Geometric interpretation of first Betti numbers of orbits of smooth functions

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Let M be a compact connected surface and P is a real line $\mathbb{R}$ or a circle S^1. Denote by $\mathcal{F}(M,P)$ the space of smooth functions $f \in C^\infty(M,P)$ satisfying the following conditions:

1) the function f takes constant value at ∂M and has no critical point in ∂M;
2) for every critical point z of f there is a local presentation $f_z\colon \mathbb{R}^2 \to \mathbb{R}$ of f near z such that f_z is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

Let X be a closed subset of M. Denote by $\mathcal{D}(M,X)$ the group of C^∞-diffeomorphisms of M fixed on X, that acts on the space of smooth functions $C^\infty(M,P)$ by the rule: $(f,h) \longmapsto f \circ h$, where $h \in D(M,X)$, $f \in C^\infty(M,P)$.

The subset $\mathcal{S}(f,X) = \{h \in D(M,X) \mid f \circ\ h = f\}$ is called the *stabilizer* of f with respect to the action above and $\mathcal{O}(f,X) = \{f \circ h \mid h \in D(M,X)$ is *orbit* of f. Denote by $\mathcal{D}_{id}(M,X)$ the identity path component of $\mathcal{D}(M,X)$ and let $\mathcal{S}'(f,X) = \mathcal{S}(f) \cap \mathcal{D}_{id}(M,X)$.

Homotopy types of stabilizers and orbits of Morse functions were calculated in a series of papers by Sergiy Maksymenko, Bohdan Feshchenko, Elena Kudryavtseva and others. Furthermore, precise algebraic structure of such groups for the case $M \neq S^2, T^2$ was described in [1]. In particular, the following theorem holds.

Theorem 1. [1] *Let M be a connected compact oriented surface except 2-sphere and 2-torus and let $f \in \mathcal{F}(M,P)$. Then $\pi_0\mathcal{S}'(f,\partial M) \in \mathcal{B}$, where $\mathcal{B}$ is a minimal class of groups satisfying the following conditions:*

1) $1 \in \mathcal{B}$;
2) *if* $A, B \in \mathcal{B}$, *then* $A \times B \in \mathcal{B}$;
3 *if* $A \in \mathcal{B}$ *and* $n \geq 1$, *then* $A \wr_n \mathbb{Z} \in \mathcal{B}$.

Note that a group G belongs to the class $\mathcal{B}$ iff G is obtained from trivial group by a finite number of operations $\times$, $\wr_n\mathbb{Z}$. It is easy to see that every group $G \in \mathcal{B}$ can be written as a word in the alphabet $\mathcal{A} = \{1, \mathbb{Z}, (,), \times, \wr_2, \wr_3, \wr_4, \dots\}$. We will call such word a *realization* of the group G in the alphabet $\mathcal{A}$.

Denote by $\beta_1(G)$ the number of symbols $\mathbb{Z}$ in the realization ω of group G. The number $\beta_1(G)$ is the rank of the center $Z(G)$ and the quotient-group $G/[G,G]$ (Theorem 1.8 [2]). Note, this number depends only on the group G, not the presentation ω. Moreover, $\beta_1(G)$ is first Betti number of $\mathcal{O}(f)$.

Edge of Γ_f will be called *external* if it is incident to the vertex of Γ_f that is corresponding to a non-degenerate critical point of f or non-fixed boundary component of ∂M with respect to the action of $S'(f,W)$ for f-adapted submanifold X which contains $W = S^1 \times 0$. Otherwise, it will be called *internal*. Denote by $\sharp Orb_{int}(M,W)$ the number of orbits of the action of $S'(f,W)$ on internal edges of $\Gamma_{f|_X}$.

Theorem 2. *Let M be a disk D^2 or a cylinder $C = S^1 \times [0,1]$ and $f \in \mathcal{F}(M, P)$. Then*

$$\sharp Orb_{int}(M, W) = \beta_1(\pi_0 S^{'}(f, \partial M)),$$

where $W = \partial M$ if $M = D^2$ or $W = S^1 \times 0$ if M is a cylinder.

References

[1] Maksymenko S. I. Deformations of functions on surfaces by isotopic to the identity diffeomorphisms. *Topology and its Applications*, vol. 282, 2020.

[2] Kuznietsova, I.V. and Soroka, Y.Y. The first Betti numbers of orbits of Morse functions on surfaces. *Ukrainian Math. Journal*, 73(2): 179–200, 2021.