Non-Lipshitz Singularities in the Malliavin Calculus: Raise of Smoothness for Infinite Dimensional Semigroups.

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ABSTRACT

In the framework of infinite dimensional analysis on Wiener space we study the raise of smoothness for the important class of not strongly continuous semigroups, associated with the second order elliptic differential operators of infinite number of variables.

The principal unboundedness of the coefficients of operator can lead to singularities in the basic integration by parts formula on Wiener space. We show how to avoid such singularities in our case.

To achieve the arbitrary order raise of smoothness in the spaces of continuously differentiable functions we derive quasi-contractive nonlinear estimates on directional derivatives, which permit us to study the smoothness of associated Wiener functionals. The influence of nonlinearity parameters is also discussed in this setting.

1 Introduction.

The original idea to consider the solutions to stochastic differential equations as the smooth distributions on Wiener space, initiated in [31, 32] and developed in [6, 40, 41, 43] and later in [4, 7, 9, 26, 27, 28, 29, 33], has already influenced the rich connection between infinite dimensional analysis and stochastic theory. The keen interpretation of Wiener integral as the adjoint operator to the stochastic directional derivative

$$\mathbf{E}\, D_u F = \mathbf{E}\, F \int_0^\infty u_s dW_s \tag{1.1}$$

has given the scope of ground-breaking applications of the analysis on Wiener space. The development of Malliavin calculus led not only to the deeper understanding of the Hörmander hypoellipticity conditions for second order operators with degenerate coefficients and regular properties of the associated semigroups, but also to serious advances in the infinite dimensional geometry, operator theory and the stochastic theory itself, see reviews [7, 10, 34, 36] and references therein.

It is a natural question what happens when the differential operator is not degenerate but has some poles in coefficients and how change the techniques of Malliavin calculus in the study of regularity properties of associated semigroups.

The main obstacle is that one faces a problem of singularities, applying the integration by parts formula following from (1.1)

$$\mathbf{E} f'(\xi_t^0)\Psi = \mathbf{E} f(\xi_t^0)\{\frac{\Psi}{D_u\xi_t^0}\int_0^t u_s dW_s - D_u\frac{\Psi}{D_u\xi_t^0}\} \tag{1.2}$$

Indeed, to achieve the raise of smoothness for semigroup $P_t f(x) = \mathbf{E} f(\xi_t^0(x))$, for example $P_t : C_b \to C_b^1,\ t > 0$, using representation

$$\frac{\partial}{\partial x}P_t f(x) = \mathbf{E} f'(\xi_t^0(x))\xi_t^{(1)}(x) = \mathbf{E} f(\xi_t^0)\{\frac{\xi_t^{(1)}}{D_u\xi_t^0}\int_0^t u_s dW_s - D_u\frac{\xi_t^{(1)}}{D_u\xi_t^0}\}$$

in terms of first variation $\xi_t^{(1)}(x) = \partial\xi_t^0(x)/\partial x$ we need good non-explosion estimates on $\xi_t^{(1)}$ and $1/D_u\xi_t^0$. The structure of principal parts of equations on $\xi_t^{(1)}$ and $1/D_u\xi_t$ requires the *global Lipschitz* assumptions on coefficients of equation on $\xi_t^0(x)$ to fulfill simultaneously the one-sided coercitivity condition for both direct and inverse equations, see e.g. [10, 17, 29, 35, 36, 38, 41, 45].

In this paper we show how to avoid such global Lipschitz conditions for important in applications class of differential operators with infinite number of variables, associated with Gibbs measures of unbounded lattice spin systems [12, 15, 16, 18, 42].

In spite of the principal unboundedness of interaction potentials in Gibbs measure we demonstrate that the associated infinite dimensional diffusions admit a choice of directions, when derivative $D_u\xi_t^0$ becomes a deterministic process and non-Lipschitz singularities in the integration by parts disappear.

We have to use the techniques of stochastic evolution equations because the corresponding semigroups fail the strong continuity in spaces of continuously differentiable functions and the Hille-Yosida approach does not work.

Let us discuss the key idea in a simple one-dimensional case. Consider semigroup

$$(P_t f)(x) = \mathbf{E} f(\xi_t^0(x))$$

associated with $I\!R^1$-valued stochastic equation $d\xi_t^0 = dW_t - F'(\xi_t^0)dt$, $\xi_t^0|_{t=0} = x$ for non-Lipschitz drift F of at most polynomial behaviour.

Due to the property of stochastic derivative $D_{\tilde{u}}\xi_t^0(x) = t$ in direction $\tilde{u}_t = 1 + tF'(\xi_t^0(x))$ we obtain an integration by parts formula

$$\mathbf{E} f'(\xi_t^0)\Psi = \frac{1}{t}\,\mathbf{E} f(\xi_t^0)\{\Psi \int_0^t \tilde{u}_s dW_s - D_{\tilde{u}}\Psi\}$$

which does not contain singularities in denominators and permits to work in the domain of *monotonicity* condition on F.

Integrating by parts the representation of derivatives of semigroup in the terms of variations $\xi_t^{(j)}(x) = \partial^{(j)}\xi_t^0(x)/(\partial x)^j$ we have the connection of derivatives of $\partial^{(n)} P_t f$ with the behaviour of initial function f

$$\partial^{(n)} P_t f(x) = \sum_{\ell=1}^{n} \sum_{j_1+\ldots+j_\ell = n} \mathbf{E} f^{(\ell)}(\xi_t^0)\xi^{(j_1)}\ldots\xi^{(j_\ell)} =$$

$$= \sum_{\ell=1}^{n} \sum_{j_1+\ldots+j_\ell = n} \frac{1}{t^\ell}\,\mathbf{E} f(\xi_t^0)\, I\!D^* \underset{\ell}{\ldots} I\!D^*(\xi^{(j_1)}\ldots\xi^{(j_\ell)})$$

where $I\!D^*\Psi = \Psi \int_0^t \tilde{u}_s dW_s - D_{\tilde{u}}\Psi$. To obtain the raise of smoothness

$$\|P_t f\|_{C^{n+m}_{\mathbf{k}}} \le \frac{1}{t^{m/2}} K e^{Mt} \|f\|_{C^n_{\mathbf{k}}}, \quad m \in I\!N$$

in the scales $\{C^m_{\mathbf{k}}\}_{m\in I\!N}$ of continuously differentiable functions we study the quasi-contractive behaviour of derivatives $I\!D^\beta \xi^{(j)} = (D_{\tilde{u}})^\beta \xi^{(j)}$ and, using the nonlinear symmetries of the associated variational equations, prove estimates

$$\rho^{n,m}(t) \le e^{Mt}\rho^{n,m}(0)$$

for nonlinear expressions like

$$\rho^{n,m}(t) = \sum_{j=1}^{n}\sum_{\beta=0}^{m} \mathbf{E}\, p_{j,\beta}(\xi_t^0) \left| \frac{I\!D^\beta \xi_t^{(j)}}{t^\beta} \right|^{m_1/j}$$

The hierarchy of weights $p_{j,\beta}$, connected with non-Lipschitz parameter $\mathbf{k}$ of map F, influences the special hierarchy of topologies in $C^n_{\mathbf{k}}$ scales.

The paper consists of five parts. In Section 2 we give necessary definitions of Malliavin calculus and prove the integration by parts formula for diffusions with essentially non-Lipschitz drift (Theorem 2.3). Section 3 is devoted to the nonlinear quasi-contractive estimate on the directional derivatives (Theorem 3.2).

Section 4 contains the raise of smoothness result in the scale of continuously differentiable functions (Theorem 4.2).

In Section 5 we provide a necessary background. In Subsection 5.1 we verify the smoothness of $\xi_t^0(x)$ as a functional over the Wiener space (Theorem 2.2). Subsection 5.2 is devoted to the correct proof of smooth properties of variations $I\!\!D^\beta \xi^{(j)}$ and their finite order differentiability (Theorem 3.1).

The main stress is put on the development of monotone methods of nonlinear analysis to the setting of Malliavin calculus.

2 Integration by parts without singularities for non-Lipschitz diffusions.

Consider semigroup

$$(P_t f)(x^0) = \mathbf{E} f(\xi^0(t, x^0)) \tag{2.1}$$

associated with solutions to the nonlinear stochastic differential equation

$$\begin{cases} \xi_k^0(t,x^0) = x_k^0 + \int_0^t dW_k - \int_0^t [F(\xi_k^0) + (B\xi^0)_k] ds, \quad k \in \mathbb{Z}^d \\ \\ x^0 \in \ell_2(a), \quad tr\, a = \sum\limits_{k \in \mathbb{Z}^d} a_k = 1, \quad a \in I\!\!P \end{cases} \tag{2.2}$$

The nonlinear diagonal map $F : I\!\!R^{\mathbb{Z}^d} \to I\!\!R^{\mathbb{Z}^d}$ is generated by smooth function $F : I\!\!R^1 \to I\!\!R^1$ such that $\exists\, \mathbf{k} \geq -1\ \forall n \in I\!\!N\ \exists\, C_n\ \forall x, y \in I\!\!R^1$

$$|F^{(i)}(x) - F^{(i)}(y)| \leq C_n |x - y| (1 + |x| + |y|)^{\mathbf{k}} \ i = 0, ..., n \tag{2.3}$$

and linear finite-diagonal map $B : I\!\!R^{\mathbb{Z}^d} \to I\!\!R^{\mathbb{Z}^d}$ is defined by

$$\exists\, r_0 \quad (Bx)_k = \sum\nolimits_{j:\ |j-k| \leq r_0} b(k-j) x_j \tag{2.4}$$

The cylinder Wiener process $W = \{W_k(t)\}_{k \in \mathbb{Z}^d}$ with values in $\ell_2(a)$ is canonically realized on measurable space $(\Omega = C_0([0,T], \ell_2(a)), \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ with canonical filtration $\mathcal{F}_t = \sigma\{W(s) | 0 \leq s \leq t\}$ and cylinder Wiener measure $\mathbf{P}$. Processes $W_k,\ k \in \mathbb{Z}^d$ are independent $I\!\!R^1$-valued Wiener processes and $\mathbf{E}$ denotes the expectation with respect to measure $\mathbf{P}$. The set of all vectors $a = \{a_k\}_{k \in \mathbb{Z}^d}$ such that $\delta_a = \sup_{|k-j|=1} |a_k / a_j| < \infty$ we denote by $I\!\!P$.

It is well known that for initial data $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$ there is a unique strong solution to equation (2.2), i.e. $\ell_2(a)$ continuous $\mathcal{F}_t$ adapted process $\xi^0(t, x^0) \in \mathcal{D}_{\ell_2(a)}(F)$ which fulfills $\mathbf{P}$ a.e. equation (2.2) in $\ell_2(a)$ and for $x^0 \in \ell_2(a)$ the *generalized solution* is obtained as a uniform on $[0,T]$ $\mathbf{P}$ a.e. limit of strong solutions [13, 15, 25, 30, 39]. Moreover $\exists\, M,\ \exists\, K \in \bigcap\limits_{p \geq 1} L^p(\Omega, \mathbf{P})$ such that $\forall\, x^0, y^0 \in \ell_2(a)$ one has $\mathbf{P}$ a.e. estimates on generalized solutions

$$\begin{gathered} \sup_{t \in [0,T]} \|\xi^0(t, x^0)\|_{\ell_2(a)} \leq e^{MT} \|x^0\|_{\ell_2(a)} + K(\omega) \\ \sup_{t \in [0,T]} \|\xi^0(t, x^0) - \xi^0(t, y^0)\|_{\ell_2(a)} \leq e^{MT} \|x^0 - y^0\|_{\ell_2(a)} \end{gathered} \tag{2.5}$$

In [3, Th.2.4] it was shown that semigroup P_t preserves certain spaces $C_\Theta(\ell_2(a))$ of continuously differentiable functions, in particular, for C^∞ cylinder function f of polynomial with derivatives behaviour at infinity the representation of partial derivatives holds:

$$\partial_\tau P_t f(x) = \sum_{\ell=1}^{|\tau|} \sum_{\gamma_1\cup...\cup\gamma_\ell=\tau} \mathbf{E} < f^{(\ell)}(\xi^0), \xi_{\gamma_1}\otimes ... \otimes \xi_{\gamma_\ell} > \tag{2.6}$$

Above we used notations $\tau = \{j_1, ..., j_n\}$, $\partial_\tau = \partial_{j_n}...\partial_{j_1}$, $\partial_j = \dfrac{\partial}{\partial x_j}$

$$< f^{(\ell)}(\xi^0), \xi_{\gamma_1}\otimes ... \otimes \xi_{\gamma_\ell} >= \sum_{j_1,...,j_\ell\in \mathbb{Z}^d} \partial_{j_1,..,j_\ell} f(\xi^0)\xi_{j_1,\gamma_1}...\xi_{j_\ell,\gamma_\ell}$$

and $\xi_{\gamma_i}(t,x) = \partial_{\gamma_i}\xi^0(t,x)$ denotes the unique strong solution to corresponding system in variations for initial equation (2.2)

$$\begin{cases} \xi_{k,\gamma} = \tilde{x}_{k,\gamma} - \int_0^t \{(F'(\xi_k^0) + B)\xi_\gamma\}_k ds - \\ \qquad - \sum\limits_{\alpha_1\cup...\cup\alpha_\ell=\gamma,\ \ell\geq 2} \int_0^t F^{(\ell)}(\xi_k^0)\xi_{k,\alpha_1}...\xi_{k,\alpha_\ell} ds \\ \tilde{x}_{k,\gamma} = \delta_{kj}\ for\ \gamma = \{j\}\ and\ \tilde{x}_{k,\gamma} = 0\ for\ |\gamma| \geq 2 \end{cases} \tag{2.7}$$

In this paper we are going to show that semigroup P_t raises the smoothness of initial function in the scale C_Θ. To control the non-Lipschitz singularities of initial infinite-dimensional SDE we write a special integration by parts formula on Wiener space. The following Definition adopts a notion of differentiable Wiener functionals to our setting.

Denote by $\mathcal{J}_{cyl}$ the set of $\mathcal{F}_t$ adapted continuous *integrable cylinder-valued* processes $u_t = \{u_{t,k}\}_{k\in\mathbb{Z}^d}$, i.e. $\exists\, \Lambda_u \subset \mathbb{Z}^d, |\Lambda_u| < \infty$ such that $\forall\, k \notin \Lambda_u \quad u_{t,k} \equiv 0,\ t\in[0,T]$ and

$$\forall\, k\in\Lambda_u \quad \forall\, p\geq 1 \qquad \mathbf{E}\int_0^T |u_{t,k}|^p dt < \infty \tag{2.8}$$

Definition 2.1. Measurable function G on Ω is *differentiable in direction* $u\in\mathcal{J}_{cyl}$ and has directional derivative D_uG if $\exists\, \varepsilon_0 > 0\ \forall\, |\varepsilon| \leq \varepsilon_0$ function $G(\omega_\bullet + \varepsilon \int_0^\bullet u_s ds)$ belongs to $\cap_{p\geq 1} L^p(\Omega, \mathbf{P})$ and there is a measurable function $D_uG \in \cap_{p\geq 1} L^p(\Omega,\mathbf{P})$ such that

$$\forall\, p\geq 1 \quad \lim_{|\varepsilon|\to 0} \mathbf{E}|\frac{G(\omega_\bullet + \varepsilon\int_0^\bullet u_s ds) - G(\omega)}{\varepsilon} - D_uG(\omega)|^p = 0 \tag{2.9}$$

We say that $G\in\mathcal{D}_{loc}(\Omega)$ iff $\forall\, j\in\mathbb{Z}^d$ there is a map $\mathbf{D}_jG \in \bigcap_{p\geq 1} L^p(\Omega,\mathbf{P},\mathcal{H})$ and $\forall\, u\in\mathcal{J}_{cyl}\ \ \exists\, D_uG$ in the sense above which admits representation

$$D_uG = \sum_{j\in\Lambda_u} < \mathbf{D}_jG, \int_0^\bullet u_{s,j} ds >_{\mathcal{H}} \tag{2.10}$$

Above $\mathcal{H}$ denotes the Cameron-Martin space of absolutely continuous functions $\gamma : [0,T]\to \mathbb{R}^1$, $\gamma(0) = 0$, equipped with the scalar product $<\gamma,\gamma>_{\mathcal{H}} = \int_0^T |\dot{\gamma}(s)|^2 ds.$

Immediately remark that the following properties hold by a slight modification of results [6, 7, 31, 32, 34, 36, 37, 41, 45]

1°. $\forall f \in C^\infty(I\!R^n, I\!R^1)$ of polynomial with all derivatives behaviour at infinity and $\forall G_1, ..., G_n \in \mathcal{D}_{loc}(\Omega)$ we have $f(G_1, ..., G_n) \in \mathcal{D}_{loc}(\Omega)$ and

$$D_u f(G_1, ..., G_n) = \sum_{i=1}^{n} [\partial_i f \circ (G_1, ..., G_n)]\, D_u G_i, \quad u \in \mathcal{J}_{cyl} \tag{2.11}$$

2°. For all real-valued $\mathcal{F}_t$ adapted continuous processes $H_t \in \mathcal{D}_{loc}(\Omega)$, $t \in [0, T]$ such that

$$\mathbf{E} \int_0^T |H_s|^p ds < \infty \ \ and \ \ \forall j \in Z\!\!Z^d \ \mathbf{E} \int_0^T \|\mathbf{D}_j H_s\|^p_{\mathcal{H}} ds < \infty, \quad \forall p \geq 1$$

we have

$$\forall t \in [0, T]\ \forall k \in Z\!\!Z^d \quad \{\int_0^t H_s ds, \ \int_0^t H_s dW_k(s)\} \in \mathcal{D}_{loc}(\Omega)$$

and $\forall u \in \mathcal{J}_{cyl}$

$$D_u \int_0^t H_s ds = \int_0^t D_u H_s ds$$

$$D_u \int_0^t H_s dW_k(s) = \int_0^t H_s u_{s,k} ds + \int_0^t D_u H_s dW_k(s) \tag{2.12}$$

The following Theorem checks Definition 2.1 for solutions to nonlinear equation (2.2).

Theorem 2.2. *$\forall x^0 \in \ell_2(a)$ the generalized solution $\xi^0(t, x^0)$ to (2.2) has coordinates $\xi^0_k(t, x^0) \in \mathcal{D}_{loc}(\Omega)$, $\forall k \in Z\!\!Z^d, t \in [0, T]$.*

Moreover $\forall u \in \mathcal{J}_{cyl}$ the derivatives $D_u \xi^0_k(t, x^0)$, $k \in Z\!\!Z^d$ form a unique strong solution to system

$$D_u \xi^0_k(t, x^0) = \int_0^t u_{s,k} ds - \int_0^t \{[F'(\xi^0_k(s, x^0)) + B]\, D_u \xi^0(s, x^0)\}_k ds, \quad k \in Z\!\!Z^d \tag{2.13}$$

The *strong solution* is understood as $\mathcal{F}_t$ adapted $\ell_2(a)$ continuous process $D_u \xi^0(t, x^0) \in \mathcal{D}_{\ell_2(a)}(F'(\xi^0(t, x^0)) + B)$ a.e. on $[0, T]$ which $\mathbf{P}$ a.e. satisfies equation (2.13) in space $\ell_2(a)$ and $\exists M$ such that

$$\sup_{t \in [0,T]} \|D_u \xi^0(t, x^0)\|^2_{\ell_2(a)} \leq e^{MT} \int_0^T \|u_s\|^2_{\ell_2(a)} ds$$

Proof of Theorem is given in Subsection 5.1. In Lemmas 5.1 and 5.2 we construct processes $\xi^0(t, x^0, \omega_\bullet + \varepsilon \int_0^\bullet u_s ds)$ and $D_u \xi^0(t, x^0)$ as solutions to stochastic equations and state their continuity with respect to $x^0 \in \ell_2(a)$ and $u \in \mathcal{J}_{cyl}$. Lemma 5.3 gives a sense to $D_u \xi^0(t, x^0)$ as derivative in direction $u \in \mathcal{J}_{cyl}$. The final verification of $\xi^0_k \in \mathcal{D}_{loc}(\Omega)$ is done in Lemma 5.4. ■

Denote by $\mathcal{P}^\infty_{cyl}(\ell_2(a))$ the set of C^∞-smooth cylinder functions of polynomial with all derivatives behaviour at infinity, i.e. $\forall f \in \mathcal{P}^\infty_{cyl}(\ell_2(a))\ \exists m_f\ \exists \Lambda = supp_{cyl} f \subset Z\!\!Z^d$, $|\Lambda| < \infty$ and $\exists h \in C^\infty(I\!R^\Lambda)$ such that $\forall x \in \ell_2(a)\ \ f(x) = h(\{x_k, k \in \Lambda\})$ and $\forall \tau \subset Z\!\!Z^d$

$$|\partial_\tau f(x)| \leq const_\tau\, (1 + \sum_{i \in \Lambda} x_i^2)^{m_f} \tag{2.14}$$

The following Theorem gives an integration by parts formula for functionals on Wiener space generated by solutions to the infinite dimensional stochastic equations with non-Lipshitz drift. In Section 4 we apply it to obtain the raise of smoothness under the action of associated not strongly continuous semigroup P_t.

