Smooth properties of non-Lipschitz differential equations and *a priori* nonlinear quasi contractive estimate.

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ABSTRACT

We investigate the non-linear differential equations with quasi-monotone non-Lipschitz's coefficients on the subject of smooth dependence with respect to the initial data.

To solve this problem we study the corresponding variations and propose to introduce a certain non-linear expression, which reflects intrinsic symmetry of this system and has non-inevitably non-Banach structure.

Quasi-contractive estimate on this expression we apply to prove the C^∞- smoothing and ergodicity in variations for the associated flows. We also deal with the infinite-dimensional essence of the problem considered.

1 Introduction.

The aim of this paper is to investigate the smooth C^∞ dependence with respect to the initial data for the solution of differential equation like

$$\begin{cases} \dfrac{dy^{(0)}}{dt} = -F(y^{(0)}) \quad \boxed{\text{f1}} \\ y^{(0)}(0) = x^0 \end{cases} \tag{1.1}$$

with essentially nonlinear right part F.

Probably (1.1) was the first problem considered as soon as the differential calculus was originated two centuries ago and now there is a vast number of articles, surveys, monographs and textbooks, where such equations and properties of its solutions are profoundly studied. Nevertheless in nonlinear essentially unbounded case it still remains certain open questions, like differentiability of solution on initial data.

For equation with globally Lipschitz's coefficient with bounded Fréchet derivatives the solution is constructed in a standard way as application of fixed point arguments and its C^∞ property is a consequence of implicit function theorem. In this case the choice of appropriate Banach topology is trivial and remains the same for solvability, continuous dependence and differentiability of solutions.

In contrary, for nonlinear equation it arises the problem of adjusting the topology on space, in which the behaviour of solution and its variations can be predictable. In the most excellent case this topology is generated by Banach norm $\|\cdot\|$ in which the solution continuously depends on the initial data and satisfies some estimate of *quasi-contractive* type $\|y(t)\| \leq e^{\omega t}\|y(0)\|$. Such estimate also gives that the problem is well-posed, i.e. the solution does not leave the space, where the initial data are taken.

In the non-Lipschitz case, the most progress has been achieved by the methods of classic nonlinear semigroups theory for monotone (or quasi-monotone) nonlinear equations. The quasi-contractive *a priori* estimates enabled to apply at least in the reduced form the fixed point arguments and use the Lipschitzness of the Yosida approximations *to construct* solutions and state their *continuous* dependence on the initial data [2]-[7], [9, 13, 15], [18]-[20], [22, 23], see also [1, 8, 21, 24] and references therein. The investigation of C^∞ *dependence* on the initial data remains by now the problem of interest in quasi-monotone infinite-dimensional case. Without speaking about the fact that in many applications even the local on balls Lipschitz's property of coefficients fails, we first note that in the associated system in variations appear unbounded operators. Moreover, these operators are controlled by the solution of initial ODE itself and this compels us to study the properties of corresponding non-autonomous equations. At last the fact that each variation (derivative of solution on the initial data) is interlaced with the lower rank variations rather complicates the selection of quasi-contractive topologies.

We show that the system in variations, obtained by the direct formal differentiation of initial nonlinear equation admits *non-linear weighted estimate of quasi-contractive type*. Here different variations are interlaced in the intrinsic non-linear manner and the essence of considered problem requires to introduce the weight, controlled by the solution itself. As a result the associated expression doesn't permit the interpretation of norm in some Banach space.

We apply the obtained quasi-contractive estimates to investigate C^∞-smoothing on the initial data and the exponential ergodicity in variations.

2 Description of the problem.

Consider the problem

$$\begin{cases} \dfrac{dy^{(0)}}{dt} = -F(y^{(0)}(t)) - By^{(0)}(t) \quad \boxed{\text{d1}} \\ y^{(0)}(0) = x^0 \end{cases} \tag{2.1}$$

where $F : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ is a non-linear diagonal map

$$F : \mathbb{R}^{\mathbb{Z}^d} \ni x = \{x_k\}_{k\in\mathbb{Z}^d} \to F(x) = \{F(x_k)\}_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$$

for $\mathbb{Z}^d$ to be a d-dimensional integer lattice. Monotone function $F \in C^\infty(\mathbb{R}, \mathbb{R})$, $F(0) = 0$ has no more than polynomial growth

$$\exists\, C \in \mathbb{R}^1,\ \mathbf{k} \geq -1 \ \ \textit{such that} \ \ \forall\, i = 1, .., n \ \ \forall\, x, y \in \mathbb{R}$$

$$|F^{(i)}(x) - F^{(i)}(y)| \leq C|x - y|(1 + |x| + |y|)^{\mathbf{k}} \boxed{\text{d2}} \tag{2.2}$$

The linear map $B : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ has a representation on the vector $x = \{x_k\}_{k \in \mathbb{Z}^d}$

$$(Bx)_k = \sum_{j \in \mathbb{Z}^d} b(k - j)x_j$$

with the finite-diagonal matrix B, i.e.

$$\exists\, r_0 \ \ \forall\, j \in \mathbb{Z}^d \ \ |j| > r_0 \quad b(j) \equiv 0 \boxed{\text{d3}} \tag{2.3}$$

where $|j| = |j_1| + .. + |j_d|$ for $j = (j_1, .., j_d) \in \mathbb{Z}^d$.

