# Group-theoretical framework for potential symmetries of evolution equations

Qing Huang,[1,2] Changzheng Qu,[1,2] and Renat Zhdanov[3,a)]
[1]*Department of Mathematics, Northwest University, Xi'an 710069, People's Republic of China*
[2]*Center for Nonlinear Studies, Northwest University, Xi'an 710069, People's Republic of China*
[3]*BIO-key International, Eagan, Minnesota 55121, USA*

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We develop algebraic approach to the problem of classification of potential symmetries of nonlinear evolution equations. It is essentially based on the recently discovered fact [R. Zhdanov, *J. Math. Phys.* **50**, 053522 (2009)], that any such symmetry is mapped into a contact symmetry. The approach enables using the classical results on classification of contact symmetries of nonlinear evolution equations by Sokolov and Magadeev to classify evolution equations admitting potential symmetries. We construct several examples of new nonlinear fourth-order evolution equations admitting potential symmetries. Since the symmetries obtained depend on nonlocal variables, they cannot be derived by the infinitesimal Lie approach. © *2011 American Institute of Physics*. [doi:10.1063/1.3554692]

## I. INTRODUCTION

The concept of group of transformations of the space of dependent and independent variables is in the core of the Lie group approach to the analysis of partial differential equations (PDEs). In the case when transformations do not involve integrals of dependent variables, the corresponding transformation group is called local. If transformation laws for dependent or independent variables do contain integrals of dependent variables, then the corresponding group is called nonlocal.

Sophus Lie developed the regular method for calculating local symmetry group of a given PDE long time ago (see, e.g., Refs. 1–4). However, there is still no systematic approach for constructing nonlocal symmetries of nonlinear differential equations. Almost every specific nonlinear PDE requires an individual approach for constructing its nonlocal symmetries.

One of the most well understood cases is the particular case of the nonlocal symmetry called "quasi-local symmetry".[5,6] The idea behind this concept is a variable transformation that involves integrals of dependent variables. In some cases such transformation maps local symmetries of a given PDE into nonlocal ones. In our recent paper[7] we describe constraints on the forms of local (Lie) symmetries enabling to identify those symmetries which lead to quasi-local ones.

Potential symmetries of PDEs constitute another class of nonlocal symmetries which got a lot of attention recently. It was Bluman who introduced the concept of potential symmetry in the early 1990s.[8–10] A potential symmetry is a special case of nonlocal symmetry realized as a Lie symmetry of the associated system of partial differential equations.[11,12]

A number of papers, devoted to studying potential symmetries of various linear and nonlinear evolution equations, have been published.[13–29] Recently, the notion of potential symmetry has been extended to accommodate nonclassical symmetries of evolution equations.[30–32] A natural connection between potential symmetries and conservation laws of evolution-type equations has been explored in the papers.[33–39]

[a)]Author to whom correspondence should be addressed. Electronic mail: renat.zhdanov@bio-key.com.

 

We have recently established that potential symmetries of evolution equations in one spatial variable boil down to quasi-local symmetries.[40] Namely, for any potential symmetry of a one-dimensional evolution equation there is a change of variables which reduces equation in question to another evolution PDE so that the potential symmetry becomes the contact symmetry of the latter. This fact is a foundation of our group-theoretical approach to classification of nonlinear evolution equations that admit potential symmetries.

In the present paper we apply the approach in question to fourth-order evolution equations of the form

$$u_t = F(x, u, u_1, u_2, u_3, u_4). \tag{1}$$

Here $u = u(t, x)$, $u_t = \partial u/\partial t$, $u_i = \partial u^i/\partial x^i$, $i = 1, 2, 3, 4$, and $F$ is sufficiently smooth real-valued function with $F_{u_4} \neq 0$.

Evolution equations of the form (1) with specific $F$ have been used to model a variety of nonlinear phenomena in physical, chemical, and biological sciences.[41–48]

In Ref. 49, we perform preliminary group classification of the class of fourth-order evolution equations of the form (1),

$$u_t = -u_4 + F(t, x, u, u_1, u_2, u_3).$$

Group classification of local and quasi-local symmetries of the more general class of fourth-order evolution equations (1),

$$u_t = F(t, x, u, u_1, u_2, u_3)u_4 + G(t, x, u, u_1, u_2, u_3),$$

is obtained in Ref. 50.

The paper is organized as follows. In Sec. II, we give the detailed description of our algebraic approach to classification of potential symmetries of evolution equations in one spatial variable. We apply this approach in Sec. III to construct fourth-order nonlinear evolution equations that admit potential symmetry. Note that we restrict our considerations to contact symmetries preserving the temporal variable $t$. Section IV contains discussion of the generalization of the approach in question enabling to utilize the most general contact symmetries. In addition, we present the examples of using contact symmetries preserving the temporal variable to solve initial-value problem for the nonlinear evolution equation of the form (1).