Theorem 2.3. *Let $\xi^0(t,x^0)$ be a generalized solution to (2.2) at $x^0 \in \ell_2(a)$. Introduce process*

$$\Gamma_t v = [Id + t(F'(\xi^0(t,x^0)) + B)]v \in \mathcal{J}_{cyl} \tag{2.15}$$

for vector $v \in I\!R^{\mathbb{Z}^d}$ with finite number of nonzero coordinates.

Then the derivative in direction $u_t = \Gamma_t v$ gives

$$D_{\Gamma v}\xi^0(t,x^0) = tv \tag{2.16}$$

Moreover, the **integration by parts formula** *holds*

$$\mathbf{E} < \partial f(\xi_t^0), v >_{\ell_2(1)} \Psi = \frac{1}{t}\mathbf{E}\, f(\xi_t^0)\{\Psi \int_0^t < \Gamma_s v, dW(s) >_{\ell_2(1)} - D_{\Gamma v}\Psi\} \tag{2.17}$$

for all $\mathcal{F}_t$ measurable $\Psi \in \mathcal{D}_{loc}(\Omega)$, $f \in \mathcal{P}^\infty_{cyl}(\ell_2(a))$, $t > 0$.

Remark. The integration by parts above uses, in fact, the set of derivatives in directions, generated by the monotonicity parameter $F'(x) + B$ of initial operator. This parameter also describes the log-concavity properties of the associated Gibbs measure with local specifications

$$d\mu_\Lambda(x) = \frac{1}{Z_\Lambda}\exp\{-\frac{1}{2}\sum_{\{k,j\}\cap\Lambda\neq\emptyset} b(k-j)x_k x_j\}\prod_{k\in\Lambda} e^{-\Phi(x_k)}dx_k, \quad \Phi(x) = \int_0^x F(y)dy$$

Proof of Theorem 2.3. Properties of generalized solution $\xi^0(t,x^0)$ (Theorem 2.2), polynomiality of F (2.3) and finite radius of B (2.4) give $\Gamma_t v \in \mathcal{J}_{cyl}$ for finite vector $v \in I\!R^{\mathbb{Z}^d}$. By Theorem 2.2 $D_{\Gamma v}\xi_k^0(t,x^0)$ satisfies

$$D_{\Gamma v}\xi_k^0(t,x^0) = \int_0^t \{\Gamma_s v\}_k ds - \int_0^t [\{F'(\xi^0(s,x^0)) + B\}D_{\Gamma v}\xi^0(s,x^0)]_k ds \tag{2.18}$$

Substitution of $D_{\Gamma v}\xi_k^0(t,x^0) = tv_k$ makes equation (2.18) hold identically and due to the uniqueness of solutions we have (2.16).

For $\mathcal{F}_t$ measurable $\Psi \in \mathcal{D}_{loc}(\Omega)$ and $f \in \mathcal{P}^\infty_{cyl}(\ell_2(a))$ by Theorem 2.2 and (2.11) we have $f(\xi^0(t,x^0))\Psi \in \mathcal{D}_{loc}(\Omega)$. Using the integration by parts Theorem 3.1 [37] for projection onto $\prod_{k\in\Lambda_u} C_0([0,T], I\!R^1)$ of product measure $\mathbf{P}$ with subsequent integration with respect to ω_k, $k \notin \Lambda_u$ we achieve by chain rule (2.11)

$$\mathbf{E}f(\xi^0(t,x^0))\Psi\int_0^t < \Gamma_s v, dW(s) >_{\ell_2(1)} =$$

$$= \mathbf{E} \sum_{j\in\Lambda_u} < \mathbf{D}_j\{f(\xi^0(t,x^0))\Psi\}, \int_0^\bullet \{\Gamma_s v\}_j ds >_{\mathcal{H}} = \mathbf{E}\, D_{\Gamma v}[f(\xi^0(t,x^0))\Psi] =$$

$$= t\mathbf{E} < \partial f(\xi^0(t,x^0)), v >_{\ell_2(1)} \Psi + \mathbf{E} f(\xi^0(t,x^0)) D_{\Gamma v}\Psi$$

which gives (2.17). Remark that we used the closure of [37, Th.3.1] from bounded cylinder-valued $\mathcal{F}_t$ adapted continuous processes u_t to any $L^p(\Omega \times [0,T])$ summable processes, $p \geq 1$, which is possible due to the property (2.10) for functionals from $\mathcal{D}_{loc}(\Omega)$. ■

3 Quasi-contractive nonlinear estimate on directional derivatives.

Applying below the integration-by-parts Theorem 2.3 we rewrite the representation of partial derivatives $\partial_\tau P_t f$ (2.6) in the terms of directional derivatives on variations ξ_γ and specify the quasi-contractive nonlinear estimates [3] to this setting. Introduce notation

$$I\!D^k = D_{\Gamma e_k} \tag{3.1}$$

for the directional derivative, generated by Γ_t (2.15) and k^{th} unit vector $e_k = (..., 0, 1_k, 0, ...) \in I\!R^{Z\!\!Z^d}$. Formula (2.17) gives for $\mathcal{F}_t$ measurable $\Psi \in \mathcal{D}_{loc}(\Omega)$

$$\forall\ f \in \mathcal{P}^\infty_{cyl}(\ell_2(a)) \quad \mathbf{E}\,\partial_k f(\xi^0_t)\Psi = \frac{1}{t}\,\mathbf{E} f(\xi^0_t) I\!D^*_k \Psi \tag{3.2}$$

with

$$I\!D^*_k\Psi = \Psi \int_0^t < \Gamma_s e_k, dW(s) >_{\ell_2(1)} - I\!D^k\Psi \tag{3.3}$$

Therefore the partial derivatives of semigroup (2.6) permit representation

$$\partial_\tau P_t f(x) = \sum_{\ell=1}^{|\tau|} \sum_{\gamma_1\cup...\cup\gamma_\ell=\tau} \sum_{j_1,...,j_\ell\in Z\!\!Z^d} \mathbf{E} f(\xi^0_t)\, \frac{I\!D^*_{j_1}...I\!D^*_{j_\ell}(\xi_{j_1,\gamma_1}...\xi_{j_\ell,\gamma_\ell})}{t^\ell} \tag{3.4}$$

and to obtain the raise of smoothness under the action of semigroup P_t we have to investigate the behaviour of derivatives

$$I\!D^\beta \xi_\tau = I\!D^{j_1}...\, I\!D^{j_\ell}\xi_\tau, \qquad \beta = \{j_1, ..., j_\ell\}$$

on solutions ξ_γ to variational equations (2.7).

Theorem 3.1. *$\forall\, x^0 \in \ell_2(a)$ the variations $\xi_\tau(t,x^0)$ (2.7) have coordinates $\forall\, k \in Z\!\!Z^d$ $\xi_{k,\tau}(t,x^0) \in \mathcal{D}_{loc}(\Omega)$ and $\forall\, \beta \subset Z\!\!Z^d$, $|\beta| \geq 1$ $\exists\ I\!D^\beta\xi_{k,\tau} \in \mathcal{D}_{loc}(\Omega)$. They are represented as a strong solutions in the scale of spaces $\ell_{m_\tau}(c_{\tau,\beta})$, $m_\tau = m_1/|\tau|$, to system*

$$\forall\, |\beta| \geq 0 \quad I\!D^\beta\xi_{k,\tau}(t) = \tilde{x}_{k;\tau,\beta} - \int_0^t [(F'(\xi^0) + B) I\!D^\beta\xi_\tau]_k ds - \int_0^t \varphi_{k;\tau,\beta}(s) ds \tag{3.5}$$

where $\tilde{x}_{\tau,\beta} = 0$, $|\beta| \geq 1$, $\tilde{x}_{\tau,\emptyset} = \tilde{x}_\tau$ (2.7),

$$\varphi_{k;\tau,\beta}(t) = \sum_{\substack{\gamma_1\cup...\cup\gamma_\ell=\tau \\ |\gamma_i|\geq 1,\ \ell\geq 1}} \sum_{\substack{\sigma_0\cup...\cup\sigma_\ell=\beta \\ |\sigma_0|\geq 2-\ell,\ |\sigma_i|\geq 0}} t^{|\sigma_0|}\delta_k^{\sigma_0} F^{(\ell+|\sigma_0|)}(\xi^0_k)\, I\!D^{\sigma_1}\xi_{k,\gamma_1}...I\!D^{\sigma_\ell}\xi_{k,\gamma_\ell} \tag{3.6}$$

and vectors $c_{\tau,\beta} \in I\!P$ *are any that fulfill hierarchy*

$$\exists\, K_c \quad \delta_k^{\sigma_0}[c_{k;\,\tau,\beta}]^{|\tau|} a_k^{-\frac{\mathbf{k}_{+1}}{2}m_1} \le K_c [c_{k;\,\gamma_1,\sigma_1}]^{|\gamma_1|} ... [c_{k;\,\gamma_\ell,\sigma_\ell}]^{|\gamma_\ell|}, \quad k \in Z\!\!Z^d \tag{3.7}$$

Above $\delta_j^\beta = \prod_{i\in\beta} \delta_j^i$ *is a product of Kronnecker symbols and the subdivisions of sets* $\tau = \gamma_1 \cup ... \cup \gamma_\ell$, $\beta = \sigma_0 \cup ... \cup \sigma_\ell$ *are such that* $1 \le \ell \le |\tau|$ *and for* $\ell = 1$, $|\sigma_0| \ge 1$; *for* $\ell \ge 2$ $|\sigma_0| \ge 0$.

Moreover, at $t = 0$ *there is asymptotic* $\forall\, R > 0\ \exists\, K_R \in \bigcap_{p\ge 1} L^p(\Omega, \mathbf{P})$ *such that*

$$\forall\, |\beta| \ge 1 \quad \|I\!D^\beta \xi_\tau(t, x^0)\|_{\ell_{m_\tau}(c_{\tau,\beta})} \le t^{|\beta|+1} K_R(\omega), \quad t \in [0, T] \tag{3.8}$$

uniformly on $\max(\|x^0\|_{\ell_2(a)}, \|\tilde{x}_{\gamma,\emptyset}\|_{\ell_{m_\gamma}(c_{\gamma,\emptyset})}, \gamma \subset \tau) \le R$.

The *strong solutions* are understood as $\mathcal{F}_t$ adapted $\ell_{m_\tau}(c_{\tau,\beta})$ continuous finite variation processes

$$[0, T] \ni t \to I\!D^\beta \xi_\tau(t, x^0) \in \mathcal{D}_{\ell_{m_\tau}(c_{\tau,\beta})}(F'(\xi^0(t, x^0)) + B),\ |\beta| \ge 0,\ |\tau| \ge 1$$

which for $\mathbf{P}$ a.e. $\omega \in \Omega$ fulfill equation (3.5) in $\ell_{m_\tau}(c_{\tau,\beta})$ a.e. on $[0, T]$.

Proof of this Theorem is quite complicated and we provide it through the set of Lemmas in Subsection 5.2. There we successively check Definition 2.1 for $I\!D^\beta \xi_\tau(t, x^0)$. In Lemmas 5.6 and 5.7 we construct processes $I\!D^\beta \xi_\tau(t, x^0, \omega_\bullet + \varepsilon \int_0^\bullet u_s ds)$ and $D_u I\!D^\beta \xi_\tau(t, x^0)$ as solutions to nonautonomous stochastic equations and state their continuity on $x^0 \in \ell_2(a)$ and $u \in \mathcal{J}_{cyl}$. Lemma 5.8 gives a sense to $D_u I\!D^\beta \xi_\tau$ as derivative in direction $u \in \mathcal{J}_{cyl}$. In Lemma 5.9 we provide a final verification of $I\!D^\beta \xi_{k,\tau} \in \mathcal{D}_{loc}(\Omega)$ and show that $D_{\Gamma e_j} I\!D^\beta \xi_{k,\tau} = I\!D^{\beta\cup\{j\}} \xi_{k,\tau}$. We also prove asymptotic (3.8). An important tool to deal with the multiplicative structure of $\varphi_{\tau,\beta}$ (3.6) is given in Lemma 5.5. ■

Remark that the equations (3.5) are obtained by direct action of $I\!D^\beta$ on equations (2.7), because (2.16) and chain rule (2.11) for F (2.3) give

$$I\!D^\beta F^{(\ell)}(\xi_j^0(t, x^0)) = \delta_j^\beta t^{|\beta|} F^{(\ell+|\beta|)}(\xi_j^0(t, x^0))$$

We see that the property $I\!D^k \xi_t^0 = t e_k$ (2.16) implies not only the simple integration by parts formula, but also a simplified structure of coefficients in equations (3.5), which depend *exclusively* on t and ξ_t^0.

Taking into account this reason we introduce, like in [2, 3], a nonlinear expression

$$\rho_{\tau,\beta}(t) = \sum_{\gamma\subset\tau,\ \sigma\subset\beta,\ \gamma\ne\emptyset} \mathbf{E} p_{\gamma,\sigma}(z_t) \| \frac{I\!D^\sigma \xi_\gamma}{t^{|\sigma|}} \|_{\ell_{m_\gamma}(c_{\gamma,\sigma})}^{m_\gamma} \tag{3.9}$$

for $m_\gamma = m_1/|\gamma|$, $z_t = \|\xi^0(t, x^0)\|^2_{\ell_2(a)}$. It accumulates the nonlinear symmetries of equation on $I\!D^\beta \xi_\tau$ (3.5), i.e. that the terms

$$F'(\xi^0) I\!D^\beta \xi_\tau,\ t^{|\beta|} F^{(1+|\beta|)}(\xi^0)\xi_\tau,\ t^{|\beta|} F^{(|\tau|+|\beta|)}(\xi^0)\xi_{j_1}...\xi_{j_n}, \quad \tau = \{j_1, ..., j_n\}$$

appear in the r.h.s. of (3.5) simultaneously.

The following Theorem gives a quasi-contractive estimate on $\rho_{\tau,\beta}$. In Section 4 it is applied to control the raise of smoothness.

Theorem 3.2. *Let F fulfill (2.3), $x^0 \in \ell_2(a)$ and vectors $c_{\gamma,\sigma} \in I\!\!P$ satisfy hierarchy (3.7). Suppose that monotone functions $p_{\gamma,\sigma} \in C^2(I\!\!R^1_+)$ are such that $\exists\, \varepsilon > 0\ \exists\, K_p > 0\ \forall\, z \in I\!\!R_+$*

$$p_{\gamma,\sigma}(z) \geq \varepsilon \ \ \textit{and} \ \ (1+z)(|p'_{\gamma,\sigma}(z)| + |p''_{\gamma,\sigma}(z)|) \leq K_p p_{\gamma,\sigma}(z) \tag{3.10}$$

and

$$[p_{\tau,\beta}]^{|\tau|}(1+z)^{\frac{\mathbf{k}+1}{2}m_1} \leq K_p [p_{\gamma_1,\sigma_1}]^{|\gamma_1|} ... [p_{\gamma_\ell,\sigma_\ell}]^{|\gamma_\ell|} \tag{3.11}$$

for any subdivision $\tau = \gamma_1 \cup ... \cup \gamma_\ell$, $\beta = \sigma_0 \cup ... \cup \sigma_\ell$ such that $2 - \ell \leq |\sigma_0|$.

Then there is a constant $M = M_{\tau,\beta} \in I\!\!R^1$ such that the **quasi-contractive nonlinear estimate** *holds*

$$\rho_{\tau,\beta}(t) \leq e^{Mt} \rho_{\tau,\beta}(0) \tag{3.12}$$

The r.h.s. limit $t = 0$ is substituted by Theorem 3.1 (3.8) and (2.7).

Proof. Let $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$. Introduce for $\tau = \{k_1, ..., k_n\}$, $\beta = \{j_1, ..., j_m\}$ function

$$h_i(t) = \sum_{\ell=0}^{i} \sum_{\substack{\sigma \subset \beta \\ |\sigma| = \ell}} \sum_{\substack{\gamma \subset \tau \\ \gamma \neq \emptyset}} \mathbf{E} p_{\gamma,\sigma}(z_t) \| \frac{I\!\!D^\sigma \xi_\gamma}{t^{|\sigma|}} \|^{m_\gamma}_{\ell_{m_\gamma}(c_{\gamma,\sigma})}$$

We will show by induction that $\forall\, i \in \{0, ..., m\}$

$$h_i(t) \leq e^{M_i t} h_i(0) \tag{3.13}$$

which at $i = m$, $h_m = \rho_{\tau,\beta}(t)$ gives the statement of Theorem. Base of induction at $i = 0$, i.e. when $\beta = \emptyset$, was proved in [3, Th.2.2]. Note that

$$h_i(t) = h_{i-1}(t) + \sum_{\sigma \subset \beta,\ |\sigma| = i} g_\sigma(t) \tag{3.14}$$

where

$$g_\sigma(t) = \sum_{\gamma \subset \tau, \gamma \neq \emptyset} \mathbf{E} p_{\gamma,\sigma}(z_t) \| \frac{I\!\!D^\sigma \xi_\gamma}{t^{|\sigma|}} \|^{m_\gamma}_{\ell_{m_\gamma}(c_{\gamma,\sigma})}$$

Therefore to obtain (3.13) it is sufficient to prove that for any $\sigma \subset \beta$, $|\sigma| = i$

$$g_\sigma(t) \leq e^{K_1 t} g_\sigma(0) + K_2 \int_0^t e^{K_1(t-s)} h_{i-1}(s) ds \tag{3.15}$$

Then due to inductive assumption and representation (3.14) we have

$$h_i(t) \leq e^{M_{i-1} t} h_{i-1}(0) +$$

$$+ \sum_{\sigma \subset \beta;\, |\sigma| = i} \{ e^{K_1 t} g_\sigma(0) + K_2 h_{i-1}(0) \int_0^t e^{K_1(t-s)} e^{M_{i-1} s} ds \} \leq$$

$$\leq e^{(M_{i-1} + K_1)t} \{1 + K_2 t\} h_i(0) \leq e^{(M_{i-1} + K_1 + K_2)t} h_i(0)$$

It remains to show (3.15). Introduce notations $X_{\gamma,\sigma} = \ell_{m_\gamma}(c_{\gamma,\sigma}), \eta_t = \dfrac{I\!D^\sigma \xi_\gamma}{t^{|\sigma|}}$, where $I\!D^\sigma\xi_\gamma$ satisfies equation (3.5). Due to (3.8) $\|\eta_0\| = 0$. Ito formula for $z_t = \|\xi_t^0\|^2_{\ell_2(a)}$ gives

$$p_{\gamma,\sigma}(z_t)\|\eta_t\|^{m_\gamma}_{X_{\gamma,\sigma}} = p_{\gamma,\sigma}(z_0)\|\eta_0\|^{m_\gamma}_{X_{\gamma,\sigma}} + 2\int_0^t p'_{\gamma,\sigma}(z_s)\|\eta_s\|^{m_\gamma}_{X_{\gamma,\sigma}}(\xi^0(s), dW(s))_{\ell_2(a)} +$$

$$+ m_\gamma \int_0^t \{p_{\gamma,\sigma}(z_s) < \frac{d\eta_s}{ds}, \eta_s^\# >_{X\gamma,\sigma} - \|\eta_s\|^{m_\gamma}_{X_{\gamma,\sigma}}[H_\mu p_{\gamma,\sigma}](z_s)\}ds =$$

$$= p_{\gamma,\sigma}(z_0)\|\eta_0\|^{m_\gamma}_{X_{\gamma,\sigma}} + 2\int_0^t p'_{\gamma,\sigma}(z_s)\|\eta_s\|^{m_\gamma}_{X_{\gamma,\sigma}}(\xi^0(s), dW(s))_{\ell_2(a)} +$$

$$+ m_\gamma \int_0^t p_{\gamma,\sigma}(z_s)\{\frac{1}{s^{|\sigma|}} < \frac{d}{ds} I\!D^\sigma \xi_\gamma, \eta^\# > - \frac{|\sigma|}{s}\|\eta_s\|^{m_\gamma}_{X_{\gamma,\sigma}}\}ds - \tag{3.16}$$

$$- m_\gamma \int_0^t \|\eta_s\|^{m_\gamma}_{X_{\gamma,\sigma}}(H_\mu p_{\gamma,\sigma})(z_s)ds$$

Here $(x, y)_{\ell_2(a)} = \sum_{k\in Z\!\!\!Z^d} a_k x_k y_k$,

$$< u, v^\# >_{\ell_m(c)} = \sum_{k\in Z\!\!\!Z^d} c_k u_k v_k |v_k|^{m-2} \tag{3.17}$$

for $v^\# = \|v\|^{m-2}_{\ell_m(c)}\mathcal{F}v$ with duality map $\mathcal{F}$ in $\ell_m(c)$ and operator H_μ acts on cylinder test functions by rule $(H_\mu f)(\bullet) = \sum_{k\in Z\!\!\!Z^d}\{-\frac{1}{2}\partial_k^2 + \beta_k\partial_k\}f(\bullet)$ with $\beta_k = F(x_k) + \sum_{j\in Z\!\!\!Z^d} b(k-j)x_j$.

