Solitary Wave Solutions for Heat Equations

Tetyana A. BARANNYK [†] *and Anatoly G. NIKITIN* [‡]

[‡] *Poltava State Pedagogical University, 2 Ostrogradsky Str., 36003 Poltava, Ukraine*
  E-mail: *slpoltva@e-mail.pl.ua*

[‡] *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
  E-mail: *nikitin@imath.kiev.ua*

We present exact solitary wave solutions for a class of nonlinear heat equations and for coupled system of heat equations.

1 Introduction

A great number of mathematical models in thermomechanics, chemistry, biology, ecology, etc., are formulated using nonlinear heat equations. Some of these models admit exact solutions which play fundamental role for theory of the related equations and have large application values, refer, e.g., to monograph [1].

There are well-known regular approaches for searching exact solutions of partial differential equations (see, e.g., monographs [2–4]) which make it possible to find specific changes of dependent and independent variables (Ansätze) reducing equations of interest to simpler ones. Usually the possibility of such reduction is caused by the symmetry (classical Lie [2], conditional [3], Lie–Bäcklund [4], etc.) of the related equations.

In the present paper we find exact solutions for the nonlinear heat equations

$$u_t - u_{xx} = f(u) \tag{1}$$

and for coupled systems of heat equations

$$\begin{aligned} u_t - u_{xx} &= f(u, v), \\ v_t - v_{xx} &= g(u, v), \end{aligned} \tag{2}$$

where $f(u)$, $f(u, v)$ and $g(u, v)$ are functions of dependent variables specified in the following. To find these solutions we use the "universal" Ansatz of the following general form

$$u = (z_x)^k \varphi(z) \tag{3}$$

and

$$v = (z_x)^k \xi(z), \tag{4}$$

where the new variable z is a function of t and x.

The Ansätze (3), (4) appear to be very effective tools for reduction of equations (1), (2) and many others, refer e.g. to [5, 6]. In particular, they enable making a wide class of reductions caused by classical Lie symmetries and conditional symmetries as well [5].

In this paper we use Ansätze (3), (4) to find special exact solutions of equations (1), (2), namely, solitary wave solutions. We present also some other solutions which appear in our analysis. To achieve our goals we select special classes of equations (1), (2) which can be reduced using the mentioned Ansätze.

2   Reduction of equations (1) with cubic nonlinearity

We start with equations (1) with the cubic nonlinearities, i.e.,

$$u_t - u_{xx} = -2u^3. \tag{5}$$

Exact solutions for (5) as well as for equation (1) with cubic polynomial nonlinearity were found in [7, 8] using the conditional symmetry approach. Here we find these solutions directly using Ansatz (3). The results presented in this section will be used to construct solutions for the system of heat equations (2).

Using Ansatz (3) with $k = 1$ we find

$$u_t = z_{tx}\varphi + z_x z_t \varphi_z, \qquad u_x = z_{xx}\varphi + z_x^2 \varphi_z,$$
$$u_{xx} = z_{xxx}\varphi + 3z_x z_{xx}\varphi_z + z_x^3 \varphi_{zz}, \qquad u^3 = z_x^3 \varphi^3. \tag{6}$$

Substituting (6) into (5) and equating coefficients for φ and φ' we come to the following reduced equations

$$z_{tx} - z_{xxx} = 0, \qquad z_t - 3z_{xx} = 0, \tag{7}$$

$$\varphi_{zz} = 2\varphi^3. \tag{8}$$

Up to constant multiplier and shifts of t and x solution of (7) have the form

$$z = 6t + x^2. \tag{9}$$

Solutions of equations (8) can be expressed via Jacobi elliptic functions. In paper [5] we present an infinite set of such solutions (and solutions for equation (1) with cubic polynomial nonlinearities as well) in explicit form.

Thus the Ansatz (3) makes it possible to find exact solutions for equation (5) in the form

$$u = 2x\varphi\left(6t + x^2\right),$$

where $\varphi(z)$ is an elliptic function satisfying (8).

In the following Section we generalize the reduction scheme (5)–(9) to find exact solutions for systems of heat equations (2).