In the coordinate form equation (2.1) can be written as *infinite system* of the ordinary differential equations, *interlacing* through the matrix B

$$\begin{cases} \dfrac{dy_k^{(0)}(t)}{dt} = -F(y_k^{(0)}(t)) - \sum\limits_{j:\ |k-j| \leq r_0} b(k - j)y_j^{(0)}(t) \\ y_k^{(0)}(0) = x_k^0, \quad k \in \mathbb{Z}^d \end{cases} \boxed{\text{d4}} \tag{2.4}$$

Henceforth we use the same notation for the diagonal map $F : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ and function $F : \mathbb{R} \to \mathbb{R}$.

At the first look the taken nonlinear system is of a very special type. However, the investigation of similar equations in ℓ_p spaces, at one time, has stimulated a remarkable progress in field of nonlinear analysis ([2]). Moreover, we should remark that for $\mathbf{k} > -1$ in (2.2) even this simple map $F : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ is non-Lipschitz's in any space $\ell_{m_0}(a)$ and has the unbounded Frechet derivatives.

From the other hand we were motivated by the problems coming from the statistical physics, where such systems describe the infinite system of statistical particles, matrix B plays the role of interaction and map F, roughly speaking, recovers the evolution of each particle. In this case the smoothing properties of evolution of such system mathematically and technically correspond to the differentiability of equations, analogous to (1.1), with respect to the initial data.

Further we will use a standard notation

$$\ell_{m_0}(a) = \{x \in \mathbb{R}^{\mathbb{Z}^d},\ \|x\|_{\ell_{m_0}(a)} = (\sum_{k \in \mathbb{Z}^d} a_k |x_k|^{m_0})^{1/m_0} < \infty\}, \qquad m_0 > 1$$

$$a = \{a_k\}_{k \in \mathbb{Z}^d}, \qquad \sum_{k \in \mathbb{Z}^d} a_k = 1, \quad \gamma_a = \sup_{|k-j|=1} |a_k/a_j| < \infty$$

The following Theorem states the existence of strong solutions for the Cauchy problem (2.1) in the space $\ell_{m_0}(a)$.

Theorem 2.1. *Suppose that the conditions (2.2),(2.3) on function F and matrix B hold.*

Then for any $x^0 \in \ell_{m_0(\mathbf{k}+1)}(a)$ there is a unique strong solution to the problem (2.1), i.e. Lipschitz continuous function $y^{(0)} \in C([0,T], \ell_{m_0}(a))$ such that

1. *$y^{(0)}(0) = x^0$, and $y^{(0)}(\,\cdot\,) \in \mathcal{D}_{\ell_{m_0}(a)}(F+B)$ a.e. on $[0,T]$;*
2. *$\exists$ a strong derivative $\ell_{m_0}(a) - \dfrac{dy^{(0)}(t)}{dt}$ a.e. on $[0,T]$;*
3. *Equation (2.1) is satisfied a.e on $[0,T]$.*

The norm $[0,T] \ni t \to \|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)} \in I\!\!R$ is a.e. on $t \in [0,T]$ differentiable and its derivative fulfills

$$\frac{d}{dt}\|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)} \le m_0\omega\|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)} \quad \text{a.e. on } [0,T] \boxed{rkrk} \tag{2.5}$$

with $\omega = \|B\|_{\ell_{m_0}(a)}$. For any initial data $x_1, x_2 \in \ell_{m_0(\mathbf{k}+1)}(a)$ the strong solutions $y^{(0)}(\,\cdot\,, x_1)$, $y^{(0)}(\,\cdot\,, x_2)$ satisfy

$$\exists\,\omega \quad \|y^{(0)}(t,x_1) - y^{(0)}(t,x_2)\|_{\ell_{m_0}(a)} \le e^{\omega t}\|x_1 - x_2\|_{\ell_{m_0}(a)} \boxed{d9} \tag{2.6}$$

For convenience of reader we give the proof of this Theorem in the Appendix to the paper.

To investigate the C^∞–smooth dependence of solution $y^{(0)}(t,x^0)$ on the initial data x^0 we write the associated system in variations.

Let $\tau = \{j_1, .., j_n\}$, $j_s \in Z\!\!\!Z^d$ be any ordered array of points from $Z\!\!\!Z^d$. To the set τ we correspond vector $y_\tau = \{y_{k,\tau}\}_{k\in Z\!\!\!Z^d}$ which we will later give a sense of derivative

$$\frac{\partial^{|\tau|} y_k^{(0)}(t,x^0)}{\partial x^0_{j_n} ... \partial x^0_{j_1}} = y_{k,\tau} \boxed{\text{d12}} \tag{2.7}$$

Differentiating (2.4) on x^0, we obtain equation for functions y_τ:

$$\begin{cases} \dfrac{dy_{k,\tau}(t)}{dt} = -F'(y_k^{(0)})y_{k,\tau} - \sum\limits_{j:\,|k-j|\le r_0} b(k-j)y_{j,\tau} - f_{k,\tau} \\ y_{k,\tau}(0) = x_{k,\tau} \end{cases} \boxed{\text{d13}} \tag{2.8}$$