## II. THEORETICAL BACKGROUND

Suppose that evolution equation

$$u_t = F(t, x, u, u_1, \ldots, u_n) \tag{2}$$

can be written in a conserved form

$$\frac{\partial}{\partial t} G(t, x, u) + \frac{\partial}{\partial x} H(t, x, u, \ldots, u_{n-1}) = 0.$$

Then it can be replaced by the equivalent system of two PDEs,

$$v_t = -H(t, x, u, \ldots, u_{n-1}), \quad v_x = G(t, x, u), \tag{3}$$

where $v = v(t, x)$ is the new dependent variable.

In addition, we assume that system (3) admits Lie transformation group

$$\begin{aligned} t' &= T(t, x, u, v, \theta), \quad x' = X(t, x, u, v, \theta), \\ u' &= U(t, x, u, v, \theta), \quad v' = V(t, x, u, v, \theta), \end{aligned} \tag{4}$$

where $\theta$ is the group parameter.

If the right-hand sides, $T$, $X$, $U$ in (4) satisfy the inequality $|T_v| + |X_v| + |U_v| \neq 0$, then initial equation (2) is invariant under the group of nonlocal transformations since the latter involves the nonlocal variable $v = \partial_x^{-1} G(t, x, u)$. This kind of symmetry is referred to as the potential symmetry.[8]

We see that the traditional approach to computation of potential symmetries implies the necessity of group analysis of an auxiliary system of differential equations of the form (3).

Recently, we have established that there is a more direct way of computing potential symmetries of evolution equations by using the old good method of transformation of variables. It is based on the fact that any potential symmetry of an equation of the form (2) can be mapped into a contact symmetry of another evolution equation from the same class (2). The mapping, **M**, is a superposition of point transformations of the space of variables $t, x, u$ and of the transformation $t \to t, x \to x, u \to u_1$ (further details can be found in Ref. 40).

Consequently, to get exhaustive classification of potential symmetries of evolution equations (2), one needs to perform classification of contact symmetries of equations of the form (2) and select those symmetries which lead to potential symmetries through sequences of transformations described above.

Not every contact symmetry leads to a potential symmetry. That is why we need criteria enabling to select contact symmetries that can be mapped by **M** into potential ones. For the case when contact symmetry is a point symmetry (which means that the transformation laws for independent and dependent variables do not include derivatives) this problem has been solved in Ref. 40. Here, we generalize the approach of Ref. 40 in order to incorporate contact symmetries.

The most general form of the infinitesimal generator of a group of contact symmetries has the form

$$Q = -\frac{\partial g}{\partial u_t}\partial_t - \frac{\partial g}{\partial u_1}\partial_x + \left(g - u_t\frac{\partial g}{\partial u_t} - u_1\frac{\partial g}{\partial u_1}\right)\partial_u + \left(\frac{\partial g}{\partial t} + u_t\frac{\partial g}{\partial u}\right)\partial_{u_t} + \left(\frac{\partial g}{\partial x} + u_1\frac{\partial g}{\partial u}\right)\partial_{u_1}, \tag{5}$$

where $g$ is an arbitrary smooth real-valued function called generating function.[2] Since the contact symmetry is fully determined by its generating function, we adopt the notation $Q = g(t, x, u, u_t, u_1)$.

Operators (5) span infinite-dimensional Lie algebra, $\mathfrak{A}$.

In what follows, we restrict our considerations to contact symmetries preserving the temporal variable $t$. Requiring that the contact symmetry generated by the operator $Q$ preserves the temporal variable $t$ yields the constraint on the form of the generating function, $g_{u_t} = 0$. The Lie algebra spanned by the operators

$$P = g(t, x, u, u_1), \tag{6}$$

is the ideal in the algebra $\mathfrak{A}$. We denote it as $\mathfrak{I}$.

The main idea of our approach is quite simple. Suppose that Eq. (2) admits $N$-dimensional Lie algebra $\mathcal{L} \subset \mathfrak{I}$ of contact symmetries with $N \geq 2$. Then provided the algebra $\mathcal{L}$ contains at least two basis elements that do not commute, evolution equation (2) can be mapped into equation of the form (2) that possesses potential symmetry. Let us consider this procedure in more detail.

Assume that Eq. (2) admits Lie algebra $\mathcal{L}$ of contact symmetries (5) such that $\mathcal{L} \subset \mathfrak{I}$. Suppose also that $[\mathcal{L}, \mathcal{L}] \neq 0$. Hereafter, we denote by the symbol $[\mathcal{L}, \mathcal{L}]$ the Lie algebra spanned by all possible commutators of the algebra $\mathcal{L}$.

