

Victor Enolski (1945–2019)

Emma Previato

Our friend and coauthor Victor Enolski passed away in June 2019, after a brief but valiant struggle with pancreatic cancer. Victor’s profound and multipronged contribution to the mathematical sciences flew under the radar, to some extent; “Victor’s modesty is legend” says one of the pieces below, and indeed he was self-effacing to a fault, promoting instead others’ work, creating interdisciplinary collaborations among scientists of diverse backgrounds, re-discovering classical work with childlike enthusiasm and breathing new life into it. We write this homage to his life with the hope of bringing his contribution to the attention of the larger mathematical community, particularly because the subjects of his main focus, the special functions of classical geometry, mathematical physics, analytic number theory and wide-ranging applications (for cosmology, e.g., see Kunz and Lämmerzahl’s piece below), are gaining more power, partly on the strength of computational differential algebra and geometry (see Eilbeck’s piece below). Buchstaber’s piece offers a broad overview of the foundational aspects of these specific special functions, a view from the top that allows you to glimpse their significance in topology, combinatorics, algebraic geometry, deformation theory, and their crowning application to integrability of non-linear PDEs, one main reason for the resurgence they enjoyed the world over starting in the 1970s (see Braden’s and Matveev’s pieces). We tried to provide a somewhat technical (although, we hope, accessible) guide to enable you to follow your specific interest and read more about Victor’s profound ideas. There is so much farther to go, for future researchers.

Victor Zelikovich Ėnol’skiĭ, as first transliterated in *Mathematical Reviews* (most recently he chose the simpler version Enolski) was born in Odessa (then USSR),

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Ukraine, on April 26, 1945. His father, Eugene (Zelik) Ėnol’skiĭ, was a major in the Soviet Army during WWII and afterwards Professor of History in Kiev State University; his mother, Elizabeth Enolskaya (née Vainrub), served in hospitals during the war as a senior lieutenant; after the war she was a medical doctor in child psychiatry. Victor earned an MSc from Kiev State University under the supervision of Professor D. Ya. Petrina, National Academy of Sciences of Ukraine (NASU), with a Master’s thesis *Differential Equations for Feynman Integrals*. Another part of his MSc research, on the Mandelstam hypothesis in elementary particle physics, is the topic of his earliest MathSciNet entry [15]. At NASU, Victor earned a PhD in Theoretical Physics and Applied Mathematics (1977) with his thesis *Analytical Properties of Feynman Integrals*, as well as a Doctor of Science degree in Theoretical Physics and Applied Mathematics (1996) with the thesis *Method of Reduction to Elliptic Functions in the Theory of Solitons*. Victor was a junior research fellow at NASU from 1975 to 1982, and a senior research fellow thereafter. In 2016, he became a Professor in the Physics and Mathematics Department of the National University of Kiev-Mohyla Academy. Victor conducted extended visits as an invited scholar at over 20 universities or research institutes in Canada, Denmark, France, Germany, Italy, Japan, Norway, Spain, Sweden, the UK and the US.

Victor moved to Germany in the late 1990s, with his wife Rena and their daughter, but he still spent most of his working time in Ukraine and untiringly applied for workshop grants so that he could bring an international community to Kiev; the last workshop he organized, “Algebraic Curves, Integrable Systems, and Cryptography,” (August 24–25, 2018) gave rise to several new collaborative projects and publications. The announcement of the workshop on the university’s webpage allows us to preview the arc of Victor’s interests we’ll follow below, as part of it reads (verbatim): “different aspects of the theory of algebraic curves (hyperelliptic and nonhyperelliptic, and some particular cases): fields of Abelian functions and addition

laws in these fields, multivariate entire functions which generate the Abelian functions.”

He spent the last few months of his life at home in Munich, where he died on June 20, 2019.

In this short article all we can do is try to give you a glimpse into Victor’s legacy to mathematics and physics. There are no words that could describe his legacy to his collaborators, of whom MathSciNet lists 51, and to his friends: since Victor could find humor under the most trying circumstances and used the English language with gusto, and in view of his expansive professional pilgrimages, we will remember him with the humorous words of the Head of the School of Mathematical and Computer Sciences at Heriot-Watt University, who had been chasing him for days with a travel-reimbursement form to sign. Victor, Chris Eilbeck and I were working in Chris’ office when he walked in and said “You are peripatetic and mischievous.”

1. Multidimensional Sigma Functions and Applications

V. M. Buchstaber

1.1. Introduction. We met with Victor Enolski in 1995. It was a time when the theory of multidimensional theta functions and methods of algebraic geometry were the focus of attention of a large mathematical community, inspired by fundamental results in a number of relevant areas of theoretical and mathematical physics. Our close collaboration and friendship continued until his last days. He was a brilliant scientist and a very good friend.

Almost immediately after the start of our collaboration, we put forward an ambitious program: to develop the theory of multidimensional sigma functions based on modern achievements of differential topology and algebraic geometry. We decided to follow Weierstrass’ approach to the theory of elliptic functions, which essentially uses the remarkable model of elliptic curves. Our theory was designed to ensure the construction of the fundamental equations of mathematical physics based on the uniformization of the Jacobians of a certain class of algebraic curves. The key goal was to obtain solutions of these equations, independent of the choice of a basis in the lattice spanned by the periods of holomorphic differentials on these curves.

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The goal was motivated, among other things, by the following problem. Let a differential equation with parameters be given. Its solution depends on the parameters and initial data. The problem is to build a new differential equation describing this dependence. Such a problem naturally arises in theoretical and mathematical physics.

In memory of Victor Enolski, I prepared, addressed to a wide mathematical audience, a review of the theory whose creation was largely due to Enolski.

Victor Enolski attained deep results on the relationship between theta and sigma functions of algebraic curves, which gave answers to long-standing questions (see [11]). In this review, multidimensional theta functions are not explicitly involved. However, it should be noted that now there are many representations of multidimensional sigma functions in the form of modified multidimensional theta functions. We can say that in essence we are talking about one class of functions, but in different “coordinates.”

In the multidimensional case, as in the elliptic, there are problems in which either theta or sigma functions can be successfully used. But in well-known problems, to obtain an effective description of solutions of equations, it is important to use one and not the other. For example, if you want to investigate the dependence on the variations of individual branch points of a curve, theta functions are preferred. But, if, for example, it is necessary to obtain results that require solutions of differential equations, including differentiation with respect to the parameters of the curve, or require limit transitions to the variety of degenerate curves, then sigma functions are preferable. At the same time, there are remarkable results that use sigma functions as modified theta functions and vice versa.

1.2. Theory of sigma functions. In the classical sense, an *Abelian function* is a meromorphic function on a complex Abelian torus $T^g = \mathbb{C}^g/\Gamma$, where $\Gamma \subset \mathbb{C}^g$ is a rank- $2g$ lattice satisfying the Riemann conditions. That is, f is an Abelian function if and only if f is a meromorphic function and $f(\mathbf{u}) = f(\mathbf{u} + \omega)$ for all $\mathbf{u} = (u_1, \dots, u_g) \in \mathbb{C}^g$ and $\omega \in \Gamma$. Abelian functions form a differential field F . Basic facts about Abelian functions:

- A_1 If $f \in F$, then $\partial_{u_i} f \in F$, $i = 1, \dots, g$.
- A_2 For any nonconstant f_1, \dots, f_{g+1} from F there exists $P \in \mathbb{C}[z_1, \dots, z_{g+1}]$ such that $P(f_1, \dots, f_{g+1}) = 0$.
- A_3 If $f \in F$ is any nonconstant function, then any $h \in F$ is a rational function of $(f, \partial_{u_1} f, \dots, \partial_{u_g} f)$.
- A_4 There exists an entire function $\theta : \mathbb{C}^g \rightarrow \mathbb{C}$ such that $\partial_{u_i} \partial_{u_j} \log \theta \in F$, $i, j = 1, \dots, g$.

The theory of Abelian functions was a central topic of 19th century mathematics. In the mid-seventies of the last century, a new wave of investigation in this field arose in response to the discovery that Abelian functions provide

solutions to a number of challenging problems of modern theoretical and mathematical physics.

