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Representations of the q-Deformed Algebra $\mathrm{so}_q(2,1)$

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We give a classification theorem for irreducible weight representations of the q-deformed algebra $U_q(\mathrm{so}_{2,1})$ which is a real form of the nonstandard deformation $U_q(\mathrm{so}_3)$ of the Lie algebra $\mathrm{so}(3,\mathbf{C})$. The algebra $U_q(\mathrm{so}_3)$ is generated by the elements I_1, I_2 and I_3 satisfying the relations $[I_1,I_2]_q := q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3$, $[I_2,I_3]_q = I_1$ and $[I_3,I_1]_q = I_2$. The real form $U_q(\mathrm{so}_{2,1})$ is determined for real q by the $*$-involution $I_1^* = -I_1$ and $I_2^* = I_2$. Weight representations of $U_q(\mathrm{so}_{2,1})$ are defined as representations T for which the operator $T(I_1)$ can be diagonalized and has a discrete spectrum. A part of the irreducible representations of $U_q(\mathrm{so}_{2,1})$ turn into irreducible representations of the Lie algebra $\mathrm{so}_{2,1}$ when $q \to 1$. Representations of the other part have no classical analogue.

1 The algebras $U_q(\mathrm{so}_3)$ and $U_q(\mathrm{so}_{2,1})$

The algebra $U_q(\mathrm{so}_3)$ is obtained by a q-deformation of the standard commutation relations $[I_1,I_2] = I_3$, $[I_2,I_3] = I_1$, $[I_3,I_1] = I_2$ of the Lie algebra $\mathrm{so}(3,\mathbf{C})$ and is defined as the complex associative algebra (with a unit element) generated by the elements I_1, I_2, I_3 satisfying the defining relations

$$[I_1,I_2]_q := q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3, \tag{1}$$

$$[I_2,I_3]_q := q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 = I_1, \tag{2}$$

$$[I_3,I_1]_q := q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 = I_2. \tag{3}$$

A Hopf algebra structure is not known on $U_q(\mathrm{so}_3)$. However, it can be embedded into the Hopf algebra $U_q(\mathrm{sl}_3)$ as a Hopf coideal (see [1]). This embedding is very important for the possible application in spectroscopy.

It follows from the relations (1)–(3) that for the algebra $U_q(\mathrm{so}_3)$ the Poincaré–Birkhoff–Witt theorem is true and this theorem can be formulated as: *The elements* $I_3^kI_2^mI_1^n$, $k,m,n = 0,1,2,\ldots$, *form a basis of the linear space* $U_q(\mathrm{so}_3)$. This theorem is proved by using the diamond lemma [2] (or its special case from Subsect. 4.1.5 in [3]).

By (1) the element I_3 is not independent: it is determined by the elements I_1 and I_2. Thus, the algebra $U_q(\mathrm{so}_3)$ is generated by I_1 and I_2, but now instead of quadratic relations (1)–(3) we must take the relations

$$I_1I_2^2 - \left(q+q^{-1}\right)I_2I_1I_2 + I_2^2I_1 = -I_1, \qquad I_2I_1^2 - \left(q+q^{-1}\right)I_1I_2I_1 + I_1^2I_2 = -I_2, \tag{4}$$

which are obtained if we substitute the expression (1) for I_3 into (2) and (3). The equation $I_3 = q^{1/2}I_1I_2 - q^{-1/2}I_2I_1$ and the relations (4) restore the relations (1)–(3).

Up to now we did not introduce $*$-involutions on $U_q(\mathrm{so}_3)$ determining real forms of this algebra. The $*$-involution $I_1 = -I_1$, $I_2 = -I_2$ determines the real form of $U_q(\mathrm{so}_3)$ which can be called a compact real form of $U_q(\mathrm{so}_3)$. The $*$-involution uniquely determined by the relations

$$I_1^* = -I_1, \qquad I_2^* = I_2 \tag{5}$$

gives a noncompact real form of $U_q(\mathrm{so}_3)$ which is denoted by $U_q(\mathrm{so}_{2,1})$. It is a q-analogue of the real form $\mathrm{so}_{2,1}$ of the complex Lie algebra $\mathrm{so}(3,\mathbf{C})$.

