>0$ $f(x)\in L_2(R^1)$.

By analogy with Eq. (4.1), it can be hoped that such a solution of Eq. (2.2) also exists, but it cannot be constructed iteratively. The existence of a solution to Eq. (2.2) remains an open problem.

For what follows, we shall need some ancillary constructions, to which we now turn.

4.1. The Set of Contributions of the Renormalized Feynman Diagrams. In D^*, we consider the set G, which consists of sequences of contributions from the renormalized Feynman diagrams. To construct such a set, we introduce the operators $A_{1,3;i_1}$, $A_{1,21;i_1}$, $A_{1,111;i_1}$, $A_{2,2;i_1,i_2}$, $A_{2,11;i_1,i_2}$, $A_{3,1;i_1,i_2,i_3}$ and $A_{4,0;i_1,\dots,i_4}$ corresponding to fixed variables $p_{i_1},\dots,p_{i_k}$, $k=1, 2, 3, 4$, in the kernels of the operators $A_{i,j}$. Then

$$(A_{i,j}f)_N{}^{\sim}=\sum_{i_1\neq\dots\neq i_k}(A_{i,j;i_1,\dots,i_k}f)_N{}^{\sim}, \tag{4.2}$$

where k = 1 corresponds to the operators $A_{1,3}$, $A_{1,21}$, and $A_{1,111}$; $k=2$, to the operators $A_{2,2}$ and $A_{2,11}$; $k=3$, to the operator $A_{3,1}$, and k = 4, to the operator $A_{4,0}$, and the summation in (4.2) is over all sets of numbers $i_1\neq i_2\neq\ldots\neq i_k$ in 1, ..., N. For brevity, we introduce the index $\alpha=(i, j; i_1,\ldots, i_k)$ and denote the operators defined above by A_α. The operator A is the sum of the operators A_α over all admissible α:

$$A=\sum_\alpha A_\alpha.$$

The formal expression $A_{\alpha_1}\ldots A_{\alpha_m}F^0$ is the contribution (possibly divergent) from a definite Feynman diagram.

We denote by $R(A_{\alpha_1}\ldots A_{\alpha_m}F^0)$ the contribution from the diagram corresponding to $A_{\alpha_1}\ldots A_{\alpha_m}F^0$ when it is renormalized by means of the R operation. We form finite sequences consisting of functions of the form

$$g_{\alpha_1,\ldots,\alpha_m}=R(A_{\alpha_1}\ldots A_{\alpha_m}F^0), \tag{4.3}$$

where the indices $\alpha_1,\ldots, \alpha_m$ take all admissible values. We define the set G as the linear hull of these sequences. It follows from the properties of the R operation [5-11] that the contributions $g_{\alpha_1,\ldots,\alpha_m}$ are locally bounded functions, and therefore the sequence g belongs to $D_0(\mathscr{E}^T_{\text{loc}})$. It is obvious that also $G\subset D_0(\mathscr{E}^T_{\text{loc}})\subset D^*$.

<u>4.2. Construction of an Equation "Close" to Eq. (2.2).</u> The operators $A_{1,3;\,i_1}$, $A_{1,21;\,i_1}$, $A_{2,2;\,i_1,i_2}$ are not defined on the whole of D*, and therefore to construct the iterative series we introduce operators that are "close" to the operators A_α.

We shall say that the operators A_α^r are "close" [27] to the operators A_α if for $f\in D(A_\alpha)$

$$A_\alpha^r f=A_\alpha f. \tag{4.4}$$

We construct the operators A_α^r such that they are defined on the complete D^*.* We do this successively in several steps. We introduce first in $\mathscr{B}^T$, $\mathscr{E}^T$, and D^* the operators Π_α. For α = 1,3; i_1

$$\Pi_\alpha=\Pi^{(i_1)},\quad (\Pi^{(i_1)}f)_{N,\sigma}(p_1,\ldots,p_N)=\frac{1}{(2\pi)^4[(p_{i_1}-p_{N-1}-p_N)^2+\mu^2]}f_{N,\sigma}(p_1,\ldots,\hat{p}_{i_1},\ldots,p_N,p_{i_1}-p_{N-1}-p_N), \tag{4.5}$$

p_{N-1}, p_N and $p_{i_1}-p_{N-1}-p_N\in\{n_i\}$, i.e., p_{N-1}, p_N and $p_{i_1}-p_{N-1}-p_N$ belong to one of the connected components.