A plane nonsingular algebraic curve defines a lattice Γ as the set of all periods of its holomorphic differentials. The resulting torus is called the Jacobian of a curve. Suppose \mathcal{B} is an open dense subset in \mathbb{C}^d . We will consider a family V of nonsingular curves, depending linearly on a parameter $b \in \mathcal{B}$. We use V to define a space of Jacobians U over \mathcal{B} . The space U is naturally fibered, $p : U \rightarrow \mathcal{B}$, where the fiber V_b over a point $b \in \mathcal{B}$ is the Jacobian J_b of the curve with the parameter b . Let $g = 1$ and $d = 2$. Then

$$V = \{(x, y, g_2, g_3) \in \mathbb{C}^2 \times \mathcal{B} \mid y^2 = 4x^3 - g_2x - g_3\}$$

is the family of Weierstrass elliptic nonsingular curves, where $\mathcal{B} = \{(g_2, g_3) \in \mathbb{C}^2 \mid g_2^3 - 27g_3^2 \neq 0\}$.

We employ the following properties of the Weierstrass σ -function $\sigma : \mathbb{C}^3 \rightarrow \mathbb{C}$ with $(g_2, g_3) \in \mathcal{B}$:

- $\sigma(u, g_2, g_3)$ is entire in $(u, g_2, g_3) \in \mathbb{C}^3$;
- $\partial_u^2 \log \sigma(u, g_2, g_3) = -\wp(u, g_2, g_3) \in F$;
- $Q_0\sigma = 0$, $Q_2\sigma = 0$, where $Q_0 = 4g_2\partial_{g_2} + 6g_3\partial_{g_3} - u\partial_u + 1$ and $Q_2 = 6g_3\partial_{g_2} + \frac{1}{3}g_2^2\partial_{g_3} - \frac{1}{2}\partial_u^2 - \frac{1}{24}g_2u^2$;
- $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, where $\wp' = \frac{\partial}{\partial u}\wp$.

The system of equations (c) allows us, using the initial condition $\sigma(u, 0) = u$, to restore the expansion coefficients of the function $\sigma(u, g_2, g_3)$ as a series in u in the form of polynomials in g_2 and g_3 .

The problem of construction of a multidimensional analogue of the Weierstrass σ -function is a classical one. In 1886, F. Klein proposed the following problem:

Modify the multidimensional θ -function $\theta(\mathbf{u}; \Gamma_{V_b})$ in order to obtain an entire function which is:

- independent* of a choice of basis in the lattice Γ_{V_b} ;
- covariant* with respect to the Möbius transformations of the curve V_b .

On this problem, Klein published three works (1886–1890). In 1923, a 3-volume collection of Klein's scientific works was published. In the preface to the works on the problem under discussion, he emphasized that the theory of hyperelliptic σ -functions is still far from complete.

The covariance requirement (2) immediately led to the need to confine ourselves to the class of hyperelliptic curves. But even this case caused unnatural difficulties.

In his work (1903), H. F. Baker disregarded requirement (2) and showed that in the case of curves of genus 2, it is possible to construct analogues of elliptic σ -functions without using θ -functions of genus 2.

In a series of works, V. Buchstaber, V. Enolski and D. Leykin (1997–2019) developed a theory and applications of multidimensional σ -functions associated with given models of plane algebraic curves.

The work in [9] and subsequent papers is devoted to the development of a method of integration of nonlinear partial differential equations on the basis of uniformization of Jacobi varieties of algebraic curves. It was motivated by the fact that when using Abelian functions, defined by the logarithmic derivatives of multidimensional σ -functions, the important equations of theoretical and mathematical physics appear naturally and explicitly.

Let $V_\lambda = \{(x, y) \in \mathbb{C}^2 : f(x, y; \lambda) = 0\}$ be a plane *nonsingular* algebraic curve of genus g , where $f(x, y; \lambda)$ is a polynomial in x and y with the coefficient vector $\lambda = (\lambda_{q_1}, \dots, \lambda_{q_d}) \in \mathcal{B}_g$, where \mathcal{B}_g is an open dense subset in \mathbb{C}^d . Denote by $\Gamma_\lambda \subset \mathbb{C}^{2g}$ the lattice of periods of holomorphic differentials on V_λ .

Problem I. Construct an entire function $\sigma(\mathbf{u}; \lambda)$ such that:

- The decomposition of $\sigma(\mathbf{u}; \lambda)$ at the origin $\mathbf{u} = 0$ is given by a power series in \mathbf{u} with polynomial coefficients in λ .
- For any i and j , the function $\wp_{i,j}(\mathbf{u}; \lambda) = -\partial_{u_i}\partial_{u_j} \ln \sigma(\mathbf{u}; \lambda)$ defines a meromorphic function on $\text{Jac } V_\lambda = \mathbb{C}^g/\Gamma_\lambda$.
- The function $\sigma(\mathbf{u}; 0)$ is a polynomial of a prescribed form.

Problem II. Obtain the uniformization of Jacobi varieties in terms of functions $\wp_{i,j}(\mathbf{u}; \lambda)$ and their derivatives with respect to \mathbf{u} .

In the case of hyperelliptic curves of genus g , in our works (1997–1999) we constructed the desired σ -function $\sigma(\mathbf{u}; \lambda)$, where $\mathbf{u} = (u_1, \dots, u_{2g-1})$, $\lambda = (\lambda_4, \dots, \lambda_{4g+2})$ and $\sigma(\mathbf{u}; 0)$ coincides up to a factor with the Adler-Moser polynomials. On this basis we obtained the key results of the theory of hyperelliptic functions of genus $g > 1$ (see below). In subsequent works, for coprime n and s , $n < s$, we introduced a class of plane (n, s) -curves of genus $g = \frac{(n-1)(s-1)}{2}$ with the coefficient vector $(\lambda_{q_1}, \dots, \lambda_{q_d})$, where $d = \frac{(n+1)(s+1)}{2} - \left\lfloor \frac{s}{n} \right\rfloor - 3$, and constructed the corresponding σ -functions such that $\sigma(\mathbf{u}; 0)$ are the so-called Schur-Weierstrass (n, s) -polynomials (see below).

A family V_λ of (n, s) -curves is defined by the polynomials

$$f(x, y; \lambda) = y^n - x^s - \sum_{i,j} \lambda_{(in+js)} x^i y^j,$$

where $0 \leq i \leq s-2$, $0 \leq j \leq n-2$ and $in + js < ns$. Hyperelliptic curves of genus g are $(2, 2g+1)$ -curves. The trigonal curves $(3, s)$ have genus $g = 3m$ at $s = 3m+1$ and genus $g = 3m+1$ at $s = 3m+2$, $m > 0$.

In the class of (n, s) -curves, cases $n = 2$ and $n = 3$ are distinguished by the fact that only for them is the condition $n \leq g < s$ satisfied: this is instrumental for an effective construction of the theory of σ -functions.

Singularity theory studies the family V of functions

$$f(x, y, \lambda) = y^n - x^s - \sum_{j=0}^{n-2} \sum_{i=0}^{s-2} \lambda_{(in+js)} x^i y^j$$

as *miniversal unfolding* of the *Pham singularity* $y^n - x^s$. Miniversal unfolding has $(n - 1)(s - 1)$ parameters λ . The number $m = \#\{\lambda_k \mid k > ns\}$ is the *modality* of f . Consider the discriminant variety of the family V :

$$\Sigma \subset \mathbb{C}^{2g} : (\lambda \in \Sigma)$$

$$\Leftrightarrow (\exists (x, y) \in \mathbb{C}^2 : f = f_x = f_y = 0 \text{ at } (x, y, \lambda)).$$

We use a construction based on V. M. Zakalyukin's theorem of holomorphic vector fields tangent to Σ . The fields define a holomorphic $\Delta(\lambda) \in \mathbb{C}(\lambda)$ such that $\Sigma = \{\lambda \in \mathbb{C}^{2g} \mid \Delta(\lambda) = 0\}$. A vector field ℓ is tangent to Σ if and only if $\ell\Delta(\lambda) = \phi(\lambda)\Delta(\lambda)$, where $\phi(\lambda) \in \mathbb{C}[[\lambda]]$. In the space of holomorphic vector fields tangent to Σ , there exists a unique basis $\mathcal{L} = (\ell_1, \dots, \ell_{2g})^t$ such that

$$\mathcal{L} = T(\lambda)\partial_\lambda, \quad T(\lambda) = T(\lambda)^t, \quad \Delta(\lambda) = \det T(\lambda),$$

where $T(\lambda)$ is the matrix of Arnold's convolution.