Note that for real q the equations $I_1^* = -I_1$ and $I_2^* = I_2$ do not mean that $I_3^* = I_3$ or $I_3^* = -I_3$:

$$I_3^* = \left(q^{1/2}I_1I_2 - q^{-1/2}I_2I_1\right)^* = q^{1/2}I_2^*I_1^* - q^{-1/2}I_1^*I_2^* = -q^{1/2}I_2I_1 + q^{-1/2}I_1I_2 \neq \pm I_3.$$

However, if $|q| = 1$ then $I_3^* = I_3$. Really,

$$I_3^* = \left(q^{1/2}I_1I_2 - q^{-1/2}I_2I_1\right)^* = q^{-1/2}I_2^*I_1^* - q^{1/2}I_1^*I_2^* = -q^{-1/2}I_2I_1 + q^{1/2}I_1I_2 = I_3.$$

In this paper we are interested in irreducible infinite dimensional representations of the algebras $U_q(\mathrm{so}_{2,1})$. Infinite dimensional irreducible representations of $U_q(\mathrm{so}_{2,1})$ are important for physical applications. For example, irreducible $*$-representations of the so called strange series (these representations were defined in [4]) are related to a certain type of Schrödinger equation [5]. Infinite dimensional representations of $U_q(\mathrm{so}_{2,1})$ appear in the theory of quantum gravity [6].

Infinite dimensional representations of $U_q(\mathrm{so}_{2,1})$ were studied in [4]. However, not all such representations were found there. Note that $*$-representations of real forms of $U_q(\mathrm{so}_3)$ different from $U_q(\mathrm{so}_{2,1})$ were studied in [7] and [8]. Irreducible representations of $U_q(\mathrm{so}_3)$ (including the case when q is a root of unity) are studied in [9–11].

2 Definition of weight representations of $U_q(\mathrm{so}_{2,1})$

From this point we assume that q is not a root of unity.

Definition 1. *By a weight representation T of $U_q(\mathrm{so}_{2,1})$ we mean a homomorphism of $U_q(\mathrm{so}_{2,1})$ into the algebra of linear operators (bounded or unbounded) on a Hilbert space $\mathcal{H}$, defined on an everywhere dense invariant subspace $\mathcal{D}$, such that the operator $T(I_1)$ can be diagonalized, has a discrete spectrum (with finite multiplicities of spectral points if T is irreducible), and its eigenvectors belong to $\mathcal{D}$. Two weight representations T and T' of $U_q(\mathrm{so}_{2,1})$ on spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively, are called (algebraically) equivalent if there exist everywhere dense invariant subspaces $V \subset \mathcal{H}$ and $V' \subset \mathcal{H}'$ and a one-to-one linear operator $A : V \to V'$ such that $AT(a)v = T'(a)Av$ for all $a \in U_q(\mathrm{so}_{2,1})$ and $v \in V$.*

Remark. Note that the element $I_1 \in U_q(\mathrm{so}_{2,1})$ corresponds to the compact part of the group $SO(2,1)$. Therefore, as in the classical case, it is natural to demand in the definition of representations of $U_q(\mathrm{so}_{2,1})$ that the operator $T(I)$ has a discrete spectrum (with finite multiplicities of spectral points for irreducible representations T). Such representations correspond to Harish–Chandra modules of Lie algebras. Note that the algebra $U_q(\mathrm{so}_{2,1})$ has irreducible representations T for which the operator $T(I_1)$ can be diagonalized and has a continuous spectrum (this follows from the results of Section 4 in [12]). We do not consider such representations in this paper.

Since we shall consider only weight representations, below speaking about weight representations we shall omit the word "weight".