For α = 1,21 the action of the operator Π_α is defined by the expression (4.5) with the only difference that the operator acts on functions for which two of the variables p_{N-1}, p_N, and $p_{i_1}-p_{N-1}-p_N$ belong to one connected component, and the third to another connected component.

For α = 2,2; i_1, i_2

$$\Pi_\alpha=\Pi^{(i_1,i_2)},\quad (\Pi^{(i_1,i_2)}f)_{N,\sigma}(p_1,\ldots,p_N)=\frac{1}{(2\pi)^4[(p_{i_1}+p_{i_2}-p_N)^2+\mu^2]}f_{N-1,\sigma}(p_1,\ldots,\hat{p}_{i_1},\ldots,\hat{p}_{i_2},\ldots,p_N,p_{i_1}+p_{i_2}-p_N),$$
$$p_N,\ p_{i_1}+p_{i_2}-p_N\in\{n_i\}. \tag{4.6}$$

For the remaining α, we set Π_α = 1.

Using the operators Π_α, we represent the operators A_α in the form

$$A_\alpha=B_\alpha\Pi_\alpha. \tag{4.7}$$

For α = 1,3; i_1 and α = 1,21; i_1

$$(B_\alpha f)_{N,\sigma}(p_1,\ldots,p_N)=4(2\pi)^4\frac{(N+2)(N+1)}{N}\int\frac{dq_1}{(2\pi)^4(q_1^2+\mu^2)}\frac{dq_2}{(2\pi)^4(q_2^2+\mu^2)}f_{N+2,\sigma}(p_1,\ldots,p_N,q_1,q_2). \tag{4.8}$$

For α = 2,2; i_1, i_2

$$(B_\alpha f)_{N,\sigma}(p_1,\ldots,p_N)=\frac{12(2\pi)^4}{N}\int\frac{dq_1}{(2\pi)^4(q_1^2+\mu^2)}f_{N+1,\sigma}(p_1,\ldots,p_N,q_1). \tag{4.9}$$

For the remaining α, $B_\alpha = A_\alpha$.

It is readily verified that the operators B_α, like A_α, are defined on a set that is everywhere dense in $\mathscr{B}^T$, namely, the set of finite sequences of functions $D_0(\mathscr{B}^T)$ that are concentrated on compact sets, and

* We recall that the operators A_α are not defined on the complete domain D^*, and therefore A_α^r and A_α do not coincide outside $D(A_\alpha)$.

that they map this set to the set of finite sequences in $\mathscr{E}^T$. By anology with the set $D^*(A_\alpha)$, we introduce the set $D^*(B_\alpha)$, which consists of vectors $f \in D^*$ and such that the vector $B_\alpha f$ belongs to one of the spaces $\mathscr{B}^T, \mathscr{E}^T$, or $\mathscr{E}^T_{loc}$. To construct the operators A^r_α, we introduce in D^* the operators of projection onto $D^*(B_\alpha)$ in accordance with the rule

$$(P_\alpha f)_N^\sim(\ldots)=\begin{cases}\tilde{f}_N^\sim(\ldots), & \text{if } \tilde{f}_N\in D^*(B_\alpha),\\ (\tilde{f}_\alpha^*)_N(\ldots), & \text{if } \tilde{f}_N=\Pi_\alpha g_{\alpha_1,\ldots,\alpha_m}, \quad g\notin D^*(A_\alpha),\\ (\tilde{f}_\alpha')_N(\ldots) & \text{in all other cases.}\end{cases} \tag{4.10}$$

Here, $(f_\alpha^*)_N$ is defined by the formula

$$(B_\alpha f_\alpha^*)_N^\sim=R(A_\alpha A_{\alpha_1}\ldots A_{\alpha_m}F^0)_N, \tag{4.11}$$

and f_α' is the part of the function f that lies in $D^*(B_\alpha)$. It is obviously sufficient to define the operators P_α on functions corresponding to trivial partitionings of the set of variables $p_1, \ldots, p_N$. For the sequences corresponding to arbitrary partitionings the operators P_α will act on the connected components on which A_α acts. We do not fix the procedure for separating f_α', since in what follows we shall deal with only vectors in G. However, we shall describe in more detail the procedure for separating f_α^*.

To define $(\tilde{f}_\alpha^*)_N$ we must in Eq. (4.11), regarded as an equation for f_α^* for known $R(A_\alpha A_{\alpha_1}\ldots A_{\alpha_m}F^0)$, invert, as it were, the operator B_α. This can be readily done by using the following arguments.