If $\lambda \notin \Sigma$, the genus of a miniversal unfolding is $\geq (n - 1)(s - 1)/2$. We impose the condition $\lambda_k = 0$ when $k > ns$ and obtain a family of curves of constant genus $g = (n - 1)(s - 1)/2$ over $\mathcal{B} = \mathbb{C}^{2g-m} \setminus (\mathbb{C}^{2g-m} \cap \Sigma)$.

In what follows we use an obvious renumbering of λ_k . Under the condition $\lambda_k = 0$ for $k < 0$,

(1) the holomorphic symmetric matrix $T(\lambda)$ becomes a matrix over $\mathbb{C}[\lambda_1, \dots, \lambda_{2g-m}]$;

(2) $\Delta(\lambda) \in \mathbb{C}[\lambda_1, \dots, \lambda_{2g-m}]$ and $\mathcal{B} = \{\lambda \in \mathbb{C}^{2g-m} \mid \Delta(\lambda) \neq 0\}$;

(3) the holomorphic frame \mathcal{L} becomes a $2g$ -dimensional basis of the $\mathbb{C}[\lambda_1, \dots, \lambda_{2g-m}]$ -module of *global* sections of the $(2g - m)$ -dimensional tangent bundle $\mathcal{T}\mathcal{B} \rightarrow \mathcal{B}$. Fix the notation for the frame

$$\mathcal{L} = (\ell_1, \dots, \ell_{2g})^t = T(\lambda)(\underbrace{0, \dots, 0}_m, \partial_{\lambda_1}, \dots, \partial_{\lambda_{2g-m}})^t$$

and its structure functions

$$[\ell_i, \ell_j] = \sum_{h=1}^{2g} c_{ij}^h(\lambda)\ell_h, \quad c_{ij}^h(\lambda) \in \mathbb{C}[\lambda].$$

The equation $f(x, y, \lambda) = 0$ in \mathbb{C}^{2+2g-m} defines the family V of (n, s) -curves over \mathcal{B} . Consider the bundle $\hat{p} : \hat{V} \rightarrow \mathcal{B}$ whose fiber is the curve

$$\hat{V}_\lambda = \{(x, y) \in \mathbb{C}^2 \mid f(x, y, \lambda) = 0\}$$

with a *puncture at infinity*. Let $H^1(\hat{V}_\lambda, \mathbb{C})$ be the linear $2g$ -dimensional vector space of holomorphic 1-forms on \hat{V}_λ . Consider associated with $\hat{p} : \hat{V} \rightarrow \mathcal{B}$ the locally trivial vector bundle $\varpi : \Omega^1 \rightarrow \mathcal{B}$ whose fiber is $H^1(\hat{V}_\lambda, \mathbb{C})$. A connection in Ω^1 is a Gauß-Manin connection on \hat{V} .

Since (∞) belongs to all curves from V , we can construct a global section of Ω^1 by taking the classical basis of Abelian differentials of first and second kind. Let $D(x, y, \lambda)$ be the vector $D(x, y, \lambda) = (D_1(x, y, \lambda), \dots, D_{2g}(x, y, \lambda))$ of canonical basis 1-forms from $H^1(\hat{V}_\lambda, \mathbb{C})$. Its matrix of periods Ω satisfies the Legendre relation (a particular case of the Riemann-Hodge relations)

$$\Omega^t J \Omega = 2\pi i J, \quad \text{where } J = \begin{pmatrix} 0_g & 1_g \\ -1_g & 0_g \end{pmatrix}.$$

The classical theory of Abelian differentials asserts that such basis exists and provides a means to construct it. The essential part of $D(x, y, \lambda)$, that is, the part that provides the Legendre relation, is defined by the classic formula

$$D(x_1, y_1)^t J D(x_2, y_2) = \{\Phi_{1,2} - \Phi_{2,1}\} dx_1 dx_2,$$

where $\Phi_{1,2} = \frac{1}{f_y(x_1, y_1, \lambda)} \frac{\partial}{\partial x_2} \left(\frac{f(x_1, y_2, \lambda)}{(x_1 - x_2)(y_1 - y_2)} \right)$. The calculation is carried out "on the curve," that is, with the assumption $f(x_i, y_i, \lambda) = 0$, $i = 1, 2$. The Christoffel coefficient of the Gauß-Manin connection

$$\Gamma_j = (\Gamma_{ji}^k), \quad i, j, k = 1, \dots, 2g,$$

associated to the field ℓ_j is uniquely defined by the following relation:

The holomorphic vector-valued 1-form $\ell_j(D(x, y, \lambda)) + \Gamma_j D(x, y, \lambda)$ is *exact* "on the curve."

Our σ -function $\sigma(\mathbf{u}, \lambda)$, $\mathbf{u} \in \mathbb{C}^g$, $\lambda \in \mathbb{C}^d$, is an entire function that solves Problem I. It is obtained as a solution to the system $Q_j \sigma(\mathbf{u}, \lambda) = 0$, $j = 1, \dots, 2g$. Here $Q_j = \ell_j - \frac{1}{2} H_j - \delta_j(\lambda)$, with $\ell_j \in \mathcal{L}$ and

$$H_j = \alpha_j^{kl}(\lambda) \partial_{u_k} \partial_{u_l} + 2\beta_{jk}^l(\lambda) u_k \partial_{u_l} + \gamma_{jkl}(\lambda) u_k u_l, \\ \delta_j(\lambda) = \frac{1}{8} \ell_j \log \Delta(\lambda) + \frac{1}{2} \beta_{jk}^k(\lambda),$$

where the summation from 1 to g extends over the repeated indices. The coefficients $\alpha_j^{kl}(\lambda) = \alpha_j^{lk}(\lambda)$, $\beta_{jk}^l(\lambda)$ and $\gamma_{jkl}(\lambda) = \gamma_{jlk}(\lambda)$ are polynomials of λ .

Solving the system of equations $H_j \varphi(\mathbf{u}) = 2\delta_j(\lambda) \varphi(\mathbf{u})$, we obtain a polynomial $\varphi(\mathbf{u}) = \sigma(\mathbf{u}, 0)$ defined up to a constant factor. Then, solving the system of equations $Q_j \sigma(\mathbf{u}, \lambda) = 0$, $j = 1, \dots, 2g$, under this initial condition $\sigma(\mathbf{u}, 0)$, we can reconstruct the expansion coefficients of the function $\sigma(\mathbf{u}, \lambda)$ in a series in \mathbf{u} in the form of polynomials in λ .

Write the system of equations

$$Q_j \sigma(\mathbf{u}, \lambda) = 0, \quad j = 1, \dots, 2g,$$

in the form of the Schrödinger equations

$$\ell_j \sigma = \left\{ \frac{1}{2} H_j + \delta_j(\lambda) \right\} \sigma,$$

of a multidimensional quantum harmonic oscillator with multiple “times.” According to the formalism of the quantum oscillator:

- H_j is a set of “quadratic Hamiltonians;”
- ℓ_j are derivatives over “times;”
- δ_j is “the energy of an oscillator mode.”

The realization of the σ -function in the form of an average of the “ground-state wave function” (a multidimensional Gaussian function) over a lattice suggests a natural interpretation of the σ -function as the “wave function of the coherent state” of the oscillator.

An operator Q_j is defined if we know the (polynomial in λ) matrices

$$\alpha_j = (\alpha_j^{kl}), \quad \beta_j = (\beta_j^{lk}), \quad \text{and} \quad \gamma_j = (\gamma_{jkl}).$$

Let

$$A_j(\lambda) = J \begin{pmatrix} \alpha_j & (\beta_j)^t \\ \beta_j & \gamma_j \end{pmatrix}.$$

Theorem 1. *The set of matrices $\{\alpha_j, \beta_j, \gamma_j\}$ defines the operators Q_j , $j = 1, \dots, 2g$, such that*

$$[Q_i, Q_j] = \sum_{h=1}^{2g} c_{ij}^h(\lambda) Q_h \quad \text{and} \quad Q_j(\sigma(u, \lambda)) = 0$$

if and only if $A_j = \Gamma_j$.

Observe that our operators Q_j and vector fields ℓ_j obey the same commutation relations.

