Some critical point results for Fréchet manifolds

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Linking techniques (see [1]) provide significant results in critical points theory. We present linking theorem and some of its corollaries, namely a mountain pass theorem and a three critical points theorem for Keller C^1-functional on C^1-Fréchet manifolds. We refer to [2] for the definitions.

Theorem 1 (Linking Theorem, [2]). *Let M be a C^1- Fréchet manifold endowed with a complete Finsler metric ρ and let $\varphi : M \to \mathbb{R}$ be a closed Keller C^1_c-functional. Suppose $\{S_0, S, C\}$ is a linking set through $\gamma \in C(S_0, \mathsf{T})$, C is closed and $\rho(\gamma(S_0), C) > 0$. Suppose the following conditions hold*

(1) $\mathbf{s} \coloneqq \sup_{\gamma(S_0)} \leqslant \inf_C \varphi =: \mathbf{i}$,

(2) φ satisfies the Palais-Smale condition at

$$c \coloneqq \inf_{h \in \mathcal{H}} \sup_{x \in S} \varphi(\gamma(x)), \tag{1}$$

where $\mathcal{H} \coloneqq \{h \in C(S, \mathsf{T}) : h|_{\partial S_0} = \gamma\}$.

Then c is a critical value and $c \geqslant \mathbf{i}$. Furthermore, if $c = \mathbf{i}$ then $\mathrm{Cr}(\varphi, c) \cap C \neq \varnothing$.

The theorem yields the following corollaries:

Theorem 2 (Mountain Pass Theorem, [2]). *Suppose that $x_0, x_1 \in M$, x_0 belongs to an open subset $U \subset M$ and $x_1 \notin \overline{U}$. Let $\varphi : M \to \mathbb{R}$ be a closed a Keller C^1_c-functional satisfying the following condition:*

(1) $\max\{\varphi(x_0), \varphi(x_1)\} \leqslant \inf_{\partial U} \varphi(x) \coloneqq \mathbf{i}$;

(2) φ satisfies the Palais-Smale condition at

$$c \coloneqq \inf_{h \in \mathcal{C}} \sup_{t \in [0,1]} \varphi(h(t)), \tag{2}$$

where $\mathcal{C} \coloneqq \{h \in C([0, 1], M) : h(0) = x_0,\ h(1) = x_1\}$.

Then c is a critical value and $c \geqslant \mathbf{i}$. If $c = \mathbf{i}$ then $\mathrm{Cr}(\varphi, c) \cap U \neq \varnothing$.

Theorem 3 (Three Critical Points Theorem, [2]). *Let M be a connected Fréchet manifold and $\varphi : M \to \mathbb{R}$ a closed a Keller C^1_c-functional satisfying the Palais-Smale condition at all levels. If φ has two minima, then φ has one more critical point.*

We apply the mountain pass theorem and the Minimax principle to prove the following theorem which provides the sufficient conditions for a local diffeomorphism to be a global one.

Theorem 4. [2] *Let M, N be connected C^1- Fréchet manifolds endowed with complete Finsler metrics δ, ρ respectively. Assume that $\varphi : M \to N$ is a local diffeomorphism of class Keller C^1_c. Let $\mathcal{I} : N \to [0, \infty]$ be a closed Keller C^1_c-functional such that $\mathcal{I}(x) = 0$ if and only if $x = 0$ and $\mathcal{I}'(x) = 0$ if and only if $x = 0$. If for any $q \in N$ the functional ϕ_q defined by*

$$\phi_q(x) = \mathcal{I}(\varphi(x) - q)$$

satisfies the Palais-Smale condition at all levels, then φ is a Keller C^1_c-global diffeomorphism.

References

[1] Martin Schetcher. Linking Method in Critical Point Theory. Berlin Birkhäuser Basel, 1999.

[2] Kaveh Eftekharinasab. *Some critical point results for Fréchet manifolds. https://arxiv.org/abs/2205.01359.*