Estimates (2.5), [3, (3.61)], (3.8) and inequality $|H_\mu p|(z_t) \le Cp(z_t)[M + K\|\xi^0(t)\|^{2(\mathbf{k}+1)}_{\ell_{2(\mathbf{k}+1)}(a)}]$ guarantee the integrability on $[0, T]\times\Omega$ of all expressions in Ito formula. Thus we have

$$\mathbf{E}p_{\gamma,\sigma}(z_t)\|\eta_t\|^{m_\gamma}_{X_{\gamma,\sigma}} = \mathbf{E}p_{\gamma,\sigma}(z_0)\|\eta_0\|^{m_\gamma}_{X_{\gamma,\sigma}} - m_\gamma\int_0^t \mathbf{E}\|\eta_s\|^{m_\gamma}_{X_{\gamma,\sigma}}[H_\mu p_{\gamma,\sigma}](z_s)ds +$$

$$+ m_\gamma\int_0^t \mathbf{E}p_{\gamma,\sigma}(z_s)[\frac{1}{s^{|\sigma|}} < \frac{d}{ds} I\!D^\sigma\xi_\gamma, \eta^\# > - \frac{|\sigma|}{s}\|\eta_s\|^{m_\gamma}_{X_{\gamma,\sigma}}]ds$$

Due to inequality $H_\mu p_{\gamma,\sigma}(z) \ge K_{\gamma,\sigma}p_{\gamma,\sigma}(z),\ z \in I\!R_+$ (see [1, Hint 9]) and property $-1/s \le 0$ we have

$$g_\sigma(t) \le g_\sigma(0) + K'_\sigma\int_0^t g_\sigma(s)ds +$$

$$+ \sum_{\gamma\subset\tau,\,\gamma\ne\emptyset} m_\gamma \int_0^t \mathbf{E}p_{\gamma,\sigma}(z_s)\frac{1}{s^{|\sigma|}} < \frac{d}{ds} I\!D^\sigma\xi_\gamma, \eta^\# > ds \tag{3.18}$$

It remains to estimate (3.18). The process $I\!D^\sigma\xi_\gamma$ satisfies equation (3.5) therefore

$$(3.18) = -\sum_{\gamma\subset\tau,\,\gamma\ne\emptyset} m_\gamma\int_0^t \mathbf{E}p_{\gamma,\sigma}(z_s) < (F' + B)\frac{I\!D^\sigma\xi_\gamma}{s^{|\sigma|}}, \eta_s^\# >_{X_{\gamma,\sigma}} -$$

$$- \sum_{\gamma\subset\tau,\,\gamma\neq\emptyset} m_\gamma \int_0^t \mathbf{E} p_{\gamma,\sigma}(z_s) < \frac{\varphi_{\gamma,\sigma}}{s^{|\sigma|}}, \eta^{\#} >_{X_{\gamma,\sigma}} \leq$$

$$\leq (K_{\tau,\sigma} + K'_{\tau,\sigma}) \int_0^t g_\sigma(s)ds + \sum_{\gamma\subset\tau,\,\gamma\neq\emptyset} \sum_{\substack{\alpha_1\cup...\cup\alpha_\ell=\gamma \\ |\alpha_i|\geq 1,\ \ell\geq 1}} \sum_{\substack{\pi_0\cup...\cup\pi_\ell=\sigma \\ |\pi_0|\geq 2-\ell,\ |\pi_i|\geq 0}}$$

$$\int_0^t \mathbf{E} p_{\gamma,\sigma}(z_s) \| \frac{s^{|\pi_0|}\delta^{\pi_0} F^{(\ell+|\pi_0|)}(\xi^0) I\!\!D^{\pi_1}\xi_{\alpha_1}...I\!\!D^{\pi_\ell}\xi_{\alpha_\ell}}{s^{|\sigma|}} \|^{m_\gamma}_{\ell_{m_\gamma}(c_{\gamma,\sigma})} ds \tag{3.19}$$

with $K_{\tau,\sigma} = \sum_{\gamma\subset\tau} m_\tau \|B\|_{\mathcal{L}(X_{\gamma,\sigma})}$. Above we used that $F'(z)\geq 0$, $z\in I\!\!R^1$, representation (3.6) of $\varphi_{\gamma,\sigma}$ and inequality

$$| < \psi, \eta^{\#} >_{\ell_m(c)} | \leq \frac{1}{m}\|\psi\|^m_{\ell_m(c)} + \frac{m-1}{m}\|\eta\|^m_{\ell_m(c)} \tag{3.20}$$

Due to the property $|\sigma| - |\pi_0| = |\pi_1| + ... + |\pi_\ell|$ we obtain

$$s^{|\pi_0|}\frac{I\!\!D^{\pi_1}\xi_{\alpha_1}...I\!\!D^{\pi_\ell}\xi_{\alpha_\ell}}{s^{|\sigma|}} = \frac{I\!\!D^{\pi_1}\xi_{\alpha_1}}{s^{|\pi_1|}}...\frac{I\!\!D^{\pi_\ell}\xi_{\alpha_\ell}}{s^{|\pi_\ell|}} \tag{3.21}$$

Moreover, by (2.3)

$$|F^{(\ell+|\pi_0|)}(x^0_k)| \leq C(1+|x^0_k|)^{\mathbf{k}+1} \leq C a_k^{-\frac{\mathbf{k}+1}{2}}(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}} \tag{3.22}$$

Using (3.21), (3.22) and hierarchies (3.7), (3.10) we have estimate on each term in (3.19)

$$\mathbf{E} p_{\gamma,\sigma}(z_s) \sum_{k\in Z\!\!\!Z^d} c_{k;\gamma,\sigma}\delta_k^{\pi_0} |F^{(\ell+|\pi_0|)}(\xi^0_k)\frac{I\!\!D^{\pi_1}\xi_{k,\alpha_1}}{s^{|\pi_1|}}...\frac{I\!\!D^{\pi_\ell}\xi_{k,\alpha_\ell}}{s^{|\pi_\ell|}}|^{m_\gamma} \leq$$

$$\leq C^{m_\gamma} K_c^{1/m_\gamma} K_p^{1/m_\gamma} \mathbf{E} \sum_{k\in Z\!\!\!Z^d} \prod_{j=1}^{\ell} \{p_{\alpha_j,\pi_j}(z_t) c_{k;\alpha_j,\pi_j} |\frac{I\!\!D^{\pi_j}\xi_{k,\alpha_j}}{s^{|\pi_j|}}|^{m_{\alpha_j}}\}^{|\alpha_j|/|\gamma|} \leq$$

$$\leq K_2 \mathbf{E} \sum_{j=1}^{\ell} \frac{|\alpha_j|}{|\gamma|} p_{\alpha_j,\pi_j}(z_s) \|\frac{I\!\!D^{\pi_j}\xi_{\alpha_j}}{s^{|\pi_j|}}\|^{m_{\alpha_j}}_{X_{\alpha_j,\pi_j}} \tag{3.23}$$

Above we applied Hölder inequality $|x_1...x_\ell| \leq \frac{|x_1|^{q_1}}{q_1} + ... + \frac{|x_\ell|^{q_\ell}}{q_\ell}$ with $q_j = \frac{|\gamma|}{|\alpha_j|}$.

Now, if $\ell = 1$ therefore $\pi_0 \neq \emptyset$ and for subdivision $\sigma = \pi_0 \cup \pi_1$ we have $|\pi_0|,\ |\pi_1| \leq |\sigma| - 1$ and for (3.23) the inductive assumption (3.15) works.

If $\ell = 2$ then even for $\pi_0 = \emptyset$ there exists at least two subsets $\pi_1 \cup \pi_2 = \sigma$ for which $|\pi_j| \leq |\sigma| - 1$ and inductive assumption is again applicable. We obtain (3.23) $\leq K_2\, h_{i-1}(t)$ which by (3.18) and (3.19) gives

$$g_\sigma(t) \leq g_\sigma(0) + K_1\int_0^t g_\sigma(s)ds + K_2\int_0^t h_{i-1}(s)ds$$

This implies (3.15) with $K_1 = K'_\sigma + K_{\tau,\sigma} + K'_{\tau,\sigma}$ and finishes the proof of nonlinear estimate (3.12) for $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$. The closure up to $x^0 \in \ell_2(a)$ is simple by Lemma 5.1 (5.2) and 5.6 (5.26) with $u^1 = u^2 = 0$. ■

4 Smoothing properties of semigroups.

Using the quasi-contractive properties of derivatives $I\!D^\beta \xi_\gamma$ governed by Γ_t (2.15) and the integration-by-parts Theorem 2.3 we show that semigroup P_t (2.1) raises the smoothness in certain scale of Banach spaces $C_\Theta(\ell_2(a))$ of continuously differentiable functions which essentially depend on the non-Lipschitz parameter $\mathbf{k}$ (2.3).

The following Proposition displays the precise behaviour of directional derivatives $I\!D^\beta \xi_{k,\tau}(t,x^0)$ (3.5) with respect to the lattice $Z\!\!\!Z^d$ and points $x^0 \in \ell_2(a)$. We apply here the nonlinear estimate (3.12).

Proposition 4.1. *Let F fulfill (2.3), $\psi \in I\!P$ and function $Q \in C^2(I\!R^1)$ satisfy (3.10). Then $\forall\, n \in I\!N$ $\exists\, M = M_n(\psi, Q)$ such that the estimate holds*

$$\forall\, t \in [0,T] \quad \mathbf{E}\, Q(\|\xi^0(t,x^0)\|^2_{\ell_2(a)}) |I\!D^\beta \xi_{k,\tau}|^{m_\tau} \le$$

$$\le \frac{t^{|\beta| m_\tau} e^{Mt} |\tau| \psi_0 Q(\|x^0\|^2_{\ell_2(a)})(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2} m_\tau (|\tau|+|\beta|-1)}}{a_k^{\frac{\mathbf{k}+1}{2} m_\tau (|\tau|-1)} \prod_{i\in\beta} a_i^{\frac{\mathbf{k}+1}{2} m_\tau} \prod_{j\in\tau} \psi_{k-j}^{m_1/|\tau|}} \tag{4.1}$$

for all $1 \le m_1 \le n$, $\{k,\tau,\beta\} \subset Z\!\!\!Z^d$, $|\tau| \le m_1$, $|\beta| \le n$ and $m_\tau = m_1/|\tau|$.

Proof. For fixed τ, β introduce

$$\tilde{p}_{\gamma,\sigma}(z) = Q(z)(1+z)^{\frac{\mathbf{k}+1}{2} m_1 (\frac{|\beta|-1}{|\tau|} - \frac{|\sigma|-1}{|\gamma|})}, \quad \gamma \subset \tau,\ \sigma \subset \beta \tag{4.2}$$

$$\tilde{c}_{k;\gamma,\sigma} = (\prod_{i\in\sigma} a_i^{\frac{\mathbf{k}+1}{2} \frac{m_1}{|\gamma|}}) a_k^{\frac{\mathbf{k}+1}{2} m_1 \frac{|\gamma|-1}{|\gamma|}} \prod_{j\in\gamma} \psi_{k-j}^{m_1/|\gamma|}, \quad \psi \in I\!P \tag{4.3}$$

where function $Q \in C^\infty(I\!R^1)$ fulfills (3.10). These weights satisfy conditions (3.7), (3.11) with uniform constants $K_{\tilde{c}} = K_{\tilde{p}} = 1$. Indeed, for $\gamma = \alpha_1 \cup ... \cup \alpha_\ell$ and $\sigma = \pi_0 \cup ... \cup \pi_\ell$

$$(1+z)^{\frac{\mathbf{k}+1}{2} m_1 (\frac{|\beta|-1}{|\tau|} - \frac{|\sigma|-1}{|\gamma|})|\gamma|} (1+z)^{\frac{\mathbf{k}+1}{2} m_1} \le (1+z)^{\frac{\mathbf{k}+1}{2} m_1 \sum_{j=1}^{\ell} (\frac{|\beta|-1}{|\tau|} - \frac{|\pi_j|-1}{|\alpha_j|})|\alpha_j|}$$

or $2-\ell \le |\pi_0|$ which holds by assumptions on hierarchy (3.11).

Substituting $\tilde{c}_{\gamma,\sigma}$ in (3.7) we have

$$\delta_k^{\pi_0} (\prod_{i\in\sigma} a_i^{\frac{\mathbf{k}+1}{2} m_1}) a_k^{\frac{\mathbf{k}+1}{2} m_1 (|\gamma|-1)} \prod_{j\in\gamma} \psi_{k-j}^{m_1} \bullet a_k^{-\frac{\mathbf{k}+1}{2} m_1} \le$$

$$\le \prod_{q=1}^{\ell} \{ (\prod_{i\in\pi_q} a_i^{\frac{\mathbf{k}+1}{2} m_1}) a_k^{\frac{\mathbf{k}+1}{2} m_1 (|\alpha_q|-1)} \prod_{j\in\alpha_q} \psi_{k-j}^{m_1} \}$$

or equivalently, because $|\gamma| = |\alpha_1| + ... + |\alpha_\ell|$,

$$\delta_k^{\pi_0} \prod_{j\in\pi_0} a_j^{\frac{\mathbf{k}+1}{2} m_1} \bullet a_k^{\frac{\mathbf{k}+1}{2} m_1 (\ell-2)} \le 1 \tag{4.4}$$

Due to $a_k \le tr\, a = 1$, inequality (4.4) is obvious for $\ell \ge 2$. For $\ell = 1$ the set $\pi_0 \ne \emptyset$ and rewriting

$$\delta_k^{\pi_0} \prod_{j\in\pi_0} a_j^{\frac{\mathbf{k}+1}{2} m_1} = \begin{cases} a_k^{\frac{\mathbf{k}+1}{2} m_1} \delta_k^{\pi_0\setminus\{k\}} \prod\limits_{j\in\pi_0\setminus\{k\}} a_j^{\frac{\mathbf{k}+1}{2} m_1}, & k\in\pi_0 \\ 0, \quad k\notin\pi_0 \end{cases}$$

the condition (4.4) is also satisfied.

Using that for $\gamma=\tau,\ \sigma=\beta$ weight $\widetilde{p}_{\tau,\beta}(z) = Q(z)$ and for $|\gamma| = 1,\ \sigma=\emptyset$ $\widetilde{p}_{\gamma,\emptyset}(z) = Q(z)(1+z)^{\frac{\mathbf{k}+1}{2} m_\tau(|\tau|+|\beta|-1)}$ we apply Theorem 3.2 and have

$$\mathbf{E}\, Q(\|\xi^0(t,x^0)\|^2_{\ell_2(a)}) \|\frac{I\!D^\beta \xi_\tau}{t^{|\beta|}}\|^{m_\tau}_{\ell_{m_\tau}(\widetilde{c}_{\tau,\beta})} \le \rho_{\tau,\beta}(t) \le e^{Mt}\rho_{\tau,\beta}(0) =$$

$$= |\tau|\psi_0 e^{Mt} Q(\|x^0\|^2_{\ell_2(a)})(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2} m_\tau(|\tau|+|\beta|-1)} \tag{4.5}$$

where we used (3.8) and (2.7) with $\|\tilde{x}_j\|_{\ell_{m_1}(c_{\{j\},\emptyset})} = \psi_0$. The coordinate form of (4.5) gives (4.1). Constant M is uniform with respect to $|\tau|, |\beta| \le n$ due to $K_{\widetilde{p}} = K_{\widetilde{c}} = 1$ and uniformity of $K_{\gamma,\alpha}$ and $\|B\|_{\mathcal{L}(\ell_{m_\tau}(c_{\tau,\beta}))}$ [3, (3.24)]. ■

Proposition just proved shows that for $I\!D^\beta\xi_\gamma$ there is a *certain ordering of behaviour* with respect to $\beta,\gamma \subset Z\!\!\!Z^d$, generated by weights

$$(\,(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}}, \{a_k^{-\frac{\mathbf{k}+1}{2}}\}_{k\in Z\!\!\!Z^d})$$

Due to the representation (3.4) it influences corresponding relations between different order derivatives of semigroup P_t and therefore requires a reduction of weights in seminorms on partial derivatives $\partial_\tau P_t f$ in C_Θ.

Let $\Theta = \Theta_0 \cup ... \cup \Theta_n$ denote any array of pairs $(p,\mathcal{G}) \in \Theta_i$ with i-tensor $\mathcal{G} = G^1 \otimes ... \otimes G^i$, constructed by vectors $G^1, ..., G^i \in I\!P$, and monotone functions $p \in C^2(I\!R^1_+)$ with property (3.10). Array Θ_0 should consist of pairs $(p,\emptyset)$ with empty tensor $\emptyset$ such that $\emptyset \otimes G = G \otimes \emptyset = G,\ G \in I\!P$. The array $\Theta = \Theta_0 \cup ... \cup \Theta_n,\ n \in I\!N$, is *quasi-contractive with parameter* $\mathbf{k}$ iff $\forall\ (p, \mathcal{G} = G^1 \otimes ... \otimes G^i) \in \Theta_i,\ i = 2, ..., n,\ \forall\, k,j \in \{1,...,i\},\ k\ne j$, there is a pair $(\tilde{p}, \widetilde{\mathcal{G}} = \widetilde{G}^1 \otimes ... \otimes \widetilde{G}^{i-1}) \in \Theta_{i-1}$ such that $\exists\, K>0\ \forall\, z \in I\!R_+$

$$(1+z)^{\frac{\mathbf{k}+1}{2}} \tilde{p}(z) \le Kp(z) \quad and \quad (\widehat{\mathcal{G}}^{\{k,j\}})^\ell \le K\widetilde{\mathcal{G}}^\ell, \quad \ell = 1,...,i-1 \tag{4.6}$$

Here $(i-1)$-tensor $\widehat{\mathcal{G}}^{\{k,j\}}$ is constructed from i-tensor $\mathcal{G} = G^1\otimes ... \otimes G^i$ by the rule $\widehat{\mathcal{G}}^{\{k,j\}} = G^1 \otimes ... \otimes G^{k-1} \otimes G^{k+1} \otimes ... \otimes G^{j-1} \otimes A^{-(\mathbf{k}+1)} G^k G^j \otimes G^{j+1} \otimes ... \otimes G^i$ with $A^{-(\mathbf{k}+1)} = \{a_k^{-(\mathbf{k}+1)}\}_{k\in Z\!\!\!Z^d}$.

We say that $f \in C_\Theta(\ell_2(a)),\ \Theta = \Theta_0 \cup ... \cup \Theta_n$ iff $f \in C(\ell_2(a))$ and $\forall\, \tau = \{k_1, ..., k_i\},\ |\tau| \le n$ there are partial derivatives $\partial_\tau f = \partial_{k_1} ... \partial_{k_i} f \in C(\ell_2(a))$ such that the following norm is finite

$$\|f\|_\Theta = \max_{i=0,...,n} |||\partial^{(i)} f|||_{\Theta_i} \tag{4.7}$$

where

$$|||\partial^{(i)} f|||_{\Theta_i} = \sup_{x\in\ell_2(a)} \max_{(p,\mathcal{G})\in\Theta_i} \frac{\|\partial^{(i)} f(x)\|_{\mathcal{G}}}{p(\|x\|^2_{\ell_2(a)})} \tag{4.8}$$

and

$$\|\partial^{(i)} f(x)\|^2_{\mathcal{G}} = \sum_{\tau=\{k_1,...,k_i\}\in\mathbb{Z}^d} G^1_{k_1}...G^i_{k_i} |\partial_\tau f(x)|^2$$

Above partial derivatives $\partial_\tau f$ are understood in the sense that $\forall\, x^0 \in \ell_2(a)\; \forall\, h \in \mathbf{X}_\infty([a,b]) = \bigcap_{p\geq 1,\, c\in\mathbb{P}} AC_\infty([a,b], \ell_p(c))$ representations hold

$$\forall\, |\tau| = 0, ..., n-1 \quad \partial_\tau f(x^0 + h(\bullet))\Big|_a^b = \int_a^b \sum_{k\in\mathbb{Z}^d} \partial_{\tau\cup\{k\}} f(x^0 + h(s)) h'_k(s) ds \tag{4.9}$$

where we used notation $AC_\infty([a,b], X) = \{h \in C([a,b], X);\ \exists\, h' \in L^\infty([a,b], X)\}$ for Banach space X.