Definition 2. *By a $*$-representation T of $U_q(\mathrm{so}_{2,1})$ we mean a representation of $U_q(\mathrm{so}_{2,1})$ in a sense of Definition 1 such that the equations $T(I_1)^* = -T(I_1)$ and $T(I_2)^* = T(I_2)$ are fulfilled on the domain $\mathcal{D}$.*

Definition 1 does not use the $*$-structure of $U_q(\mathrm{so}_{2,1})$. This means that representations of Definition 1 are in fact representations of $U_q(\mathrm{so}_3)$.

3 Representations of the principal series

Let us study irreducible infinite dimensional representations of the algebra $U_q(\mathrm{so}_{2,1})$ which were constructed in [4] and [11].

Let $q = e^\tau$ and ϵ be a fixed complex number such that $0 \le \mathrm{Re}\,\epsilon < 1$ and $\epsilon \ne \pm\mathrm{i}\pi/2\tau$. Let $\mathcal{H}_\epsilon$ be a complex Hilbert space with the orthonormal basis

$$|m\rangle, \qquad m = n+\epsilon, \quad n = 0, \pm1, \pm2, \ldots.$$

To every complex number a there corresponds the representation $R_{a\epsilon}$ of $U_q(\mathrm{so}_{2,1})$ on the Hilbert space $\mathcal{H}_\epsilon$ defined by the formulas

$$R_{a\epsilon}(I_1)|m\rangle = \mathrm{i}[m]|m\rangle, \tag{6}$$

$$R_{a\epsilon}(I_2)|m\rangle = \frac{1}{q^m+q^{-m}}\left\{[a-m]|m+1\rangle - [a+m]|m-1\rangle\right\}, \tag{7}$$

$$R_{a\epsilon}(I_3)|m\rangle = \frac{\mathrm{i}q^{1/2}}{q^m+q^{-m}}\left\{q^m[a-m]|m+1\rangle + q^{-m}[a+m]|m-1\rangle\right\}. \tag{8}$$

(Everywhere below, under considering representations of $U_q(\mathrm{so}_{2,1})$, we do not give the operator corresponding to I_3 since it can be easily calculated by using formula (3).)

Note that we excluded the cases $\epsilon = \pm\mathrm{i}\pi/2\tau$ since for these ϵ the coefficients in (7) and (8) are singular.

If $\epsilon = -\mathrm{i}\pi/2\tau + \sigma$ and $q^\sigma = \lambda$, then the representation $R_{a\epsilon}$ can be reduced to the following form:

$$R_{a\epsilon}(I_1)|n\rangle = \frac{\lambda q^n + \lambda^{-1}q^{-n}}{q-q^{-1}}|n\rangle,$$

$$R_{a\epsilon}(I_2)|n\rangle = \frac{-1}{\lambda q^n - \lambda^{-1}q^{-n}}\left(\frac{\lambda q^{n-a}+\lambda^{-1}q^{-n+a}}{q-q^{-1}}|n{+}1\rangle + \frac{\lambda q^{n+a}+\lambda^{-1}q^{-n-a}}{q-q^{-1}}|n{-}1\rangle\right),$$

where the basis elements $|n+\epsilon\rangle$ are denoted by $|n\rangle$, $n = 0, \pm1, \ldots$. In particular, if $a = \pm\mathrm{i}\pi/2\tau$ and $q^\sigma = \lambda$, $0 \le \mathrm{Re}\,\sigma < 1$, then after rescaling the basis vectors the representation $R_{a\epsilon}$ (we denote it in this case as Q_λ^+) takes the form

$$Q_\lambda^+(I_1)|m\rangle = \frac{\lambda q^m + \lambda^{-1}q^{-m}}{q-q^{-1}}|m\rangle, \qquad Q_\lambda^+(I_2)|m\rangle = \frac{1}{q-q^{-1}}|m+1\rangle + \frac{1}{q-q^{-1}}|m-1\rangle.$$