The right-hand side of Eq. (4.10) is the contribution from the renormalized diagram and is defined in accordance with the R operation as the result of subtraction from the function equal to the product of the Feynman propagators of functions equal to the values of a definite number of propagators and their derivatives at fixed points and integration of the resulting rational function $Q(\ldots)$ with respect to the independent momentum variables* $q_1, \ldots, q_l$, i.e.,

$$R(A_\alpha A_{\alpha_1}\ldots A_{\alpha_m}F^0)_N(p_1,\ldots,p_N)=\delta(p_1+\ldots+p_N)\int\frac{dq_1}{(2\pi)^4(q_1^2+\mu^2)}\cdots\frac{dq_l}{(2\pi)^4(q_l^2+\mu^2)}Q_{\alpha,\alpha_1,\ldots,\alpha_m}(p_1,\ldots,p_N,q_1,\ldots,q_l). \tag{4.12}$$

For simplicity, we consider here the case when the expression $A_\alpha A_{\alpha_1}\ldots A_{\alpha_m}F^0$ corresponds to a connected Feynman graph. Since the expression $R(A_\alpha A_{\alpha_1}\ldots A_{\alpha_m}F^0)$ corresponds to the renormalized Feynman contribution, the integrals over $q_1, \ldots, q_l$ converge absolutely. In the first part of the expression (4.12), we now separate the operator B_α. Then what remains is f_α'. We do this for the example of the operator $B_{1,3;i_1}$, i.e., $\alpha = 1,3; i_1$. Using (4.8), we find

$$f^*_{N+2}(p_1,\ldots,p_N,q_1,q_2)=\frac{N}{4(2\pi)^4(N+2)(N+1)}\int\frac{dq_3}{(2\pi)^4(q_3^2+\mu^2)}\cdots\frac{dq_l}{(2\pi)^4(q_l^2+\mu^2)}Q_{\alpha,\alpha_1,\ldots,\alpha_m}(p_1,\ldots,p_N,q_1,q_2,q_3,\ldots,q_l). \tag{4.13}$$

Similarly, we obtain $B_{2,2;i_1,i_2}$.

Finally, we define the operators A^r_α "close" to A_α by the formula

$$A_\alpha{}^r=B_\alpha P_\alpha \Pi_\alpha \tag{4.14}$$

and the operator

$$A_r=\sum_\alpha A_\alpha{}^r.$$

Then the equation

$$F=-\lambda A_r F+F^0 \tag{4.15}$$

is "close" to Eq. (2.2).

We formulate our main theorem.

<u>THEOREM 4.1.</u> A formal solution of Eq. (4.15) is the iterative series

$$F_r=\sum_{n=0}^{\infty}(-\lambda)^n A_r{}^n F^0. \tag{4.16}$$

Each term of the series (4.16) belongs to G. The series (4.16) is equivalent to the renormalized perturbation

* The rational function $Q(\ldots)$ depends both on the external momenta $p_1, \ldots, p_N$ as well as on the internal independent momenta $q_1, \ldots, q_l$, where l is the number of independent cycles of the Feynman diagram corresponding to the contribution $A_\alpha A_{\alpha_1}\ldots A_{\alpha_m}F^0$.

series for the S matrix in the model of Euclidean quantum field theory with the Lagrangian (1.1).

Proof. We prove the theorem by induction. For $n = 0$, the theorem is trivial, since $F^0 \in G$ by the definition of the set G.

For $n = 1$,

$$A_r F^0 = \sum_\alpha B_\alpha P_\alpha \Pi_\alpha F^0.$$

For $\alpha=4,0$; $i_1, \ldots, i_4$, $\alpha=3,1$; i_1, i_2, i_3, $\alpha=2,11$; i_1, i_2 and $\alpha=1,111$; i_1 $F^0 \in D(A_\alpha = B_\alpha)$, and therefore $B_\alpha P_\alpha \Pi_\alpha F^0 = A_\alpha F^0$, and for $\alpha=1,3$; i_1, $\alpha=1,21$; i_1 and $\alpha=2,2$; i_1, i_2 $B_\alpha P_\alpha \Pi_\alpha F^0 = R(A_\alpha F^0) = g_\alpha \in G$ in accordance with the definitions (4.10) and (4.11).