Given the above conditions, there exist basis elements $P \in \mathcal{L}$ and $Q \in \mathcal{L}$ such that $[P, Q] \neq 0$. As the operator $P$ is of the form (6) it can be reduced to operator (6) with $g = 1$ by a contact transformation,

$$\bar{t} = t, \quad \bar{x} = \bar{X}(t, x, u, u_1), \quad \bar{u} = \bar{U}(t, x, u, u_1), \quad \bar{u}_{\bar{x}} = \bar{V}(t, x, u, u_1). \tag{7}$$

In what follows, we denote operator (6) with $g = 1$ as **1**. The operator **1** can be equivalently represented in the form of the generator of the one-parameter displacement group by $u$, namely, $\mathbf{1} = \partial_u$. That is why, rewriting the initial equation (2) in the new variables and dropping the bars yield,

$$u_t = f(x, u_1, \ldots, u_n). \tag{8}$$

Now, we differentiate Eq. (8) with respect to $x$ and make the nonlocal change of variables

$$\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u_1. \tag{9}$$

After dropping the bars we get the evolution equation in the form of conservation law,

$$u_t = \frac{\partial}{\partial x} f(x, u, u_1, \ldots, u_{n-1}). \tag{10}$$

What we claim is that Eq. (10) either admits a potential symmetry or can be reduced to another evolution equation of the form (10) that possesses potential symmetry.

Let us rewrite the image of the contact symmetry $Q$ under the mapping (7) in the form of Lie vector field,

$$-\frac{\partial g}{\partial u_1}\partial_x + \left(g - u_1\frac{\partial g}{\partial u_1}\right)\partial_u + \left(\frac{\partial g}{\partial x} + u_1\frac{\partial g}{\partial u}\right)\partial_{u_1}. \tag{11}$$

We remind that the contact symmetry $P$ has been mapped into $\mathbf{1}$. What is more, the operators $P$ and $Q$ do not commute. Hence, we get that at least one of the expressions,

$$\partial_u g_{u_1}, \quad \partial_u(g - u_1 g_{u_1}), \quad \partial_u(g_x + u_1 g_u),$$

does not vanish identically.

*Case 1 :* $(\partial_u g_{u_1})^2 + (\partial_u(g_x + u_1 g_u))^2 \neq 0.$

The finite transformation group generated by (11) reads

$$\begin{aligned} &t' = t, \quad x' = X(x, u, u_1, \theta), \\ &u' = U(x, u, u_1, \theta), \quad u'_{x'} = V(x, u, u_1, \theta). \end{aligned} \tag{12}$$

Here $X, U, V$ are solutions of the initial-value problems

$$\begin{aligned} \frac{dX}{d\theta} &= -\frac{\partial g}{\partial u_1}, & X|_{\theta=0} &= x, \\ \frac{dU}{d\theta} &= \left(g - u_1\frac{\partial g}{\partial u_1}\right), & U|_{\theta=0} &= u, \\ \frac{dV}{d\theta} &= \left(\frac{\partial g}{\partial x} + u_1\frac{\partial g}{\partial u}\right), & V|_{\theta=0} &= u_x, \end{aligned}$$

and $\theta$ is the group parameter.

Now if at least one of the expressions $\partial X/\partial u = 0$, $\partial V/\partial u = 0$ does not vanish identically, then group (12) is mapped by transformation (9) into potential symmetry,

$$t' = t, \quad x' = X(x, \partial_x^{-1}u, u, \theta), \quad u' = V(x, \partial_x^{-1}u, u, \theta), \tag{13}$$

of Eq. (10). Here $\partial_x^{-1}$ is the inverse of $\partial_x$, i.e., $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1$.

If we expand the functions $\partial X/\partial u = 0$ and $\partial V/\partial u = 0$ into the Taylor series with respect to $\theta$, then the coefficients by $\theta^1$ will be $\partial_u g_{u_1}$ and $\partial_u(g_x + u_1 g_u)$, correspondingly. Hence we conclude that $(\partial X/\partial u)^2 + (\partial V/\partial u)^2 \neq 0$, which means that (13) is the potential symmetry of Eq. (10).

*Case 2 :* $\partial_u g_{u_1} = 0,\ \partial_u(g_x + u_1 g_u) = 0,\ \partial_u(g - u_1 g_{u_1}) \neq 0.$

Integrating the system of PDEs, $\partial_u g_{u_1} = 0,\ \partial_u(g_x + u_1 g_u) = 0$ yields that

$$P = \partial_u, \quad Q = -u_1 h_{u_1u_1}\partial_x + (wu + h - u_1 h_{u_1} - u_1^2 h_{u_1u_1})\partial_u + wu_1\partial_{u_1},$$

where $w = w(t)$ and $h = h(t, u_1)$ are arbitrary smooth functions. There exists contact transformation (7) reducing the operator $Q$ to the form $wu\partial_u$, while the operator $P$ is not altered. Making the change of variables

$$\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = w^{-1}\ln u,$$

reduces $Q$, $P$ to the form $Q = \partial_u$ and $P = w^{-1}\exp(-wu)\partial_u + u\exp(-wu)\partial_{u_1}$. Since $[Q, P] \neq 0$, we have arrived at case 1 where one should replace $Q$ with $P$ and $P$ with $Q$. Consequently, case 2 leads to evolution equations possessing potential symmetry as well.