In [3, Th.2.4] it was shown that the semigroup P_t (2.1) preserves spaces $C_\Theta(\ell_2(a))$ and fulfills estimate $\exists\, M_\Theta, K_\Theta$: $\forall\, f \in C_\Theta(\ell_2(a))$

$$\|P_t f\|_{C_\Theta} \leq K_\Theta e^{M_\Theta t} \|f\|_{C_\Theta} \tag{4.10}$$

if the array Θ is quasi-contractive with parameter $\mathbf{k}$ (2.3). In particular the partial derivatives of semigroup P_t fulfill representation (2.6) for all $f \in C_\Theta$.

Introduce

$$T_{\mathbf{k}}\Theta = \{(\,(1+z^2)^{\frac{\mathbf{k}+1}{2}} p(z),\ sym(\mathcal{G} \otimes A^{\mathbf{k}+2})\,);\ (p,\mathcal{G}) \in \Theta\} \tag{4.11}$$

and denote

$$(\Theta)^m = \bigcup_{i=0}^{m} T^i_{\mathbf{k}}\Theta, \quad (\Theta)^0 = \Theta$$

Remark that for quasi-contractive with parameter $\mathbf{k}$ array Θ the array $(\Theta)^m$ is also quasi-contractive. This follows from $(\Theta)^i = (\Theta)^{i-1} \cup T_{\mathbf{k}}(\Theta)^{i-1}$ and ordering (4.6).

Next Theorem gives the raise of smoothness in scale C_Θ under the action of semigroup P_t. Denote by $\mathcal{D}_\Theta$ the closure in C_Θ of $f \in \mathcal{P}^\infty_{cyl}(\ell_2(a))$ such that $\|f\|_{C_\Theta} < \infty$.

Theorem 4.2. *Let Θ be quasi-contractive array with parameter $\mathbf{k}$ (2.3). Then $\forall\, m \geq 1\ \exists\, K_{\Theta,m}, M_{\Theta,m}$ such that $\forall\, f \in \mathcal{D}_\Theta$ we have $P_t f \in C_{(\Theta)^m}$, $t > 0$ and*

$$\|P_t f\|_{C_{(\Theta)^m}} \leq \frac{1}{t^{m/2}} K_{\Theta,m} e^{M_{\Theta,m} t} \|f\|_{C_\Theta}, \quad t > 0 \tag{4.12}$$

Proof. Let $\Theta = \Theta_0 \cup ... \cup \Theta_n$ be a quasi-contractive array. Consider $f \in \mathcal{P}^\infty_{cyl}(\ell_2(a))$ such that $\|f\|_{C_\Theta} < \infty$. Due to cylindricity and property (2.14) the norms are finite $\|f\|_{C_{(\Theta)^m}} < \infty$, $m \geq 1$. By [3, Th.2.4] $P_t f \in C_{(\Theta)^m}$ and $\|P_t f\|_{C_{(\Theta)^m}} < \infty$.

I. At first we show $\forall\, f \in \mathcal{P}^\infty_{cyl}(\ell_2(a))$

$$\forall\, t > 0 \quad \|P_t f\|_{C_{(\Theta)^1}} \leq \frac{1}{\sqrt{t}} K e^{Mt} \|f\|_{C_\Theta} \tag{4.13}$$

Definition (4.7) implies

$$\|P_t f\|_{C_{(\Theta)^1}} = \max(\, \|P_t f\|_{C_\Theta}, \max_{i=0,\dots,n} |||\partial^{(i+1)} P_t f|||_{T_{\mathbf{k}}\Theta_i}\,)$$

and due to (4.10) estimate (4.13) will simply follow if we show that $\forall\, i \in \{0, ..., n\}$ $\forall\, t > 0$

$$|||\partial^{(i+1)} P_t f|||_{T_{\mathbf{k}}\Theta_i} \le \frac{1}{\sqrt{t}} K e^{Mt} \|f\|_{C_\Theta} \tag{4.14}$$

Now we prove inductively that the next estimate: $\forall\, i = 0, ..., n$ $\forall\, \Theta_i \in \Theta$

$$|||\partial^{(i+1)} P_t f|||_{T_{\mathbf{k}}\Theta_i} \le K e^{Mt} \{ \frac{1}{\sqrt{t}} |||\partial^{(i)} f|||_{\Theta_i} + \max_{\ell=1,\dots,i} (|||\partial^{(\ell)} f|||_{\Theta_\ell}, |||\partial^{(\ell)} f|||_{T_{\mathbf{k}}\Theta_{\ell-1}}) \tag{4.15}$$

implies (4.14). Inductive base for (4.14) holds by (4.15) at $i = 0$. Let for $i \le i_0$ estimate (4.14) be fulfilled. Due to the $P_t f = P_{t/2} P_{t/2} f$ and (4.15) we have

$$|||\partial^{(i_0+2)} P_t f|||_{T_{\mathbf{k}}\Theta_{i_0+1}} = |||\partial^{(i_0+2)} P_{t/2} P_{t/2} f|||_{T_{\mathbf{k}}\Theta_{i_0+1}} \le$$

$$\le K e^{Mt/2} \{ \frac{\sqrt{2}}{\sqrt{t}} |||\partial^{(i_0+1)} P_{t/2} f|||_{\Theta_{i_0+1}} +$$

$$+ \max_{\ell=1,\dots,i_0+1} (|||\partial^{(\ell)} P_{t/2} f|||_{\Theta_\ell}, |||\partial^{(\ell)} P_{t/2} f|||_{T_{\mathbf{k}}\Theta_{\ell-1}}) \} \le \frac{K' e^{M' t}}{\sqrt{t}} \|f\|_{C_\Theta}$$

Above we used (4.10), inductive assumption (4.14) and the structure of seminorms in space C_Θ. Therefore (4.14) and (4.13) are proved.

To obtain (4.12) we represent $P_t f = P_{t/m} ... P_{t/m} f$ for $f \in \mathcal{P}^\infty_{cyl}(\ell_2(a))$ and apply m times (4.13).

Consider a sequence of functions $\{g_n\} \in \mathcal{P}^\infty_{cyl}(\ell_2(a))$ such that $\|g_n\|_{C_\Theta} < \infty$ and $g_n \to g^* \in \mathcal{D}_\Theta$, $n \to \infty$ in C_Θ, $\Theta = \Theta_0 \cup ... \cup \Theta_n$. By [3, Th.2.4] and property $\|g_m\|_{C_{(\Theta)^\ell}} < \infty$, $\ell \ge 1$, we have that semigroup $P_t g_m \in C_{(\Theta)^\ell}$, $\ell \ge 1$, and possesses integral representations (4.9) up to order $n + \ell$. Substituting $f = g_m - g_n$ in (4.12) we have

$$\|P_t g_m - P_t g_n\|_{C_{(\Theta)^\ell}} \le \frac{1}{t^{\ell/2}} K_{\Theta,\ell}\, e^{M_{\Theta,\ell} t} \|g_m - g_n\|_{C_\Theta} \tag{4.16}$$

Estimate (4.16) implies the uniform on balls on $x^0 \in \ell_2(a)$ convergence of partial derivatives $\partial_\tau P_t g_m$ to some continuous functions on $\ell_2(a)$. They fulfill the representations (4.9) up to $(n+\ell)^{th}$ order and give the partial derivatives of $P_t g^*$ in sense (4.10). This finishes the proof of Theorem and gives that $\forall\, g^* \in \mathcal{D}_\Theta$ $\;P_t g^* \in C_{(\Theta)^\ell}$, $\ell \ge 1$ and fulfills an estimate (4.12).

II. *It remains to show (4.15).* Due to integration by parts formula (3.2) and representation (2.6) for partial derivatives $\{\partial^{(i+1)} P_t f\}_{k_1,\dots,k_{i+1}} = \partial_{k_1} ... \partial_{k_{i+1}} P_t f$ of semigroup P_t we have

$$\partial_{k_{i+1}} ... \partial_{k_1} P_t f = \sum_{\ell=1}^{i} \sum_{\gamma_1 \cup ... \cup \gamma_\ell = \{k_1,\dots,k_{i+1}\}} \mathbf{E} < \partial^{(\ell)} f(\xi^0), \xi_{\gamma_1} \otimes ... \otimes \xi_{\gamma_\ell} > + \tag{4.17}$$

$$+ \frac{1}{t} \sum_{j_1,\dots,j_{i+1} \in \mathbb{Z}^d} \mathbf{E}\, \partial_{j_i} ... \partial_{j_1} f(\xi^0) \xi_{j_1,k_1} ... \xi_{j_{i+1},k_{i+1}} \int_0^t < \Gamma_s e_{j_{i+1}}, dW(s) >_{\ell_2(1)} - \tag{4.18}$$

$$-\frac{1}{t}\sum_{\ell=1}^{i+1}\sum_{j_1,...,j_{i+1}\in\mathbb{Z}^d}\mathbf{E}\,\partial_{j_i}...\partial_{j_1}f(\xi^0)\xi_{j_1,k_1}...\mathbb{D}^{j_{i+1}}\xi_{j_\ell,k_\ell}...\xi_{j_{i+1},k_{i+1}} \tag{4.19}$$

In the proof of Theorem 3.9 [3] we have shown by application of nonlinear estimates that for any quasi-contractive array $\Psi = \Psi_0 \cup ... \cup \Psi_n$, $\forall\,\ell \le i \in \{1,...,n\}$ and $|\gamma_1| + ... + |\gamma_\ell| = i$

$$|||\sum_{\gamma_1\cup...\cup\gamma_\ell}\mathbf{E} < \partial^{(\ell)}f(\xi^0), \xi_{\gamma_1}\otimes ...\otimes \xi_{\gamma_\ell} > |||_{\Psi_i} \le Ke^{Mt}|||\partial^{(\ell)}f|||_{\Psi_\ell}$$

For quasi-contractive array $\Theta = \Theta_0 \cup ... \cup \Theta_n$ the array $T_{\mathbf{k}}\Theta_i$ is a subset of quasi-contractive array $(\Theta)^1 = \Theta \cup T_{\mathbf{k}}\Theta = \Psi_0 \cup ... \cup \Psi_{n+1}$ with $\Psi_0 = \Theta_0$, $\Psi_{n+1} = T_{\mathbf{k}}\Theta_n$ and $\Psi_\ell = \Theta_\ell \cup T_{\mathbf{k}}\Theta_{\ell-1}$. Therefore we have $\forall\, i \in \{1,...,n\}$

$$|||(4.17)|||_{T_{\mathbf{k}}\Theta_i} \le Ke^{Mt}\max_{\ell=1,...,i}(|||\partial^{(\ell)}f|||_{\Theta_\ell}, |||\partial^{(\ell)}f|||_{T_{\mathbf{k}}\Theta_{\ell-1}})$$

To finish the proof of (4.15) it is sufficient to show that $\forall\,(p,\mathcal{G}) \in \Theta_i$

$$|||(4.18)|||_{T_{\mathbf{k}}(p,\mathcal{G})} \le \frac{1}{\sqrt{t}}Ke^{Mt}|||\partial^{(i)}f|||_{\Theta_i} \tag{4.20}$$

$$|||(4.19)|||_{T_{\mathbf{k}}(p,\mathcal{G})} \le Ke^{Mt}|||\partial^{(i)}f|||_{\Theta_i} \tag{4.21}$$

with $T_{\mathbf{k}}(p,\mathcal{G}) = (\,(1+z)^{\frac{\mathbf{k}_{+1}}{2}}p(z), \mathcal{G}\otimes A^{\mathbf{k}_{+2}})$ (4.11).

Estimate (4.20). Applying Hölder inequality we have

$$|||(4.18)|||_{T_{\mathbf{k}}(p,\mathcal{G})} \le \frac{\|\sum_{j_1,...,j_{i+1}\in\mathbb{Z}^d}(\mathbf{E}\frac{|\partial_{j_i}...\partial_{j_1}f(\xi^0)|^2}{p^2(z_t)})^{1/2}A_{j_1,...,j_{i+1}}\|_{\mathcal{G}\otimes A^{\mathbf{k}_{+2}}}}{t\,(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}_{+1}}{2}}p(\|x^0\|^2_{\ell_2(a)})} \tag{4.22}$$

with $z_t = \|\xi^0(t,x^0)\|^2_{\ell_2(a)}$ and

$$A^{k_1,...,k_{i+1}}_{j_1,...,j_{i+1}} = (\mathbf{E}p^2(z_t)\,|\xi_{j_1,k_1}...\xi_{j_{i+1},k_{i+1}}\int_0^t < \Gamma_s e_{j_{i+1}}, dW(s) >_{\ell_2(1)}|^2)^{1/2}$$

A simple consequence of finite dimensional Ito formula

$$\forall\, v\in\mathcal{J}_{cyl}(\Omega) \qquad \mathbf{E}(\int_0^t < v_s, dW(s) >_{\ell_2(1)})^{2n} \le (n(2n-1))^n t^{n-1}\mathbf{E}\int_0^t \|v_s\|^{2n}_{\ell_2(1)}ds$$

gives

$$(\mathbf{E}(\int_0^t < \Gamma_s e_{j_{i+1}}, dW(s) >_{\ell_2(1)})^{2(i+2)})^{1/2(i+2)} \le Ke^{Mt}\sqrt{t}a_{j_{i+1}}^{-\frac{\mathbf{k}_{+1}}{2}}(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}_{+1}}{2}} \tag{4.23}$$

with Γ_s introduced in (2.15). We used $\|B\|_{\mathcal{L}(\ell_2(1))} < \infty$, property (2.5) of process $\xi^0(t,x^0)$ and estimate

$$|F'(\xi^0_j)| \le C(1+|\xi^0_j|)^{\mathbf{k}_{+1}} \le Ca_j^{-\frac{\mathbf{k}_{+1}}{2}}(1+\|\xi^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}_{+1}}{2}}$$

The successive application of Hölder inequality with $q_\ell = 1/(i+2)$, (4.23) and nonlinear estimate (4.1) with $|\beta| = 0,\ \tau = \{j\}$ lead to

$$A^{k_1,\ldots,k_{i+1}}_{j_1,\ldots,j_{i+1}} \leq \prod_{\ell=1}^{i+1} (\mathbf{E} p^{2\frac{i+2}{i+1}} |\xi_{j_\ell,k_\ell}|^{2(i+2)})^{1/2(i+2)} \{\mathbf{E}(\int_0^t < \Gamma_s e_{j_{i+1}}, dW >)^{2(i+2)}\}^{1/2(i+2)} \leq$$

$$\leq K e^{Mt} \sqrt{t} (1 + \|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}_{+1}}{2}} p(\|x^0\|^2_{\ell_2(a)}) \frac{a_{j_{i+1}}^{-\frac{\mathbf{k}_{+1}}{2}}}{\prod_{\ell=1}^{i+1} \psi_{j_\ell - k_\ell}} \tag{4.24}$$

Substituting (4.24) in (4.22) we obtain

$$|||(4.18)|||_{T_{\mathbf{k}}(p,\mathcal{G})} \leq \frac{K e^{Mt}}{\sqrt{t}} (\sum_{k_1,\ldots,k_{i+1}} G^1_{k_1} \ldots G^i_{k_i} a^{\mathbf{k}_{+2}}_{k_{i+1}} \bullet$$

$$\bullet | \sum_{j_1,\ldots,j_{i+1}} (\mathbf{E} \frac{|\partial_{j_i} \ldots \partial_{j_1} f(\xi^0)|^2}{p^2(z_t)})^{1/2} \frac{a_{j_{i+1}}^{-\frac{\mathbf{k}_{+1}}{2}}}{\prod_{\ell=1}^{i+1} \psi_{j_\ell - k_\ell}} |^2)^{1/2} \leq$$

$$\leq \frac{K e^{Mt}}{\sqrt{t}} (tr\, a)^{1/2} \sum_{j \in \mathbb{Z}^d} \frac{\delta_a^{\frac{\mathbf{k}_{+1}}{2}|j|}}{\psi_j} (\sum_{k_1,\ldots,k_i} G^1_{k_1} \ldots G^i_{k_i} \bullet$$

$$\bullet | \sum_{j_1,\ldots,j_i} (\mathbf{E} \frac{|\partial_{j_i} \ldots \partial_{j_1} f(\xi^0)|^2}{p^2(z_t)})^{1/2} \prod_{\ell=1}^{i} \frac{1}{\psi_{j_\ell - k_\ell}} |^2)^{1/2} \leq$$

$$\leq \frac{K' e^{Mt}}{\sqrt{t}} K_{G,\psi} |||\partial^{(i)} f|||_{(p,\mathcal{G})} \leq \frac{K' e^{Mt}}{\sqrt{t}} K_{G,\psi} |||\partial^{(i)} f|||_{\Theta_i} \tag{4.25}$$

Above we used $a_{j_{i+1}}^{-\frac{\mathbf{k}_{+1}}{2}} \leq \delta_a^{\frac{\mathbf{k}_{+1}}{2}|j_{i+1} - k_{i+1}|} a_{k_{i+1}}^{-\frac{\mathbf{k}_{+1}}{2}}$ with $\delta_a = \sup_{|k-j|=1} |a_k/a_j|$ and Proposition 4.3 with $x_{j_1 \ldots j_i} = (\mathbf{E} \frac{|\partial_{j_i} \ldots \partial_{j_1} f(\xi^0)|^2}{p^2(z_t)})^{1/2}$ and $b(k) = 1/\psi_k$.

The choice of vector $\psi \in I\!\!P$ makes the constants $K_{G,\psi} = \prod_{\ell=1}^{i} (\sum_{k \in \mathbb{Z}^d} \delta_{G^\ell}^{|k|/2} / \psi_k)$ in Proposition 4.3 and sum $\sum_{k \in \mathbb{Z}^d} \delta_a^{\frac{\mathbf{k}_{+1}}{2}|k|} / \psi_k$ be finite which finishes proof of (4.20).

Estimate (4.21) can be done in a similar way.

$$|||(4.19)|||_{T_{\mathbf{k}}(p,\mathcal{G})} \leq \sum_{\ell=1}^{i} \frac{\| \sum_{j_1,\ldots,j_{i+1} \in \mathbb{Z}^d} (\mathbf{E} \frac{|\partial_{j_i} \ldots \partial_{j_1} f(\xi^0)|^2}{p^2(z_t)})^{1/2} B^\ell_{j_1,\ldots,j_{i+1}} \|_{\mathcal{G} \otimes A^{\mathbf{k}_{+2}}}}{t\,(1 + \|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}_{+1}}{2}} p(\|x^0\|^2_{\ell_2(a)})}$$

with

$$B^{\ell;k_1,..,k_{i+1}}_{j_1,\ldots,j_{i+1}} = (\mathbf{E} \{ \prod_{m=1, m\neq \ell}^{i+1} p^{2/(i+1)}(z_t) |\xi_{j_m,k_m}|^2 \} p^{2/(i+1)}(z_t) |I\!\!D^{j_{i+1}} \xi_{j_\ell,k_\ell}|^2)^{1/2} \leq$$

$$\leq \prod_{m=1, m\neq \ell}^{i+1} (\mathbf{E} p^2(z_t) |\xi_{j_m,k_m}|^{2(i+1)})^{1/2(i+1)} \bullet (\mathbf{E} p^2(z_t) |I\!\!D^{j_{i+1}} \xi_{j_\ell,k_\ell}|^{2(i+1)})^{1/2(i+1)} \leq$$

$$\leq K e^{Mt} t (1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}} p(\|x^0\|^2_{\ell_2(a)}) \frac{a_{j_{i+1}}^{-\frac{\mathbf{k}+1}{2}}}{\prod_{\ell=1}^{i+1} \psi_{j_\ell - k_\ell}} \tag{4.26}$$

Above we applied nonlinear estimate (4.1) for $|\beta| \in \{0,1\}$, $\tau = \{j\}$.

As the expression (4.26) up to the factor $\sqrt{t}$ coincides with (4.24) we proceed further like in (4.25) and obtain (4.21). ∎

Below we give a simple Proposition on convolutional estimates, which was used in the proof of Theorem 4.2.

