One-dimensional Monotone Non-autonomous Dynamical Systems and Strange Nonchaotic Attractors

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This work is devoted to the study of the dynamics of one-dimensional monotone non-autonomous (cocycle) dynamical systems and strange nonchaotic attractors. A description of the structure of their invariant sets, omega limit sets, Bohr/Levitan almost periodic and almost automorphic motions, global attractors, pinched and minimal sets is given. An application of our general results is given to scalar differential and difference equations. Below we give some of our results for discrete dynamical systems generated by scalar difference equations.

Below we will use the terminology and notation from [1]. Let (Y, d) be a complete metric space and $(Y, \mathbb{Z}, \sigma)$ be a dynamical system on the space Y and $C(\mathbb{Z} \times Y, \mathbb{R})$ be the space of all continuous functions $f : \mathbb{Z} \times Y \to \mathbb{R}$ equipped with the compact-open topology.

Consider the scalar difference equation

$$u(t + 1) = f(\sigma(t, y), u), \quad (y \in Y) \tag{1}$$

where $f \in C(Y \times \mathbb{Z}, \mathbb{R})$. Denote by $\varphi(t, u, y)$ a unique solution of equation (1) passing through the point $u \in \mathbb{R}$ at the initial moment $t = 0$.

From the general properties of solutions of equation (1) we have

a. $\varphi(0, u, y) = u$ for any $u \in \mathbb{R}$ and $y \in Y$;
b. $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for any $t, \tau \in \mathbb{Z}_+$, $u \in \mathbb{R}$ and $y \in Y$;
c. the mapping $(t, u, y) \to \varphi(t, u, y)$ from $\mathbb{Z}_+ \times \mathbb{R} \times Y \to \mathbb{R}$ is continuous;
d. if the function f is monotonically increasing in $u \in \mathbb{R}$ uniformly with respect to $y \in Y$, then one has $\varphi(t, u_1, y) \le \varphi(t, u_2, y)$ for any $t \in \mathbb{Z}_+$ and $y \in Y$.

Taking in consideration $a.-b.$ we can conclude that every equation (1) with monotonically increasing right hand side f generates a monotone cocycle $\langle R, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ with discrete time $\mathbb{Z}_+$.

Quasi-periodically forced monotone maps. An m-dimensional torus is denoted by $\mathcal{T}^m := \mathbb{R}^m / 2\pi\mathbb{Z}^m$. Let $(\mathcal{T}^m, \mathbb{T}, \sigma)$ be an irrational winding of $\mathcal{T}^m$ with the frequency $\nu = (\nu_1, \nu_2, \dots, \nu_m) \in \mathbb{R}^m$. Consider difference equation

$$u(t + 1) = f(\sigma(t, \omega), u), \tag{2}$$

where $f \in C(\mathcal{T}^m \times \mathbb{R}, \mathbb{R})$, $\omega \in \mathcal{T}^m$ and $(\mathcal{T}^m, \mathbb{T}, \sigma)$ is an irrational winding of $\mathcal{T}^m$ with the frequency $\nu = (\nu_1, \nu_2, \dots, \nu_m) \in \mathbb{R}^m$. Denote by $\varphi(t, u, \omega)$ the unique solution of equation (2) passing through the point $u \in \mathbb{R}$ ate the initial moment $t = 0$. If the function f is monotonically increasing in $u \in \mathbb{R}$ uniformly with respect to $\omega \in \mathcal{T}^m$, then the mapping $\varphi : \mathbb{Z}_+ \times \mathbb{R} \times \mathcal{T}^m \to \mathbb{R}$ $((t, u, \omega) \to \varphi(t, u, \omega))$ possesses the properties $a. - d.$

Theorem 1. *Let $f \in C(\mathbb{Z} \times \mathbb{R}, \mathbb{R})$. Assume that the following conditions hold:*

(1) *there exist a solution $\varphi(t, u_0, f)$ of equation*

$$x' = f(t, x) \tag{3}$$

bounded on $\mathbb{Z}_+$;