Let us introduce the Schur-Weierstrass (n, s) -polynomials, which allow us to obtain an explicit form for the polynomial $\sigma(\mathbf{u}, 0)$.

Let, as above, n and s be a pair of coprime integers such that $s > n \geq 2$. Let us consider the set of all nonnegative integers of the form $an + bs$, where a, b are nonnegative integers. Natural numbers that cannot be represented in this form give the Weierstrass gap sequence. Such a sequence, ordered in ascending order, will be denoted by $\mathbf{W}_{n,s}$. The number of these integers in the sequence is called its length.

The elements w of the Weierstrass sequence $\mathbf{W}_{n,s}$ can be represented in the form $w = -\alpha n + \beta s$, where α and β are integers such that $\alpha > 0$ and $n > \beta > 0$, and so that the numbers α and β are uniquely defined. The length g of the Weierstrass sequence $\mathbf{W}_{n,s} = \{w_1, \dots, w_g\}$ is equal to $(n-1)(s-1)/2$. Its maximal element w_g is equal to $2g-1 = -n + s(n-1)$, and $i \leq w_i \leq 2i-1$ for all $i = 1, \dots, g$. We denote by WS_g the set of all Weierstrass sequences $\mathbf{W}_{n,s} = \{w_1, \dots, w_g\}$ of length g .

For $\mathbf{W}_{n,s}$ and any $r \geq 0$, rational analogues of the Abel mapping

$$A_{n,s}^r : \mathbb{C}^r \longrightarrow \mathbb{C}^g : A_{n,s}(\mathbf{x}) = (p_{w_1}(\mathbf{x}), \dots, p_{w_g}(\mathbf{x}))$$

are defined. Here $p_k(\mathbf{x}) = \sum_{j=1}^r x_j^k$ is the k th Newton polynomial.

A partition π of length g is a nonincreasing set of g positive integers π_i . In the classical theory of invariants, each partition π is matched with the Schur polynomial S_π . It is known that the polynomials S_π satisfy the Kadomtsev-Petviashvili hierarchy.

Denote by Par_g the set of all partitions of length g . The formula $\chi(\mathbf{w}) = \pi$, where $\pi_k = w_{g-k+1} + k - g$, defines an embedding $\chi : WS_g \rightarrow \text{Par}_g$. A partition π that is the image of any Weierstrass sequence under the mapping χ is called the Weierstrass partition. Set $\pi_{n,s} = \chi(\mathbf{W}_{n,s})$.

We denote by $S_{n,s}$ the Schur polynomials corresponding to Weierstrass partitions $\pi_{n,s}$. We have

$$S_{n,s} = S_{n,s}(e_1, \dots, e_{2g-1}), \quad \text{where} \quad g = (n-1)(s-1)/2.$$

Here $g = (n-1)(s-1)/2$ and e_k are elementary symmetric polynomials in x_1, \dots, x_{2g-1} .

Theorem 2. *In the representation via the Newton polynomials, any polynomial $S_{n,s}$ is a polynomial in g variables $\{p_{w_1}, \dots, p_{w_g}\}$ only, where $\{w_1, \dots, w_g\} = \mathbf{W}_{n,s}$.*

A polynomial $\sigma_{n,s}(z_1, \dots, z_g)$ in g variables, which exists by Theorem 2 and is given by the identity $\sigma_{n,s}(p_{w_1}, \dots, p_{w_g}) \equiv S_{n,s}$, is called a Schur-Weierstrass polynomial.

For an arbitrary Weierstrass sequence $\mathbf{W}_{n,s} = \{w_1, \dots, w_g\}$ and for the corresponding Schur-Weierstrass polynomial $\sigma_{n,s}(z_1, \dots, z_g)$ under a chosen value of the vector $\mathbf{z} = (z_1, \dots, z_g) \in \mathbb{C}^g$, we introduce the following polynomial in $x \in \mathbb{C}$:

$$R_{n,s}(x; \mathbf{z}) = \sigma_{n,s}(A_{n,s}^1(x) + \mathbf{z}).$$

Theorem 3. *For a given $\xi \in \mathbb{C}^g$, the polynomial $R_{n,s}(x; \xi)$ either does not depend on x or has at most q roots $\{x_1, \dots, x_q\}$, $q \leq g$.*

Let a polynomial $P(\xi_1, \dots, \xi_g)$ in g variables be given. For every $\mathbf{z} \in \mathbb{C}^g$, using the Weierstrass sequence $\mathbf{W}_{n,s}$ of length g , to this polynomial we put into correspondence a polynomial in x of the form $R_P(x; \mathbf{z}) = P(\mathbf{z} - A_{n,s}^1(x))$.

We say that a polynomial $P(\xi_1, \dots, \xi_g)$ satisfies the (n, s) -analogue of the Riemann vanishing theorem for polynomials if the polynomial $R_P(\mathbf{x}; \mathbf{z})$ for $\mathbf{z} = A_{n,s}^g(\mathbf{x})$ either has exactly g roots $\{t_1, \dots, t_g\} = \{x_1, \dots, x_g\}$ or is identically zero.

Theorem 4. *If a polynomial P satisfies the (n, s) -analogue of Riemann’s vanishing theorem for polynomials, then it is equal to the Schur-Weierstrass polynomial $\sigma_{n,s}$ up to a constant factor.*

For any $g \geq 1$ in the hyperelliptic case, the polynomial $\sigma(\mathbf{u}, 0) = \sigma_{2,2g+1}(\mathbf{u})$ coincides, up to a factor, with the Adler-Moser polynomial.

Corollary 5. A polynomial P satisfies the $(2, 2g + 1)$ -analog of Riemann's vanishing theorem for polynomials if and only if P is the Adler-Moser polynomial up to a constant factor.

2.1. Applications. We denote by $\pi : \mathcal{U}_g \rightarrow \mathcal{B}_g$ the universal bundle of Jacobian varieties $J_\lambda = \text{Jac}V_\lambda$ of hyperelliptic curves. Let us consider the mapping $\varphi : \mathcal{B}_g \times \mathbb{C}^g \rightarrow \mathcal{U}_g$ such that

$$\begin{array}{ccc} \mathcal{B}_g \times \mathbb{C}^g & \xrightarrow{\varphi} & \mathcal{U}_g \\ \downarrow & & \downarrow \\ \mathcal{B}_g & \xlongequal{\quad} & \mathcal{B}_g \end{array}$$

which defines the projection $\lambda \times \mathbb{C}^g \rightarrow \mathbb{C}^g / \Gamma_\lambda$ for any $\lambda \in \mathcal{B}_g$. We denote by $\mathcal{F} = \mathcal{F}_g$ the field of functions on \mathcal{U}_g such that for any $f \in \mathcal{F}$ the function $\varphi^*(f)$ is meromorphic, and its restriction to the fiber J_λ is an Abelian function for any point $\lambda \in \mathcal{B}_g$, that is, $f(\mathbf{u} + 2\Omega) = f(\mathbf{u})$ for any $2\Omega \in \Gamma_\lambda$.

Let $\mathbf{u} = \mathbf{t}, t_1 = x, f'(\mathbf{t}) = \frac{\partial}{\partial x} f(\mathbf{t}), \partial_{2k-1} = \frac{\partial}{\partial t_{2k-1}}$, and

$$\omega = \begin{pmatrix} j_1 & \dots & j_s \\ 2k_1 - 1 & \dots & 2k_s - 1 \end{pmatrix},$$

where $1 \leq k_1 < \dots < k_s, 1 \leq s \leq g, j_q > 0, q = 1, \dots, s$, and $j_1 + \dots + j_s \geq 2$. Set $\wp_\omega = \wp_\omega(\mathbf{t}) = -\partial_{2k_1-1}^j \dots \partial_{2k_s-1}^s \ln \sigma(\mathbf{t})$. Let \mathcal{P} denote the subring over \mathbb{Q} in the field \mathcal{F} generated by the functions \wp_ω for all ω .

The following theorem is one of the central results in our theory of hyperelliptic functions. It is a nontrivial generalization of the Dubrovin-Novikov theorem on the unirationality of the space of hyperelliptic Jacobians.