If $a = \pm\mathrm{i}\pi/2\tau$ and $q^\sigma = -\lambda$, $0 \le \mathrm{Re}\,\sigma < 1$, then we obtain the representation $R_{a\epsilon}$ (we denote it in this case as Q_λ^-) in the form

$$Q_\lambda^-(I_1)|m\rangle = -\frac{\lambda q^m + \lambda^{-1}q^{-m}}{q-q^{-1}}|m\rangle, \qquad Q_\lambda^-(I_2) = Q_\lambda^+(I_2).$$

Since the representations $R_{a\epsilon}$ are determined for $\epsilon \neq \pm \mathrm{i}\pi/2\tau$, then the representations $Q_\lambda^\pm$ are determined for $\lambda \neq 1$. However, the operators $Q_\lambda^\pm(I_j)$, $j = 1, 2, 3$, are well defined also for $\lambda = \pm 1$ and satisfy the defining relations (1)–(3). Thus, the representations $Q_\lambda^\pm$ are determined for all complex values of λ.

Theorem 1. *The representation $R_{a\epsilon}$ is irreducible if and only if $a \not\equiv \pm\epsilon \,(\mathrm{mod}\, \mathbf{Z})$ or if $\epsilon \not\equiv \pm \mathrm{i}\pi/2\tau + 1/2$ or if (a, ϵ) does not coincide with one of four couples $(\pm\mathrm{i}\pi/2\tau, \pm\mathrm{i}\pi/2\tau + 1/2)$. The representation $Q_\lambda^\pm$ is irreducible if and only if $\lambda \neq \pm 1, \pm q^{1/2}$.*

This theorem follows from Theorem 1 in [4] and the results of Section 7 in [11].

There exist equivalence relations between irreducible representations $R_{a\epsilon}$. They are completely described in [4].

In the excluded cases of Theorem 1, representations $R_{a\epsilon}$ and $Q_\lambda^\pm$ are reducible. In particular, the representations $Q_\lambda^\pm$, $\lambda = \pm 1, \pm q^{1/2}$, are reducible (see [11]) and leads to the irreducible representations which are described as follows.

Let V_1 and V_2 be the vector spaces with the bases

$$|m\rangle', \quad m = 0, 1, 2, \ldots, \qquad \text{and} \qquad |m\rangle'', \quad m = 1, 2, 3, \ldots,$$

respectively. Then the operators $Q_1^{1,\pm}(I_1)$, $Q_1^{1,\pm}(I_2)$, $Q_1^{2,\pm}(I_1)$, $Q_1^{2,\pm}(I_2)$ given by the formulas

$$Q_1^{1,\pm}(I_1)|m\rangle' = \pm\frac{q^m + q^{-m}}{q - q^{-1}}\,|m\rangle', \qquad Q_1^{2,\pm}(I_1)|m\rangle'' = \pm\frac{q^m + q^{-m}}{q - q^{-1}}\,|m\rangle'',$$

$$Q_1^{1,\pm}(I_2)|0\rangle = \frac{\sqrt{2}}{q - q^{-1}}\,|1\rangle', \qquad Q_1^{2,\pm}(I_2)|1\rangle'' = \frac{1}{q - q^{-1}}\,|2\rangle'',$$

$$Q_1^{1,\pm}(I_2)|1\rangle' = \frac{\sqrt{2}}{q - q^{-1}}\,|0\rangle' + \frac{1}{q - q^{-1}}\,|2\rangle', \qquad Q_1^{2,\pm}(I_2)|2\rangle' = \frac{1}{q - q^{-1}}\,|1\rangle' + \frac{1}{q - q^{-1}}\,|3\rangle',$$

$$Q_1^{1,\pm}(I_2)|m\rangle' = \frac{1}{q - q^{-1}}\,|m+1\rangle' + \frac{1}{q - q^{-1}}\,|m-1\rangle', \qquad m > 1,$$

$$Q_1^{2,\pm}(I_2)|m\rangle'' = \frac{1}{q - q^{-1}}\,|m+1\rangle'' + \frac{1}{q - q^{-1}}\,|m-1\rangle'', \qquad m > 2,$$

determine irreducible representations of $U_q(\mathrm{so}_{2,1})$ which are denoted by $Q_1^{1,\pm}$ and $Q_1^{2,\pm}$, respectively.