(2) *the function f is strongly Poisson stable in $t \in \mathbb{Z}$ uniformly with respect to u on every compact subset of $\mathbb{R}$.*

Then the following statements hold:

(1) *the ω-limit set ω_{x_0} ($x_0 := (u_0, f) \in \mathbb{R} \times H(f)$) of point x_0 is a nonempty, conditionally compact and invariant set of skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$;*
(2) *$h(\omega_{x_0}) = Y := H(f)$;*
(3) *the set ω_{x_0} contains at least one but at most two minimal sets;*
(4) *if $\mathcal{M} \subseteq \omega_{x_0}$ is a minimal set, then every point $x = (u, f) \in \mathcal{M}$ is strongly Poisson stable;*
(5) *if the function f is almost recurrent (respectively, recurrent) in $t \in \mathbb{Z}$ uniformly with respect to u on every compact subset of $\mathbb{R}$ and $\mathcal{M} \subseteq \omega_{x_0}$ is a minimal set, then every point $x = (u, f) \in \mathcal{M}$ is almost recurrent (respectively, recurrent);*
(6) *if the function f is almost automorphic in $t \in \mathbb{Z}$ uniformly with respect to u on every compact subset of $\mathbb{R}$, then the minimal set $\mathcal{M} \subseteq \omega_{x_0}$ is almost automorphic.*

Theorem 2. *Assume that equation (3) is uniformly dissipative, then the following statements hold:*

(1) *the cocycle $\langle \mathbb{R}, \varphi, (H(f), \mathbb{Z}, \sigma) \rangle$ associated by equation (3) admits a compact global attractor* [2] *$I = \{I_g | \ g \in H(f)\}$;*
(2) *$\alpha(g), \beta(g) \in I_g$, and hence, $I_g \subseteq [\alpha(g), \beta(g)]$, where*

$$\alpha(g) := \inf\{u \in I_g\} \quad \textit{and} \quad \beta(g) := \sup\{u \in I_g\};$$

(3) *the scalar function $\beta : H(f) \to \mathbb{R},\ g \to \beta(g)$ (respectively, $\alpha : H(f) \to \mathbb{R},\ g \to \alpha(g)$) is upper semi-continuous (respectively, lower semi-continuous);*
(4)

$$\varphi(t, \alpha(g), g) = \alpha(\sigma(t, g)) \tag{4}$$

(respectively,

$$\varphi(t, \beta(g), g) = \beta(\sigma(t, g)) \ ) \tag{5}$$

for any $t \in \mathbb{Z}$ and $g \in H(f)$;
(5) *if the function f is strictly Poisson stable in $t \in \mathbb{Z}$ uniformly with respect to u on every compact subset of $\mathbb{R}$, then there exists a residual subset $G \subseteq H(f)$ such that for any $g \in G$ the solution $\varphi(t, \alpha(g), g)$ (respectively, $\varphi(t, \beta(g), g)$) of equation*

$$x' = g(t, x) \quad (g \in G \subseteq H(f)) \tag{6}$$

is compatible;
(6) *$I_g = [\alpha(g), \beta(g)]$ for any $g \in H(f)$.*

Remark 3. Suppose that $\alpha(g_0) = \beta(g_0)$ for some $g_0 \in H(f)$. Then $\alpha(g) = \beta(g)$ for a residual set $G \subseteq H(f)$ of $g \in G$. This type of attractors are called *Strange Nonchaotic Attractors* (see, for example, [3, Ch.I] and the bibliography therein).

References

[1] David N. Cheban, *Nonautonomous Dynamics:* Nonlinear Oscillations and Global Attractors. *Springer Nature Switzerland AG 2020,* xxii+ 434 pp.

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[3] Ulrike Feudel, Sergey Kuznetsov and Arkady Pikovsky. *STRANGE NONCHAOTIC ATTRACTORS.* Dynamics between Order and Chaos in Quasiperiodically Forced Systems. *World Scientific Series on Nonlinear Science,* Series A, Vol.56. *World Scientific,* New Jersey· London· Singapore· Beijing· Shanghai· Hong Kong· Taipei· Chennai, 2006, xi+213 pp.