Recent progress in Iwasawa theory of knots and links

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We briefly survey joint works with Ryoto Tange, Hyuga Yoshizaki, and Sohei Tateno.

Twisted Iwasawa invariants of knots [1]. Let K be a knot in S^3 with $\pi_K = \pi_1(S^3 - K)$ and let $X_n \to X = S^3 - K$ denote the $\mathbb{Z}/n\mathbb{Z}$-cover for each $n \in \mathbb{Z}_{>0}$. Let p be a prime number and let $m \in \mathbb{Z}$ with $p \nmid m$. Let $\rho : \pi_K \to \mathrm{GL}_N(O_\mathfrak{p})$ be a representation over a finite extension $O_\mathfrak{p}$ of the p-adic number field $\mathbb{Q}_p$ and let $\Delta_\rho(t)$ denote the twisted Alexander polynomial. Then we have the following.

Theorem 1. *Let (K, p, m, ρ) be as above. Then there exists some $\lambda, \mu, \nu \in \mathbb{Z}$ such that for any $n \gg 0$, $|H_1(X_{mp^n}, \rho)_{\mathrm{tor}}| = p^{\lambda n + \mu p^n + \nu}$ holds. (We have $\mathrm{Nr}\Delta_\rho(T) \doteq p^\mu(\lambda + p(\textit{lower terms}))$ in $\mathbb{Z}_p[[T]]$.)*

For each (K, p, ρ), there exists some m such that $\lambda/[O_\mathfrak{p} : \mathbb{Z}_p] = \deg\Delta_\rho(t)$. Hence for each K, there exists some (p, ρ, m) such that λ coincides with the genus of K.

For each (p, K, ρ), μ's and λ's determine whether $\Delta_\rho(t)$ is monic in $O_\mathfrak{p}[t]$ and whether K is fibered.

Example 2. (1) The λ's of the lifts $\rho^\pm_{\mathrm{hol}} : \pi_K \to \mathrm{SL}_2(O)$ of the holonomy representation of the figure eight knot $K = 4_1$.

(2) For any SL_2-representations of the twist knots $J(2, 2k)$ $(k \in \mathbb{Z})$, we have $\mu = 0$. We may expect that if $k \neq 0, \pm 1$, then there exists some ρ of $J(2, 2k)$ with $\mu > 0$.

Weber's class number problem for knots [2]. Weber's class number problem for number fields is mostly unsolved for 200 years. Yoshizaki [3] recently pointed out that the sequence of the class numbers converges in the ring of p-adic integers $\mathbb{Z}_p$. In the knot theory side, we obtain the following.

Theorem 3. *Let K be a knot in S^3 and let p be a prime number. Then the sizes of the p-torsion subgroups of $H_1(X_{p^n}; \mathbb{Z})$ converges in $\mathbb{Z}_p$. The limit value is given by the roots of unity that are close to the roots of the Alexander polynomial $\Delta_K(t)$.*

Example 4. The limit values for the torus knot $T_{a,b}$ $(a, b \in \mathbb{Z}$; coprime$)$ and the twist knot $J(2, 2k)$ $(k \in \mathbb{Z})$.

Iwasawa invariants of the ${\mathbb{Z}_p}^d$-covers of links [4]. Cuoco–Monsky gave a variant of the Iwasawa class number formula for ${\mathbb{Z}_p}^d$-extensions of number fields and pointed out the existence of the term $O(1)$. In our side, we have the following.

Theorem 5. *Let L be a d-component link in a rational homology 3-sphere M and let $Y_n \to X = M - L$ denote the $\mathbb{Z}/n\mathbb{Z}^d$-cover. Then there exists some λ, μ such that the size of p-torsion subgroup of $H_1(Y_n, \mathbb{Z})$ is given by $p^{p^{(d-1)n}(\mu p^n + \lambda n + O(1))}$, where $O(1)$ is the Bachmann–Landau notation. If M is an integral homology 3-sphere, then the ${\mathbb{Z}_p}^d$-cover is Greenberg, namely, $O(1)$ is a constant.*

Example 6. The values μ, λ, and $O(1)$ of Solomon's link 4^2_1 and the twisted Whitehead link W_{2k-1} $(k \in \mathbb{Z})$. We have a link with $O(1) \neq 0$ and a link with any $\mu \in \mathbb{Z}_{\geq 0}$.

References

[1] Ryoto Tange and Jun Ueki. *Twisted Iwasawa invariants of knots*, preprint, 2022. arXiv:2203.03239

[2] Jun Ueki and Hyuga Yoshizaki. *On Weber's class number problem for knots*, preprint, 2022.

[3] Hyuga Yoshizaki, *A New Continued Fraction Expansion and Weber's Class Number Problem*, in preparation, 2020. arXiv: 2010.06399

[4] Sohei Tateno and Jun Ueki. *Iwasawa invariants of ${\mathbb{Z}_p}^d$-covers of links*, in preparation, 2022.