Let W_1 and W_2 be the vector spaces spanned by the basis vectors

$$|m + \frac{1}{2}\rangle', \quad m = 0, 1, 2, \ldots, \qquad \text{and} \qquad |m + \frac{1}{2}\rangle'', \quad m = 0, 1, 2, \ldots,$$

respectively. Then the operators $Q_{\sqrt{q}}^{1,\pm}(I_1)$, $Q_{\sqrt{q}}^{1,\pm}(I_2)$, $Q_{\sqrt{q}}^{2,\pm}(I_1)$, $Q_{\sqrt{q}}^{2,\pm}(I_2)$ given by the formulas

$$Q_{\sqrt{q}}^{1,\pm}(I_1)|m + \tfrac{1}{2}\rangle' = \pm\frac{q^{m+1/2} + q^{-m-1/2}}{q - q^{-1}}\,|m + \tfrac{1}{2}\rangle',$$

$$Q_{\sqrt{q}}^{2,\pm}(I_1)|m + \tfrac{1}{2}\rangle'' = \pm\frac{q^{m+1/2} + q^{-m-1/2}}{q - q^{-1}}\,|m + \tfrac{1}{2}\rangle''$$

and

$$Q_{\sqrt{q}}^{1,\pm}(I_2)|\tfrac{1}{2}\rangle' = -\frac{1}{q - q^{-1}}\,|\tfrac{1}{2}\rangle' + \frac{1}{q - q^{-1}}\,|\tfrac{3}{2}\rangle',$$

$$Q^{1,\pm}_{\sqrt{q}}(I_2)|m+\tfrac12\rangle' = \frac{1}{q-q^{-1}}\,|m+\tfrac32\rangle' + \frac{1}{q-q^{-1}}\,|m-\tfrac12\rangle', \qquad m>0,$$

$$Q^{2,\pm}_{\sqrt{q}}(I_2)|\tfrac12\rangle'' = \frac{1}{q-q^{-1}}\,|\tfrac12\rangle'' + \frac{1}{q-q^{-1}}\,|\tfrac32\rangle'',$$

$$Q^{2,\pm}_{\sqrt{q}}(I_2)|m+\tfrac12\rangle'' = \frac{1}{q-q^{-1}}\,|m+\tfrac32\rangle'' + \frac{1}{q-q^{-1}}\,|m-\tfrac12\rangle'', \qquad m>0,$$

determine irreducible representations of $U_q(\mathrm{so}_{2,1})$ which are denoted by $Q^{1,\pm}_{\sqrt{q}}$ and $Q^{2,\pm}_{\sqrt{q}}$, respectively. We have

$$Q^{\pm}_1 = Q^{1,\pm}_1 \oplus Q^{2,\pm}_1, \qquad Q^{\pm}_{\sqrt{q}} = Q^{1,\pm}_{\sqrt{q}} \oplus Q^{2,\pm}_{\sqrt{q}}.$$

The representations $R_{a\epsilon}$ with $\epsilon = \pm\mathrm{i}\pi/2\tau + \frac12$ are also reducible. They lead to the following irreducible representations. For any complex number a we define the representations $R^{(\mathrm{i},\pm)}_a$ and $R^{(-\mathrm{i},\pm)}_a$ of $U_q(\mathrm{so}_{2,1})$ acting on the Hilbert space $\mathcal{H}$ with the orthonormal basis $|n\rangle$, $n=1,2,3,\ldots$, by the formulas

$$R^{(\mathrm{i},\pm)}_a(I_1)|k\rangle = -\frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}|k\rangle,$$

$$R^{(\mathrm{i},\pm)}_a(I_2)|1\rangle = \pm\frac{[a]}{q^{1/2}-q^{-1/2}}|1\rangle + \mathrm{i}\frac{[a-1]}{q^{1/2}-q^{-1/2}}|2\rangle,$$

$$R^{(\mathrm{i},\pm)}_a(I_2)|k\rangle = \mathrm{i}\frac{[a-k]}{q^{k-1/2}-q^{-k+1/2}}|k+1\rangle + \mathrm{i}\frac{[a+k-1]}{q^{k-1/2}-q^{-k+1/2}}|k-1\rangle, \qquad k\neq 1,$$

and by the formulas

$$R^{(-\mathrm{i},\pm)}_a(I_1)|k\rangle = \frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}|k\rangle, \qquad R^{(-\mathrm{i},\pm)}_a(I_2) = -R^{(\mathrm{i},\pm)}_a(I_2).$$