Theorem 6. There exists an isomorphism

$$\mathcal{P} \cong \mathbb{Q}[\wp(1), \dots, \wp(g)],$$

where $\wp(k) = (\wp_{2k}, \wp'_{2k}, \wp''_{2k})$ and $\wp_{2k} = -\partial_1 \partial_{2k-1} \ln \sigma(\mathbf{t})$.

As a development of D. Mumford's results on hyperelliptic analogs of Weierstrass elliptic functions, we gave an explicit description of all algebraic relations between our functions \wp_ω . We also showed that for any $g \geq 1$ there are only two types of algebraic relations connecting the functions \wp_ω and the parameters $\lambda_4, \dots, \lambda_{4g-2}$ of a hyperelliptic curve of genus g .

Let $\wp_{2i-1, 2k-1} = \wp_{2i-1, 2k-1}(\mathbf{t}) = -\partial_{2i-1} \partial_{2k-1} \ln \sigma(\mathbf{t})$, where $i \neq 1$ or $k \neq 1$.

Relations of the first type:

$$\wp''_{2i} = 6(\wp_{2i+2} + \wp_2 \wp_{2i}) - 2(\wp_{3, 2i-1} - \lambda_{2i+2} \delta_{i,1}). \quad (1)$$

Here $\delta_{i,k}$ is the Kronecker symbol.

Corollary 7. For all $g \geq 1$, we have the following relations:

1. Setting $i = 1$ in (1), we obtain

$$\wp''_2 = 6\wp_2^2 + 4\wp_4 + 2\lambda_4. \quad (2)$$

2. Setting $i = 2$ in (1), we obtain

$$\wp''_4 = 6(\wp_2 \wp_4 + \wp_6) - 2\wp_{3,3}. \quad (3)$$

We have $\wp'_{2i} = \partial_{2i-1} \wp_2$. Then from (2) we obtain:

Corollary 8. For any $g > 1$, the function $u = 2\wp(\mathbf{t})$ is a solution of the KdV equation

$$u''' = 6uu' + 4\dot{u}, \text{ where } \dot{u} = 2\partial_3 \wp_2.$$

Relations of the second type are cubic. Here we will give only one example of such relations for all $g \geq 1$:

$$(\wp'_2)^2 = 4[\wp_2^3 + (\wp_4 + \lambda_4)\wp_2 + \wp_{3,3} - \wp_6 + \lambda_6]. \quad (4)$$

From all relations of the second type directly follow the equations that specify the parameters $\lambda_{2k}, k > 2$, in the form of polynomials in \wp_ω .

The solution of the hyperelliptic Problem II is the following theorem.

Theorem 9. (1) The projection of the universal bundle $\pi_g : \mathcal{U}_g \rightarrow \mathcal{B}_g \subset \mathbb{C}^{2g}$ is given by the polynomials $\lambda_{2k} = \lambda_{2k}(\wp(1), \dots, \wp(g)) \in \mathcal{P}, k = 2, \dots, 2g + 1$, of degree at most 3.

(2) Equations $\lambda_{2k}(\wp(1), \dots, \wp(g)) = \lambda_{2k} = \text{const}$ provide the uniformization of the Jacobian $\text{Jac}V_\lambda$ in the form of an affine algebraic variety in \mathbb{C}^{3g} .

In the case of $(3, s)$ -curves, a similar result was obtained in [10]. In this case, the uniformization of the Jacobi variety led to solutions of the Boussinesq equation.

The next result is a fruitful generalization of Gelfand-Dikii's result on the connection between the Korteweg-de Vries hierarchy and the resolvent of the Sturm-Liouville equation.

The field $\mathcal{F} = \mathcal{F}_g$ contains the coordinate ring $\Lambda = \mathbb{Q}[\lambda_4, \dots, \lambda_{4g+2}]$ of the parameter space \mathcal{B}_g . Denote by \mathcal{P}_Λ the algebra of polynomials over Λ generated by the hyperelliptic function \wp_2 and all its derivatives with respect to x .

Theorem 10. For all $k \geq 1$, the hyperelliptic functions \wp_{2k} belong to the ring \mathcal{P}_Λ , that is, there exist differential polynomials $\Psi_{2k}(\wp_2, \wp'_2, \dots, \wp_2^{(2k-2)})$ over the ring Λ such that

$$2\wp_{2k} = \Psi_{2k}(\wp_2, \wp'_2, \dots, \wp_2^{(2k-2)}).$$

By definition $\wp'_{2k} = \partial_{2k-1} \wp_2$.

Corollary 11. For any $g \geq 1$, the hyperelliptic function $u(t_1, \dots, t_{2g-1}) = 2\wp_2(\mathbf{t})$ satisfies the hierarchy

$$\partial_{2k-1} \wp_2 = \partial_1 \Psi_{2k}(\wp_2, \wp'_2, \dots, \wp_2^{(2k-2)})$$

which is equivalent to the g -stationary hierarchy of the KdV equation.

2. Integrable Systems

Vladimir B. Matveev

I first met Victor Zelikovich Enolski, and his longtime collaborator and friend Eugeny Dmitrievich Belokolos, on the occasion of the Soviet-American symposium held in Kiev in 1979. All of the participants were lodged in the same hotel, "Feofania," owned by the National Academy of Sciences of Ukraine. The informal, friendly and relaxed atmosphere of the meeting stimulated new contacts and interactions between the participants. That meeting marked the beginning of a friendship between myself, my colleague Alexandre Its, and Victor and Eugeny.

After 1979 the algebro-geometric approach to integrable systems and the theory of Abelian functions and algebraic curves together with their applications in physics became the main focus of Victor's scientific activities, continuing into his very last days.

Victor's first step in this direction was the joint paper with Belokolos [4] based on the Weierstrass-Poincaré theory of reduction to elliptic integrals for Abelian integrals associated to algebraic curves of genus $g > 1$. That article revealed that the generalized Lamb Ansatz formula, containing the physically interesting genus-2 solutions of the sine-Gordon equation, could be expressed by Jacobi elliptic theta functions with arguments depending linearly on space and time variables.

In a subsequent article [16], Victor obtained similar formulas by an appropriate reduction of the genus-2 case of the Its-Matveev formula for the theta-function solutions of the KdV equation with initial condition the Lamé potential $u(x) = 6\wp(x) + c$, as well as a new kind of solutions of the sine-Gordon equation, generalizing the results of [4]. In the same article, Victor finds new nondegenerate genus-2 strictly periodic solutions for the Kowalevski top equations of motion, expressed by means of elliptic functions.

The meeting in Kiev gave rise to our scientific collaboration, resulting in the appearance in 1986 of the review paper [2], dedicated to the reduction of the solutions expressed by means of multidimensional Riemann theta functions of given genus g to theta functions of lower genera and, in particular, to elliptic theta functions, where we combined an approach of Victor and Eugeny with an alternative approach to solving reduction problems, proposed by Babich, Bobenko and myself [1].

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These reductions in general are obtained by appropriate specifications of spectral curves having a nontrivial automorphism group or an appropriate covering structure.¹ Notably, the nontriviality of these reductions is related to the fact that the B -periods matrix of the algebraic curve of genus $g > 1$ cannot be diagonal for any choice of the canonical basis of cycles.

The Springer volume [3] co-authored by Belokolos, Bobenko, Enolski, Its and myself, was also a product of our long-standing scientific exchanges. It is impossible to overestimate the contribution made by Victor while writing this book.

Many new results connected with reductions of Abelian functions were later achieved in a large review comprised of two consecutive articles by Belokolos and Enolski [6], summarizing the state of the art in 2002.

The scale of various developments and applications of the algebro-geometric approach to integrable systems obtained by Victor, ranging from the theory of finite-dimensional classical and quantum systems to nonlinear PDEs, connected with various branches of physics, ranging from integrable gravity to the theory of magnetic monopoles, and including Bose-condensates, crystalline systems and Josephson junctions for superconductors, is immense.

He published more than one hundred research articles (more than the 96 indexed in MathSciNet which omits ones that appeared in strictly physics venues), making a strong and lasting impact in mathematics, theoretical physics and the theory of algebraic curves and Abelian functions.

The profile in Google Scholar (search for Enolski Victor) provides a panoramic view of the majority of Victor's scientific results, mentioning 3886 citations of his works on March 25, 2020.

Victor had the courage to maintain a very intense scientific activity until the very last; in fact, several of his papers appeared posthumously.