3   Reduction and exact solutions for equations (2)

Consider a subclass of equations (2) where f and g are homogeneous functions of order 3, i.e.,

$$f(\lambda u, \lambda v) = \lambda^3 f(u, v), \qquad g(\lambda u, \lambda v) = \lambda^3 g(u, v). \tag{10}$$

To find exact solutions of these equations we need nothing but a direct generalization of the procedure described in the previous Section. Indeed, all such equations can be reduced to systems of ordinary differential equations using the Ansätze (3), (4) for $k = 1$. In analogy with (5)–(9) we obtain

$$u = 2x\varphi\left(6t + x^2\right), \qquad v = 2x\xi\left(6t + x^2\right),$$
$$\varphi'' = f(\varphi, \xi), \qquad \xi'' = g(\varphi, \xi). \tag{11}$$

Thus to find exact solutions for system (2), (10) it is sufficient to solve the system of ordinary differential equations (11). For some particular form of functions f and g this system can be integrated in quadratures. Here we consider three examples.

1. Let $f = 2\left(u^3 - 3u^2v + 2uv^2\right)$, $g = 0$. Then equations (11) admit the following particular solutions: $\varphi = 1 - \tanh(z)$, $\xi = 1$. Using them we come to the following solutions for the related equations (2)

$$u = 2x\left(1 - \tanh\left(x^2 + 6t\right)\right), \qquad v = 2x. \tag{12}$$

For any finite interval $x_0 < x < x_1$ solutions (12) are bounded. Moreover, the pattern u has a solitary wave sharp. The plot of function u (12) is given in Fig. 1.
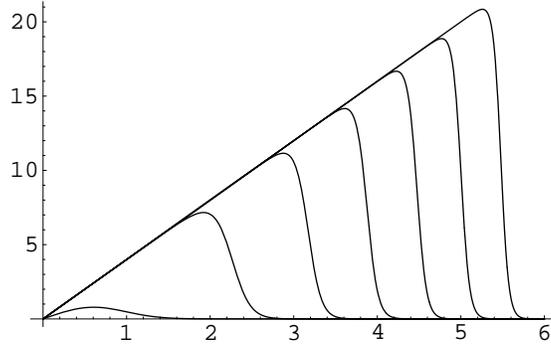
Figure 1. Solutions u (12) for $t = 0, -1/6, -1/3, -1/2, -3/2, -5/6, -1$.

We see that when the time variable t runs from $t = -1$ to $t = 0$ our solitary wave moves from the point $x = 5.5$ to $x = 0.6$ and its amplitude decreases.

2. Consider now the system (2) with nonlinearities of the following general form

$$f = \frac{\varepsilon}{2}\left(n^2 u^{n+3} v^{-n} - n(2n+m)v^m u^{3-m}\right),$$

$$g = \frac{\varepsilon}{2}\left((n+2)^2 u^{n+2} v^{1-n} - (n+2)(2n+m+2)v^{m+1} u^{2-m}\right), \tag{13}$$

where $\varepsilon = \pm 1$, m and n are arbitrary parameters satisfying $m + n \neq 0$, $n \neq 0$, -2.

The Ansatz (4) reduces (2), (13) to the following system of ordinary equations

$$\varphi'' = \varepsilon\left(n^2 \varphi^{n+3}\xi^{-n} - n(2n+m)\varphi^m \xi^{-m+3}\right),$$

$$\xi'' = \varepsilon\left((n+2)^2 \varphi^{n+2}\xi^{1-n} - (n+2)(2n+m+2)\xi^{m+1}\varphi^{2-m}\right)$$

whose solutions for $\varepsilon = -1$ and $\varepsilon = 1$, are expressed via trigonometric and hyperbolic functions respectively. Thus for $\varepsilon = -1$ we come to the following exact solutions for (2), (13)

$$u = \frac{2\mu x}{n+m}\left[\cos\left(\mu\left(6t + x^2\right)\right)\right]^{-\frac{n}{n+m}}, \qquad v = \frac{2\mu x}{n+m}\left[\cos\left(\mu\left(6t + x^2\right)\right)\right]^{-\frac{(2+n)}{n+m}}. \tag{14}$$

For $\frac{n}{n+m} < 0$ and $\frac{2+n}{n+m} < 0$ solutions (14) are bounded on any finite interval $x_0 \leq x \leq x_1$. The plots of these solutions for $n = 2$, $m = -1$ are given in Figs. 2, 3.