For $\epsilon = \pm\mathrm{i}\pi/2\tau + \frac12$ we have

$$R_{a,\pm\mathrm{i}\pi/2\tau+1/2} = R^{(\mathrm{i},\pm)}_a \oplus R^{(-\mathrm{i},\pm)}_a.$$

Note that for $a=1/2$ the representations $R^{(\pm\mathrm{i},\pm)}_a$ are equivalent to the corresponding representations $Q^{1,\pm}_{\sqrt{q}}$ and $Q^{2,\pm}_{\sqrt{q}}$.

The algebra $U_q(\mathrm{so}_{2,1})$ has also irreducible infinite dimensional representations with highest weights or with lowest weights which are classified in the paper [4]. They are subrepresentations of the corresponding representations $R_{a\epsilon}$. We give a list of these representations.

Let $l = \frac12, 1, \frac32, 2, \ldots$. We denote by R^+_l the representation of $U_q(\mathrm{so}_3)$ acting on the Hilbert space $\mathcal{H}_l$ with the orthonormal basis $|m\rangle$, $m = l, l+1, l+2, \ldots$, and given by formulas (6)–(8) with $a=-l$. By R^-_l we denote the representation of $U_q(\mathrm{so}_3)$ acting on the Hilbert space $\hat{\mathcal{H}}_l$ with the orthonormal basis $|m\rangle$, $m=-l,-l-1,-l-2,\ldots$, and given by formulas (6)–(8) with $a=l$.

Now let $a\neq 0$ (mod $\mathbf{Z}$) and $a\neq\frac12$ (mod $\mathbf{Z}$). We denote by $\mathcal{H}_a$ the Hilbert space with the orthonormal basis $|m\rangle$, $m=-a,-a+1,-a+2,\ldots$. On this space the representation R^+_a acts which is given by formulas (6)–(8). On the Hilbert space $\hat{\mathcal{H}}_a$ with the orthonormal basis $|m\rangle$, $m=a,a-1,a-2,\ldots$, the representation R^-_a acts which is given by (6)–(8).

4 Other infinite dimensional representations of $U_q(\mathrm{so}_{2,1})$

Let us construct additional two series of infinite dimensional irreducible representations of $U_q(\mathrm{so}_{2,1})$ which cannot be obtained from the representations $R_{a\epsilon}$. Let $\mathcal{H}$ be the complex Hilbert space with the basis $|m\rangle$, $m = 0, \pm 1, \pm 2, \ldots$. Let a and b be complex numbers such that $a^2 + b^2 = 1$, $a \neq 0$, $b \neq 0$ and $a \neq b$. We define on the operators $\hat{Q}^{\pm}_{ab}(I_1)$ and $\hat{Q}^{\pm}_{ab}(I_2)$ determined by the formulas

$$\hat{Q}^{\pm}_{ab}(I_1)|m\rangle = \pm \frac{q^m + q^{-m}}{q - q^{-1}}\, |m\rangle,$$

$$\hat{Q}^{\pm}_{ab}(I_2)|m\rangle = \frac{1}{q - q^{-1}}\, |m-1\rangle + \frac{1}{q - q^{-1}}\, |m+1\rangle, \qquad m \neq 0, \pm 1,$$

$$\hat{Q}^{\pm}_{ab}(I_2)|0\rangle = \frac{b\sqrt{2}}{q - q^{-1}}\, |1\rangle + \frac{a\sqrt{2}}{q - q^{-1}}\, |-1\rangle,$$

$$\hat{Q}^{\pm}_{ab}(I_2)|1\rangle = \frac{b\sqrt{2}}{q - q^{-1}}\, |0\rangle + \frac{1}{q - q^{-1}}\, |2\rangle,$$

$$\hat{Q}^{\pm}_{ab}(I_2)|-1\rangle = \frac{a\sqrt{2}}{q - q^{-1}}\, |0\rangle + \frac{1}{q - q^{-1}}\, |-2\rangle.$$