In (2.8) the inhomogeneous part $f_\tau = \{f_{k,\tau}(y^{(0)}; y_\gamma, \gamma \subset \tau, \gamma \ne \tau)\}_{k\in Z\!\!\!Z^d}$ depends on the lower rank variations $\{y_\gamma\}_{\gamma\subset\tau,\ \gamma\ne\tau}$. It is defined reccurently by

$$\begin{cases} f_{k,\tau} \equiv 0, \quad |\tau| = 1 \\ f_{k,\tau\cup i} = F''(y_k^{(0)})y_{k,i}y_{k,\tau} + \dfrac{\partial f_{k,\tau}(y_k^{(0)};\, y_{k,\gamma},\, \gamma\subset\tau,\, \gamma\ne\tau)}{\partial y_k^{(0)}} y_{k,i} + \\ \qquad + \sum\limits_{\gamma\subset\tau, \gamma\ne\tau} \dfrac{\partial f_{k,\tau}(y_k^{(0)};\, y_{k,\gamma},\, \gamma\subset\tau,\, \gamma\ne\tau)}{\partial y_{k,\gamma}} y_{k,\gamma\cup i}, \quad |\tau| \ge 1 \end{cases} \boxed{\text{d15}} \tag{2.9}$$

or can be written as

$$f_{k,\tau} = \begin{cases} 0, \ |\tau| = 1 \\ \sum\limits_{\gamma_1 \cup .. \cup \gamma_s = \tau, \ s \geq 2} F^{(s)}(y_k^{(0)}) y_{k,\gamma_1} ... y_{k,\gamma_s}, \quad |\tau| \geq 2 \end{cases} \boxed{\text{d16}} \qquad (2.10)$$

The summation above

$$\sum_{\gamma_1 \cup .. \cup \gamma_s = \tau, \, s \geq 2}$$

runs on all possible subdivisions of the set $\tau = \{j_1, .., j_n\}, \quad j_i \in \mathbb{Z}^d$ on the non-intersecting subsets $\gamma_1, .., \gamma_s \subset \tau$, with $|\gamma_1| + .. + |\gamma_s| = |\tau|$, $s \geq 2$, $|\gamma_i| \geq 1$.

The strong solution to problem (2.8) we understand in the following sence.

Definition 2.2. Let $m_\tau > 1$. Function $y_\tau(t), \ \tau = \{j_1, .., j_n\}, \ j_s \in \mathbb{Z}^d$ is a *strong solution of the system in variations in space* $\ell_{m_\tau}(c_\tau)$ iff the map

$$[0, T] \ni t \to y_\tau(t) \in \ell_{m_\tau}(c_\tau)$$

is Lipschitz continuous and fulfills

1. $y_{k,\tau}(0) = x_{k,\tau}$, and $y_\tau(t) \in \mathcal{D}_{\ell_{m_\tau}(c_\tau)}(F'(y^{(0)}(t)) + B)$ a.e. on $[0, T]$;
2. $\exists$ a strong $\ell_{m_\tau}(c_\tau)$ derivative $\dfrac{dy_\tau(t)}{dt}$ a.e. on $[0, T]$;
3. Equation (2.8) is satisfied in $\ell_{m_\tau}(c_\tau)$ a.e on $[0, T]$ with $f_\tau(t) \in \ell_{m_\tau}(c_\tau)$ a.e. on $[0, T]$.

Theorem 4.5 later gives sufficient conditions for existence and uniqueness of such solutions. We remark that the due to the nonlinear multiplicative structure of functions $\{f_\tau\}$ in (2.10) it is rather complicated to check assumption $f_\tau \in \ell_{m_\tau}(c_\tau)$ imposed in Definition 2.2. Thus we use the idea of the nonlinear estimate of Theorem 3.3 to guess the proper relations between topologies $\{\ell_{m_\gamma}(c_\gamma)\}_{\gamma \subset \tau}$ on the strong solutions.

At last note that the interpretation (2.7) in $\ell_p(c)$ sense is only possible at the special choice of "zero-one" initial data (Theorem 5.3):

$$x_{k,\tau} = \begin{cases} \delta_{k,j}, \quad |\tau| = 1, \ \tau = \{j\} \subset \mathbb{Z}^d \\ 0, \quad |\tau| > 1 \end{cases} \boxed{\text{d14}} \qquad (2.11)$$

3 *A priori* nonlinear quasi-contractive estimate on variations.

Before to give the definition of nonlinear expression and prove the quasi-contractive estimate for it, we need some notation which reflects the special *hierarchy* between the spaces for variations.

Let $I\!\!P$ denote the set of vectors $c = \{c_k\}_{k\in Z\!\!\!Z^d} \in (I\!\!R_+)^{Z\!\!\!Z^d}$ satisfying

$$\gamma_c = \sup_{|k-j|=1} |c_k/c_j| < \infty \boxed{\text{d5}} \tag{3.1}$$

Definition 3.1. Fix $m_1 > 1$. The family $\{c_\tau\}_\tau \subset I\!\!P$, $c_\tau = \{c_{k,\tau}\}_{k\in Z\!\!\!Z^d}$, enumerated by all finite ordered arrays $\tau = \{j_1, .., j_n\}$ of points $j_s \in Z\!\!\!Z^d$, is called a *vector weight*, iff it satisfies the following condition:

$\forall\, \tau = \{j_1, .., j_n\},\ j_i \in Z\!\!\!Z^d$ and for any subdivision of the set τ on non-empty subsets $\gamma_1, .., \gamma_s,\ s \geq 2$

$$\tau = \gamma_1 \cup .. \cup \gamma_s, \quad |\gamma_1| + .. + |\gamma_s| = |\tau|$$

$\exists\ const_{\tau,\gamma_1,..,\gamma_s}$ such that $\forall\, k \in Z\!\!\!Z^d$

$$[c_{k,\tau}]^{|\tau|} a_k^{-\frac{\mathbf{k}+1}{m_0} m_1} \leq const_{\tau,\gamma_1,..,\gamma_s} [c_{k,\gamma_1}]^{|\gamma_1|} ... [c_{k,\gamma_s}]^{|\gamma_s|} \boxed{\text{d17}} \tag{3.2}$$

Upper indexes mean the powers, $|\gamma|$ denotes the number of points in γ and the parameter of nonlinearity $\mathbf{k}$ is introduced in (2.2).

Remark 3.2. (1) In the Definition 3.1 we do not require that all points of the set τ are different. In this case we understand actually the subdivision of the set τ as generated by a subdivision of the set $\{1, .., n\}$.

(2) *The set of vectors, which satisfy condition (3.2), is not empty.*

Indeed, fix some $d = \{d_k\}_{k\in Z\!\!\!Z^d} \in I\!\!P$. Introduce vectors $s_{k,\tau} = d_k [c_{k,\tau}]^{-1}$. Then in the terms of s_τ condition (3.2) adopts the following form

$$[s_{k,\gamma_1}]^{|\gamma_1|} ... [s_{k,\gamma_s}]^{|\gamma_s|} \leq const_{\tau,\gamma_1..\gamma_s}\, a_k^{+\frac{\mathbf{k}+1}{m_0} m_1} [s_{k,\tau}]^{|\tau|}$$

Starting from vectors $s_\tau,\ |\tau| = 1$, we can always choose such weights inductively on the number of points in τ. Moreover for the weight $\{c_\tau\}$ and arbitrary vector $d \in I\!\!P$ we have that $\{d \cdot c_\tau\}$ also form a weight. For arbitrary $d \in I\!\!P$ the example of vector weight can be given by:

$$c_\tau^{a,d} = \begin{cases} d, & |\tau| = 1 \\ d \cdot a^{+\frac{\mathbf{k}+1}{m_0} \frac{n-1}{n} m_1}, & |\tau| = n \end{cases} \boxed{\text{cad}} \tag{3.3}$$

Note that in the Lipschitz's case $\mathbf{k} = -1$ we can simply put $c_\tau^{a,d} \equiv d$ at any τ.

Theorem 3.3. *Let the maps F and B fulfill conditions (2.2) and (2.3) and functions $y^{(0)}(t),\ \{y_\gamma(t),\ \gamma \subset \tau\}$ be the strong solutions to the problems (2.1) and (2.8) correspondingly. Fix $m_1 \in [2n, \infty),\ n \in I\!\!N$.*

For any $\tau = \{j_1, .., j_n\},\ j_i \in Z\!\!\!Z^d$ introduce the non-linear expression

$$\rho_\tau(y; t) = \sum_{s=1}^{n} \{\, p_s(z) \sum_{\gamma\subset\tau,\ |\gamma|=s} \|y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} \} \boxed{\textit{d18}} \tag{3.4}$$

with $z = \|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)}$.

Suppose that:

1. *The powers* $m_\gamma = m_1/|\gamma|$*;*
2. *The family* $\{c_\gamma\}$ *is a vector weight;*
3. *The functions* $p_i(z) \in C^\infty(I\!R^1),\ i = 1,..,n$ *satisfy:*
   (a) $\exists\, \varepsilon > 0\ \ p_i(z) \geq \varepsilon, \quad z \in I\!R_+$ *and* $\exists\, K > 0\ \ |zp_i'(z)| \leq Kp_i(z)$*;*
   (b) $\exists\, K_p\ \ \forall\, j = 2,..,n\ \forall\, i_1,..,i_s:\ i_1 + ... + i_s = j,\ s \geq 2$

$$[p_j(z)]^j (1+z)^{\frac{\mathbf{k}+1}{m_0} m_1} \leq K_p [p_{i_1}(z)]^{i_1} ... [p_{i_s}(z)]^{i_s},\ z \geq 0 \boxed{d19} \tag{3.5}$$

Then $\exists\, M_\tau = M(\tau, \omega, \mathbf{k}, m_0, m_1, \{c_\gamma\}_{\gamma\subset\tau}, \{p_i\}_{i=1}^n)$ *such that* $\forall\, t \geq 0$ *the* **nonlinear quasi-contractive estimate** *on the strong solutions* $\{y_\gamma\}_{\gamma\subset\tau}$ *of the system in variations (2.8) is true*

$$\rho_\tau(y;t) \leq e^{M_\tau t} \rho_\tau(y;0) \boxed{d20} \tag{3.6}$$

Proof. For $z = \|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)}$ introduce functions

$$h^i_\tau(y;t) = \begin{cases} 0, & i = 0 \\ \sum\limits_{s=1}^{i} \{p_s(z) \sum\limits_{\gamma\subset\tau,\ |\gamma|=s} \|y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}\}, & i \geq 1 \end{cases}$$