If $\varepsilon = 1$ then the related solutions can be found in the form

$$u = \frac{2\mu x}{n+m}\left[\frac{1}{\mathrm{ch}(\mu(6t + x^2))}\right]^{\frac{n}{n+m}}, \qquad v = \frac{2\mu x}{n+m}\left[\frac{1}{\mathrm{ch}(\mu(6t + x^2))}\right]^{\frac{n+2}{n+m}}. \tag{15}$$

For $\frac{n}{n+m} > 0$ and $\frac{n+2}{n+m} > 0$ formulae (15) present solitary waves. We notice that solutions (15) are not of plane wave type and their amplitudes decrease if t goes from $-t_0$ to 0 ($t_0 > 0$). The plots of these solutions for $n = 1$, $m = 0$ are given in Figs. 4 and 5.

Like in the case of solutions (12), when t runs from $t = -1$ to $t = 0$, our solitary waves move from the point $x = 2.3$ to $x \approx 0.7$ and their amplitudes decrease. When variable t becomes positive and grows the amplitudes tend to zero.
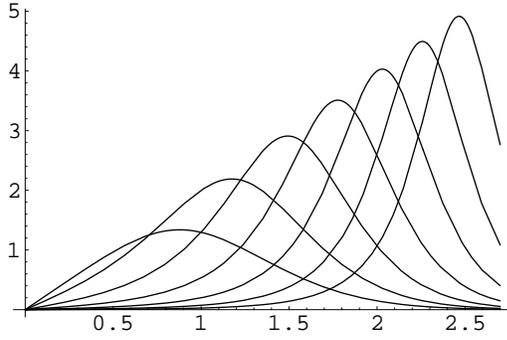
In plots given by Figs. 2–5 the value of amplitudes decrease with increasing of time variable.
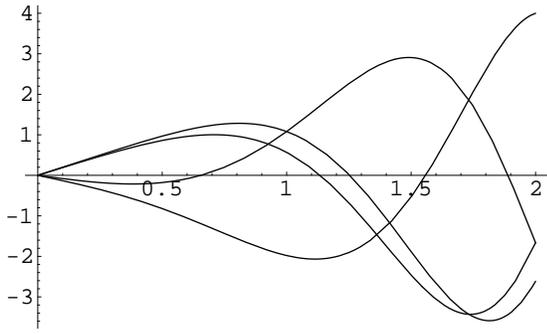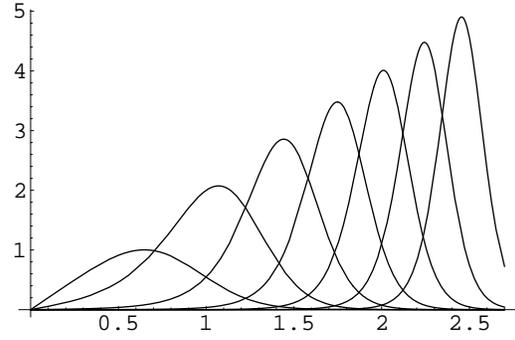
Figure 2. Solutions u (14) for $t = 0, -1/3,$ $-3/2, -1.$
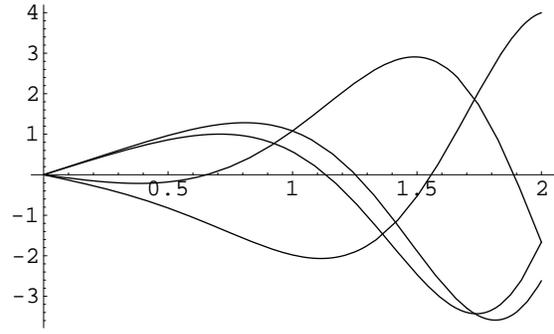
Figure 3. Solutions v (14) for $t = 0, -1/3,$ $-3/2, -1.$

Figure 4. Solutions u (14) for $n = 1$, $m = 0$, and $t = 0, -1/6, -1/3, -1/2, -3/2, -5/6, -1.$

Figure 5. Solutions v (14) for $n = 1$, $m = 0$, $t = 0, -1/3, -3/2, -1.$

4   Solitary wave solutions for equation (1)

Let us return to equation (1) and choose the nonlinearity $f(u)$ in the form

$$f(u) = -\frac{n+1}{n-1}u^n + \lambda_1 u + \lambda_2 u \frac{n+1}{2} + \lambda_3 u^{\frac{3-n}{2}} + \lambda_4 u^{2-n}, \tag{16}$$

where n is an arbitrary parameter.