A direct computation shows that these operators satisfy the determining relations (1)–(3) and therefore determine a representation of $U_q(\mathrm{so}_{2,1})$ which is denoted by $\hat{Q}^{\pm}_{ab}$.

Let now $\mathcal{H}'$ be the complex Hilbert space with the basis $|k\rangle$, $k = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$. Let a and b be complex numbers such that $a^2 + b^2 = 1$, $a \neq 0$, $b \neq 0$. We define on the space $\mathcal{H}'$ the operators $\breve{Q}^{\pm}_{ab}(I_1)$ and $\breve{Q}^{\pm}_{ab}(I_2)$ determined by the formulas

$$\breve{Q}^{\pm}_{ab}(I_1)|k\rangle = \frac{q^k + q^{-k}}{q - q^{-1}}\, |k\rangle,$$

$$\breve{Q}^{\pm}_{ab}(I_2)|k\rangle = \frac{1}{q - q^{-1}}\, |k-1\rangle + \frac{1}{q - q^{-1}}\, |k+1\rangle, \qquad k \neq \pm\frac{1}{2},$$

$$\breve{Q}^{\pm}_{ab}(I_2)|\tfrac{1}{2}\rangle = \frac{a}{q - q^{-1}}\, |\tfrac{1}{2}\rangle + \frac{1}{q - q^{-1}}\, |\tfrac{3}{2}\rangle + \frac{b}{q - q^{-1}}\, |-\tfrac{1}{2}\rangle,$$

$$\breve{Q}^{\pm}_{ab}(I_2)|-\tfrac{1}{2}\rangle = -\frac{a}{q - q^{-1}}\, |-\tfrac{1}{2}\rangle + \frac{b}{q - q^{-1}}\, |\tfrac{1}{2}\rangle + \frac{1}{q - q^{-1}}\, |-\tfrac{3}{2}\rangle.$$

A direct computation shows that these operators also determine representations of $U_q(\mathrm{so}_{2,1})$ which are denoted by $\breve{Q}^{\pm}_{ab}$.

Thus, we have constructed the following classes of irreducible infinite dimensional representations of the algebra $U_q(\mathrm{so}_{2,1})$:

(a) The representations $R_{a\epsilon}$ with the exclusions of Theorem 1.
(b) The representations $R^{\pm \mathrm{i},\pm}_a$, $a \in \mathbf{C}$.
(c) The representations $R^{\pm}_l$, $l = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and $R^{\pm}_a$, $a \neq 0 \,(\mathrm{mod}\, \mathbf{Z})$, $a \neq \frac{1}{2} \,(\mathrm{mod}\, \mathbf{Z})$.
(d) The representations $Q^{1,\pm}_1$ and $Q^{2,\pm}_1$.
(e) The representations $Q^{1,\pm}_{\sqrt{q}}$ and $Q^{2,\pm}_{\sqrt{q}}$.
(f) The representations $\hat{Q}^{\pm}_{ab}$ and $\breve{Q}^{\pm}_{ab}$, $a^2 + b^2 = 1$, $a \neq 0$, $b \neq 0$, and $a \neq b$ for $\hat{Q}^{\pm}_{ab}$.