Note that for $|\tau| = n$ $\rho_\tau(y;t) = h^n_\tau(y;t)$ and

$$h^i_\tau(y;t) = h^{i-1}_\tau(y;t) + \sum_{\gamma\subset\tau,\ |\gamma|=i} g_\gamma(t) \boxed{\text{d21-1}} \tag{3.7}$$

with

$$g_\gamma(t) = p_i(z) \|y_\gamma(t)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}, \quad |\gamma| = i \boxed{\text{d21}} \tag{3.8}$$

We prove inductively that

$$\forall\, i = 0,..,n\ \ \exists\, M_i \in I\!R \quad h^i_\tau(y;t) \leq e^{M_i t} h^i_\tau(y;0) \boxed{\text{kako}} \tag{3.9}$$

which at $i = n$ provides the statement of Theorem. The base of induction at $i = 0$ is obvious.

If for any $\gamma \subset \tau,\ |\gamma| = i$ we prove inequality

$$\frac{dg_\gamma(t)}{dt} \leq K_1 g_\gamma(t) + K_2 h^{i-1}_\tau(y;t) \boxed{\text{d22}} \tag{3.10}$$

thus we obtain the estimate on $g_\gamma(t)$

$$g_\gamma(t) \leq e^{K_1 t} g_\gamma(0) + K_2 \int_0^t e^{K_1(t-s)} h^{i-1}_\tau(y;s) ds \boxed{\text{d22-1}} \tag{3.11}$$

Using the inductive assumption, representation (3.7) and (3.11) we obtain

$$h_\tau^i(y;t) \le e^{M_{i-1}t} h_\tau^{i-1}(y;0) +$$

$$+ \sum_{\gamma\subset\tau,\ |\gamma|=i} \{e^{K_1 t} g_\gamma(0) + K_2 \int_0^t e^{K_1(t-s)} e^{M_{i-1}s} h_\tau^{i-1}(y;0) ds\} \le \boxed{\text{d22-3}} \tag{3.12}$$

$$\le e^{(M_{i-1}+K_1)t}(1 + 2^{|\tau|}K_2 t) h_\tau^i(y;0) \le e^{(M_{i-1}+K_1+2^{|\tau|}K_2)t} h_\tau^i(y;0)$$

and this gives the inductive step.

It remains to prove (3.10). The norm of a strong solution y_γ of the system in variations (2.8) is differentiable a.e. on $[0,T]$ ([2, 18, 19, 20], see also (7.2)). Moreover, by Theorem 2.1 function $z = \|y^{(0)}(t)\|_{\ell_{m_0}(a)}$ is also differentiable a.e. on $[0,T]$. Simple calculations together with (2.5), assumption 3(a) and (3.15) give

$$\frac{dg_\gamma(t)}{dt} \le (m_0 K \|B\|_{\mathcal{L}(\ell_{m_0}(a))} + m_\gamma \|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))}) g_\gamma(t) + \boxed{\text{d25}} \tag{3.13}$$

$$+ m_\gamma p_i(\|y^{(0)}\|^{m_0}_{\ell_{m_0}(a)}) | < f_\gamma, [y_\gamma]^\# > | \boxed{\text{d26}} \tag{3.14}$$

Above we also used inequality

$$\frac{d}{dt}\|y_\gamma(t)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} = m_\gamma < \frac{d}{dt} y_\gamma(t), [y_\gamma(t)]^\# > \le$$

$$\le m_\gamma \|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))} \|y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} + m_\gamma | < f_\gamma, y_\gamma^\# > | \boxed{\text{d17-2}} \tag{3.15}$$

which is obtained due to $F'(x) \ge 0,\ x \in \mathbb{R}$ and boundedness of map B in any space $\ell_p(c)$ with $1 < p < \infty$ and $c \in \mathbb{P}$ (see (7.7)). For $x \in \ell_p(c)$ we denote $x^\# = \|x\|^{p-2}_{\ell_p(c)} \mathcal{F} x$ with duality map $\mathcal{F}$ defined in the space $\ell_p(c),\ 1 < p < \infty$ by

$$(\mathcal{F}x)_k = \frac{x_k |x_k|^{p-2}}{\|x\|^{p-2}_{\ell_p(c)}}$$

If $|\gamma| = 1$ then (3.10) is already proved, because $f_\gamma = 0$. If $|\gamma| > 1$ using the representation (2.10) and inequality

$$| < z, x^\# > | \le \|z\|_{\ell_m(c)} \|x^\#\|_{\ell^*_m(c)} \le \frac{1}{m}\|z\|^m_{\ell_m(c)} + \frac{m-1}{m}\|x\|^m_{\ell_m(c)} \boxed{\text{egeg}} \tag{3.16}$$

we have

$$(3.14) \le C_\gamma (m_\gamma - 1) g_\gamma(t) +$$

$$+ \sum_{\alpha_1\cup..\cup\alpha_s=\gamma,\ s\ge 2} p_i(\|y^{(0)}\|^{m_0}_{\ell_{m_0}(a)}) \|F^{(s)}(y^{(0)}) y_{\alpha_1} ... y_{\alpha_s}\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} \boxed{\text{d28}} \tag{3.17}$$

Above the summation runs on all possible subdivisions $\alpha_1, .., \alpha_s,\ s \ge 2$ of the set $\gamma,\ |\gamma| = i$ on the non-intersecting subsets.