We summarize the above reasonings in the assertion below.

**Theorem 1**: *Let $\mathcal{L} \subset \mathfrak{I}$ be the algebra of contact symmetries of evolution equation (2) such that*

1. $\dim(\mathcal{L}) \geq 2$,
2. $[\mathcal{L},\ \mathcal{L}] \neq 0$.

*Then Eq. (2) can be mapped into another evolution equation of the form (2) possessing potential symmetry.*

*Corollary 1*: *Any linear evolution equation can be reduced to a nonlinear PDE of the form (2) admitting potential symmetry.*

Indeed, an arbitrary linear partial differential equation $u_t = Lu + a(t, x)$ admits the two-dimensional Lie algebra $\langle v(t, x)\partial_u, (u + w(t, x))\partial_u\rangle \subset \mathfrak{I}$, where $v(t, x)$ is an arbitrary solution of the equation in question and $w(t, x)$ is a solution of the homogeneous equation $w_t = Lw$. Now the validity of the assertion follows from the fact that the operators $v(t, x)\partial_u, (u + w(t, x))\partial_u$ do not commute.

Our algebraic approach to classification of nonlinear evolution equations admitting potential symmetries is based on Theorem 1. As a prerequisite, we need the list of evolution equations possessing nontrivial contact symmetries together with the corresponding symmetry algebras. With this list in hand, we select those algebras $\mathcal{L}$ which have nontrivial ideals $\mathcal{M} \in \mathfrak{I}$ with $[\mathcal{M}, \mathcal{M}] \neq 0$.

As a result, we get the list of Lie algebras of contact symmetries, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, ..., such that the corresponding invariant equations can be transformed into evolution equations (2) admitting potential symmetries.

By construction, the algebra $\mathcal{L}_i$ contains two basis elements $P \in \mathfrak{I}$ and $Q \in \mathfrak{I}$ such that, $R = [P, Q] \neq 0$. Now if $R \neq r(t, x, u)P$ we reduce the basis operator $P$ to the canonical form **1** and rewrite $Q$ in the new variables to become operator (11), while the initial Eq. (2) takes the form (8). Differentiating (8) with respect to $x$ and performing variable transformation (9) map the symmetry $Q$ into the potential symmetry of the obtained evolution equation.

Provided $R = r(t, x, u)P$, we reduce the second operator $Q$ to the canonical form **1**. Differentiating (8) with respect to $x$ and making the change of variables (9) reduces operator $P$ to the potential symmetry of the transformed equation (10).

In Sec. III, we apply the above approach to construct fourth-order nonlinear evolution equations possessing potential symmetries.

## III. POTENTIAL SYMMETRIES

In the sequel, we consider contact symmetries from $\mathfrak{I}$ that are independent of the temporal variable $t$,

$$P = g(x, u, u_1). \tag{14}$$

We denote the class of contact symmetries of the form as $\mathfrak{S}$.

Since $\partial g/\partial u_t = 0$ the transformation group generated by symmetries from $\mathfrak{S}$ preserves the temporal variable $t$.

The class $\mathfrak{S}$ of contact symmetries (14) has been extensively studied. Sokolov[51] obtained exhaustive classification of evolution equations

$$u_t = F(x, u, u_1, u_2, \ldots, u_n), \quad n \geqslant 2, \tag{15}$$

admitting symmetries from $\mathfrak{S}$ (see, also, Refs. 52 and 53).

We summarize below the relevant results of Ref. 51.

**Theorem 2**: *$\mathfrak{S}$ is finite-dimensional, and* $\dim S \leqslant n+3$ *for any Eq. (15). If* $\dim S = n+3$, *then the right-hand side of (15) has the form*

$$F = (a_1 u_n + a_0)^{\frac{1-n}{1+n}}, \quad a_1 \neq 0.$$

**Theorem 3**: *Any finite-dimensional subalgebra over $\mathbb{C}$ of $\mathfrak{S}$ is equivalent to a subalgebra of one of the algebras $H_m$, $A_2$, $C_2$, $G_W$, $F_U$ given below,*

$$\begin{aligned}
H_m &= \langle 1, x, \ldots, x^m, u, xu_1, u_1, xu - \frac{1}{m}x^2 u_1\rangle,\\
A_2 &= \langle 1, u, u^2 - u_1^2, u_1, (\mathrm{e}^x u_1)^{\frac{1}{2}}, (\mathrm{e}^{-x} u_1)^{\frac{1}{2}}, (u - u_1)(\mathrm{e}^x u_1)^{\frac{1}{2}},\\
&\qquad (u + u_1)(\mathrm{e}^{-x} u_1)^{\frac{1}{2}}\rangle,\\
C_2 &= \langle 1, u, u^2, u_1, xu_1, x^2 u_1, u_1^{\frac{1}{2}}, xu_1^{\frac{1}{2}}, xuu_1^{\frac{1}{2}}, uu_1^{\frac{1}{2}}\rangle,\\
G_W &= \langle u, f_1(x), f_2(x), \ldots, f_m(x)\rangle,\\
F_U &= \langle u, u_1, \ldots, g_1(x), \ldots, g_m(x)\rangle.
\end{aligned}$$