Now suppose $A_r^m F^0 \in F$, i.e.,

$$A_r^m F^0 = \sum_{\alpha_1,\ldots,\alpha_m} g_{\alpha_1,\ldots,\alpha_m} = \sum_{\alpha_1,\ldots,\alpha_m} R(A_{\alpha_1} \ldots A_{\alpha_m} F^0).$$

Then, using (4.10) and (4.11), we obtain

$$A_r^{m+1} F^0 = \sum_{\alpha,\alpha_1,\ldots,\alpha_m} B_\alpha P_\alpha \Pi_\alpha g_{\alpha_1,\ldots,\alpha_m} = \sum_{\alpha,\alpha_1,\ldots,\alpha_m} B_\alpha \overset{*}{g}_{\alpha,\alpha_1,\ldots,\alpha_m} = \sum_{\alpha,\alpha_1,\ldots,\alpha_m} R(A_\alpha A_{\alpha_1} \ldots A_{\alpha_m} F^0),$$

which completes the proof of the theorem.

4.3. Regularization and Quasisolutions. We show that the renormalized perturbation series (4.16), which is a solution of Eq. (4.15), is a formal quasisolution of the original equation (2.2). For this, we represent Eq. (2.2) in the form

$$F = -\lambda \sum_\alpha B_\alpha (P_\alpha + Q_\alpha) \Pi_\alpha F + F^0, \quad Q_\alpha = 1 - P_\alpha, \tag{4.17}$$

and substitute in Eq. (4.17) the solution (4.16) of Eq. (4.15). We obtain

$$F_r = -\lambda \sum_\alpha B_\alpha P_\alpha \Pi_\alpha F_r - \lambda \sum_\alpha B_\alpha Q_\alpha \Pi_\alpha F_r + F^0 \tag{4.18}$$

or

$$F_r = -\lambda A_r F_r + F^0 - \lambda \sum_\alpha B_\alpha Q_\alpha \Pi_\alpha F_r. \tag{4.19}$$

Since F_r satisfies Eq. (4.15), for F_r to be a solution of Eq. (2.2) it is necessary that

$$\sum_\alpha B_\alpha Q_\alpha \Pi_\alpha F_r = 0. \tag{4.20}$$

Therefore, $\sum_\alpha B_\alpha Q_\alpha \Pi_\alpha F_r$ serves as a measure of the deviation of the solution of Eq. (4.15) from the exact solution of Eq. (2.2). If $\sum_\alpha B_\alpha Q_\alpha \Pi_\alpha F_r \neq 0$, then F_r is a formal "quasisolution" [28]. Note that if there exists an exact solution F of Eq. (2.2), then for it the condition (4.20) is satisfied.

Besides the proposed method of solution of Eq. (2.2), there also exists another approach based on preliminary regularization of the equation and construction of a regularized solution. As an example, let us consider Pauli–Villars regularization. The essence of this regularization is that the functions $G_0(p) = (2\pi)^{-4} (p^2+\mu^2)^{-1}$ are replaced by the regularized functions

$$G_0^M(p) = G_0(p) + \sum_i \frac{c_i}{(2\pi)^4 (p^2 + M_i^2)}, \tag{4.21}$$

where the coefficients c_i and the masses M_i are chosen such that the function $G_0^M(p)$ decreases sufficiently rapidly as $p \to \infty$, and as $M_i \to \infty$

$$G_0^M(p) \to G_0(p). \tag{4.22}$$

After the substitution (4.21), Eq. (2.2) takes the form

$$F=-\lambda A(M)F+F^0, \tag{4.23}$$

where the operator A(M) is obtained from A by the replacement (4.21). Since the functions decrease sufficiently rapidly, the operator A(M) and all its powers are defined on F^0, and Eq. (4.27) admits formal solution in the form of an iterative series. However, each term of this series will diverge as $M_i \to \infty$. Therefore, the next stage is the application to it of the R operation.

Pauli–Villars regularizations (and other regularizations) of the equations of quantum field theory are conceptually close to Tikhonov's method of solution of improperly posed problems of mathematical physics. At the beginning of the sixties, Tikhonov gave a definition of the solution of an improperly posed problem by means of a regularizing family of operators and developed methods for constructing such solutions [17]. The essence of Tikhonov's method consists of replacing the original equation, which belongs to the class of improperly posed problems, by an equation in which the original operator is replaced by a regularizing family of operators, and the equation with the regularizing family is a properly posed problem. The regularizing family depends on a definite parameter, and when this parameter tends to zero (to infinity) the regularizing family of operators tends to the original operator. By solution of the improperly posed problem, one understands the limit of the solution of the regularized equations when the parameter tends to zero (to infinity).

As we have seen above, Pauli–Villars regularization (and other regularizations) of the equations of quantum field theory and the subsequent lifting of the regularization by means of the R operation have much in common with these methods.

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