We will hold him in our memory forever.

Appendix [E.P.] The late J.-L. Verdier together with Armando Treibich investigated the beautiful number-theoretic, geometric and dynamical properties of the "elliptic solitons" and their generalizations, a theme of interest for many other scholars as well. Armando remembers Victor in the following terms: I met Victor, together with his friend mathematician Eugeny Belokolos, back in March 1998, at a congress organized by Fritz Gesztesy and

¹Note [E.P.]: The algebro-geometric significance of these moduli spaces was developed by J.-L. Verdier [5], together with his student A. Treibich; cf. Appendix. In his article [20], Matveev points out that the stationary solution had occurred in the work of Darboux, the reason why they now go by the name of Darboux-Treibich-Verdier potentials; cf. Braden's piece.

Rudi Weikard at the University of Alabama. Since then we kept meeting from time to time, inviting each other, in France, Germany, Italy or Scotland, despite never collaborating on a project: I would say that we did it just for the pleasure of having a simple conversation about Mathematics, the Soviet Union and life in general. Victor's warmth was such that as soon as you met him you felt that you had been friends for years. And I had counted on years to come, for him to keep providing me with references and technical advice. I remember in 2018 asking him for help with computer algebra to resolve a problem I was stuck on; he immediately started to work on it and after two months of unsuccessful calculations, he was able to explain to me why Maple (or any other symbolic system) could not provide a complete answer. And this allowed me to move on and look for a different insight. How sad, how predictable yet unpredictable life can be. Victor's passing is a major loss for the entire mathematical community.

3. Computer Algebra

J. C. Eilbeck

Richard Bellman, in his 1961 book *A Brief Introduction to Theta Functions*, wrote "The theory of elliptic functions is the fairyland of mathematics. The mathematician who once gazes upon this enchanting and wondrous domain crowded with the most beautiful relations and concepts is forever captivated." I am forever grateful to Victor Enolski for introducing me to this enthralling world.

I first met Victor in 1989, when we were both attending a conference in Lyngby organised by Alwyn Scott and Peter Christiansen. At the time, Victor and I were both involved with AI in a project comparing different methods for calculating quantum states [19]. I had written a small package in Mathematica to facilitate the calculations, and I had with me a primitive laptop with the software installed. Victor was intrigued by the possibility of automating tedious algebraic calculations, although at the time he had not encountered either portable computers or computer algebra systems.

He was, in his own words, "frantically eager" to calculate the cover of an elliptic curve relevant to a certain Lamé potential. This involved calculating the resultant of two polynomials of moderate size, a calculation he had been pursuing for several years. A few lines of code and about 60 seconds CPU time led to the required formula for the

resultant, and revealed that Victor's incomplete hand calculations had several errors. This was the start of a long collaboration with him totalling 27 papers, often with other coworkers. We switched to Maple at an early stage as it was, at the time, much faster at working with polynomials.

Our initial work was using computer algebra to find new solutions to integrable cases of various physical systems of PDEs which reduce in the travelling wave case to Hamiltonian ODE systems with cubic or quartic potentials. This work also extended to some cases of solutions using R-matrix algebras where the R-matrix depends on the dynamical variables (cf. [13]).

Computer algebra became an essential part of Victor's mathematics and he thought of his mathematical niche as bringing together fundamental results of 19th century classical mathematics to investigate the theta-functional and \wp formulae of finite-gap integration with the aid of computer algebra. This not only included Weierstrass-Poincaré reduction theory, but also much of his later collaborations (with Belokolos, Braden, Eilbeck, Yu. Fedorov, T. Grava and Previato) where multivariable theta functions were decomposed to lower genera theta functions. These works make significant use of the Poincaré theorem on complete reducibility, Rosenhain-Thomae formulae expressing periods of hyperelliptic integrals in terms of theta functions, and the Riemann-Jacobi derivative formulae.

Our Lyngby meeting led to many visits by Victor to Edinburgh in the early 1990s. On one of these visits another chance meeting led to a very fruitful collaboration: while in Harry Braden's office he met Victor Buchstaber. At the time Dmitry Leykin was Victor's PhD student (1991–1994) and they were studying an integrable dynamical system whose corresponding spectral curve had a genus greater than the number of degrees of freedom. They were seeking a more effective procedure than standard theta-functional integration for such a divisor with deficiency. Such problems had been observed by H. F. Baker in a 1903 memoir.

Baker's manuscript dealt with genus-two sigma and \wp functions, which represented a natural generalization of Weierstrass's genus-one functions and inherited their principal properties. Enolski and Leykin adopted this multivariable sigma function approach to integrate solitonic equations. These functions turned out to be a convenient language to speak about integrability, the addition of divisors, to differentiate Abelian functions and the periods of Abelian integrals over the parameters of the curve, and other issues. Buchstaber brought a number of new perspectives to this area.

Buchstaber, Enolski and Leykin wrote a seminal guide to the higher-genus sigma function in the case of hyperelliptic curves in 1997, and began to look at the trigonal case. In 2000, together with Eilbeck, Enolski and Leykin

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found an explicit solution of the Jacobi inversion problem for the genus-three trigonal case. Further work led to a detailed description of the sigma function equations and those for the corresponding 3-variable \wp functions in this case [14]. This was later extended to a number of many other higher-genus nonhyperelliptic cases, giving, for example, multiperiodic solutions of the Boussinesq and KP hierarchies.

Our most recent work, with Keno Eilers of Oldenburg, was on finding expressions of periods of the second kind in terms of periods of the first kind and various theta constants. We had completed this problem in the case of genus two in 2013 ([12]), and we were working on higher genus cases up to a few weeks before Victor's untimely death. He will be much missed by all who worked with him.

4. Monopoles and Finite Gap Integration

Harry Braden

Victor would cite two turning points of his scientific career: the first, in attending the 1979 Soviet-American Symposium held in Kiev, and the second, his encounter with computer algebra via Chris Eilbeck at the Lyngby meeting in 1989.

The Kiev meeting introduced Victor to integrability, particularly solitons, and to many future leaders of this science. Victor already knew Eugeny Belokolos, and at the meeting both were inspired by the new ideas. In his talk, Alwyn Scott applied a special solution of the sine-Gordon equation,² built from elliptic functions ("Lamb ansatz"), in seeking to describe the volt-ampere characteristics of a Josephson junction. Belokolos and Victor understood the Lamb ansatz as a particular case of a finite-gap solution and they wished to find a spectral curve for it, possibly of higher genus. They began their study with an almost complete lack of literature, having just one copy of J. Fay's book that had been found by B. Dubrovin in Moscow. Fortunately, Belokolos found, in the comments to the collected works of Sophie Kowalevski (written by P. Ya. Polubarinova-Kochina), reference to a letter of Weierstrass to Fuchs explaining Kowalevski's dissertation in terms of reduction of genus-three hyperelliptic integrals to

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²Note [E.P.]: See Matveev's piece.

elliptic.³ Weierstrass' statement about the structure of the Riemann period matrix responsible for such reduction, just one line of the letter, was a key point in the modern theory. This careful reading of 19th century mathematicians characterized Victor's approach to mathematics. In particular, Victor and Belokolos were able to obtain the results of Darboux-Treibich-Verdier from the general formula of Its-Matveev for finite-gap solutions for the KdV equation. They used the Weierstrass-Poincaré theory on reduction of Abelian functions of algebraic curves to elliptic functions. Despite the deep and beautiful mathematics that ensued, Victor would note that they were never able to complete the initial Josephson junction problem.

The Lyngby meeting with Chris Eilbeck led to many visits to Edinburgh in the early 1990s and I first met Victor on such an occasion. The University of Edinburgh has a very good library, and Victor would visit me to seek out and sometimes discuss various papers, books and other gems in our collection.