*Here $\{f_1(x), \ldots, f_m(x)\}$ is a basis of an arbitrary finite-dimensional space $W$ of functions of $x$; the functions, $g_i(x)$, form a basis of the space $U$ of functions in $x$, invariant under $\partial_x$, i.e., $U = Ker P$, where $P$ is a differential operator of the form $\sum c_i \partial_x^i$, $c_i \in \mathbb{C}$.*

Note that only algebras $A_2$ and $C_2$ contain contact symmetries which are not generators of point transformation groups. It is a common knowledge that the Lie algebra $A_2$ over $\mathbb{R}$ is isomorphic to the Lie algebra sl(3, $\mathbb{R}$). Classification of inequivalent subalgebras of the algebra sl(3, $\mathbb{R}$) has been performed by Winternitz.[54] Classification of inequivalent subalgebras of the algebra $C_2 \cong o(2,3)$ has been obtained in the paper.[55] We intent to utilize these classification results to describe nonlinear evolution equations of the form (1) that admit potential symmetries in one of our future publications. Here, we present several examples of application of our classification approach.

*Example 1 :* Consider the Lie algebra of contact symmetries $\mathcal{L} = \langle 1, u, u^2 - u_1^2, u_1\rangle$. To construct Eq. (1) invariant under the algebra $\mathcal{L}$ we need to apply the infinitesimal Lie method.[2,3] After some involved calculations, we establish that Eq. (1) is invariant under the contact symmetry $Q = g(x, u, u_1)$ if the generating function $g$ satisfies the determining equation

$$\begin{aligned}
&g_u F + g_{u_1} F_x + (g_{u_1} u_1 - g) F_u - (g_x + g_u u_1) F_{u_1} - [g_{uu} u_1^2 + 2 g_{uu_1} u_1 u_2\\
&\quad + g_{u_1 u_1} u_2^2 + 2 g_{xu} u_1 + (g_u + 2 g_{xu_1}) u_2 + g_{xx}] F_{u_2} - [g_{uuu} u_1^3 + 3 g_{uuu_1} u_1^2 u_2\\
&\quad + 3 g_{uu_1u_1} u_1 u_2^2 + g_{u_1u_1u_1} u_2^3 + 3 g_{xuu} u_1^2 + 3(g_{uu} + 2 g_{xuu_1}) u_1 u_2\\
&\quad + 3(g_{uu_1} + g_{xu_1u_1}) u_2^2 + 3 g_{uu_1} u_1 u_3 + 3 g_{u_1u_1} u_2 u_3 + 3 g_{xxu} u_1\\
&\quad + 3(g_{xxu_1} + g_{xu}) u_2 + (3 g_{xu_1} + g_u) u_3 + g_{xxx}] F_{u_3} - [g_{uuuu} u_1^4\\
&\quad + 4 g_{uuuu_1} u_1^3 u_2 + 6 g_{uuu_1u_1} u_1^2 u_2^2 + 4 g_{uu_1u_1u_1} u_1 u_2^3 + g_{u_1u_1u_1u_1} u_2^4\\
&\quad + 4 g_{xuuu} u_1^3 + 6(g_{uuu} + 2 g_{xuuu_1}) u_1^2 u_2 + 12(g_{xuu_1u_1} + g_{uuu_1}) u_1 u_2^2\\
&\quad + 2(2 g_{xu_1u_1u_1} + 3 g_{uu_1u_1}) u_2^3 + 6 g_{uuu_1} u_1^2 u_3 + 12 g_{uu_1u_1} u_1 u_2 u_3\\
&\quad + 6 g_{u_1u_1u_1} u_2^2 u_3 + 6 g_{xxuu} u_1^2 + 12(g_{xuu} + g_{xxuu_1}) u_1 u_2\\
&\quad + 3(g_{uu} + 4 g_{xuu_1} + 2 g_{xxu_1u_1}) u_2^2 + 4(3 g_{xuu_1} + g_{uu}) u_1 u_3
\end{aligned}$$

$$+ 3g_{u_1u_1}u_3^2 + 4g_{uu_1}u_1u_4 + 2(5g_{uu_1} + 6g_{xu_1u_1})u_2u_3$$
$$+ 4g_{u_1u_1}u_2u_4 + 4g_{xxxu}u_1 + 2(2g_{xxxu_1} + 3g_{xxu})u_2$$
$$+ 2(3g_{xxu_1} + 2g_{xu})u_3 + (g_u + 4g_{xu_1})u_4 + g_{xxxx}]F_{u_4} = 0. \tag{16}$$

