Structures of optimal flows on the Boy's and Girl's surfaces

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For a closed oriented surface, the Morse-Smale flows with a minimum number of fixed points (optimal ms-flow) has a single source and sink, is defined by a chord diagram, and can be embedded in R^3 [3]. For the projective plane, the optimal flow has three critical points, but it cannot even be mapped on any immersion in R^3. The simplest immersions with one triple point are Boy's and Girl's surfaces [1, 2]. Each of the surfaces has a natural stratification (cellular structure). It consists of one 0-strata, three 1-strata (A, B, C) and four 2-strata. In the Boy's surface 2-strata are set by their boundaries: A, B, C, $ABA^{-1}CAC^{-1}BCB^{-1}$. On the Girl's surface, the boundaries of 2-strata are as follows: A, B, $ABA^{-1}CB^{-1}$, $AC^{-1}C{-}1BC$.

We describe all possible structures of flows on these surfaces with respect to the homeomorphism (*isotopy*) of the surface using separatrix diagrams and methods of papers [4, 5, 6, 7].

For the flows with one isolated point and a minimum number of separatrices, there are 18 (*108*) structures per Boy's surface (with one separatrix) and 3 (*6*) structures per Girl's surface (without separatrices).

For optimal ms-flows on the surfaces as stratified sets, there are 342 (*2004*) and 534 (*1058*) flows, respectively. These flows have by 4 fixed points: 0-strata and by one point on each 1-strata.

There are 80 (*438*) and 118 (*230*) different structures for the ms-flows on the projective plane that are mapping on these surfaces. The flows have by 3 sources, 3 sinks and 5 saddles (0-strata has triple counting and points from 1-stratas have double counting).

References

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