On packing and lattice packing of Minkowski-Chebyshev balls

Nikolaj Glazunov
(Glushkov Institute of Cybernetics NASU, Kiev)
E-mail: `glanm@yahoo.com`

The Minkowski hypothesis was formulated in [1] and refined in [2, 3, 4]. Regarding the concepts of the geometry of numbers, see [5].

Let

$$D_p = \{(x,y),\ p > 1\} \subset \mathbb{R}^2 \tag{1}$$

be the 2-dimension region:

$$|x|^p + |y|^p < 1. \tag{2}$$

Let

$$\Delta(p,\sigma) = (\tau+\sigma)(1+\tau^p)^{-\frac{1}{p}}(1+\sigma^p)^{-\frac{1}{p}}, \tag{3}$$

be the function defined in the domain

$$\mathcal{M}:\ \infty > p > 1,\ 1 \leq \sigma \leq \sigma_p = (2^p-1)^{\frac{1}{p}}, \tag{4}$$

of the $\{p,\sigma\}$ plane, where σ is some real parameter; here $\tau = \tau(p,\sigma)$ is the function uniquely determined by the conditions

$$A^p + B^p = 1,\ 0 \leq \tau \leq \tau_p,$$

where

$$A = A(p,\sigma) = (1+\tau^p)^{-\frac{1}{p}} - (1+\sigma^p)^{-\frac{1}{p}}, \tag{5}$$

$$B = B(p,\sigma) = \sigma(1+\sigma^p)^{-\frac{1}{p}}\tau(1+\tau^p)^{-\frac{1}{p}}, \tag{6}$$

τ_p is defined by the equation

$$2(1-\tau_p)^p = 1+\tau_p^p,\ 0 \leq \tau_p \leq 1. \tag{7}$$

Proposition 1. *The function $\Delta(p,\sigma)$ in region $\mathcal{M}$ determines the moduli space of admissiblel lattices of the rigion D_p each of which contains three pairs of points on the boundary of D_p.*

Proposition 2. *Let $\Delta(D_p)$ be the critical determinant of the region $|x|^p+|y|^p<1$. Let $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$ be two D_p-admissible lattices each of which contains three pairs of points on the boundary of D_p and with the property that $(1,0) \in \Lambda_p^{(0)}$, $(-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)}$. Under these conditions the lattices are uniquely defined.*

Let $d(\Lambda_p^{(0)}), d(\Lambda_p^{(1)})$ be determinants of the lattices. Let $\Delta_p^{(1)} = \Delta(p,1) = 4^{-\frac{1}{p}}\frac{1+\tau_p}{1-\tau_p}$, $\Delta_p^{(0)} = \Delta(p,\sigma_p) = \frac{1}{2}\sigma_p$.

Proposition 3. $d(\Lambda_p^{(0)}) = \Delta(p,\sigma_p),\ d(\Lambda_p^{(1)}) = \Delta(p,1)$.

Remark 4. For example in the case $p=2$ the lattice $\Lambda_2^{(0)}$ has the determinant $d(\Lambda_2^{(0)}) = \frac{\sqrt{3}}{2}$ and is defined by generators $a_1 = (1,0), a_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Theorem 5. [6]

$$\Delta(D_p) = \begin{cases} \Delta(p,1), \ 1 < p \leq 2, \ p \geq p_0, \\ \Delta(p,\sigma_p), \ 2 \leq p \leq p_0; \end{cases}$$

here p_0 is a real number that is defined unique by conditions $\Delta(p_0,\sigma_p) = \Delta(p_0,1), \ 2,57 \leq p_0 \leq 2,58$.

Definition 6. In two-dimensional case we will call geometric figures of the form $|x|^p + |y|^p \leq R, R > 0$, with $p \geq p_0$ the two-dimensional Minkowski-Chebyshev balls.
In cases of dimension grater then two, when the constant p_0 is unknown, we will call geometric figures of the form $|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p \leq R, R > 0$, the n-dimensional Minkowski-Chebyshev balls if p is a sufficiently large.

We investigate packing and lattice packing by equal Minkowski-Chebyshev balls of n-dimensional Euclidean spaces and also of corresponding spheres.

Proposition 7. *Let $\mathbb{Z}^2$ be the integer lattice in $\mathbb{R}^2$ with a point in the origin. Then the density of packing by two-dimensional open Minkowski-Chebyshev balls over the lattice $\mathbb{Z}^2$ tends to unity as p tends to infinity*

Conjecture 8. *Let Λ be the integer $(n > 2)$-dimensional lattice in $\mathbb{R}^n$ with a point in the origin. Then the density of packing by n-dimensional open Minkowski-Chebyshev balls over the lattice Λ tends to unity as p tends to infinity*

Problem 9. Is there an analogue of Theorem 5 in the case of geometric bodies of the form

$$|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p < 1, n > 2,$$

.

Problem 10. If there exists an analogue of Theorem 5 in the case of geometric bodies of the form

$$|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p < 1, n > 2,$$

what is the value of the constant p_0 .

References

[1] H. Minkowski, *Diophantische Approximationen*, Leipzig: Teubner (1907).
[2] C. Davis, Note on a conjecture by Minkowski, *J. London Math. Soc.*, **23**, 172–175 (1948).
[3] H. Cohn, Minkowski's conjectures on critical lattices in the metric $\{|\xi|^p + |\eta|^p\}^{1/p}$, *Annals of Math.*, **51**, (2), 734–738 (1950).
[4] G. Watson, Minkowski's conjecture on the critical lattices of the region $|x|^p + |y|^p \leq 1$, (I), (II), *Jour. London Math. Soc.*, **28**, (3, 4), 305–309, 402–410 (1953).
[5] J. Cassels, *An Introduction to the Geometry of Numbers*, Berlin: Springer-Verlag (1997).
[6] N. Glazunov, A. Golovanov, A. Malyshev,
Proof of Minkowski's hypothesis about the critical determinant of $|x|^p + |y|^p < 1$ domain , Research in Number Theory 9. Notes of scientific seminars of LOMI. **151**(1986), Nauka, Leningrad, 40–53.