Proposition 4.3. *Let $G^\ell, b \in I\!P$, $b_k \leq 1$. Suppose that*

$$K_{G,b} = \prod_{\ell=1}^{i} \Big(\sum_{j \in \mathbb{Z}^d} b_j \delta_{G^\ell}^{|j|/2} \Big) < \infty$$

Then

$$\Big(\sum_{k_1,\dots,k_i \in \mathbb{Z}^d} G^1_{k_1} \dots G^i_{k_i} \Big| \sum_{j_1,\dots,j_i \in \mathbb{Z}^d} x_{j_1,\dots,j_i} \prod_{\ell=1}^{i} b(k_\ell - j_\ell) \Big|^2 \Big)^{1/2} \leq$$
$$\leq K_{G,b} \Big(\sum_{k_1,\dots,k_i \in \mathbb{Z}^d} G^1_{k_1} \dots G^i_{k_i} |x_{k_1,\dots,k_i}|^2 \Big)^{1/2}$$

Proof. Indeed

$$\Big(\sum_{k_1,\dots,k_i \in \mathbb{Z}^d} G^1_{k_1} \dots G^i_{k_i} \Big| \sum_{j_1,\dots,j_i \in \mathbb{Z}^d} x_{j_1,\dots,j_i} \prod_{\ell=1}^{i} b(k_\ell - j_\ell) \Big|^2 \Big)^{1/2} =$$
$$= \Big(\sum_{k_1,\dots,k_i \in \mathbb{Z}^d} G^1_{k_1} \dots G^i_{k_i} \Big| \sum_{m_1,\dots,m_i \in \mathbb{Z}^d} x_{m_1+k_1,\dots,m_i+k_i} b_{m_1} \dots b_{m_i} \Big|^2 \Big)^{1/2} \leq$$
$$\leq \sum_{m_1,\dots,m_i \in \mathbb{Z}^d} b_{m_1} \dots b_{m_i} \Big(\sum_{k_1,\dots,k_i \in \mathbb{Z}^d} G^1_{k_1} \dots G^i_{k_i} |x_{m_1+k_1,\dots,m_i+k_i}|^2 \Big)^{1/2} \leq$$
$$\leq K_{G,b} \Big(\sum_{k_1,\dots,k_i \in \mathbb{Z}^d} G^1_{k_1} \dots G^i_{k_i} |x_{k_1,\dots,k_i}|^2 \Big)^{1/2}$$

where we used $G_k \leq \delta_G^{|m|} G_{m+k}$. ∎

5 C^∞-smoothness of Wiener functionals.

It remains to prove Theorems 2.2 and 3.1. To do this we discuss in this Section how the nonlinearity parameter $\mathbf{k}$ influences the smoothness of ξ^0 and its variations ξ_τ as Wiener functionals. We apply the monotone methods of nonlinear analysis to work with the unbounded nonautonomous coefficients of corresponding equations.

5.1 Proof of Theorem 2.2.

In successive Lemmas 5.1-5.4 we check that $\xi_k^0(t,x^0) \in \mathcal{D}_{loc}(\Omega)$ and prove Theorem 2.2.

Lemma 5.1. *$\forall\, x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$ $\forall\, u \in \mathcal{J}_{cyl}$ process $\xi^\varepsilon(t,x^0,u,\omega) = \xi^0(t,x^0,\omega_\bullet + \varepsilon\int_0^\bullet u_s ds)$ is represented as a unique strong solution to equation*

$$\xi_k^\varepsilon(t,x^0,u) = x_k^0 + W_k(t) + \varepsilon\int_0^t u_{s,k}ds - \int_0^t [F(\xi_k^\varepsilon(s,x^0,u)) + \{B\xi^\varepsilon(s,x^0,u)\}_k]ds \tag{5.1}$$

i.e. $\ell_2(a)$ continuous $\mathcal{F}_t$ adapted process $\xi^\varepsilon(t,x^0) \in \mathcal{D}_{\ell_2(a)}(F)$, $t \in [0,T]$, $\mathbf{P}$ a.e. $\omega \in \Omega$, which fulfills equation (5.1) in $\ell_2(a)$ $\mathbf{P}$ a.e.

Moreover $\forall\, x^0 \in \ell_2(a)$ there is a unique generalized solution $\xi^\varepsilon(t,x^0,u)$ to equation (5.1), which is a uniform on $[0,T]$ $\mathbf{P}$ a.e. limit of strong solutions and fulfills estimate: $\exists\, M$ $\forall\, x^0, y^0 \in \ell_2(a)$, $u^1, u^2 \in \mathcal{J}_{cyl}$

$$\sup_{|\varepsilon|\le\varepsilon_0}\sup_{t\in[0,T]} \|\xi_t^\varepsilon(x^0,u^1) - \xi_t^\varepsilon(y^0,u^2)\|^2_{\ell_2(a)} \le e^{MT}(\|x^0-y^0\|^2_{\ell_2(a)} + \varepsilon_0^2\int_0^T \|u_s^1 - u_s^2\|^2_{\ell_2(a)}ds) \tag{5.2}$$

$$\sup_{|\varepsilon|\le\varepsilon_0}\sup_{t\in[0,T]} \|\xi_t^\varepsilon(x^0,u)\|^2_{\ell_2(a)} \le K(\omega) + e^{MT}(\|x^0\|^2_{\ell_2(a)} + \varepsilon_0^2 K\int_0^T \|u_s\|^2_{\ell_2(a)}ds) \tag{5.3}$$

with $K \in \bigcap_{p\ge1} L^p(\Omega,\mathbf{P})$.

Proof. Like in [14, 15] due to representation

$$\xi_k^\varepsilon(t,x^0,u) = \eta_k^\varepsilon(t,x^0,u) + W_k(t) + \varepsilon\int_0^t u_{s,k}ds \tag{5.4}$$

strong solvability of equation (5.1) is equivalent to the solvability of equation

$$\eta_k^\varepsilon(t) = x_k^0 - \int_0^t [(F+B)(\eta^\varepsilon(s) + W(s) + \varepsilon\int_0^s u_\tau d\tau)]_k ds \tag{5.5}$$

For $u \in \mathcal{J}_{cyl}$ process $\int_0^s u_\tau d\tau$ is $\ell_p(a)$ continuous for any $p \ge 1$. By [14, Th.4.1] due to $\ell_p(a)$ continuity of $W(s)$ (see e.g. [15]) we have that for $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$ equation (5.5) is strongly solvable in space $\ell_2(a)$ and its solution $\eta^\varepsilon(t)$ is $\ell_{2(\mathbf{k}+1)}(a)$-valued uniformly on $t \in [0,T]$ bounded finite variation process. Furthermore for $I\!\!P$ a.e. $\omega \in \Omega$

$$(F+B)(\eta^\varepsilon(\bullet) + W(\bullet) + \varepsilon\int_0^\bullet u_\tau d\tau) \in L_\infty([0,T],\ell_2(a))$$

and due to representation (5.5) function $[0,T] \ni t \to \eta^\varepsilon(t,x^0,u)$ is $\mathbf{P}$ a.e. absolutely continuous in $\ell_2(a)$ and thus for a.e. $t \in [0,T]$ differentiable. This guarantees the correctness of the following differentiation.

Denote $\xi_1^\varepsilon = \xi^\varepsilon(t,x^0,u^1)$, $\xi_2^\varepsilon = \xi^\varepsilon(t,y^0,u^2)$ for $x^0, y^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$ and $u^1, u^2 \in \mathcal{J}_{cyl}$ then

$$\frac{d}{dt}\|\xi_1^\varepsilon - \xi_2^\varepsilon\|^2_{\ell_2(a)} = \frac{d}{dt}\|\eta_t^\varepsilon(x^0,u^1) - \eta_t^\varepsilon(y^0,u^2) + \varepsilon\int_0^t (u_s^1 - u_s^2)ds\|^2_{\ell_2(a)} =$$

$$= -2 < \xi_1^\varepsilon - \xi_2^\varepsilon, [F(\xi_1^\varepsilon) - F(\xi_2^\varepsilon)] + B(\xi_1^\varepsilon - \xi_2^\varepsilon) + \varepsilon(u_t^1 - u_t^2) > \le$$

$$\leq (2\|B\|+1)\|\xi_1^\varepsilon - \xi_2^\varepsilon\|^2_{\ell_2(a)} + \varepsilon^2\|u_t^1 - u_t^2\|^2_{\ell_2(a)}$$

Above we used the monotonicity of map $F : \ell_2(a) \to \ell_2(a)$. This gives (5.2) for $x^0, y^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$. Choosing $y^0 = x^0$ and $u^2 \equiv 0$ as a consequence of (5.2) we have

$$\sup_{t\in[0,T]} \|\xi_t^\varepsilon(x^0,u^1) - \xi_t^0(x^0)\|^2_{\ell_2(a)} \leq \varepsilon^2 e^{MT}\int_0^T \|u_s\|^2_{\ell_2(a)}ds$$

which together with (2.5) gives (5.3). The closure of estimates (5.2) and (5.3) up to $x^0, y^0 \in \ell_2(a)$ proves the statement. ■

Lemma 5.2. $\forall\, x^0 \in \ell_2(a)\ \forall\, u \in \mathcal{J}_{cyl}$ *the equation*

$$D_u\xi_k^0(t,x^0) = \int_0^t u_{s,k}ds - \int_0^t \{[F'(\xi_k^0(s,x^0)) + B]D_u\xi^0(s,x^0)\}_k ds \tag{5.6}$$

has a unique strong solution in $\ell_2(a)$. *Moreover*

$$\sup_{t\in[0,T]} \|D_u\xi^0(t,x^0)\|^2_{\ell_2(a)} \leq e^{MT}\int_0^T \|u_s\|^2_{\ell_2(a)}ds \tag{5.7}$$

$$\sup_{t\in[0,T]} \|D_u\xi^0(t,x^0) - D_u\xi^0(t,y^0)\|^2_{\ell_2(a^{\mathbf{k}+2})} \leq e^{MT}K_R(\omega)\|x^0-y^0\|^2_{\ell_2(a)}\int_0^T \|u_s\|^2_{\ell_2(a)}ds \tag{5.8}$$

with $K_R \in \bigcap_{p\geq 1} L^p(\Omega,\mathbf{P})$, $R = \max(\|x^0\|_{\ell_2(a)}, \|y^0\|_{\ell_2(a)})$.

Proof. For any $u \in \mathcal{J}_{cyl}$ the inhomogeneous part of equation (5.6) $u_\bullet \in C([0,T],\ell_2(c))$ $\mathbf{P}$ a.e. for all $c \in I\!P$. Following the lines of Th.3.1 proof in [3] we have that linear non-autonomous equation (5.6) has a unique $\ell_2(a)$ continuous $\mathcal{F}_t$ adapted strong solution $D_u\xi^0(t,x^0) \in \mathcal{D}_{\ell_2(a)}(F'(\xi^0(t,x^0)))$ $\mathbf{P}$ a.e., which possesses a strong $\ell_2(a)$ derivative $\frac{d}{dt}D_u\xi^0(t,x^0)$ and has representation like (3.12) in [3]

$$D_u\xi^0(t,x^0,\omega) = \int_0^t U^\omega_{x^0}(t,s)u_s ds \tag{5.9}$$

with $\mathcal{F}_t$ adapted strongly continuous in $\ell_2(a)$ evolution system $\{U^\omega_{x^0}(t,s),\ 0 \leq s \leq t\}$ generated by $A(t,x^0,\omega) = F'(\xi^0(t,x^0)) + B$.

Inequality (5.7) simply follows from $D_u\xi^0(0,x^0) = 0$ and estimate

$$\frac{d}{dt}\|D_u\xi_t^0(x^0)\|^2_{\ell_2(a)} = 2 < D_u\xi_t^0(x^0), u_t - (F'(\xi^0)+B)D_u\xi_t^0 > \leq$$

$$\leq (2\|B\|_{\mathcal{L}(\ell_2(a))}+1)\|D_u\xi_t^0(x^0)\|^2_{\ell_2(a)} + \|u_t\|^2_{\ell_2(a)}$$

To obtain (5.8) we write

$$\frac{d}{dt}\|D_u\xi_t^0(x^0) - D_u\xi_t^0(y^0)\|^2_{\ell_2(a^{\mathbf{k}+2})} =$$

$$= -2 < D_u\xi^0(x^0) - D_u\xi^0(y^0), B[D_u\xi^0(x^0) - D_u\xi^0(y^0)] > -$$

$$-2 < D_u\xi^0(x^0) - D_u\xi^0(y^0), F'(\xi^0(x^0))D_u\xi^0(x^0) - F'(\xi^0(y^0))D_u\xi^0(y^0) >_{\ell_2(a^{\mathbf{k}+2})} \leq$$

$$\leq (2\|B\|+1)\|D_u\xi^0(x^0) - D_u\xi^0(y^0)\|^2_{\ell_2(a^{\mathbf{k}+2})} +$$

$$+\|[F'(\xi^0(x^0)) - F'(\xi^0(y^0))]D_u\xi^0(x^0)\|^2_{\ell_2(a^{\mathbf{k}+2})} \leq$$

$$\leq M\|D_u\xi^0(x^0) - D_u\xi^0(y^0)\|^2_{\ell_2(a^{\mathbf{k}+2})} +$$

$$+K(1+\|\xi^0(x^0)\|_{\ell_2(a)} + \|\xi^0(y^0)\|_{\ell_2(a)})^{2\mathbf{k}}\|\xi^0(x^0) - \xi^0(y^0)\|^2_{\ell_2(a)}\|D_u\xi^0_t(x^0)\|^2_{\ell_2(a)} \tag{5.10}$$

Above we used that $F'(z) \geq 0,\ \forall\, z \in I\!R^1$, and inequality $\forall\, c \in I\!P$

$$\|[F'(\xi) - F'(\zeta)]y\|^2_{\ell_2(c)} \leq K^2 \sum_{k\in Z\!\!Z^d} c_k|(\xi_k - \zeta_k)(1+|\xi_k| + |\zeta_k|)^{\mathbf{k}} y_k|^2 \leq$$

$$\leq K^2 \sum_{k\in Z\!\!Z^d} \frac{c_k}{a_k^{\mathbf{k}+1}}|y_k|^2 \bullet \|\xi - \zeta\|^2_{\ell_2(a)}(1+\|\xi\|_{\ell_2(a)} + \|\zeta\|_{\ell_2(a)})^{2\mathbf{k}} \leq$$

$$\leq K^2\|\xi - \zeta\|^2_{\ell_2(a)}(1+\|\xi\|_{\ell_2(a)} + \|\zeta\|_{\ell_2(a)})^{2\mathbf{k}}\|y\|^2_{\ell_2(dc)} \tag{5.11}$$

with $d_k \geq a_k^{-\mathbf{k}+1}$, which is a simple consequence of property (2.3) for map F. Properties of process ξ^0 (2.5) and estimates (5.10), (5.7) give (5.8). ■

Lemma 5.3. *$\forall\, u \in \mathcal{J}_{cyl}$ the solutions $\xi^\varepsilon, D_u\xi^0$ to equations (5.1), (5.6) fulfill: $\forall\, R > 0\ \exists\, K_{u,R} \in \cap_{p\geq 1} L^p(\Omega, \mathbf{P})$ such that $\forall\, \|x^0\|_{\ell_2(a)} \leq R$*

$$\sup_{|\varepsilon|\leq\varepsilon_0}\sup_{t\in[0,T]} \left\|\frac{\xi^\varepsilon_t(x^0,u) - \xi^0_t(x^0)}{\varepsilon} - D_u\xi^0_t(x^0)\right\|_{\ell_2(a^{\mathbf{k}+2})} \leq \varepsilon_0 K_{u,R}(\omega) \tag{5.12}$$

Proof. Denote $\Delta_\emptyset(t) = \dfrac{\xi^\varepsilon_t(x^0,u) - \xi^0_t(x^0)}{\varepsilon} - D_u\xi^0_t(x^0)$. Due to representation (5.4) and properties of strong solution $D_u\xi^0_t(x^0)$ to equation (5.6) the differentiation below is justified. Using

$$f(y) - f(x) = f'(y-x) + \int_0^1 [f'(x+\lambda(y-x)) - f'(x)](y-x)d\lambda$$

notation $\zeta_\lambda = \xi^0 + \lambda(\xi^\varepsilon - \xi^0)$ and monotonicity of F: $F'(z) \geq 0,\ z \in I\!R^1$ we have

$$\frac{d}{dt}\|\Delta_\emptyset\|^2_{\ell_2(a^{\mathbf{k}+2})} = -2 < \Delta_\emptyset, B\Delta_\emptyset >_{\ell_2(a)} -$$

$$-2 < \Delta_\emptyset, F(\xi^\varepsilon) - F(\xi^0) + u_t - u_t - F'D_u\xi^0_t >_{\ell_2(a)} \leq$$

$$\leq (2\|B\|+1)\|\Delta_\emptyset\|^2_{\ell_2(a^{\mathbf{k}+2})} - 2\int_0^1 \|[F'(\zeta_\lambda) - F'(\xi^0)]\frac{\xi^\varepsilon - \xi^0}{\varepsilon}\|^2_{\ell_2(a^{\mathbf{k}+2})} d\lambda \leq$$

$$\leq K\|\Delta_\emptyset\|^2_{\ell_2(a^{\mathbf{k}+2})} + 2\int_0^1 K'(1+\|\xi^0\|_{\ell_2(a)} + \|\zeta_\lambda\|_{\ell_2(a)})^{2\mathbf{k}}\|\xi^0 - \zeta_\lambda\|^2_{\ell_2(a)}\|\frac{\xi^\varepsilon - \xi^0}{\varepsilon}\|^2_{\ell_2(a)} d\lambda \leq$$

$$\leq K\|\Delta_\emptyset\|^2 + \varepsilon^2 K''_u(\|x^0\|^2_{\ell_2(a)}) \tag{5.13}$$

At last two steps we used estimates (5.11), (2.5) and properties of ξ^0 and ξ^ε (Lemma 5.1).

Due to $\Delta_\emptyset(0) = 0$ and (5.13) we obtain (5.12) for $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$. By (5.2), (5.8) we can close this estimate up to $x^0 \in \ell_2(a)$ for $\mathbf{P}$ a.e. $\omega \in \Omega$. $\blacksquare$

Lemma 5.4. *$\forall\, x^0 \in \ell_2(a)$ $\forall\, u \in \mathcal{J}_{cyl}$ $\forall\, k \in \mathbb{Z}^d$ the coordinates $\xi^0_k(t, x^0) \in \mathcal{D}_{loc}(\Omega)$, $t \in [0, T]$.*

Proof. Formula (5.12) gives: $\forall\, x^0 \in \ell_2(a),\ u \in \mathcal{J}_{cyl},\ k \in \mathbb{Z}^d$

$$\lim_{|\varepsilon|\to 0} \mathbf{E} \sup_{t\in[0,T]} \Big|\frac{\xi^\varepsilon_k(t, x^0, u) - \xi^0_k(t, x^0)}{\varepsilon} - D_u \xi^0_k(t, x^0)\Big|^p = 0, \quad \forall\, p \geq 1 \qquad (5.14)$$

By (5.3) and (5.7) we have that $D_u \xi^0_k(t, x^0)$ is derivative of $\xi^0_k(t, x^0)$ in sense of Definition 2.1.

Representation (5.9) leads to the property (2.10): $\forall\, u \in \mathcal{J}_{cyl}$

$$D_u \xi^0_k(t, x^0) = \int_0^t \sum_{j\in\mathbb{Z}^d} [U^\omega_{x^0}(t, s)]_{kj} u_{s,j} ds = \sum_{j\in\Lambda_u} < \mathbf{D}_j \xi^0_k(t, x^0), \int_0^\bullet u_{s,j} ds >_{\mathcal{H}}$$

with $\mathcal{F}_t$ adapted $\mathbf{D}_j \xi^0_k(t, x^0) = \int_0^\bullet \chi_{s\leq t} [U^\omega_{x^0}(t, s)]_{kj} ds \in \mathcal{H}$. From estimate (5.7) we have $\mathbf{P}$ a.e.

$$\forall\, u \in \mathcal{J}_{cyl} \quad |D_u \xi^0_k(t, x^0)|^2 \leq \frac{1}{a_k} e^{Mt} \int_0^t \|u_s\|^2_{\ell_2(a)} ds$$

therefore

$$\int_0^t \|\, [U^\omega_{x^0}(t, s)]_{k\bullet}\|^2_{\ell_2(a^{-1})} ds \leq \frac{1}{a_k} e^{Mt}$$

This implies

$$\|\mathbf{D}_j \xi^0_k(t, x^0)\|^2_{\mathcal{H}} = \int_0^t |\, [U^\omega_{x^0}(t, s)]_{k,j}|^2 ds \leq \frac{a_j}{a_k} e^{Mt}$$

and $\mathbf{D}_j \xi^0_k(t, x^0) \in \bigcap_{p\geq 1} L^p(\Omega, \mathbf{P}, \mathcal{H})$. $\blacksquare$

5.2 Proof of Theorem 3.1.

The correct proof of Theorem 3.1 requires some bookkeeping. The following Lemma gives a tool to work with the multiplicative structure of $\varphi_{\tau,\beta}$ in (3.5).

