Equivalence classes of inner mappings built with Galois connection

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Definition 1. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. A function $f : P \to Q$ is *monotone* (or *order-preserving*) if $x \leq_P y \implies f(x) \leq_Q f(y)$ for all $x, y \in P$.

Definition 2. A *partially ordered set* (or *poset*) is a pair $(P, \leq)$ consisting of a set P and a binary relation $\leq$ on P such that for all $x, y, z \in P$:

(1) **Reflexivity:** $x \leq x$;
(2) **Antisymmetry:** if $x \leq y$ and $y \leq x$, then $x = y$;
(3) **Transitivity:** if $x \leq y$ and $y \leq z$, then $x \leq z$.

The relation $\leq$ is called a *partial order*.

Definition 3. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. A pair of monotone maps $f : P \to Q$, $g : Q \to P$ forms a *Galois connection* (or *adjunction between posets*) if for every $x \in P$ and $y \in Q$, $f(x) \leq_Q y \Leftrightarrow x \leq_P g(y)$.

In this case, f is called the *left adjoint* of g, and g is called the *right adjoint* of f. One writes $f \dashv g$.

A map is called inner mapping if it is open and isolated (the preimage of a point consists of isolated points). Yuriy Trokhimchuk studied inner mappings a lot during his life and published a book [4]. Topological properties of dynamical systems generated by inner mappings were studied in [2].

The usual equivalence relation for discrete dynamical systems is the topological conjugacy. Dynamical systems generated by maps $f : M \to M$ and $g : M \to M$ are equivalent iff there exists a homeomorphism $h : M \to M$ such that $f \circ h = h \circ g$. However, in the case of inner mappings the topological conjugacy creates continuum of equivalence classes even in the simplest cases such as a wandering set subset in a basin of attraction.

To define an equivalence of dynamical systems generated by maps $f : M \to M$ and $g : M \to M$ via a Galois connection:

(1) Fix a Galois setup $(\mathcal{A}(f), \mathcal{B}(f))$ associated to (M, f) (e.g., invariant sets $\leftrightarrow$ initial conditions, or open sets $\leftrightarrow$ limit sets).
(2) Let c_f and d_f be the induced closure operators.
(3) Declare $(M, f) \sim (N, g)$ if there exist poset isomorphisms $\phi : \mathcal{A}(f) \to \mathcal{A}(g)$ and $\psi : \mathcal{B}(f) \to \mathcal{B}(g)$ that commute with the Galois maps and the closure operators.
(4) This $\sim$ is reflexive, symmetric, transitive, and yields equivalence classes coarser than topological conjugacy.

We discuss the properties of such equivalence classes for inner mappings of surfaces.

References

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