Condition (2.2) on the function F gives

$$|F^{(s)}(y_k^{(0)})| \leq C a_k^{-\frac{\mathbf{k}+1}{m_0}} (1+\|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)})^{\frac{\mathbf{k}+1}{m_0}}$$

Thus each term in (3.17) is estimated by

$$C \sum_{k\in \mathbb{Z}^d} c_{k,\gamma} a_k^{-\frac{\mathbf{k}+1}{m_0} m_\gamma} \; p_i(\|y^{(0)}\|^{m_0}_{\ell_{m_0}(a)})(1+\|y^{(0)}\|^{m_0}_{\ell_{m_0}(a)})^{-\frac{\mathbf{k}+1}{m_0} m_\gamma} |y_{k,\alpha_1}|^{m_\gamma} ... |y_{k,\alpha_s}|^{m_\gamma} \boxed{\text{d29}} \tag{3.18}$$

Because $m_\gamma = m_\alpha \frac{|\alpha|}{|\gamma|}$ we have $|y_{k,\alpha_1}|^{m_\gamma}...|y_{k,\alpha_s}|^{m_\gamma} = [|y_{k,\alpha_1}|^{m_{\alpha_1}}]^{|\alpha_1|/|\gamma|}...[|y_{k,\alpha_s}|^{m_{\alpha_s}}]^{|\alpha_s|/|\gamma|}$. Conditions 2 and 3(b) of Theorem imply

$$(3.18) \leq K_p \, C_{\gamma,\alpha_1..\alpha_s} \sum_{k\in\mathbb{Z}^d} \prod_{i=1}^{s} \{p_{|\alpha_i|}(\|y^{(0)}\|^{m_0}_{\ell_{m_0}(a)}) c_{k,\alpha_i} |y_{k,\alpha_i}|^{m_{\alpha_i}}\}^{|\alpha_i|/|\gamma|} \boxed{\text{d33}} \tag{3.19}$$

Finally we apply to (3.19) inequality $|x_1...x_s| \leq |x_1|^{q_1}/q_1 + ... + |x_s|^{q_s}/q_s$ with $q_j = |\gamma|/|\alpha_j|$, $1/q_1 + .. + 1/q_s = 1$ and obtain

$$(3.19) \leq K_p \, C_{\gamma,\alpha_1..\alpha_s} \sum_{j=1}^{s} \frac{|\alpha_j|}{|\gamma|} p_{|\alpha_j|}(\|y^{(0)}\|^{m_0}_{\ell_{m_0}(a)}) \|y_{\alpha_j}\|^{m_{\alpha_j}}_{\ell_{m_{\alpha_j}}(c_{\alpha_j})} \leq$$

$$\leq K_p \, C_{\gamma,\alpha_1,..,\alpha_s} h_\tau^{i-1}(y;t) \boxed{\text{d34}} \tag{3.20}$$

Above we have used that $s \geq 2$ in representation $\gamma = \alpha_1 \cup .. \cup \alpha_s$, $\gamma \subset \tau$, $|\gamma| = i$ and so $|\alpha_j| \leq i-1$, $j = 1,..,s$.

Therefore the inequality (3.10) is proved with constants

$$K_1 = m_0 \, K \, \|B\|_{\mathcal{L}(\ell_{m_0}(a))} + \sup_{\gamma\subset\tau}(m_\gamma \|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))} + (m_\gamma - 1) 2^{|\gamma|^2}),$$

$$K_2 = K_p \, C \, 2^{|\tau|^2} \max_{\alpha_1\cup..\cup\alpha_s=\gamma, \; \gamma\subset\tau} C_{\gamma,\alpha_1...\alpha_s}$$

■

4 Strong solvability of variational system.

The main result of this sections is the following theorem, which states the solvability of system (2.8) with inhomogeneous part (2.10).

Theorem 4.5. *Fix arbitrary* $\tau = \{j_1,..,j_n\}$, $j_s \in \mathbb{Z}^d$ *such that* $|\tau| = n \leq m_1/2$, $m_1 > 1$. *Let* $x_0 \in \ell_{m_0(\mathbf{k}+1)}(a)$, $\{c_\alpha\}$, $\alpha \subset \tau$ *be a vector weights and* $d = \{d_k\}_{k\in\mathbb{Z}^d} \in \mathbb{P}$ *satisfy*

$$d_k \geq a_k^{-\frac{(\mathbf{k}+1)}{m_0} m_1} \geq 1 \boxed{\textit{d51}} \tag{4.1}$$

Then for all $\alpha \subset \tau$ if initial data $x_\alpha \in Y_\alpha$, there is a unique function

$$y_\alpha \in C([0,T], X_\alpha) \cap L^\infty([0,T], Y_\alpha)$$

which is a strong solution in the space X_α for the system (2.8). Above $X_\alpha = \ell_{m_\alpha}(c_\alpha)$ and $Y_\alpha = \ell_{m_\alpha}(d \cdot c_\alpha)$, with $m_\alpha = m_1/|\alpha|$.

First of all we remark that (2.8) is an infinite dimensional system of linear on y_τ equations of the type

$$\begin{cases} \dfrac{dy_\tau(t)}{dt} = -A(t)y_\tau(t) + f_\tau(t) \\ y_\tau(0) = x_\tau \end{cases}$$

with non-autonomous coefficient $A(t) = F'(y^{(0)}(t)) + B$. These equations are not only interlaced through the matrix B, like in (2.1), but also have inhomogeneous parts f_τ, which essentially depend on the lower rank variations $\{y_\gamma\}_{\gamma \subset \tau}$.