Lemma 5.5. *Let vectors $\{c_{\gamma,\sigma}\}_{\gamma\subset\tau,\sigma\subset\beta}$ satisfy hierarchy (3.7) and function Q such $\exists\, K > 0$ $\forall\, x, y \in \mathbb{R}^1$ $|Q(x) - Q(y)| \leq K|x - y|(1 + |x| + |y|)^{\mathbf{k}}$. Then for $d_k = a_k^{-\frac{\mathbf{k}+1}{2}m_1}$ we have*

$$\exists\, C\ \forall\, x, y \in \ell_2(a) \quad \|Q(y)\|_{\mathcal{L}(\ell_{m_\tau}(dc_{\tau,\beta}),\ell_{m_\tau}(c_{\tau,\beta}))} \leq C(1 + \|y\|_{\ell_2(a)})^{\mathbf{k}+1} \qquad (5.15)$$

$$\|\, [Q(y) - Q(x)]u\|_{\ell_{m_\tau}(c_{\tau,\beta})} \leq C\|y - x\|_{\ell_2(a)}(1 + \|y\|_{\ell_2(a)} + \|x\|_{\ell_2(a)})^{\mathbf{k}} \|u\|_{\ell_{m_\tau}(dc_{\tau,\beta})} \qquad (5.16)$$

for diagonal map $[Q(y)u]_k = Q(y_k)u_k$ *and* $m_\tau = m_1/|\tau| \geq 1$.

Consider subdivisions $\tau = \gamma_1 \cup ... \cup \gamma_\ell$, $\beta = \sigma_0 \cup ... \cup \sigma_\ell$ *with* $|\sigma_0| \geq 2 - \ell$ *and* $|\gamma_i| \geq 1$, $|\sigma_i| \geq 0$ *for* $i = 1, ..., \ell$, $\ell \geq 1$. *Then* $\exists\, K\ \forall\, n = 1, ..., \ell$

$$\|\delta^{\sigma_0}[Q(y^0)y_{\gamma_1,\sigma_1}...y_{\gamma_n,\sigma_n} - Q(x^0)x_{\gamma_1,\sigma_1}...x_{\gamma_n,\sigma_n}]u_{\gamma_{n+1},\sigma_{n+1}}...u_{\gamma_\ell,\sigma_\ell}\|_{\ell_{m_\tau}(c_{\tau,\beta})} \leq$$

$$\leq K(1 + \|y^0\|_{\ell_2(a)} + \|x^0\|_{\ell_2(a)})^{\mathbf{k}+1} \prod_{j=1}^{n}(1 + \|y_{\gamma_j,\sigma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})} + \|x_{\gamma_j,\sigma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})})\times$$

$$\times\{\|y^0 - x^0\|_{\ell_2(a)} + \sum_{j=1}^{n} \|y_{\gamma_j,\sigma_j} - x_{\gamma_j,\sigma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})}\} \prod_{j=n+1}^{\ell} \|u_{\gamma_j,\sigma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})} \quad (5.17)$$

where $\delta_k^{\sigma_0} = \prod_{j\in\sigma_0} \delta_k^j$ *is a product of Kronnecker symbols and vector*

$$[\delta^{\sigma_0}Q(y^0)y_{\gamma_1,\sigma_1}...y_{\gamma_n,\sigma_n}u_{\gamma_{n+1},\sigma_{n+1}}...u_{\gamma_\ell,\sigma_\ell}]_k =$$

$$= \delta_k^{\sigma_0}Q(y_k^0)y_{k;\gamma_1,\sigma_1}...y_{k;\gamma_n,\sigma_n}u_{k;\gamma_{n+1},\sigma_{n+1}}...u_{k;\gamma_\ell,\sigma_\ell}$$

Proof. Inequalities (5.15) and (5.16) simply follow from estimate

$$\|\,[Q(y) - Q(x)]u\|_{\ell_m(c)}^m \leq K^m \sum_{k\in\mathbb{Z}^d} c_k|(y_k - x_k)(1 + |y_k| + |x_k|)^{\mathbf{k}}u_k|^m \leq$$

$$\leq K^m(\sum_{k\in\mathbb{Z}^d} \frac{c_k}{a_k^{\frac{\mathbf{k}+1}{2}m}}|u_k|^m)\,\|y - x\|_{\ell_2(a)}^m(1 + \|y\|_{\ell_2(a)} + \|x\|_{\ell_2(a)})^{m\mathbf{k}}$$

To prove (5.17) first remark that

$$\|\delta^{\sigma_0}Q(y^0)y_{\gamma_1,\sigma_1}...y_{\gamma_\ell,\sigma_\ell} - \delta^{\sigma_0}Q(x^0)x_{\gamma_1,\sigma_1}...x_{\gamma_\ell,\sigma_\ell}\|_{\ell_{m_\tau}(c_{\tau,\beta})} \leq$$

$$\leq \|\delta^{\sigma_0}[Q(y^0) - Q(x^0)]y_{\gamma_1,\sigma_1}...y_{\gamma_\ell,\sigma_\ell}\|_{\ell_{m_\tau}(c_{\tau,\beta})}+ \quad (5.18)$$

$$+\sum_{j=1}^{\ell} \|\delta^{\sigma_0}Q(x^0)x_{\gamma_1,\sigma_1}...x_{\gamma_{j-1},\sigma_{j-1}}(y_{\gamma_j,\sigma_j} - x_{\gamma_j,\sigma_j})y_{\gamma_{j+1},\sigma_{j+1}}...y_{\gamma_\ell,\sigma_\ell}\|_{\ell_{m_\tau}(c_{\tau,\beta})} \quad (5.19)$$

Using hierarchy (3.7) we estimate (5.18)

$$(5.18)^{m_\tau} = \|\delta^{\sigma_0}[Q(y^0) - Q(x^0)]y_{\gamma_1,\sigma_1}...y_{\gamma_\ell,\sigma_\ell}\|_{\ell_{m_\tau}(c_{\tau,\beta})}^{m_\tau} \leq$$

$$\leq K^{m_\tau} \sum_{k\in\mathbb{Z}^d} \delta_k^{\sigma_0}c_{k;\tau,\beta}|(y_k^0 - x_k^0)(1 + |y_k^0| + |x_k^0|)^{\mathbf{k}}y_{k;\gamma_1,\sigma_1}...y_{k;\gamma_\ell,\sigma_\ell}|^{m_\tau} \leq$$

$$\leq K^{m_\tau}\|y^0 - x^0\|_{\ell_2(a)}^{m_\tau}(1 + \|y^0\|_{\ell_2(a)} + \|x^0\|_{\ell_2(a)})^{\mathbf{k}m_\tau}\bullet$$

$$\bullet \sum_{k\in\mathbb{Z}^d} \delta_k^{\sigma_0}c_{k;\tau,\beta}a_k^{-\frac{\mathbf{k}+1}{2}m_\tau} \prod_{j=1}^{\ell}(|y_{k;\gamma_j,\sigma_j}|^{m_{\gamma_j}})^{|\gamma_j|/|\tau|} \leq$$

$$\leq K^{m_\tau}K_c^{1/|\tau|}\|y^0 - x^0\|_{\ell_2(a)}^{m_\tau}(1 + \|y^0\|_{\ell_2(a)} + \|x^0\|_{\ell_2(a)})^{\mathbf{k}m_\tau}\bullet$$

$$\bullet \sum_{k\in\mathbb{Z}^d}\prod_{j=1}^{\ell}(c_{k;\gamma_j,\sigma_j}|y_{k;\gamma_j,\sigma_j}|^{m_{\gamma_j}})^{|\gamma_j|/|\tau|} \le$$

$$\le K^{m_\tau}K_c^{1/|\tau|}\|y^0-x^0\|^{m_\tau}_{\ell_2(a)}(1+\|y^0\|_{\ell_2(a)}+\|x^0\|_{\ell_2(a)})^{\mathbf{k}m_\tau}\prod_{j=1}^{\ell}\|y_{\gamma_j,\sigma_j}\|^{m_\tau}_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})} \tag{5.20}$$

In analogous way we have

$$(5.19)^{m_\tau} = \|\delta^{\sigma_0}Q(x^0)x_{\gamma_1,\sigma_1}...x_{\gamma_{j-1},\sigma_{j-1}}(y_{\gamma_j,\sigma_j}-x_{\gamma_j,\sigma_j})y_{\gamma_{j+1},\sigma_{j+1}}...y_{\gamma_\ell,\sigma_\ell}\|^{m_\tau}_{\ell_{m_\tau}(c_{\tau,\beta})} \le$$

$$\le K^{m_\tau}(1+|Q(0)|)^{m_\tau}K_c^{1/|\tau|}(1+\|x^0\|_{\ell_2(a)})^{(\mathbf{k}+1)m_\tau}\|y_{\gamma_j,\sigma_j}-x_{\gamma_j,\sigma_j}\|^{m_\tau}_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})}\bullet$$

$$\bullet(\prod_{i=1}^{j-1}\|x_{\gamma_i,\sigma_i}\|^{m_\tau}_{\ell_{m_{\gamma_i}}(c_{\gamma_i,\sigma_i})})(\prod_{i=j+1}^{\ell}\|y_{\gamma_i,\sigma_i}\|^{m_\tau}_{\ell_{m_{\gamma_i}}(c_{\gamma_i,\sigma_i})}) \tag{5.21}$$

Estimate (5.17) follows from (5.20), (5.21) with $y_{\gamma_j,\sigma_j} = x_{\gamma_j,\sigma_j} = u_{\gamma_j,\sigma_j}$ for $j = n+1,...,\ell$ and by enlarging (where necessary) the multiplicators and powers. ∎

Lemma 5.6. *Let vectors $\{c_{\tau,\beta}\}\subset \mathbb{P}$ fulfill hierarchy (3.7). Then $\forall\, x^0\in\ell_2(a)$ $\forall\, u\in\mathcal{J}_{cyl}$ $\forall\,\varepsilon\in\mathbb{R}^1$ there is a set of unique strong solutions $\mathbb{D}^\beta\xi^\varepsilon_\tau(t,x^0,u)$ in the scale of spaces $\{\ell_{m_\gamma}(c_{\gamma,\sigma})\}_{\gamma\subset\tau,\sigma\subset\beta}$, $m_\gamma = m_1/|\gamma|$, to system*

$$\forall\,|\beta|\ge 0 \quad \mathbb{D}^\beta\xi^\varepsilon_{k,\tau}(t) = \tilde{x}_{k;\tau,\beta} - \int_0^t\{(F'(\xi^\varepsilon_\emptyset)+B)\mathbb{D}^\beta\xi^\varepsilon_\tau\}_k ds - \int_0^t\varphi^\varepsilon_{k;\tau,\beta}(s)ds \tag{5.22}$$

with $\tilde{x}_{\tau,\beta}\equiv 0$, $|\beta|\ge 1$, $\tilde{x}_{\tau,\emptyset} = \tilde{x}_\tau$ (2.7), $\xi^\varepsilon_\emptyset \overset{df}{=} \xi^\varepsilon(t,x^0,u)$ introduced in (5.1) and

$$\varphi^\varepsilon_{k;\tau,\beta}(t) = \sum_{\substack{\gamma_1\cup...\cup\gamma_\ell=\tau\\ |\gamma_i|\ge 1,\ \ell\ge 1}}\ \sum_{\substack{\sigma_0\cup...\cup\sigma_\ell=\beta\\ |\sigma_0|\ge 2-\ell,\ |\sigma_i|\ge 0}} t^{|\sigma_0|}\delta_k^{\sigma_0}F^{(\ell+|\sigma_0|)}(\xi^\varepsilon_{k,\emptyset})\mathbb{D}^{\sigma_1}\xi^\varepsilon_{k,\gamma_1}...\mathbb{D}^{\sigma_\ell}\xi^\varepsilon_{k,\gamma_\ell} \tag{5.23}$$

In particular case at $\varepsilon = 0$ the process $\mathbb{D}^\beta\xi^\varepsilon_\tau(t,x^0,u)|_{\varepsilon=0}$ gives a solution $\mathbb{D}^\beta\xi_\tau(t,x^0)$ to (3.5) and there is a connection

$$\mathbb{D}^\beta\xi^\varepsilon_\tau(t,x^0,u,\omega) = \mathbb{D}^\beta\xi_\tau(t,x^0,\omega_\bullet+\varepsilon\int_0^\bullet u_s ds) \tag{5.24}$$

Moreover $\forall\, R>0$ $\exists\, K_R\in\cap_{p\ge1}L^p(\Omega,\mathbf{P})$ and polynomial P such that for all $x^0, y^0\in\ell_2(a)$, $u^1,u^2\in\mathcal{J}_{cyl}$

$$\sup_{|\varepsilon|\le\varepsilon_0}\sup_{t\in[0,T]}\|\mathbb{D}^\beta\xi^\varepsilon_\tau(t,x^0,u)\|_{\ell_{m_\tau}(c_{\tau,\beta})} \le K_R(\omega)P(\varepsilon_0\int_0^T\|u_s\|^2_{\ell_2(a)}ds) \tag{5.25}$$

$$\sup_{|\varepsilon|\le\varepsilon_0}\sup_{t\in[0,T]}\|\mathbb{D}^\beta\xi^\varepsilon_\tau(x^0,u^1)-\mathbb{D}^\beta\xi^\varepsilon_\tau(y^0,u^2)\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\tau,\beta})} \le$$

$$\le K_R(\omega)(\|x^0-y^0\|^2_{\ell_2(a)}+\varepsilon_0^2\int_0^T\|u^1_s-u^2_s\|^2_{\ell_2(a)}ds)^{1/2}P(\varepsilon_0\int_0^T\|u^1_s\|^2_{\ell_2(a)}+\|u^2_s\|^2_{\ell_2(a)}ds) \tag{5.26}$$

uniformly on $\max(\|x^0\|_{\ell_2(a)},\|y^0\|_{\ell_2(a)},\|\tilde{x}_{\gamma,\emptyset}\|_{\ell_{m_\gamma}(c_{\gamma,\emptyset})},\gamma\subset\tau)\le R$.

Proof. The strong solvability of linear equation (5.22) for initial data $\tilde{x}_{\tau,\beta} \in \bigcap_{p\geq 1, c\in I\!P} \ell_p(c)$ can be easily obtained with application of Kato criterions [20, 21, 22, 23, 24, 19]. The similar procedure was already done in [3]. In analogue to Theorem 3.1 [3] it is sufficient to show that inhomogeneous part

$$\varphi^\varepsilon_{\tau,\beta} \in C([0,T], \ell_{m_\tau}(c_{\tau,\beta})) \cap L_\infty([0,T], \ell_{m_\tau}(dc_{\tau,\beta})) \quad with\ d_k = a_k^{-\frac{\mathbf{k}+1}{2}m_1}$$

It is proved by induction with application of pathwise continuity of process $\xi^\varepsilon(t, x^0, u)$ (Lemma 5.1) and Lemma 5.5 instead of Proposition 3.3 in [3]. As a result the solutions to system (5.22) fulfill $\mathbf{P}$ a.e.

$$I\!D^\beta \xi^\varepsilon_\tau \in C([0,T], \ell_{m_\tau}(c_{\tau,\beta})) \cap L_\infty([0,T], \ell_{m_\tau}(dc_{\tau,\beta})) \tag{5.27}$$

possess strong $\ell_{m_\tau}(c_{\tau,\beta})$ derivatives $\frac{d}{dt} I\!D^\beta \xi^\varepsilon_\tau(t, x^0)$ a.e. on $[0,T]$ and have representations [3, (3.12)]

$$I\!D^\beta \xi_\tau(t, x^0) = -\int_0^t U^\omega_{x^0}(t,s)\varphi_{\tau,\beta}(s)ds \tag{5.28}$$

in the terms of $\mathcal{F}_t$ adapted evolution system $U^\omega_{x^0}(t,s)$, $0 \leq s \leq t$, generated by $F'(\xi^0(t, x^0, \omega)) + B$, which fulfills estimate

$$\sup_{t\in[0,T]} \|U^\omega_{x^0}(t,s)\|_{\mathcal{L}(\ell_p(c))} \leq \exp(T\|B\|_{\mathcal{L}(\ell_p(c))}) \tag{5.29}$$

Estimate (5.25) we prove by induction. Using $F' \geq 0$, notation (3.17) and inequality (3.20) we have

$$\frac{d}{dt}\|I\!D^\beta \xi^\varepsilon_\tau(t)\|^{m_\tau}_{\ell_{m_\tau}(c_{\tau,\beta})} = m_\tau < \frac{d}{dt} I\!D^\beta \xi^\varepsilon_\tau, (I\!D^\beta \xi^\varepsilon_\tau)^\# >_{\ell_{m_\tau}(c_{\tau,\beta})} \leq$$

$$\leq (m_\tau\|B\| + m_\tau - 1)\|I\!D^\beta \xi^\varepsilon_\tau(t)\|^{m_\tau}_{\ell_{m_\tau}(c_{\tau,\beta})} + \|\varphi^\varepsilon_{\tau,\beta}\|^{m_\tau}_{\ell_{m_\tau}(c_{\tau,\beta})} \tag{5.30}$$

Representation of $\varphi^\varepsilon_{\tau,\beta}$ (5.23) leads to

$$\|\varphi^\varepsilon_{\tau,\beta}\|_{\ell_{m_\tau}(c_{\tau,\beta})} \leq \sum_{\{\gamma,\sigma\}} t^{|\sigma_0|} \|\delta^{\sigma_0} F^{(\ell+|\sigma_0|)}(\xi^\varepsilon_\emptyset) I\!D^{\sigma_1}\xi^\varepsilon_{\gamma_1} \ldots I\!D^{\sigma_\ell}\xi^\varepsilon_{\gamma_\ell}\|_{\ell_{m_\tau}(c_{\tau,\beta})} \leq$$

$$\leq K \sum_{\{\gamma,\sigma\}} t^{|\sigma_0|}(1 + \|\xi^\varepsilon_\emptyset\|_{\ell_2(a)})^{\mathbf{k}+1} \prod_{j=1}^{\ell} (1 + \|I\!D^{\sigma_j}\xi^\varepsilon_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})}) \tag{5.31}$$

where we used Lemma 5.5 (5.17). The inductive assumption and property (5.3) applied to (5.30), (5.31) give the required estimate (5.25). We also used that $I\!D^\beta \xi^\varepsilon_\tau(0) = \tilde{x}_{\tau,\beta}$ (5.22) and for inductive base $\varphi^\varepsilon_{\{j\},\emptyset} \equiv 0$.

Estimate (5.26). Introduce notations $\xi^1_\tau = \xi^\varepsilon_\tau(x^0, u^1)$, $\xi^2_\tau = \xi^\varepsilon_\tau(y^0, u^2)$. In analog to (5.30) for $x^0, y^0 \in \ell_2(a)$ and $u^1, u^2 \in \mathcal{J}_{cyl}$ we have

$$\frac{d}{dt}\|I\!D^\beta \xi^1_\tau - I\!D^\beta \xi^2_\tau\|^{m_\tau}_{\ell_{m_\tau}(d^{-1}c_{\tau,\beta})} \leq m_\tau \|B\|_{\mathcal{L}(\ell_{m_\tau}(d^{-1}c_{\tau,\beta}))} \|I\!D^\beta \xi^1_\tau - I\!D^\beta \xi^2_\tau\|^{m_\tau}_{\ell_{m_\tau}(d^{-1}c_{\tau,\beta})} -$$

$$-m_\tau < (I\!\!D^\beta\xi^1_\tau - I\!\!D^\beta\xi^2_\tau)^\#, F'(\xi^1_\emptyset)I\!\!D^\beta\xi^1_\tau - F'(\xi^2_\emptyset)I\!\!D^\beta\xi^2_\tau >_{\ell_{m_\tau}(d^{-1}c_{\tau,\beta})} -$$

$$-m_\tau < (I\!\!D^\beta\xi^1_\tau - I\!\!D^\beta\xi^2_\tau)^\#, \varphi^\varepsilon_{\tau,\beta}(x^0,u^1) - \varphi^\varepsilon_{\tau,\beta}(y^0,u^2) >_{\ell_{m_\tau}(d^{-1}c_{\tau,\beta})} \le$$

$$\le (m_\tau\|B\| + 2(m_\tau - 1))\|I\!\!D^\beta\xi^1_\tau - I\!\!D^\beta\xi^2_\tau\|^{m_\tau}_{\ell_{m_\tau}(d^{-1}c_{\tau,\beta})} +$$

$$+\|\{F'(\xi^1_\emptyset) - F'(\xi^2_\emptyset)\}I\!\!D^\beta\xi^1_\tau\|^{m_\tau} + \|\varphi^\varepsilon_{\tau,\beta}(x^0,u^1) - \varphi^\varepsilon_{\tau,\beta}(y^0,u^2)\|^{m_\tau} \qquad (5.32)$$

Above we added and substracted term $F'(\xi^2_\emptyset)I\!\!D^\beta\xi^1_\tau$ and used property $F' \ge 0$ and inequality (3.20).