Theorem 2. *Every irreducible infinite dimensional weight representation of the algebra* $U_q(\mathrm{so}_{2,1})$ *is equivalent to one of the representations of classes* (a)–(f) *describe above.*

A proof of this theorem is long and will be given in a separate paper. In particular, the proof uses the following proposition:

Proposition. *Let* $|q| \neq 1$. *If* $b \neq \frac{1}{2}$ *and* $b \neq 1$, *then the set*

$$\frac{q^{b+m} + q^{-b-m}}{q - q^{-1}}, \qquad m \in \mathbf{Z},$$

has no coinciding numbers. If $b = \frac{1}{2}$, *then this set consists only of pairs of coinciding numbers. If* $b = 1$, *then this set consists of the point 0 and pairs of coinciding numbers.*

This proposition show for which representations the operator $R(I_1)$ has multiple eigenvalues.

5 $*$-representations of $U_q(\mathrm{so}_{2,1})$

In the previous section we described all irreducible infinite dimensional representations of $U_q(\mathrm{so}_{2,1})$. The aim of this section is to separate $*$-representations of $U_q(\mathrm{so}_{2,1})$ from the set of the representations (a)–(f).

Note that $*$-representations of the universal enveloping algebra $U(\mathrm{so}_{2,1})$ correspond to unitary representations of the Lie group $SO(2,1)$. Irreducible $*$-representations of $U_q(\mathrm{so}_{2,1})$ can be found by using the method described, for example, in Section 6.4 of [13]. The same method is used for separation of $*$-representations in the set of the representations (a)–(f). Let us give the result of this separation.

Theorem 3. *Let* $q = e^h$, $h \in \mathbf{R}$. *Then the following representations from the set (a)–(f) are $*$-representations or equivalent to $*$-representations:*

(a) *the representations* $R_{a\epsilon}$, $a = \mathrm{i}\rho - 1/2$, $\rho \in \mathbf{R}$, $\epsilon = c + \mathrm{i}n\pi/h$, $0 \leq c < 1$, $n = 0, 1$ *(the principal series);*

(b) *the representations* $R_{a\epsilon}$, $a \in \mathbf{R}$, $\epsilon = c + \mathrm{i}n\pi/h$, $0 \leq c < 1$, $n = 0, 1$, *such that* $-c < a < c-1$ *for* $c > 1/2$ *and* $c - 1 < a < -c$ *for* $c < 1/2$ *(the supplementary series);*

(c) *the representations* $R_{a\epsilon}$, $\mathrm{Im}\, a = \pi/2h$, $\epsilon = c + \mathrm{i}n\pi/h$, $0 \leq c < 1$, $n = 0, 1$ *(the strange series);*

(d) *all the representations* R_a^+, $a \geq -1/2$, *and* R_a^-, $a \leq 1/2$ *(the discrete series).*

This list of irreducible $*$-representations of $U_q(\mathrm{so}_{2,1})$ coincides with that of [4].

Theorem 4. *Let* $q = e^{\mathrm{i}\varphi}$, $0 < \varphi \leq 2\pi$. *We suppose that q is not a root of unity. The following representations from the set (a)–(f) are $*$-representations or equivalent to $*$-representations:*

(a) *the representations* $R_{a\epsilon}$, $a = \mathrm{i}\rho - 1/2$, $\rho \in \mathbf{R}$, $0 \leq \epsilon < 1$, *if*

$$\cos(\epsilon + n)\varphi \cdot \cos(\epsilon + n + 1)\varphi > 0 \qquad \text{for all} \quad n \in \mathbf{Z};$$

(b) *the representations* $R_{a\epsilon}$, $\mathrm{Re}\, a = \pi/2\varphi$, $0 \leq \epsilon < 1$, *if*

$$\sin(\epsilon + n - a)\varphi \cdot \sin(\epsilon + n + a + 1)\varphi \cdot \cos(\epsilon + n)\varphi \cdot \cos(\epsilon + n + 1)\varphi > 0 \qquad \text{for all} \quad n \in \mathbf{Z};$$

(c) *the representations* $R_a^{\pm\mathrm{i},\pm}$ *if*

$$\sin(a - n)\varphi \cdot \sin(a + n)\varphi \cdot \sin(n - 1/2)\varphi \cdot \sin(n + 1/2)\varphi < 0 \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

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