Integrating (16) for each of the basis elements of $\mathcal{L}$ yields the following class of fourth-order evolution equations:

$$u_t = u_1F(\omega_1, \omega_2), \tag{17}$$

where

$$\omega_1 = \frac{(u_3 - u_1)u_1^2}{(u_2^2 - u_1^2)^{\frac{3}{2}}},$$

$$\omega_2 = \frac{u_1^2}{(u_2^2 - u_1^2)^3}[u_4u_1^3 - 4u_3u_2u_1^2 + u_2(3u_3^2 - u_4u_2 + 3u_2^2)u_1 - 2u_3u_2^3].$$

The invariance algebra of the above equation contains contact symmetries, $Q = \mathbf{1}$ and $P = u^2 - u_1^2$, which do not commute. Consequently, conditions of Theorem 1 are met.

Integrating the Lie equations we obtain the final transformation group generated by the contact symmetry $u^2 - u_1^2$,

$$t' = t, \quad x' = \ln\left|\frac{\theta(u - u_1) - 1}{\theta(u + u_1) - 1}\right| + x,$$
$$u' = \frac{\theta(u_1^2 - u^2) + u}{\theta^2(u^2 - u_1^2) - 2\theta u + 1}, \quad u'_{x'} = \frac{u_1}{\theta^2(u^2 - u_1^2) - 2\theta u + 1}. \tag{18}$$

Differentiating equation (17) with respect to $x$, making the nonlocal change of variables (9), and dropping the bars yield the evolution equation

$$u_t = u_1F + \frac{u^2\sigma_1}{(u_1^2 - u^2)^{\frac{5}{2}}}F_{\omega_1} + \frac{u^2\sigma_2}{(u_1^2 - u^2)^4}F_{\omega_2}, \tag{19}$$

where $F = F(\omega_1, \omega_2)$ with

$$\omega_1 = \frac{u^2}{(u_1^2 - u^2)^{\frac{3}{2}}}(u_2 - u),$$

$$\omega_2 = \frac{u^2}{(u_1^2 - u^2)^3}[u_3u^3 - 4u_2u_1u^2 + (3u_1u_2^2 - u_3u_1^2 + 3u_1^3)u - 2u_1^3u_2],$$

$$\sigma_1 = -u_3u^3 + 4u_2u_1u^2 + (-3u_1u_2^2 + u_3u_1^2 - 3u_1^3)u + 2u_1^3u_2,$$

$$\sigma_2 = -u_4u^6 + (5u_1u_3 + 4u_2^2)u^5 + (-17u_2u_1^2 - 3u_2^3 + 2u_4u_1^2 - 10u_3u_2u_1)u^4$$
$$+ (35u_2^2u_1^2 + 9u_1^4)u^3 + (10u_3u_2u_1^3 - 15u_1^2u_2^3 - u_4u_1^4 - 33u_1^4u_2)u^2$$
$$+ (15u_2^2u_1^4 + 9u_1^6 - 5u_1^5u_3)u - 4u_1^6u_2.$$

The group of contact symmetries (18) turns into the group of nonlocal transformations

$$t' = t, \quad x' = \ln\left|\frac{\theta(v - u) - 1}{\theta(v + u) - 1}\right| + x, \quad u' = \frac{u}{\theta^2(v^2 - u^2) - 2\theta v + 1},$$

where $v = \partial_x^{-1}u$.

*Example 2:* Consider the Lie algebra of contact symmetries

$$\mathcal{L} = \langle 1, uu_1^{\frac{1}{2}}, u_1^{\frac{1}{2}}, u_1, u, xu_1 \rangle.$$

Integrating determining equation (16) for each of the basis elements of the algebra $\mathcal{L}$ we get the corresponding invariant equation

$$u_t = \frac{u_2^3 u_1^2 (2u_3 u_1 - 3u_2^2)^4}{[4(u_2 u_4 - 3u_3^2)u_1^2 + 20 u_3 u_2^2 u_1 - 15 u_2^4]^3}. \tag{20}$$

The algebra $\mathcal{L}$ contains two basis operators $Q = \mathbf{1}$ and $P = uu_1^{\frac{1}{2}}$ that do not commute. Hence the conditions of Theorem 1 are satisfied and we can transform Eq. (20) into evolution equation possessing potential symmetry.