Lemma 5.5 (5.16) leads to the estimate on the first term in (5.32)

$$\|\{F'(\xi^1_\emptyset) - F'(\xi^2_\emptyset)\}I\!\!D^\beta\xi^1_\tau\|_{\ell_{m_\tau}(d^{-1}c_{\tau,\beta})} \le K\|\xi^1_\emptyset - \xi^2_\emptyset\|_{\ell_2(a)} \times$$

$$\times(1 + \|\xi^1_\emptyset\|_{\ell_2(a)} + \|\xi^2_\emptyset\|_{\ell_2(a)})^{\mathbf{k}}\|I\!\!D^\beta\xi^1_\tau\|_{\ell_{m_\tau}(c_{\tau,\beta})} \qquad (5.33)$$

Lemma 5.5 (5.17) gives an estimate on the second term in (5.32)

$$\|\varphi^\varepsilon_{\tau,\beta}(x^0,u^1) - \varphi^\varepsilon_{\tau,\beta}(y^0,u^2)\|_{\ell_{m_\tau}(d^{-1}c_{\tau,\beta})} \le \sum_{\substack{\gamma_1\cup\ldots\cup\gamma_\ell = \tau \\ |\gamma_i|\ge 1,\ \ell\ge 1}} \sum_{\substack{\sigma_0\cup\ldots\cup\sigma_\ell = \beta \\ |\sigma_0|\ge 2-\ell,\ |\sigma_i|\ge 0}} t^{|\sigma_0|} \bullet$$

$$\bullet\|\delta^{\sigma_0}F^{(\ell+|\sigma_0|)}(\xi^1_\emptyset)I\!\!D^{\sigma_1}\xi^1_{\gamma_1}\ldots I\!\!D^{\sigma_\ell}\xi^1_{\gamma_\ell} - \delta^{\sigma_0}F^{(\ell+|\sigma_0|)}(\xi^2_\emptyset)I\!\!D^{\sigma_1}\xi^2_{\gamma_1}\ldots I\!\!D^{\sigma_\ell}\xi^2_{\gamma_\ell}\| \le$$

$$\le K\sum_{\gamma,\sigma}\{\|\xi^1_\emptyset - \xi^2_\emptyset\|_{\ell_2(a)} + \sum_{j=1}^{\ell}\|I\!\!D^{\sigma_j}\xi^1_{\gamma_j} - I\!\!D^{\sigma_j}\xi^2_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})}\}\bullet$$

$$\bullet(1+\|\xi^1_\emptyset\|_{\ell_2(a)}+\|\xi^2_\emptyset\|_{\ell_2(a)})^{\mathbf{k}}\bullet\prod_{j=1}^{\ell}(1+\|I\!\!D^{\sigma_j}\xi^1_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})}+\|I\!\!D^{\sigma_j}\xi^2_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})}) \quad (5.34)$$

where we used $\|\bullet\|_{\ell_{m_\tau}(d^{-1}c_{\tau,\beta})} \le \|\bullet\|_{\ell_{m_\tau}(c_{\tau,\beta})}$.

Substituting (5.33), (5.34) into (5.32) and using inductive assumption and properties (5.25) and (5.2), (5.3) we obtain the required estimate (5.26). We also applied that $(I\!\!D^\beta\xi^1_\tau - I\!\!D^\beta\xi^2_\tau)|_{t=0} = 0$.

Finally, the property (5.24) is a consequence of uniqueness of solutions to (5.22) and corresponding relation between $\xi^\varepsilon(t,x^0,u)$ and $\xi^0(t,x^0)$ (Lemma 5.1).
■

Lemma 5.7. *Let $x^0 \in \ell_2(a)$, vectors $\{c_{\tau,\beta}\} \subset I\!\!P$ be hierarchied by (3.7) and $I\!\!D^\beta\xi_\tau$ denote the strong solutions in $\ell_{m_\tau}(c_{\tau,\beta})$ to equation (5.22) at $\varepsilon = 0$.*

Then $\forall\, u \in \mathcal{J}_{cyl}$ there is a set of unique strong solutions $D_u I\!\!D^\beta\xi_\tau(t,x^0)$ in the scale of spaces $\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1}c_{\tau,\beta})$ to system

$$\forall\,|\beta|\ge 0 \quad D_u I\!\!D^\beta\xi_{k,\tau}(t) = -\int_0^t[(F'(\xi^0_k) + B)D_u I\!\!D^\beta\xi_\tau]_k - \int_0^t \psi^u_{k;\tau,\beta}(s)ds \quad (5.35)$$

where

$$\psi^u_{k;\tau,\beta} = F''(\xi^0_k)D_u\xi^0_k I\!\!D^\beta\xi_{k,\tau} +$$

$$+\sum_{\{\gamma,\sigma\}} t^{|\sigma_0|}\delta_k^{\sigma_0} F^{(\ell+|\sigma_0|+1)}(\xi_k^0) D_u\xi_k^0\, I\!D^{\sigma_1}\xi_{k,\gamma_1}...I\!D^{\sigma_\ell}\xi_{k,\gamma_\ell}+ \tag{5.36}$$

$$+\sum_{\{\gamma,\sigma\}}\sum_{j=1}^{\ell} t^{|\sigma_0|}\delta^{\sigma_0} F^{(\ell+|\sigma_0|)}(\xi_k^0) I\!D^{\sigma_1}\xi_{k,\gamma_1}...(D_u I\!D^{\sigma_j}\xi_{k,\gamma_j})...I\!D^{\sigma_\ell}\xi_{k,\gamma_\ell}$$

The summation $\sum_{\{\gamma,\sigma\}}$ *is like in (5.23).*

Moreover, $\forall\, R>0\ \exists\, K_R \in \cap_{p\geq 1} L^p(\Omega,\mathbf{P})$ *such that*

$$\sup_{t\in[0,T]} \|D_u I\!D^{\beta}\xi_\tau(t,x^0)\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1}c_{\tau,\beta})} \leq K_R(\omega)\Big(\int_0^T \|u_s\|^2_{\ell_2(a)}ds\Big)^{1/2} \tag{5.37}$$

uniformly on $\max(\|x^0\|_{\ell_2(a)}, \|\tilde{x}_{\gamma,\emptyset}\|_{\ell_{m_\gamma}(c_{\gamma,\emptyset})}, \gamma\subset\tau)\leq R$.

Proof. The strong solvability of linear equation (5.35) is obtained with application of Kato criterions [20, 21, 22, 23, 24, 19]. Like in Lemma 5.6, following the scheme of Theorem 3.1 in [3] it is sufficient to show that the inhomogeneous part

$$\psi^u_{\tau,\beta} \in C([0,T],\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1}c_{\tau,\beta}))\cap L^\infty([0,T],\ell_{m_\tau}(a^{m_1/2}c_{\tau,\beta})) \tag{5.38}$$

As a consequence there is a set of unique $\mathcal{F}_t$ adapted strong solutions to equation (5.35) in the scale $\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1}c_{\tau,\beta})$ which fulfill $\mathbf{P}$ a.e. property

$$D_u I\!D^{\beta}\xi_\tau \in C([0,T],\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1}c_{\tau,\beta}))\cap L^\infty([0,T],\ell_{m_\tau}(a^{m_1/2}c_{\tau,\beta})) \tag{5.39}$$

possess strong $\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1}c_{\tau,\beta})$ derivatives $\frac{d}{dt}D_u I\!D^{\beta}\xi_\tau$ a.e. on $[0,T]$ and have representations like [3, (3.12)]

$$D_u I\!D^{\beta}\xi_\tau(t,x^0,\omega) = -\int_0^t U^{\omega}_{x^0}(t,s)\psi^u_{\tau,\beta}(s,x^0)ds \tag{5.40}$$

in terms of $\mathcal{F}_t$ adapted evolution system $U^{\omega}_{x^0}(t,s)$, $0\leq s\leq t$, generated by $F'(\xi^0(t,x^0,\omega))+B$.

Property (5.38) is proved inductively like in [3, Th.3.1] with additional application of inequality

$$\forall\, k\in Z\!\!\!Z^d \quad |D_u\xi^0_k(t,x^0)| \leq \frac{1}{\sqrt{a_k}}\|D_u\xi^0(t,x^0)\|_{\ell_2(a)} \tag{5.41}$$

and pathwise continuity of $D_u\xi^0$ (Lemma 5.2). Indeed, for the first term in (5.36)

$$\sup_{t\in[0,T]}\|F''(\xi^0)D_u\xi^0 I\!D^{\beta}\xi_\tau\|_{\ell_{m_\tau}(a^{m_1/2}c_{\tau,\beta})} \leq \sup_{t\in[0,T]}\|D_u\xi^0\|_{\ell_2(a)}\|F''(\xi^0)I\!D^{\beta}\xi_\tau\|_{\ell_{m_\tau}(c_{\tau,\beta})}\leq$$

$$\leq K\sup_{t\in[0,T]}\|D_u\xi^0(t)\|_{\ell_2(a)}(1+\|\xi^0(t)\|_{\ell_2(a)})^{\mathbf{k}+1}\|I\!D^{\beta}\xi_\tau(t)\|_{\ell_{m_\tau}(dc_{\tau,\beta})}<\infty \tag{5.42}$$

where we used (5.41), Lemma 5.5 (5.15), property (5.27) with $d_k = a_k^{-\frac{\mathbf{k}+1}{2}m_1}$ and the pathwise continuity of processes ξ^0, $D_u\xi^0$.

Property $\psi^u_{\tau,\beta} \in L^\infty$ (5.38) follows from (5.42) and analogous estimate on second and third terms in (5.36), where one should use Lemma 5.5 (5.17) and inductive assumption (5.40) on $D_u I\!D^\sigma \xi_\gamma$, $\sigma \subset \beta$, $\gamma \subset \tau$. A simple modification with additional use of inequality

$$\forall\, k \in \mathbb{Z}^d \quad |D_u\xi^0(t) - D_u\xi^0(s)| \leq \frac{1}{\sqrt{t}} \|D_u\xi^0(t) - D_u\xi^0(s)\|_{\ell_2(a)}$$

and pathwise continuity of process $D_u\xi^0$ (Lemma 5.2) leads to $\psi^u_{\tau,\beta} \in C$ (5.38) and finishes the inductive proof of (5.38), because for inductive base $\psi^u_{\{j\},\emptyset} \equiv 0$.

Estimate (5.37) we prove inductively

$$\frac{d}{dt}\|D_u I\!D^\beta \xi_\tau\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1} c_{\tau,\beta})} = m_\tau < \frac{d}{dt} D_u I\!D^\beta \xi_\tau, (D_u I\!D^\beta \xi_\tau)^\# > \leq$$

$$\leq (m_\tau \|B\| + K_{|\tau|,|\beta|}) \|D_u I\!D^\beta \xi_\tau\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1} c_{\tau,\beta})} + \|F''(\xi^0) D_u\xi^0\, I\!D^\beta \xi_\tau\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1} c_{\tau,\beta})} \tag{5.43}$$

$$+ \sum_{\{\gamma,\sigma\}} t^{|\sigma_0| m_\tau} \|\delta^{\sigma_0} F^{(\ell+|\sigma_0|+1)}(\xi^0) D_u\xi^0 I\!D^{\sigma_1}\xi_{\gamma_1} ... I\!D^{\sigma_\ell}\xi_{\gamma_\ell}\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1} c_{\tau,\beta})} + \tag{5.44}$$

$$+ \sum_{\{\gamma,\sigma\}} \sum_{j=1}^{\ell} t^{|\sigma_0| m_\tau} \|\delta^{\sigma_0} F^{(\ell+|\sigma_0|)}(\xi^0) I\!D^{\sigma_1}\xi_{\gamma_1} ... (D_u I\!D^{\sigma_j}\xi_{\gamma_j}) ... I\!D^{\sigma_\ell}\xi_{\gamma_\ell}\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1} c_{\tau,\beta})} \tag{5.45}$$

Above we used the monotonicity of map F and (3.20).

To estimate (5.43) we remark that property (5.7) implies

$$\forall\, k \in \mathbb{Z}^d \quad \sup_{t\in[0,T]} |D_u\xi^0_k(t, x^0)| \leq \frac{1}{\sqrt{a_k}} e^{MT/2} \left(\int_0^T \|u_s\|^2_{\ell_2(a)} ds\right)^{1/2} \tag{5.46}$$

Using (5.46), Lemma 5.5 (5.15) and (5.3), (5.25) at $\varepsilon_0 = 0$ we have

$$\sup_{t\in[0,T]} \|F''(\xi^0(t)) D_u\xi^0(t) I\!D^\beta \xi_\tau(t)\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+2}{2}m_1} c_{\tau,\beta})} \leq$$

$$\leq e^{MT/2} \left(\int_0^T \|u_s\|^2_{\ell_2(a)} ds\right)^{1/2} \|F''(\xi^0) I\!D^\beta \xi_\tau\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_1} c_{\tau,\beta})} \leq$$

$$\leq K e^{MT/2} \left(\int_0^T \|u_s\|^2_{\ell_2(a)} ds\right)^{1/2} (1 + \|\xi^0\|_{\ell_2(a)})^{\mathbf{k}+1} \|I\!D^\beta \xi_\tau\|_{\ell_{m_\tau}(c_{\tau,\beta})} \leq$$

$$\leq K_R(\omega) \left(\int_0^T \|u_s\|^2_{\ell_2(a)} ds\right)^{1/2}$$

Estimation of (5.44), (5.45) is done by application of (5.46), Lemma 5.5 (5.17), inductive assumption (5.37) and (5.3), (5.25) at $\varepsilon_0 = 0$.

Finally (5.43)-(5.45) transforms to

$$\frac{d}{dt}\|D_u I\!D^\beta \xi_\tau\|^{m_\tau} \leq K' \|D_u I\!D^\beta \xi_\tau\|^{m_\tau} + K_R(\omega) \left(\int_0^T \|u_s\|^2_{\ell_2(a)} ds\right)^{m_\tau/2}$$

which by $D_u I\!D^\beta \xi_\tau|_{t=0} = 0$ leads to (5.37). We used $\psi^u_{\{j\},\emptyset} \equiv 0$ for inductive base. ■

Lemma 5.8. *Let vectors* $\{c_{\tau,\beta}\} \subset I\!P$ *satisfy hierarchy (3.7). Then* $\forall\, u \in \mathcal{J}_{cyl}$ *the solutions* $I\!D^\beta \xi^\varepsilon_\tau$, $D_u I\!D^\beta \xi_\tau$ *to equations (5.22), (5.35) fulfill estimate:* $\forall\, R > 0\ \exists\, K_{u,R} \in \cap_{p\geq 1} L^p(\Omega, \mathbf{P})$ *such that*

$$\sup_{|\varepsilon|\leq\varepsilon_0} \sup_{t\in[0,T]} \| \frac{I\!D^\beta \xi^\varepsilon_\tau - I\!D^\beta \xi_\tau}{\varepsilon} - D_u I\!D^\beta \xi_\tau \|_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1} c_{\tau,\beta})} \leq \varepsilon_0 K_{u,R}(\omega) \qquad (5.47)$$

uniformly on $\max(\|x^0\|_{\ell_2(a)}, \|\tilde{x}_{\gamma,\emptyset}\|_{\ell_{m_\gamma}(c_{\gamma,\emptyset})}, \gamma \subset \tau) \leq R$.

Proof. Denote by $\xi_\emptyset = \xi^0(t, x^0)$, $\xi^\varepsilon_\emptyset = \xi^\varepsilon(t, x^0, u)$ and

$$\Delta_{\tau,\beta}(t) = \frac{I\!D^\beta \xi^\varepsilon_\tau - I\!D^\beta \xi_\tau}{\varepsilon} - D_u I\!D^\beta \xi_\tau$$

Then

$$\frac{d}{dt}\|\Delta_{\tau,\beta}(t)\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1} c_{\tau,\beta})} = m_\tau < \frac{d}{dt}\Delta_{\tau,\beta}, \Delta^\#_{\tau,\beta} >_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1} c_{\tau,\beta})} \leq$$

$$\leq m_\tau \|B\| \cdot \|\Delta_{\tau,\beta}\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1} c_{\tau,\beta})} - m_\tau < \Upsilon, \Delta^\#_{\tau,\beta} >_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1} c_{\tau,\beta})} \qquad (5.48)$$

where

$$\Upsilon = \frac{1}{\varepsilon} F'(\xi^\bullet_\emptyset) I\!D^\beta \xi^\bullet_\tau \Big|_0^\varepsilon - F'(\xi_\emptyset) D_u I\!D^\beta \xi_\tau - F''(\xi_\emptyset) D_u \xi_\emptyset I\!D^\beta \xi_\tau +$$

$$+ \sum_{\gamma,\sigma} t^{|\sigma_0|} \delta^{\sigma_0} \{ \frac{1}{\varepsilon} F^{(\ell+|\sigma_0|)}(\xi^\bullet_\emptyset) I\!D^{\sigma_1} \zeta^\bullet_{\gamma_1} ... I\!D^{\sigma_\ell} \zeta^\bullet_{\gamma_\ell} \Big|_0^\varepsilon -$$

$$- F^{(\ell+|\sigma_0|+1)}(\xi_\emptyset) D_u \xi_\emptyset I\!D^{\sigma_1} \xi_{\gamma_1} ... I\!D^{\sigma_\ell} \xi_{\gamma_\ell} -$$

$$- \sum_{j=1}^{\ell} F^{(\ell+|\sigma_0|)}(\xi_\emptyset) I\!D^{\sigma_1} \xi_{\gamma_1} ... (D_u I\!D^{\sigma_j} \xi_{\gamma_j}) ... I\!D^{\sigma_\ell} \xi_{\gamma_\ell} \}$$

and summation $\sum_{\gamma,\sigma}$ is like in (5.23), (5.35). Using formula

$$f(y_0, .., y_\ell) - f(x_0, ..., x_\ell) = \sum_{j=0}^{\ell} \partial_j f(\vec{x})(y_i - x_i) +$$

$$+ \sum_{j=0}^{\ell} \int_0^1 [\partial_i f(\vec{x} + \lambda(\vec{y} - \vec{x})) - \partial_j f(\vec{x})](y_i - x_i) d\lambda$$

and notation $\zeta^\lambda_\tau = \xi_\tau + \lambda(\xi^\varepsilon_\tau - \xi_\tau)$ with corresponding sense of derivatives $I\!D^\beta \zeta^\lambda_\tau$ we rewrite expression Υ in the form

$$\Upsilon = F'(\xi_\emptyset)\Delta_{\tau,\beta} + F''(\xi_\emptyset)\Delta_{\emptyset,\emptyset} I\!D^\beta \xi_\tau + \qquad (5.49)$$

$$+ \int_0^1 F''(\zeta^\bullet_\emptyset) I\!D^\beta \zeta^\bullet_\tau \Big|_0^\lambda \frac{\xi^\varepsilon_\emptyset - \xi_\emptyset}{\varepsilon} d\lambda + \int_0^1 F'(\zeta^\bullet_\emptyset) \Big|_0^\lambda \frac{I\!D^\beta \xi^\varepsilon_\tau - I\!D^\beta \xi_\tau}{\varepsilon} d\lambda + \qquad (5.50)$$

$$+ \sum_{\gamma,\sigma} t^{|\sigma_0|} \delta^{\sigma_0} F^{(\ell+|\sigma_0|+1)}(\xi_\emptyset) \Delta_{\emptyset,\emptyset} I\!D^{\sigma_1} \xi_{\gamma_1} ... I\!D^{\sigma_\ell} \xi_{\gamma_\ell} + \qquad (5.51)$$

$$+\sum_{\gamma,\sigma}\sum_{j=1}^{\ell} F^{(\ell+|\sigma_0|)}(\xi_\emptyset)\mathbb{D}^{\sigma_1}\xi_{\gamma_1}...(\Delta_{\gamma_j,\sigma_j})...\mathbb{D}^{\sigma_\ell}\xi_{\gamma_\ell}+ \tag{5.52}$$

$$+\sum_{\gamma,\sigma}\int_0^1 F^{(\ell+|\sigma_0|+1)}(\zeta^\bullet_\emptyset)\mathbb{D}^{\sigma_1}\zeta^\bullet_{\gamma_1}...\mathbb{D}^{\sigma_\ell}\zeta^\bullet_{\gamma_\ell}\Big|_0^\lambda \frac{\xi^\varepsilon_\emptyset-\xi_\emptyset}{\varepsilon}d\lambda+ \tag{5.53}$$

$$+\sum_{\gamma,\sigma}\sum_{j=1}^{\ell}\int_0^1 F^{(\ell+|\sigma_0|)}(\zeta^\bullet_\emptyset)\mathbb{D}^{\sigma_1}\zeta^\bullet_{\gamma_1}...(\frac{\mathbb{D}^{\sigma_j}\xi^\varepsilon_{\gamma_j}-\mathbb{D}^{\sigma_j}\xi_{\gamma_j}}{\varepsilon})...\mathbb{D}^{\sigma_\ell}\zeta^\bullet_{\gamma_\ell}\Big|_0^\lambda d\lambda \tag{5.54}$$

Using the monotonicity of $F:\ F'\geq 0$ we continue (5.48)

$$\frac{d}{dt}\|\Delta_{\tau,\beta}(t)\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})} \leq (m_\tau\|B\|+const)\|\Delta_{\tau,\beta}(t)\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})}+$$
$$+\|(5.49)_2\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})}+...+\|(5.54)\|^{m_\tau}_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})} \tag{5.55}$$

where notation $(5.49)_2$ means the second term in (5.49). To estimate $(5.49)_2$ we use coordinate form of (5.12)

$$\forall\, k\in\mathbb{Z}^d \quad \sup_{|\varepsilon|\leq\varepsilon_0}\sup_{t\in[0,T]}|\Delta_{\emptyset,\emptyset}(t)|\leq \varepsilon_0 a_k^{-\frac{\mathbf{k}+2}{2}}K_{u,R}(\omega) \tag{5.56}$$

and Lemma 5.5 (5.15)

$$\|(5.49)_2\| = \|F''(\xi_\emptyset)\Delta_{\emptyset,\emptyset}\mathbb{D}^\beta\xi_\tau\|_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})}\leq$$
$$\leq \varepsilon_0 K_{u,R}(\omega)\|F''(\xi_\emptyset)\mathbb{D}^\beta\xi_\tau\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\tau,\beta})}\leq$$
$$\leq \varepsilon_0 K'_{u,R}(\omega)(1+\|\xi_\emptyset\|_{\ell_2(a)})^{\mathbf{k}+1}\|\mathbb{D}^\beta\xi_\tau\|_{\ell_{m_\tau}(c_{\tau,\beta})}\leq\varepsilon_0 K_{u,R}(\omega) \tag{5.57}$$

where at last step we used (5.3) and (5.25) at $\varepsilon_0=0$.