Although Victor and I shared many interests, it took some time for a problem to arise that led to actual collaboration. The problem centered on magnetic monopoles. Monopoles are a three-dimensional reduction of the self-dual Yang-Mills equation and, along with the two-dimensional reductions of Hitchin systems, are deeply connected with integrability. The 1980s saw significant results by R. S. Ward, W. Nahm, S. K. Donaldson, M. F. Atiyah and N. J. Hitchin, with the latter giving three necessary and sufficient algebro-geometric conditions on an associated spectral curve to provide a monopole. One condition gives the curve a real structure, a second constrains the periods of the curve, while the third condition says that a real line in the (complex) Jacobian doesn't intersect the theta divisor in a specified interval. Despite the efforts of many mathematicians no analytic formulae for the gauge data beyond the spherically symmetric (or along specific axes/planes for charge 2) had been produced. Work by N. M. Ercolani and A. Sinha had shown finite-gap integration was applicable, but as they noted "there are obvious technical difficulties in almost every step of the ... construction" and analytic results gave way in the late 1980s and 1990s to a number of numerical studies. Indeed, although explicit solutions were initially sought, their seeming intractability led to other perspectives being studied.

³Note [E.P.]: Sophie Kowalevski, her chosen spelling on her paper on reduction, "Über die Reduction einer bestimmten Klasse abel'scher Integrale dritten Ranges auf elliptische Integrale" (*Acta Math.* 4 (1884), 393–414), gave a necessary and sufficient condition for a curve of genus three to cover an elliptic curve. This was one of the three results each of which Weierstrass "without any hesitation (...) would have accepted (...) as a doctoral dissertation," as told in D. H. Kennedy's *Little Sparrow: A Portrait of Sophia Kovalevsky*, Ohio University Press, Athens, OH, 1983.

Our initial problem was to construct new spectral curves that described monopoles. In due course that became a programme, aimed to reconstruct the gauge and other data associated with BPS monopoles via finite-gap integration; in particular, over many years, we worked concretely with the charge-2 monopole as a test case. The project grew from communicating small progresses by email to become all-consuming, occupying much of Victor's final fifteen or so years. During this time Victor was able to spend more than three full years in Edinburgh, and we managed long visits (at least once per year) beyond that, continuing even after his diagnosis of cancer. Regarding the general programme, we succeeded in overcoming the many technical problems, showing one could *effectively* reconstruct the monopole from a *given* spectral curve; using this, we obtained the analytic solution of the charge-2 monopole and successfully compared our analytical expressions with those coming from the many numerical studies [8]. Victor was particularly pleased with the movie we produced that replicated the (then supercomputer) numerical solution of Atiyah and others of the 90° monopole scattering.

Given a spectral curve we see that we are in a good position: what then about our initial question? Prior to our work (and after nearly a quarter of a century) only nine explicit (families of) curves that satisfy Hitchin's conditions were known. Most of these are configurations of monopoles associated with the symmetries of the Platonic solids, where the spectral curve may be reduced to an elliptic curve. The genus of a spectral curve for a charge- n $su(2)$ Euclidean monopole grows quadratically, $g = (n-1)^2$, and imposing symmetry leads to a quotient of this. Our studies involving group actions uncovered some beautiful constructions. We showed how Hitchin's constraints may naturally be expressed on the quotient curve, while a further theorem of R. D. M. Accola and J. Fay on the factorization of theta functions allowed one to very concretely relate the objects needed by integrable systems on both curves. Initially we chose a C_3 family of curves studied in the 19th century⁴ that included the known tetrahedrally symmetric monopole and whose periods we could calculate. Remarkably, Hitchin's second constraint on the periods of the curve could be solved making use of work of Ramanujan. At this stage Victor and I were elated: we had a countable number of new curves satisfying two of Hitchin's conditions. Our disappointment followed. By using Humbert theory and results of O. Bolza we showed that only the tetrahedrally symmetric monopole in this family satisfied Hitchin's third constraint. By enlarging the class of curves to the general C_3 symmetric charge-three curve, Victor, Antonella D'Avanzo (one of my students) and I were

⁴This curve had been studied by J. Wellstein in 1899 and K. Matsumoto in 2000.

able to construct those curves associated with monopoles and provided a new family of examples [7]. Unfortunately the curve is described in terms of a vanishing period implicitly relating coefficients of the quotient curve. Although we were unable to proceed further analytically, the Richelot correspondence (a genus-two analogue of the arithmetic-geometric mean) did allow successful checking with existing numerical studies.

The upshot: monopole spectral curves are interesting, encoding real as well as complex algebraic geometry and number theory but they are difficult to lay hands on. A critical question that seems beyond present theory is deciding whether a real interval intersects the theta divisor. Victor's love of classical curves had him always looking for new candidates, and he would be delighted if progress could be made here.

5. General Relativity and Astronomy

Jutta Kunz and Claus Lämmerzahl

Victor first came to the Northwest of Germany in the early years of the 21st century to collaborate with the late Peter Richter, professor of theoretical physics at the University of Bremen. Working on the double pendulum they utilized algebro-geometric methods, inverting Abelian integrals on the Θ -divisor of the Jacobian variety.

Inspired by this exciting work, a little later at the Center of Applied Space Technology and Microgravity (ZARM) of the University of Bremen, Eva Hackmann, a doctoral student at ZARM, started employing algebro-geometric methods to integrate the geodesic equations in black hole space-times with an underlying hyperelliptic curve of genus two. Based on the earlier work of Victor together with Richter [18] Eva was able to explicitly solve the geodesic equation in a Schwarzschild-de Sitter space-time for the first time. This work then paved the way for producing analytic solutions of geodesic equations and of their observables in a variety of space-times of Einstein's General Relativity and generalized space-times, which have been found in our groups and also by other colleagues worldwide. For her doctoral thesis Eva received an award of the German Physical Society. Her results are playing a major role in our Deutsche Forschungsgemeinschaft (DFG) Research

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IDENTIFIZIERUNG UND ANWENDUNG MATHEMATISCHER METHODEN
FÜR SPEZIELLE FRAGESTELLUNGEN IN DER PHYSIK
- EIN KOOPERATIVES PROJEKT ZWISCHEN PHYSIK UND MATHEMATIK -

Institut für Physik



Prof. Dr. Jutta Kunz



Prof. Dr. Claus Lämmerzahl

Interdisziplinär



Prof. Dr. Victor Enolskii
(Gastprofessor)



M.Sc. Kenjo Eilers

Institut für Mathematik



Prof. Dr. Daniel Grieser



Prof. Dr. Florian Hess

Die Verbindung zweier Fächer

„Die Naturphilosophie ist in dem großen Buch zu finden, das jederzeit offen vor unseren Augen liegt: Ich habe das Weltall im Sinn. Wir werden es aber nicht lesen können, bevor wir die Sprache nicht gelernt und uns mit den Zeichen vertraut gemacht haben, in denen es geschrieben ist. Es ist dies die Sprache der Mathematik.“

Galileo Galilei¹

Spätestens seit Galilei wird die physikalische Erkenntnis in der Sprache der Mathematik codiert und kommuniziert. Oft genug geschah die Forschung in beiden Fächern in *Per-*

sonalunion: Was mathematische Forschung, was physikalische Forschung ist, war schwer zu unterscheiden.

In jüngerer Zeit ist diese Trennlinie klarer geworden und beide Fächer gewannen an Eigenständigkeit – zu Lasten der einstigen *fruchtbaren Wechselswirkung*. Wir wollen die ursprüngliche Symbiose wiederherstellen: Die Untersuchung der physikalischen Anwendbarkeit neuerer mathematischer Resultate ist eines unserer Forschungsziele, die resultierende intensivste Kooperation zweier Institute der Universität ein weiteres Ziel.

Forschungsorientierte Lehre

Physik ist viel zu schwer für Physiker. Dieser Meinung soll der große Mathematiker David Hilbert gewesen sein. Obwohl er in vielem Recht hatte, wollen wir erreichen, dass er in diesem Punkt irrt. Zwar ist die Physikausbildung zu einem großen Teil auch mathematische Ausbildung, aber der Vollständigkeit sind natürliche Grenzen gesetzt. *Forschungsfeldabhängig* müssen dann in der Master- oder Promotionsphase speziellere Mathematikmethoden selbstständig erarbeitet werden.

An diesem Punkt setzen unsere Lehrziele ein. In einem für Mathematik- und Physikstudenten zugänglichen Rahmen sollen Inhalte gelehrt werden, die nicht zur kanonischen Ausbildung gehören, dennoch aber *große Forschungsrelevanz* haben. Das angestrebte *hohe Niveau* und die *Interdisziplinarität* werden insbesondere daran deutlich, dass Studenten beider Institute teilnehmen und *miteinander* und *voneinander* lernen sollen.