First we derive the form of the final transformation group generated by the contact symmetry $P$,

$$t' = t, \quad x' = x - \theta \frac{u}{2u_1^{\frac{1}{2}}}, \quad u' = \frac{2u}{2 - \theta u_1^{\frac{1}{2}}}, \quad u'_{x'} = \frac{4u_1}{(2 - \theta u_1^{\frac{1}{2}})^2}. \tag{21}$$

Differentiating equation (20) with respect to $x$, making the nonlocal change of variables (9) and dropping the bars we arrive at PDE,

$$u_t = \frac{u_1^2(3u_1^2 - 2u_2 u)^3 u}{(-12u_2^2 u^2 + 4u^2 u_3 u_1 + 20 u_1^2 u_2 u - 15 u_1^4)^4} \Big( (72 u_2^4 - 48 u_2^2 u_3 u_1 + 24 u_1^2 u_4 u_2 - 32 u_3^2 u_1^2) u^4 + (-276 u_2^3 u_1^2 + 272 u_2 u_1^3 u_3 - 36 u_1^4 u_4) u^3 + (54 u_2^2 u_1^4 - 108 u_1^5 u_3) u^2 + 165 u_2 u u_1^6 - 90 u_1^8 \Big).$$

Transforming group (21) accordingly yields the corresponding potential symmetry

$$t' = t, \quad x' = x - \theta \frac{v}{2u^{\frac{1}{2}}}, \quad u' = \frac{4u}{(2 - \theta u^{\frac{1}{2}})^2},$$

with $v = \partial_x^{-1}$.

One of the possible uses for potential symmetries is solving nonlinear PDEs by the reduction approach. Consider, as an example, the contact symmetry, $Q = u_x^2 - u^2 - bu - c$, $\{b, c\} \subset \mathbb{R}$, admitted by Eq. (17).

Solving the invariance condition $u_x^2 - u^2 - bu - c = 0$ yields the ansatz for the function $u$,

$$u(t,x) = \frac{1}{4} \exp(-x - \varphi(t)) \Big( b^2 - 4c - 2b \exp(x + \varphi(t)) + \exp(2x + 2\varphi(t)) \Big), \tag{22}$$

where $\varphi(t)$ is an arbitrary smooth function.

Inserting (22) into (17) we get the ordinary differential equation for the unknown function $\varphi(t)$,

$$\varphi' = F(0,0). \tag{23}$$

Note that both $\omega_1$ and $\omega_2$ from Eq. (17) vanish when $u$ has the form (22).

Solving (23) and inserting $\varphi(t) = C_0 + F(0,0)t$ into (22) yields the solution of nonlinear PDE (17),

$$u(t,x) = \frac{1}{4} \exp(-x - C_0 - F(0,0)t) \Big( b^2 - 4c - 2b \exp(x + C_0 - F(0,0)t) + \exp(2x + 2C_0 + 2F(0,0)t) \Big),$$

where $C_0 \in \mathbb{R}$.

By force of Theorem 1 symmetry $Q = u_x^2 - u^2 - bu - c$ is mapped into potential symmetry of the nonlinear evolution equation (19). The corresponding ansatz has the form

$$u(t,x) = \frac{1}{4}\exp(-x-\varphi(t))\Big(-b^2+4c+\exp(2x+2\varphi(t))\Big).$$

The above ansatz is invariant under the potential symmetry of Eq. (17) and reduces the latter to ordinary differential equation (23).

## IV. DISCUSSION

In the present paper, we explore the connection between contact and potential symmetries to develop Lie algebraic approach for classification of potential symmetries of nonlinear evolution equations in one spatial variable. Note that a different symmetry approach has been suggested in Ref. 13, where the potential symmetry of the linear Fokker–Planck equation has been interpreted in terms of second-order nonclassical symmetry (see, formulas (5.8)–(5.10) from Ref. 13). The problem with this approach, however, is that computation of nonclassical symmetries requires solving nonlinear determining equations.[56] Within the framework of our approach we always deal with linear determining equations, since only classical symmetries are involved.

Note that we restrict our considerations to the subclass of contact symmetries $\Im$ preserving the temporal variable $t$. This is an important constraint, since Theorem 1 is not valid for the general contact symmetry $g(t, x, u, u_t, u_x)$. Consider, for example, the fourth-order PDE,

$$u_t = F\left(\frac{u_2}{u_1}, \frac{u_3}{u_1}, \frac{u_4}{u_1}\right). \tag{24}$$

It admits the two-dimensional Lie algebra of contact symmetries $\langle \partial_u, t\partial_t + u\partial_u \rangle$. Evidently, the basis operators $P = \partial_u$ and $Q = t\partial_t + u\partial_u$ do not commute. However, transformation (9) maps the symmetry operator $Q$ into the local (Lie) symmetry.

Still there is a strong evidence that the noncommutativity condition is a necessary one for the equation under study to be reducible to an evolution equation with potential symmetry. However, for the general contact symmetry this condition is no longer sufficient. We intend to devote one of the future publications to generalization of the approach of Sec. I to accommodate the most general form of the contact symmetry.

