Extension theorems for holomorphic bundles on complex manifolds with boundary

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We begin with the following important result due to Donaldson [Do] for Kähler, and Xi [Xi] for general Hermitian complex manifolds with boundary:

Theorem 1. *Let $\bar{X}$ be a compact complex manifold with non-empty boundary $\partial\bar{X}$, g be a Hermitian metric on $\bar{X}$ and $\mathcal{E}$ be a holomorphic bundle on $\bar{X}$. Let h be a Hermitian metric on the restriction $\mathcal{E}|_{\partial X}$. There exists a unique Hermitian metric H on $\mathcal{E}$ satisfying the conditions*

$$\Lambda_g F_H = 0,\ \ H|_{\partial X} = h,$$

where $F_H \in A^2(\bar{X}, \mathrm{End}(\mathcal{E}))$ denotes the curvature of the Chern connection associated with H.

Note that the map $H \mapsto \Lambda_g F_H$ is a non-linear second order elliptic differential operator, so the system $\Lambda_g F_H = 0$, $H|_{\partial\bar{X}} = h$ can be viewed as a non-linear Dirichlet problem. The theorem of Donaldson and Xi states that this non-linear Dirichlet problem is always uniquely sovable.

Note also that the analogue statement for closed manifolds (i.e. in the case $\partial\bar{X} = \varnothing$) does not hold. Indeed, the classical Kobayashi-Hitchin correspondence states that, for a holmorphic bundle $\mathcal{E}$ on a closed Hermitian manifold (X, g), the equation $\Lambda_g F_H = 0$ is solvable if and only if $\deg_g(\mathcal{E}) = 0$ (which is a topological condition if g is Kählerian) and $\mathcal{E}$ is polystable with respect to g (see [LT]).

Recall that a unitary connection ∇ on a Hermitian differentiable bundle (E, H) on $\bar{X}$ is called Hermitian Yang-Mills if $\Lambda_g F_\nabla = 0$, $F^{02}_\nabla = 0$. In the classical case $\dim_{\mathbb{C}}(X) = 2$ – which plays a fundamental role in Donaldson theory – these conditions are equivalent to the anti-self-duality condition $F^+_\nabla = 0$.

In [Do] Donaldson shows that Theorem 1 has important geometric consequences:

Corollary 2. *Let $\bar{X}$ be a compact complex manifold with non-empty boundary, g be a Hermitian metric on $\bar{X}$ and (E, H) be a Hermitian differentiable bundle on $\bar{X}$. There exists a natural bijection between:*

(1) the moduli space of pairs $(\mathcal{E}, \theta)$ consisting of a holomorphic structure $\mathcal{E}$ on E and a differentiable trivialization θ of $E|_{\partial\bar{X}}$,
(2) the moduli space of pairs (∇, τ) consisting of a Hermitian Yang-Mills connection on (E, H) and a differentiable unitary trivialization τ of $E|_{\partial\bar{X}}$.

In other words, the moduli space of boundary framed holomorphic structures on E can be identified with the moduli space of boundary framed Hermitian Yang-Mills connection on (E, H).

In the special case when $\bar{X}$ is the closure of a strictly pseudoconvex domain (with smooth boundary) in $\mathbb{C}^n$, Donaldson states the following result which gives an interesting geometric interpretation of the quotient $\mathcal{C}^\infty(\partial\bar{X}, \mathrm{GL}(r, \mathbb{C}))/\mathcal{O}^\infty(\bar{X}, \mathrm{GL}(r, \mathbb{C}))$ of the group of smooth maps $\partial\bar{X} \to \mathrm{GL}(r, \mathbb{C})$ by the subgroup formed by those such maps which extend smoothly and formally holomorphically to $\bar{X}$:

Corollary 3. *Let $\mathcal{O}^\infty(\bar{X}, \mathrm{GL}(r, \mathbb{C}))$ be the group of smooth, formally holomorphic $\mathrm{GL}(r, \mathbb{C})$-valued maps on $\bar{X}$, identified with a subgroup of $\mathcal{C}^\infty(\partial\bar{X}, \mathrm{GL}(r, \mathbb{C}))$ via the restriction map.*

There exists a natural bijection between the moduli space of boundary framed Hermitian Yang-Mills connections on the trivial $\mathrm{U}(r)$-bundle on $\bar{X}$ and the quotient $\mathcal{C}^\infty(\partial\bar{X}, \mathrm{GL}(r, \mathbb{C}))/\mathcal{O}^\infty(\bar{X}, \mathrm{GL}(r, \mathbb{C}))$.

The idea of proof: Taking into account Corollary 2, it suffices to construct a bijection between the quotient $\mathcal{C}^\infty(\partial\bar{X}, \mathrm{GL}(r,\mathbb{C}))/\mathcal{O}^\infty(\bar{X}, \mathrm{GL}(r,\mathbb{C}))$ and the moduli space of boundary framed holomorphic structures on the trivial differentiable bundle $\bar{X}\times\mathbb{C}^r$. The construction is very natural: one maps the congruence class $[f]$ of a smooth map $f:\partial\bar{X}\to\mathrm{GL}(r,\mathbb{C})$ to the gauge class of the pair (the trivial holomorphic structure on $\bar{X}\times\mathbb{C}^r, f$). The main difficulty is to prove the surjectivity of the map obtained in this way. This follows from the following existence result:

Proposition 4. *Let $\bar{X}$ be the closure of a strictly pseudoconvex domain (with smooth boundary) in $\mathbb{C}^n$ and $\mathcal{E}$ be a smooth, topologically trivial holomorphic bundle on $\bar{X}$. Then $\mathcal{E}$ admits a global smooth trivialization on $\bar{X}$ which is holomorphic on X.*

The statement follows using Grauert's classification theorem for bundles on Stein manifolds and the following extension theorem, which is proved in [Do] only for $n=2$:

Proposition 5. *Let $\bar{X}$ be the closure of a relatively compact strictly pseudoconvex domain (with smooth boundary) in $\mathbb{C}^n$ and $\mathcal{E}$ be a smooth, topologically trivial holomorphic bundle on $\bar{X}$. Then $\mathcal{E}$ extends holomorphically to an open neighborhood U of $\bar{X}$ in $\mathbb{C}^n$.*

In my talk I will explain the idea of proof of the following general extension theorem (see [T]):

Theorem 6. *Let M be a complex manifold, $X\subset M$ an open submanifold of M whose closure $\bar{X}$ has smooth, strictly pseudoconvex boundary in M. Let G be a complex Lie group, $\pi: Q\to M$ a differentiable principal G-bundle on M and J a holomorphic structure on the restriction $\bar{P} \coloneqq Q|_{\bar{X}}$.*

There exists an open neighborhood M' of $\bar{X}$ in M and a holomorphic structure J' on $Q|_{M'}$ which extends J.

The proof uses methods and techniques introduced in [HiNa] and [Ca1].

In the special case when $M=\mathbb{C}^n$ and $G=\mathrm{GL}(r,\mathbb{C})$ one obtains as corollary Proposition 5 (and hence Corollary 3) in full generality. Moreover, one also obtains the following generalization of this corollary:

Theorem 7. *Let $G=K^{\mathbb{C}}$ be the complexification of a compact Lie group K, $\bar{X}$ be a compact Stein manifold with boundary and g be a Hermitian metric g on $\bar{X}$. The moduli space of boundary framed Hermitian Yang-Mills connections on the trivial K-bundle on $(\bar{X}, g)$ can be identified with the quotient $\mathcal{C}^\infty(\partial\bar{X}, G)/\mathcal{O}^\infty(\bar{X}, G)$.*

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