Der physikalische Startpunkt

Zurzeit liegt das Forschungsinteresse der Arbeitsgruppe *Relativistische Feldtheorie* auf der Bestimmung der *Bewegungsbahnen* um verschiedene astrophysikalische Objekte (siehe unten links), da einerseits diese Bahnen die astronomisch zugänglichen Daten sind, andererseits aber – auf Satellit-/Erde-Systeme angewendet – für die *Raumfahrt* wichtige Ergebnisse darstellen. Die sich aus Einsteins Allgemeiner Relativitätstheorie ergebenden Gleichungen sind allerdings *gekoppelte nichtlineare Differentialgleichungen*. Ihre exakten Lösungen sind mit elementarer Mathematik nicht zu erhalten.

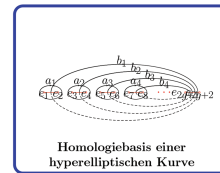
Das gleiche Schema ist auf viele weitere Situationen anwendbar und nicht beschränkt auf Astrophysik. Nichtlineares Verhalten taucht ebenso in der Festkörperphysik wie in der Hochenergiephysik oder Hydrodynamik auf: *Plasmaanregungen* in Fusionsreaktoren, Verhalten von *magnetischen Monopolen*, Bewegung von *Tsunamis*, vermutlich sogar *Nervenprozesse* der Neurobiologie sind auf einer neuen Ebene beschreibbar, wenn die dahinter stehenden nichtlinearen Differentialgleichungen gelöst werden.

Der mathematische Weg

Neben der erwähnten Riemann'schen Geometrie ist überraschenderweise die Theorie der *algebraischen Kurven* eine Hauptzutat der Lösung des physikalischen Problems. In diesen Kontext gerückt, können die Differentialgleichungen mittels Kurven ausgedrückt und ihre komplizierte Struktur als *Riemann'sche Fläche* visualisiert werden. Nach diesem Modellierungsschritt verbleibt eine rein mathematische Aufgabe zu lösen. Die Hauptarbeit ist hierbei, auf dieser Fläche eine geeignete *Homologiebasis* zu definieren (siehe unten rechts) und die *Periodenmatrizen* zu kalkulieren. Diese Matrizen sind nötig, um die Lösungsfunktionen zu bestimmen. Große Mathematiker wie Abel, Jacobi und Riemann haben diesen Weg vorbereitet, auf dem wir nun exakte Lösungen erarbeiten können. Die Verknüpfung der Details zu diesem Zweck ist aber immer noch Neuland, so dass auch die mathematische Forschung profitiert.

Der effiziente Computereinsatz

Zwar ist der vorgesehene Weg mathematisch gangbar, aber nicht ohne *effizienten Computereinsatz* zu bewerkstelligen. Da alle Schritte analytisch und exakt sein sollen, ist die Verwendung von Computer-Algebra-Systemen (CAS) im Gegensatz zu rein numerisch arbeitender Software geplant. Neben OpenSource-Produkten bietet die Universität Oldenburg Zugang zu zwei verschiedenen Systemen: *Maple* und *Magma*. Das in der physikalischen Forschung recht etablierte Maple bietet viele komplett implementierte Algorithmen aus allen Bereichen der Mathematik und ist für viele Studenten die erste Wahl zur Überprüfung eigener Resultate. Dagegen ist das vollständig auf Algebra basierende Magma mathematisch *forschungsnäher* und übertrifft Maple bei Weitem in punkto Schnelligkeit. Welches dieser beiden Programme im Endeffekt nützlicher für unsere Zwecke ist, soll unter anderem eruiert werden, ist aber sicherlich von der momentanen Schwerpunktsetzung abhängig. Da die Forschung in beiderlei Instituten ohne CAS kaum denkbar ist, bieten sich Lehrinhalte bereits auf Bachelor-Niveau zu beiden Programmen an.



¹ Aus: Il Saggiatore, 1623; nach Simonyi, K.; Kulturgeschichte der Physik, Frankfurt 2001

Figure 1. Models of Gravity Research Programme.

Training Group (RTG) *Models of Gravity*. Thus, Victor's work had a crucial impact on our scientific activities; the 2012 poster (Figure 1) introduces our group.

Soon after, Valeria Kagramanova, a doctoral student at the close-by University of Oldenburg, involved in a fruitful collaboration with Eva, joined this venture, applying these methods to a wide class of spherically and axially symmetric space-times in 4 and higher dimensions.

Valeria recalls "I worked intensively with Victor for a long time. He always arrived on the dot, turned on the computer, and while the computer started we discussed what we would be doing that day precisely. When Maple was ready, Victor started. The functions of all kinds and genera always did what Victor wanted; that brought me to marvel again and again, because with me they were often unruly and obstinate. Victor always managed to tame them and Maple as well. He had a clear structure in mind, and his Maple sheets were just as structured, however, at a level that the nonfunction lovers had no chance to penetrate. It was very nice to work with Victor. With his humorous and warmhearted nature, he created a great atmosphere and had a lot of fun 'tormenting' the algebraic functions until they gave the desired results or yielded exciting surprises. Then we worked even harder until Maple had to let off steam or explode, and sometimes, our heads as well. But with funny stories in between, he always generated a good mood and the desire for more function adventures."

The study of the geodesics and their analytic solutions in space-times equipped with different metrics is essential to completely understand the mathematical and physical properties of these space-times. Furthermore, the study of the motion of particles and light rays in gravitational fields has significant applications, ranging from astrophysical and satellite tests of General Relativity all the way to geodesy, clocks in space, positioning, etc. With appropriate analytical methods, arbitrary accuracy can, in principle, be reached in the determination of these orbits and the time elapsed.

Therefore the collaboration of our groups with Victor has been an essential element in the extension and application of algebro-geometric methods in this area of gravitational physics. In particular, based on Victor's insights, we could start addressing geodesics in black hole solutions, where higher-genus hyperelliptic curves arise; moreover, together we could successfully address an interesting alternative theory of gravity, Hořava-Lifshitz gravity, that is a power-counting renormalizable theory [17].

In 2010 Victor was awarded a fellowship by the Hanse-Wissenschaftskolleg (HWK), our local Institute of Advanced Studies, where he spent a fruitful and happy year (Figure 2). The HWK is a truly interdisciplinary endeavor with research in the areas of energy, climate, the human



Figure 2. Victor at the Hanse-Wissenschaftskolleg, Institute for Advanced Study, Delmenhorst, Germany, in 2010.

brain and sociology. And all this is completed by activities of artists doing painting, sculpturing and writing. In his fellow talk at HWK with this diverse audience Victor was speculating why a subject of geometry—the Riemann surface—became so popular in modern science, and discussing it beautifully with all the fellows from all these different areas. He even contemplated writing a book with one of the artists there.

Subsequently Victor became a long-term visitor in our DFG RTG *Models of Gravity*, where he also taught courses, gave seminars and spent a lot of time in teaching our PhD students the fundamentals and the subtleties of the field. During his stays in Oldenburg we would always meet on Friday afternoon, discussing and contemplating the state of affairs in our science and in the world.

Victor's modesty is legend. With him we organized several international meetings with renowned speakers in the fields of mathematics and theoretical physics, notably, the 515th WE-Heraeus Seminar "Algebro-geometric Methods in Fundamental Physics," held in Bad Honnef, Germany, September 3–7, 2012 (organizers, C. Lämmerzahl, J. Kunz, and V. Enolskii). But when asked to give the introductory or final address, Victor would always be hesitant, fearing he would not have anything of importance to say. When finally convinced, however, he would give an inspiring and entertaining speech.

In 2018 Victor visited Oldenburg for the last time. He attended the defense of Keno Eilers, the last of the PhD students in our RTG co-supervised by him. In fact, based on these algebro-geometric methods we had obtained a special innovative mathematics-physics grant for Keno's PhD

thesis in Oldenburg. While the thesis started out to explore the geodesics of various relevant space-times that led to trigonal or quartic curves, the thesis developed dynamically under Victor's guidance into other exciting directions, putting the original questions aside to be addressed later. We still had so many plans for enthralling scientific endeavours together, based on Victor's ingenuity and brilliance. Victor, we miss you.

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Credits

Figure 1 is courtesy of Keno Eilers.

Figure 2 is courtesy of HWK©2011.