Reaction-diffusion equations of the general form (1), (16) can be effectively reduced using Ansatz (3) [5]. Choosing $\varphi = z^{-k}$, $k = \frac{2}{n-1}$ we transform (1), (16) to the form

$$z\left(z_x z_{xx} - z_x z_{xxx} - (k-1)z_{xx}^2 - \lambda_4 z^2 - \lambda_3 z_x z\right) = z_x^2(z_x - (2k+1)z_{xx} + \lambda_1 z + \lambda_2 z_x). \tag{17}$$

Equation (17) is homogeneous in dependent variable and is more convenient for searching for exact solutions than the initial equation (1), (16).

Let us search for solutions of (17) in the form $z(t, x) = U(\xi)$ where $\xi = \mu t + x$ and μ is an arbitrary (nonzero) constant. Then we come to the following ordinary differential equation for U

$$U[U'(\mu U'' - U''' - \lambda_3 U) - \lambda_4 U^2 - (k-1)(U'')^2]$$
$$= (U')^2[(\mu + \lambda_2)U' + \lambda_1 U - (2k+1)U''], \tag{18}$$

where $U' = \frac{dU}{d\xi}$.

Using the idea of Fan [9] we express U in the form

$$U = \nu_0 + \nu_1 \varphi + \nu_2 \varphi^2 + \cdots, \tag{19}$$

where $\nu_0, \nu_1, \ldots$ are constants and φ satisfies the equation of the following general form

$$\varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + \cdots}. \tag{20}$$

In order (19) to be compatible with (18) we have to equate separately the terms which include odd and even powers of the square root given by (20). As a result we obtain

$$U'(\mu U U'' - \lambda_3 U^2) = (U')^3(\mu + \lambda_2), \tag{21}$$

$$U(U'U''' + \lambda_4 U^2 + (k-1)(U'')^2) = (U')^2((2k+1)U'' - \lambda_1 U). \tag{22}$$

Dividing any term in (21) by $\mu U^2 U'$ we come to the Riccati equation

$$Y' - \frac{\lambda_2}{\mu}Y^2 = \frac{\lambda_3}{\mu}$$

for $Y = \frac{U'}{U}$, whose general solutions are

$$Y = \sqrt{\frac{-\lambda_3}{\lambda_2}} \tanh\left(\frac{\sqrt{-\lambda_2\lambda_3}}{\mu}\xi + C\right),$$

$$Y = \sqrt{\frac{-\lambda_3}{\lambda_2}} \left(\tanh\left(\frac{\sqrt{-\lambda_2\lambda_3}}{\mu}\xi + C\right)\right)^{-1}, \qquad \text{if} \quad \lambda_2\lambda_3 < 0, \tag{23}$$

$$Y = \sqrt{\frac{\lambda_3}{\lambda_2}} \tan\left(\frac{\sqrt{\lambda_2\lambda_3}}{\mu}\xi + C\right), \qquad \text{if} \quad \lambda_2\lambda_3 > 0, \tag{24}$$

$$Y = -\frac{\mu}{\lambda_2(\xi + C)}, \qquad \text{if} \quad \lambda_3 = 0, \tag{25}$$

where C is the integration constant.

