The symplectic invariants of 3-webs

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The classical web geometry ([1],[2],[4]) studies invariants of foliation families with respect to pseudogroup of diffeomorphisms. Thus for the case of planar 3-webs the basic semi invariant is the Blaschke curvature ([3]). It is also curvature of the Chern connection ([4]) that are naturally associated with a planar 3-web.

Let $\mathbf{D} \subset \mathbb{R}^2$ be a connected and simply connected domain in the plane, equipped with symplectic structure given by differential 2-form $\Omega = dx \wedge dy$ in the standard coordinates on the plane.

Remind that a 3-web in the domain is a family of three foliations being in general position. We'll assume that these foliations are integral curves of differential 1-forms ω_i, $i = 1, 2, 3$, and write

$$W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle ,$$

where $\omega_i \in \Omega^1(\mathbf{D})$ are such differential 1-forms that $\omega_i \wedge \omega_j \neq 0$ in $\mathbf{D}$, if $i \neq j$.

Definition 1. We say that two planar 3-webs W_3 and $\widetilde{W_3}$ given in domains $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ respectively are symplectively equivalent if there is a symplectomorphism $\phi : \mathbf{D} \to \widetilde{\mathbf{D}}$, such that $\phi(W_3) = \widetilde{W_3}$.

Proposition 2. *Let $W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle$ and $\widetilde{W_3} = \langle \widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3 \rangle$ be two planar 3-webs in domains $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ respectively given by normalized*

$$\omega_1 + \omega_2 + \omega_3 = 0. \qquad (1)$$

differential forms. Then a diffeomorphism $\phi : \mathbf{D} \to \widetilde{\mathbf{D}}$ establishes a symplectic equivalence of 3-webs if and only if

$$\phi^*(\widetilde{\omega}_i) = \varepsilon \omega_{\sigma(i)},$$

where $(\sigma, \varepsilon) \in \mathbb{A}_3 \times \mathbb{Z}_2$, and $\mathbb{A}_3 \subset \mathbb{S}_3$ is the subgroup of even permutations and $\mathbb{Z}_2 = \{1, -1\}$.

In our case normalization (1) and the above proposition shows that the Chern form γ is itself symplectic invariant of 3-webs.

Let's write down γ in following form

$$\gamma = x_1\omega_1 + x_2\omega_2 + x_3\omega_3,$$

where functions $x_i \in C^\infty(\mathbf{D})$ are barycentric coordinates of γ, i.e.

$$x_1 + x_2 + x_3 = 1.$$

Then we have

$$\begin{aligned} d\omega_1 &= (x_3 - x_2)\, \omega_1 \wedge \omega_2, \\ d\omega_2 &= (x_1 - x_3)\, \omega_1 \wedge \omega_2, \\ d\omega_3 &= (x_2 - x_1)\, \omega_1 \wedge \omega_2. \end{aligned}$$

Using the second normalization (1) condition we'll rewrite these relations in the following form

$$\begin{aligned} d\omega_i &= \lambda_i \Omega, \quad i = 1, 2, 3, \\ \lambda_1 &= x_3 - x_2, \lambda_2 = x_1 - x_3, \lambda_3 = x_2 - x_1, \end{aligned} \qquad (2)$$

and

$$x_1 = \frac{1}{3}(1 + \lambda_2 - \lambda_3), x_2 = \frac{1}{3}(1 + \lambda_3 - \lambda_1), x_3 = \frac{1}{3}(1 + \lambda_1 - \lambda_2).$$

Theorem 3. *Functions*

$$\begin{aligned} J_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ J_2 &= \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \\ J_w &= \left(\lambda_2^2 - \lambda_1^2\right)\left(\lambda_3^2 - \lambda_1^2\right)\left(\lambda_3^2 - \lambda_2^2\right) \\ J_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned}$$

are symplectic invariants of 3-webs.

References

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