To solve the Cauchy problem (2.8) we use the following result [10, Thm.2.1 and 2.2], which is a development of the standard criterions [11, 12, 14, 16, 17].

Theorem 4.1 *Let $Y \subset X$ be Banach spaces with continuous embedding operator, such that they possess the closed ball property, i.e. the closed balls in Y are closed in the strong topology of X.*

Suppose that

1. *$\forall\, t \in [0,T]$ the operator $A(t)$ is a generator of a strongly continuous semigroup $\{e^{-sA(t)}\}_{s \in \mathbb{R}_+}$ of bounded linear operators in X with*

   $$\exists\,\omega\ \forall\, t \in [0,T] \quad \|e^{-sA(t)}\|_{\mathcal{L}(X)} \le e^{\omega s}, \quad s \in \mathbb{R}_+$$

   The corresponding semigroup preserve Y: $\forall\, t \in [0,T],\ s \in \mathbb{R}_+,\ e^{-sA(t)}Y \subset Y$ and

   $$\exists\,\omega_1 \quad \forall\, t \in [0,T],\ u \in Y \quad \|e^{-sA(t)}u\|_Y \le e^{\omega s}\|u\|_Y, \quad s \in \mathbb{R}_+$$

2. *The operator-valued function*

   $$A(\cdot) \upharpoonright_Y \in C([0,T], \mathcal{L}(Y,X))$$

   and fulfills $\forall\, t \in [0,T]$

   $$Y \in \mathcal{D}_X(A(t))$$

3. *Function $f_\tau \in C([0,T], X) \cap L^\infty([0,T], Y)$*

Then for any initial data $u_0 \in Y$ the Cauchy problem

$$\begin{cases} u' = -A(t)u(t) + f(t) \quad \boxed{d35} \\ u(0) = u_0 \end{cases} \tag{4.2}$$

has a unique solution

$$u \in C([0,T], X) \cap L^\infty([0,T], Y)$$

which possesses the strong X - derivative $u'(t) = X - \dfrac{d}{dt}u(t)$ and fulfills (4.2) a.e. on $t \in [0,T]$.

Thus for solvability of the system in variations (2.8) we should cope with a series of problems. We not only need to construct the spaces X_τ and $Y_\tau \subset X_\tau$ such that $Y_\tau \subset \mathcal{D}_{X_\tau}(A(t))$ but also guarantee smoothness of f_τ in X_τ and boundedness in essential in Y_τ. Recall that functions f_τ in their turn (ÿ бÿ®о ®зГаГ¤м) interlace the lowe order variations $\{y_\gamma\}_{\gamma\subset\tau}$ and this impose conditions on the structure of spaces X_γ, Y_γ, $|\gamma| \le |\tau|$ for lower variations.

We are going to show that spaces $\{\ell_{m_\gamma}(c_\gamma)\}_{\gamma\subset\tau}$ introduced in the nonlinear estimate (3.6) are specially adopted for these Cauchy problems.

To begin with we need the following Proposition, permits to verify the conditions of Theorem. This result we also use in the Section 5 to obtain C^∞ – differentiability of y_0 on initial data.

Proposition 3.3 * Let $a \in I\!\!P$, $tr\, a = 1$, $|\tau| < m_1$, vectors $\{c_\gamma\}_{\gamma\subset\tau}$ satisfy (3.2) and $d \in I\!\!P$ with $d_k \ge a_k^{-\frac{\mathbf{k}+1}{2}m_1}$. Denote $X_\gamma = \ell_{m_\gamma}(c_\gamma)$, $Y_\gamma = \ell_{m_\gamma}(dc_\gamma)$, $m_\gamma = m_1/|\gamma|$, $\gamma \subset \tau$. Then for function Q, such that $\exists K \quad |Q(x) - Q(y)| \le K|x-y|(1+|x|+|y|)^{\mathbf{k}}$, $\forall x, y \in I\!\!R^1$ the map $\ell_2(a) \ni x \to Q(x) \in \mathcal{L}(Y_\tau, X_\tau)$, $[Q(y)u]_k = Q(y_k)u_k$ is continuous and

$$\exists C\, \forall x, y \qquad \|Q(x)\|_{\mathcal{L}(Y_\tau,X_\tau)} \le C(1+|Q(0)|)(1+\|x\|_{\ell_2(a)})^{\mathbf{k}+1} \boxed{\text{EE1}} \tag{4.3}$$

$$\|\,[Q(x)-Q(y)]\,u\|_{X_\tau} \le C\|x-y\|_{\ell_2(a)}(1+\|x\|_{\ell_2(a)}+\|y\|_{\ell_2(a)})^{\mathbf{k}}\|u\|_{Y_\tau} \boxed{\text{EEE}} \tag{4.4}$$