A peculiar feature of contact symmetries from the class $\Im$ is that they leave the initial surface $t = t_0$, $t_0 \in \mathbb{R}$ invariant. That is why they can be used to perform symmetry reduction of the initial-value problem

$$u_t = F(x, u, u_1, u_2, \ldots, u_n), \qquad f(x, u, u_1)\Big|_{t=t_0} = 0,$$

in the fashion it has been done in Refs. 57, 58 and 59.

Consider as an example Eq. (20). It admits the symmetry $Q = xu_1 - ku$. The most general first-order PDE invariant under the one-parameter transformation group

$$t' = t, \quad x' = xe^{\theta}, \quad u' = ue^{k\theta}, \tag{25}$$

generated by the operator $Q$ has the form $f(t, x^{-k}u, x^{1-k}u_1) = 0$. Hence it follows that the initial-value problem

$$\begin{aligned} u_t &= \frac{u_2^3u_1^2(2u_3u_1 - 3u_2^2)^4}{[4(u_2u_4 - 3u_3^2)u_1^2 + 20u_3u_2^2u_1 - 15u_2^4]^3}, \\ f(t_0&, x^{-k}u(t_0,x), x^{1-k}u_1(t_0,x)) = 0 \end{aligned} \tag{26}$$

is not altered by transformations (25). Consequently, we can apply the symmetry reduction method to solve problem (26). The solution invariant with respect to transformation group (25) is of the form

$$u(t,x) = x^k\varphi(t), \tag{27}$$

where $\varphi(t)$ is an arbitrary smooth real-valued function.

Inserting $u(t, x) = x^k\varphi(t)$ into (26) yields the initial-value problem for the ordinary differential equation

$$(3k-1)^3\varphi' + k(k^2-1)\varphi = 0, \quad f(t_0, \varphi(t_0), k\varphi(t_0)) = 0.$$

Integrating equation for $\varphi$ and inserting its solution, $\varphi(t) = \exp((3k-1)^{-3}\, k(1-k^2)t)C$, $C \in \mathbb{R}$, into (27) yields the exact solution of the initial-value problem for the nonlinear evolution equation (20),

$$u(t, x) = \exp((3k-1)^{-3}k(1-k^2)t)x^k C,$$

where $C$ is a solution of the equation

$$f\Big(t_0, C\exp((3k-1)^{-3}k(1-k^2)t_0), kC\exp((3k-1)^{-3}k(1-k^2)t_0)\Big) = 0.$$

Any contact symmetry from the class $\mathfrak{I}$ can be used in this fashion to solve the corresponding invariant initial-value problems for nonlinear evolution equations.

As we noted in Ref. 59 higher-order symmetries can be utilized to perform reduction of initial value problems. Consider, as an example, the contact symmetry, $Q = uu_1^{1/2}$, which is not point symmetry. The most general first-order PDE invariant under the contact transformation group generated by $Q$ has the form $f(t, u_1^2u^{-1}, 2u_1 - xu_1^2u^{-1}) = 0$.

Consequently, the initial-value problem

$$u_t = \frac{u_2^3u_1^2(2u_3u_1 - 3u_2^2)^4}{[4(u_2u_4 - 3u_3^2)u_1^2 + 20u_3u_2^2u_1 - 15u_2^4]^3},$$

$$f\Big(t_0, (u_1(t_0, x))^2(u(t_0, x))^{-1}, 2u_1(t_0, x) - x(u_1(t_0, x))^2(u(t_0, x))^{-1}\Big) = 0, \tag{28}$$

admits symmetry $Q$. Consider the following ansatz, $u(t, x) = (\varphi_1(t)x + \varphi_2(t))^2$, where $\varphi_1(t)$ and $\varphi_2(t)$ are arbitrary smooth functions. Inserting the ansatz for $u$ into (28) reduces it to the initial-value problem for the system of ordinary differential equations

$$125\varphi_1' + 3\varphi_1 = 0, \quad 125\varphi_2' + 3\varphi_2 = 0,$$

$$f\Big(t_0, 4(\varphi_1(t_0))^2, 2\varphi_1(t_0)\varphi_2(t_0)\Big) = 0.$$

Integrating equations for $\varphi_1$ and $\varphi_2$ and inserting the obtained expressions into the ansatz for $u(t, x)$ we obtain the exact solution of initial-value problem (28),

$$u(t, x) = \exp(-6t/125)(C_1x + C_2)^2,$$

where $C_1, C_2$ are solutions of the equation

$$f\Big(t_0, 4C_1^2\exp(-6t_0/125), 2C_1C_2\exp(-6t_0/125)\Big) = 0.$$

We intend to devote one of the future publications to reduction of initial-value problems for nonlinear evolution equations with the help of contact and potential symmetries.

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