Estimation of $(5.50)_1$ is done in a similar way with application of coordinate form of (5.2) with $x^0=y^0$ and $u^2=0$

$$\|(5.50)_1\|\leq\sup_{\lambda\in[0,1]}\|F''(\zeta^\bullet)\mathbb{D}^\beta\zeta^\bullet_\tau\Big|_0^\lambda\frac{\xi^\varepsilon_\emptyset-\xi_\emptyset}{\varepsilon}\|_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})}\leq$$
$$\leq e^{MT/2}(\int_0^T\|u_s\|^2_{\ell_2(a)}ds)^{1/2}\|F''(\zeta^\bullet_\emptyset)\mathbb{D}^\beta\zeta^\bullet_\tau\Big|_0^\lambda\|_{\ell_{m_\tau}(a^{(\mathbf{k}+1)m_1}c_{\tau,\beta})}\leq$$
$$\leq Ke^{MT/2}(\int_0^T\|u_s\|^2_{\ell_2(a)}ds)^{1/2}(1+\|\xi_\emptyset\|_{\ell_2(a)}+\|\zeta^\lambda_\emptyset\|_{\ell_2(a)})^{\mathbf{k}+1}\bullet$$
$$\bullet(1+\|\mathbb{D}^\beta\xi_\tau\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\tau,\beta})}+\|\mathbb{D}^\beta\zeta^\lambda_\tau\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\tau,\beta})})\bullet$$
$$\bullet\{\|\zeta^\lambda_\emptyset-\xi_\emptyset\|_{\ell_2(a)}+\|\mathbb{D}^\beta\zeta^\lambda_\tau-\mathbb{D}^\beta\xi_\tau\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\tau,\beta})}\}\leq\varepsilon_0 K_{u,R}(\omega) \tag{5.58}$$

where we successively used Lemma 5.5 (5.15),(5.16), estimates (5.2), (5.3) and (5.25), (5.26) with $x^0=y^0$.

Estimate of $(5.50)_2$ is simple from Lemma 5.5 (5.15)

$$\|(5.50)_2\| \le \sup_{\lambda\in[0,1]} \|F'(\zeta_\emptyset^\bullet)\Big|_0^\lambda \frac{I\!\!D^\beta \xi_\tau^\varepsilon - I\!\!D^\beta \xi_\tau}{\varepsilon}\|_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})} \le$$

$$\le \sup_{\lambda\in[0,1]} K\|\zeta_\emptyset^\lambda - \xi_\emptyset\|_{\ell_2(a)}(1+\|\zeta_\emptyset^\lambda\|_{\ell_2(a)} + \|\xi_\emptyset\|_{\ell_2(a)})^{\mathbf{k}}\bullet$$

$$\bullet\|\frac{I\!\!D^\beta \xi_\tau^\varepsilon - I\!\!D^\beta \xi_\tau}{\varepsilon}\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\tau,\beta})} \le \varepsilon_0 K_{u,R}(\omega) \tag{5.59}$$

where at last step we applied (5.2), (5.3) and (5.26) with $x^0 = y^0$.

To estimate each term in (5.51) we use Lemma 5.5 (5.17) and (5.56)

$$\|(5.51)\| = \|\delta^{\sigma_0}F^{(\ell+|\sigma_0|+1)}(\xi_\emptyset)\Delta_{\emptyset,\emptyset}I\!\!D^{\sigma_1}\xi_{\gamma_1}...I\!\!D^{\sigma_\ell}\xi_{\gamma_\ell}\|_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})} \le$$

$$\le \varepsilon_0 K_{u,R}(\omega)\|\delta^{\sigma_0}F^{(\ell+|\sigma_0|+1)}(\xi_\emptyset)I\!\!D^{\sigma_1}\xi_{\gamma_1}...I\!\!D^{\sigma_\ell}\xi_{\gamma_\ell}\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\tau,\beta})} \le$$

$$\le \varepsilon_0 K_{u,R}(\omega)(1+\|\xi_\emptyset\|_{\ell_2(a)})^{\mathbf{k}+1}\prod_{j=1}^{\ell}(1+\|I\!\!D^{\sigma_j}\xi_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})}) \le \varepsilon_0 K_{u,R}(\omega) \tag{5.60}$$

where we also applied $\|\bullet\|_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\tau,\beta})} \le \|\bullet\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})}$, (2.5), (5.25) and that the weights $\{a^{\frac{\mathbf{k}+1}{2}m_1}c_{\gamma,\sigma}\}_{\gamma\subset\tau,\sigma\subset\beta}$ fulfills hierarchy (3.7).

The estimate of (5.52) is done by Lemma 5.5 (5.17) and inductive assumption (5.47) for $\{\gamma,\sigma\}\subset\{\tau,\beta\}$

$$\|(5.52)\| = \|F^{(\ell+|\sigma_0|)}(\xi_\emptyset)I\!\!D^{\sigma_1}\xi_{\gamma_1}...(\Delta_{\gamma_j,\sigma_j})...I\!\!D^{\sigma_\ell}\xi_{\gamma_\ell}\|_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})} \le$$

$$\le K(1+\|\xi_\emptyset\|)^{\mathbf{k}+1}\|\Delta_{\gamma_j,\sigma_j}\|_{\ell_{m_{\gamma_j}}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\gamma_j,\sigma_j})}\bullet$$

$$\bullet\prod_{i=1,i\ne j}^{\ell}(1+\|I\!\!D^{\sigma_i}\xi_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\gamma_i,\sigma_i})}) \le \varepsilon_0 K_{u,R}(\omega) \tag{5.61}$$

where we also applied (2.5) and (5.25).

Expression (5.53) we estimate by Lemma 5.5 (5.17) and coordinate form of inequality (5.2) with $x^0 = y^0$, $u^2 = 0$

$$\|(5.53)\| \le \sup_{\lambda\in[0,1]} \|F^{(\ell+|\sigma_0|+1)}(\zeta_\emptyset^\bullet)I\!\!D^{\sigma_1}\zeta_{\gamma_1}^\bullet...I\!\!D^{\sigma_\ell}\zeta_{\gamma_\ell}^\bullet\Big|_0^\lambda \frac{\xi_\emptyset^\varepsilon - \xi_\emptyset}{\varepsilon}\|_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})} \le$$

$$\le e^{MT/2}(\int_0^T \|u_s\|^2_{\ell_2(a)}ds)^{1/2}\sup_{\lambda\in[0,1]}\|F^{(\ell+|\sigma_0|+1)}(\zeta_\emptyset^\bullet)I\!\!D^{\sigma_1}\zeta_{\gamma_1}^\bullet...I\!\!D^{\sigma_\ell}\xi_{\gamma_\ell}^\bullet\Big|_0^\lambda\|_{\ell_{m_\tau}(a^{(\mathbf{k}+1)m_1}c_{\tau,\beta})} \le$$

$$\le Ke^{MT/2}(\int_0^T \|u_s\|^2_{\ell_2(a)}ds)^{1/2}\sup_{\lambda\in[0,1]}(1+\|\zeta_\emptyset^\lambda\|_{\ell_2(a)} + \|\xi_\emptyset\|_{\ell_2(a)})^{\mathbf{k}}\bullet$$

$$\bullet\prod_{j=1}^{\ell}(1+\|I\!\!D^{\sigma_j}\xi_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(a^{(\mathbf{k}+1)m_1}c_{\gamma_j,\sigma_j})} + \|I\!\!D^{\sigma_j}\zeta_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(a^{(\mathbf{k}+1)m_1}c_{\gamma_j,\sigma_j})})\bullet$$

$$\bullet\{\|\zeta_\emptyset^\lambda-\xi_\emptyset\|_{\ell_2(a)}+\sum_{j=1}^{\ell}\|I\!\!D^{\sigma_j}\xi_{\gamma_j}-I\!\!D^{\sigma_j}\zeta^\lambda_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(a^{(\mathbf{k}+1)m_1}c_{\gamma_j,\sigma_j})}\}\le\varepsilon_0K_{u,R}(\omega)\quad(5.62)$$

We applied (5.2), (5.3), (5.25) and (5.26) and $\|\bullet\|_{\ell_{m_\gamma}(a^{(\mathbf{k}+1)m_1}c_{\gamma,\sigma})}\le\|\bullet\|_{\ell_{m_\gamma}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\gamma,\sigma})}$.

Finally the estimation of (5.54) follows from Lemma 5.5 (5.17) and inequality (5.26) with $x^0=y^0$ and $u^2=0$

$$\|(5.54)\|\le\sup_{\lambda\in[0,1]}\|F^{(\ell+|\sigma_0|)}(\zeta^\bullet_\emptyset)I\!\!D^{\sigma_1}\zeta^\bullet_{\gamma_1}...\frac{I\!\!D^{\sigma_j}\xi^\varepsilon_{\gamma_j}-I\!\!D^{\sigma_j}\xi_{\gamma_j}}{\varepsilon}...I\!\!D^{\sigma_\ell}\zeta^\bullet_{\gamma_\ell}\Big|_0^\lambda\|_{\ell_{m_\tau}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\tau,\beta})}\le$$

$$\le K\sup_{\lambda\in[0,1]}(1+\|\zeta^\lambda_\emptyset\|_{\ell_2(a)}+\|\xi_\emptyset\|_{\ell_2(a)})^{\mathbf{k}+1}\bullet$$

$$\bullet\prod_{i=1,i\ne j}^{\ell}(1+\|I\!\!D^{\sigma_i}\zeta^\lambda_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\gamma_i,\sigma_i})}+\|I\!\!D^{\sigma_i}\xi_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\gamma_i,\sigma_i})})\bullet$$

$$\bullet\{\|\zeta^\lambda_\emptyset-\xi_\emptyset\|_{\ell_2(a)}+\sum_{i=1,i\ne j}^{\ell}\|I\!\!D^{\sigma_i}\zeta^\lambda_{\gamma_i}-I\!\!D^{\sigma_i}\xi_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\gamma_i,\sigma_i})}\}\bullet$$

$$\bullet\|\frac{I\!\!D^{\sigma_j}\xi^\varepsilon_{\gamma_j}-I\!\!D^{\sigma_j}\xi_{\gamma_j}}{\varepsilon}\|_{\ell_{m_{\gamma_j}}(a^{\frac{2\mathbf{k}+3}{2}m_1}c_{\gamma_j,\sigma_j})}\le\varepsilon_0K_{u,R}(\omega)\qquad(5.63)$$

where at the last step we used (5.2), (5.3) and (5.25), (5.26).

Substituting estimates (5.57)-(5.63) into (5.55) and using that $\Delta_{\tau,\beta}|_{t=0}=0$ we have (5.47). ■

Lemma 5.9. *$\forall x^0\in\ell_2(a)$ $\forall u\in\mathcal{J}_{cyl}$ $\forall k\in Z\!\!Z^d$, $\tau,\beta\subset Z\!\!Z^d$ the coordinates of solutions to equation (5.22) satisfy property $I\!\!D^\beta\xi_{k,\tau}(t,x^0)\in\mathcal{D}_{loc}(\Omega)$, $\forall|\beta|\ge0$, $|\tau|\ge1$, $t\in[0,T]$. Moreover $\forall j\in Z\!\!Z^d$ derivative $D_{\Gamma e_j}I\!\!D^\beta\xi_{k,\tau}$ in direction Γe_j (3.1) coincides with the coordinate of strong solution $I\!\!D^{\beta\cup\{j\}}\xi_{k,\tau}$ to equation (5.22) and estimate (3.8) holds.*

Proof. Formula (5.47) gives

$$\lim_{|\varepsilon|\to0}\mathbf{E}\sup_{t\in[0,T]}|\frac{I\!\!D^\beta\xi^\varepsilon_{k,\tau}-I\!\!D^\beta\xi_{k,\tau}}{\varepsilon}-D_uI\!\!D^\beta\xi_{k,\tau}|^p=0,\quad\forall p\ge1,\ k\in Z\!\!Z^d\qquad(5.64)$$

By coordinate form of (5.25) and (5.37) we have that $D_uI\!\!D^\beta\xi_{k,\tau}(t,x^0)$ is derivative of $I\!\!D^\beta\xi_{k,\tau}(t,x^0)$ in the sense of Definition 2.1 for $u\in\mathcal{J}_{cyl}$.

Representations (5.9), (5.36) and inductive use of representation (5.40) give

$$D_uI\!\!D^\beta\xi_{k,\tau}(t,x^0,\omega)=\int_0^t\sum_{j\in Z\!\!Z^d}[V^{\tau,\beta}_{x^0}(t,s)]_{kj}u_{j,s}ds$$

with some $\mathcal{F}_t$ adapted function $[V^{\tau,\beta}_{x^0}(t,s)]_{kj}$, $0\le s\le t$. The coordinate form of estimate (5.37)

$$\forall u\in\mathcal{J}_{cyl}\quad|D_uI\!\!D^\beta\xi_{k,\tau}(t,x^0)|\le\frac{K_R(\omega)}{a_k^{\frac{\mathbf{k}+2}{2}|\tau|}c^{1/m_\tau}_{k;\tau,\beta}}(\int_0^t\|u_s\|^2_{\ell_2(a)}ds)^{1/2}$$

implies that

$$\int_0^t \| [V_{x^0}^{\tau,\beta}(t,s)]_{k\bullet}\|^2_{\ell_2(a^{-1})} ds \le \frac{K_R(\omega)}{a_k^{\frac{\mathbf{k}+2}{2}|\tau|} c_{\tau,\beta}^{1/m_\tau}}$$

Therefore for $\mathbf{D}_j I\!D^\beta \xi_{k,\tau}(t,x^0) \stackrel{df}{=} \int_0^\bullet \chi_{s\le t}[U_{x^0}^{\tau,\beta}(t,s)]_{kj} ds$

$$\|\mathbf{D}_j I\!D^\beta \xi_{k,\tau}(t,x^0)\|^2_{\mathcal{H}} = \int_0^t | [V_{x^0}^{\tau,\beta}(t,s)]_{kj}|^2 ds \le \frac{a_j K_R(\omega)}{a_k^{\frac{\mathbf{k}+2}{2}|\tau|} c_{k;\tau,\beta}^{1/m_\tau}}$$

and we have representation (2.10) in the form

$$D_u I\!D^\beta \xi_{k,\tau}(t,x^0) = \sum_{j\in\Lambda_u} < \mathbf{D}_j I\!D^\beta \xi_{k,\tau}(t,x^0), \int_0^\bullet u_{s,j} ds >_{\mathcal{H}}$$

with $\mathbf{D}_j I\!D^\beta \xi_{k,\tau}(t,x^0) \in \cap_{p\ge 1} L^p(\Omega,\mathbf{P},\mathcal{H})$. This implies $I\!D^\beta \xi_{k,\tau} \in \mathcal{D}_{loc}(\Omega)$.

Choosing $u = \Gamma e_j \in \mathcal{J}_{cyl}$ in (5.64) and using property $D_u \xi^0_k(t,x^0) = t\delta^j_k$ (2.16) in the coefficients of equation (5.35) one has property $D_{\Gamma e_j} I\!D^\beta \xi_{k,\tau} = I\!D^{\beta\cup\{j\}} \xi_{k,\tau}$. Here the strong solutions to (5.22) are understood in any space $\ell_{m_\tau}(z^n c_{\tau,\beta})$, $n \in I\!N$, for $z_k = a_k^{\frac{\mathbf{k}+1}{2} m_1}$ due to the initial data $\tilde{x}_{\tau,\beta} \in \bigcap_{p\ge 1, c\in I\!P} \ell_p(c)$.

To obtain estimate (3.8) we use representation (5.28), property (5.29) and Lemma 5.5 (5.17)

$$\|I\!D^\beta \xi_\tau(t,x^0)\|_{\ell_{m_\tau}(c_{\tau,\beta})} \le te^{Mt} \sup_{s\in[0,T]} \|\varphi_{\tau,\beta}(s,x^0)\|_{\ell_{m_\tau}(c_{\tau,\beta})} \le$$

$$\le e^{Mt} \sum_{\gamma,\sigma} t^{|\sigma_0|+1} \|\delta^{\sigma_0} F^{(\ell+|\sigma_0|)}(\xi^0) I\!D^{\sigma_1}\xi_{\gamma_1} ... I\!D^{\sigma_\ell}\xi_{\gamma_\ell}\|_{\ell_{m_\tau}(c_{\tau,\beta})} \le$$

$$\le K e^{Mt} \sum_{\gamma,\sigma} t^{|\sigma_0|+1} (1 + \|\xi^0\|_{\ell_2(a)})^{\mathbf{k}+1} \prod_{j=1}^{\ell} (1 + \|I\!D^{\sigma_j}\xi_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(c_{\gamma_j,\sigma_j})}) \tag{5.65}$$

where summation $\sum_{\gamma,\sigma}$ was introduced in (3.6).

Suppose that for $\forall\, \sigma \subset \beta$, $|\sigma| < |\beta|$ inequality (3.8) is proved. Then by estimate (2.5) and inductive assumption we have

$$\forall\, t \in [0,T] \quad (5.65) \le e^{Mt} K_R(\omega) \sum_{\gamma,\sigma} t^{|\sigma_0|+1}\, t^{\sum_{j=1}^{\ell}(|\sigma_j|+1)} \le e^{Mt} K_R(\omega) t^{|\beta|+1}$$

because $|\sigma_0| + \sum_{j=1}^{\ell} |\sigma_j| = |\beta|$. To finish the proof it remains to show the inductive base at $|\beta| = 1$. By Lemma 5.5 (5.15) we have

$$\|I\!D^k \xi_\tau(t)\|_{\ell_{m_\tau}(c_{\tau,\{k\}})} \le te^{Mt} \sup_{s\in[0,T]} \|\varphi_{\tau,\{k\}}(s)\|_{\ell_{m_\tau}(c_{\tau,\{k\}})} \le$$

$$\le te^{Mt} \sup_{s\in[0,T]} \| \sum_{\gamma_1\cup...\cup\gamma_\ell=\tau, \ell\ge 2} s\delta^{\{k\}} F^{(\ell+1)}(\xi^0)\xi_{\gamma_1}...\xi_{\gamma_\ell} + s\delta^{\{k\}} F''(\xi^0)\xi_\tau +$$

$$+ \sum_{\gamma_1\cup...\cup\gamma_\ell=\tau, \ell\ge 2} \sum_{j=1}^{\ell} F^{(\ell)}(\xi^0)\xi_{\gamma_1}...(I\!D^k\xi_{\gamma_j})...\xi_{\gamma_\ell}\|_{\ell_{m_\tau}(c_{\tau,\{k\}})} \le$$

$$\leq t^2 K_R(\omega) \sum_{\gamma\subset\tau} \|\xi_\gamma\|_{\ell_{m_\gamma}(c_{\gamma,\emptyset})} + tK_R(\omega) \sum_{\gamma\subset\tau,\gamma\neq\tau} \|I\!\!D^k \xi_\gamma\|_{\ell_{m_\gamma}(c_{\gamma,\{k\}})} \tag{5.66}$$

At $\tau = \{j\},\ j \in \mathbb{Z}^d$, i.e. $|\tau| = 1$, we simply have from (5.66)

$$\|I\!\!D^k \xi_{\{j\}}(t, x^0)\|_{\ell_{m_1}(c_{\{j\},\{k\}})} \leq t^2 K_R(\omega)$$

Iterating (5.66) on $|\tau| \geq 1$ we have base of induction (3.8) at $|\beta| = 1$. $\blacksquare$

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