Solutions (23), (24) and (25) are compatible with (22) provided

$$\mu = -\lambda_2, \qquad \lambda_1 = -k\frac{\lambda_3}{\lambda_2}, \qquad \lambda_4 = (1-k)\left(\frac{\lambda_3}{\lambda_2}\right)^2, \qquad \lambda_2\lambda_3 \neq 0$$

and

$$\lambda_1 = \lambda_4 = 0, \qquad \lambda_3 = 0$$

respectively. Using variables

$$\tau = \frac{2}{(n-1)^2}t, \qquad y = \frac{\sqrt{2}}{n-1}x, \qquad \sigma = -\lambda_2(n-1), \qquad \nu = \frac{\lambda_3}{\lambda_2}$$

we can rewrite the related equation (1), (16) as follows:

$$u_\tau - u_{yy} = \left(1 + \nu u^{1-n}\right)\left(-(n+1)u^n + \nu(n-3)u + \sigma u^{\frac{n+1}{2}}\right). \tag{26}$$

The corresponding solutions for equation (26) have the following form

$$u = (-\nu)^{\frac{1}{n-1}}\left(\tanh\left(b\left(y - \frac{\sigma}{\sqrt{2}}t\right) + C\right)\right)^{\frac{2}{n-1}}, \tag{27}$$

$$u = (-\nu)^{\frac{1}{n-1}}\left(\tanh\left(b\left(y - \frac{\sigma}{\sqrt{2}}t\right) + C\right)\right)^{\frac{2}{1-n}}, \tag{28}$$

where $\nu < 0$ and $b = (n-1)\sqrt{\frac{-\nu}{2}}$,

$$u = (\nu)^{\frac{1}{n-1}}\left(\tan\left(b\left(y - \frac{\sigma}{\sqrt{2}}t\right) + C\right)\right)^{\frac{2}{n-1}}, \tag{29}$$

where $\nu > 0$ and $b = (n-1)\sqrt{\frac{\nu}{2}}$, and

$$u = 2^{n-1}\left((n-1)\left(y - \frac{\sigma}{\sqrt{2}}\tau + C\right)\right)^{\frac{2}{1-n}} \tag{30}$$

if $\nu = 0$.

For $\frac{2}{n-2} > 1$ formula (27) presents solitary wave solutions propagating with the velocity $\frac{\sigma}{\sqrt{2}}$. In the case $n = 2$ we come to the bell-shaped solitary wave.

If $\frac{2}{n-2} < -1$ then equation (26) again admits solitary wave solutions which are given by relation (28). In contrast with (14) they are bell-shaped plane waves.

5   Discussion

We present a collection of solitary wave solutions for nonlinear heat equations. All these solutions are obtained using the Ansatz (3), (4).

It is possible to show that all reductions used in the present paper are caused by the conditional symmetry of the related equations (1), (2). Instead of investigation of conditional symmetries we prefer the direct use of the universal Anzätse which seams to be the easiest way to find equations admitting exact solutions of the desired solitary wave type.

Nevertheless the very existence of extended classes of exact solutions for systems of heat equations shows that the problem of classification of conditional symmetries for these systems is quite interesting. We plane to consider this problem elsewhere.

[1] Samarskiĭ A.A., Galaktionov V.A., Kurdyumov S.P. and Mikhailov A.P., Peaking modes in problems for quasilinear parabolic equations, Moscow, Nauka, 1987 (in Russian).

[2] Olver P.J., Application of Lie groups to differential equations, New York, Springer, 1986.

[3] Fushchych W.I., Shtelen W.I. and Serov M.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989 (in Russian); Kluwer Publishers, 1993 (in English).

[4] Anderson R.L. and Ibragimov N.H., Lie–Bäcklund transformations in application, Philadelphia, SIAM, 1979.

[5] Nikitin A.G. and Barannyk T.A., Solitary wave and other solutions for nonlinear heat equations, math-ph/0303004.

[6] Barannyk A.F. and Yuryk I.I., Construction of exact solutions of diffusion equation, Contribution in these Proceedings.

[7] Fushchych W.I. and Serov M.I., Conditional invariance and reduction of the nonlinear heat equation, *Dokl. Akad. Nauk Ukr. SSR, Ser. A*, 1990, N 7, 24–27 (in Russian).

[8] Clarkson P.A. and Mansfeld E.L., Symmetry reduction and exact solutions of a class of nonlinear heat equations, *Physica D*, 1994, V.70, 250–288.

[9] Fan E., Multiple travelling wave solutions of nonlinear evolution equations using a unified algebraic method, *J. Phys. A: Math. Gen.*, 2002, V.35, 6853–6872.