Moreover, introduce $\{Q(y^{(0)})y_{\gamma_1}\dots y_{\gamma_n}\}_k = Q(y_k^{(0)})y_{k,\gamma_1}\dots y_{k,\gamma_n}$, $k \in Z\!\!\!Z^d$ for $n \ge 2$, $\gamma_1 \cup \dots \cup \gamma_n = \tau$ with the corresponding action of $Q(y^{(0)})y_{\gamma_1}\dots y_{\gamma_\ell}$ in the space $\bigotimes_{i=\ell+1}^{n} X_{\gamma_i}$. Then for any fixed $\ell = 1,..,n$ the map

$$\ell_2(a) \times \mathop{\times}_{i=1}^{\ell} X_{\gamma_i} \ni (y^{(0)}, y_{\gamma_1},..,y_{\gamma_\ell}) \to Q(y^{(0)})y_{\gamma_1}...y_{\gamma_\ell} \in \mathcal{L}(\bigotimes_{i=\ell+1}^{n} X_{\gamma_i}, X_\tau)$$

is continuous and the estimate holds

$$\begin{aligned}\exists K \qquad &\|\,[Q(y^{(0)})y_{\gamma_1}...y_{\gamma_\ell} - Q(z^{(0)})z_{\gamma_1}...z_{\gamma_\ell}\,]\,u_{\gamma_{\ell+1}}...u_{\gamma_n}\|_{X_\tau} \le \\ &\le K\{1+\|y^{(0)}\|_{\ell_2(a)}+\|z^{(0)}\|_{\ell_2(a)}\}^{\mathbf{k}+1}\prod_{i=1}^{\ell}[1+\|y_{\gamma_i}\|_{X_{\gamma_i}}+\|z_{\gamma_i}\|_{X_{\gamma_i}}]\times \\ &\times\{\|y^{(0)}-z^{(0)}\|_{\ell_2(a)}+\sum_{i=1}^{\ell}\|y_{\gamma_i}-z_{\gamma_i}\|_{X_{\gamma_i}}\}\prod_{i=\ell+1}^{n}\|u_{\gamma_i}\|_{X_{\gamma_i}} \boxed{\text{E12}}\end{aligned} \tag{4.5}$$

Proposition 4.2. *Fix m_0, $p > 1$. Let function $G : I\!\!R \to I\!\!R$ satisfy*

$$\exists K, \mathbf{k} \quad \forall x, y \in I\!\!R \qquad |G(x) - G(y)| \le K|x-y|(1+|x|+|y|)^{\mathbf{k}} \boxed{d36\text{-}2} \tag{4.6}$$

and vector $d = \{d_k\}_{k\in Z\!\!\!Z^d} \in I\!\!P$ be such that

$$1 \le a_k^{-\frac{\mathbf{k}+1}{m_0}p} \le const\, d_k \boxed{d36\text{-}3} \tag{4.7}$$

Then for any vector $c=\{c_k\}_{k\in\mathbb{Z}^d}\in I\!P$ *the map*

$$\ell_{m_0}(a)\ni x\to G(x)\in\mathcal{L}(\ell_p(dc),\ell_p(c))$$

is a continuous and the estimate is valid

$$\|(G(x)-G(y))u\|_{\ell_p(c)}\le$$

$$\le const\|x-y\|_{\ell_{m_0}(a)}(1+\|x\|_{\ell_{m_0}(a)}+\|y\|_{\ell_{m_0}(a)})^{\mathbf{k}}\|u\|_{\ell_p(dc)}$$

where the operator $G(x)$ *is defined by coordinate representation* $[G(x)u]_k=G(x_k)u_k$.

Moreover this map is bounded on balls in $\ell_{m_0}(a)$

$$\|G(x)\|_{\mathcal{L}(\ell_p(dc),\ell_p(c))}\le K(1+|G(0)|)(1+\|x\|_{\ell_{m_0}(a)})^{\mathbf{k}+1}$$

Proof. Using (4.6) we have

$$\sum_{k\in\mathbb{Z}^d}c_k|(G(x_k)-G(y_k))u_k|^p\le$$

$$\le const\sum_k c_k|(x_k-y_k)(1+|x_k|+|y_k|)^{\mathbf{k}}u_k|^p\le$$

$$\le const\sum_k\frac{c_k}{(a_k^{1/m_0}a_k^{\mathbf{k}/m_0})^p}\times$$

$$|(a_k(x_k-y_k)^{m_0})^{1/m_0}\cdot(a_k^{1/m_0}+(a_k|x_k|^{m_0})^{1/m_0}+(a_k|y_k|^{m_0})^{1/m_0})^{\mathbf{k}}u_k|^p\le$$

$$\le const\|x-y\|^p_{\ell_{m_0}(a)}(1+\|x\|_{\ell_{m_0}(a)}+\|y\|_{\ell_{m_0}(a)})^{\mathbf{k}p}\sum_k c_kd_k|u_k|^p\boxed{\text{d37}}\qquad(4.8)$$

which follows from $a_k^{-p(\mathbf{k}+1)/m_0}\le const\,d_k$. Taking the p^{th} root in (4.8) we obtain

$$\|(G(x)-G(y))u\|_{\ell_p(c)}\le$$

$$\le const\,\|x-y\|_{\ell_{m_0}(a)}(1+\|x\|_{\ell_{m_0}(a)}+\|y\|_{\ell_{m_0}(a)})^{\mathbf{k}}\|u\|_{\ell_p(cd)}$$

The second statement is obvious. ■

Now we control conditions 1,2 of Theorem 4.1.

Theorem 4.3. *Let* $y^{(0)}$ *be a strong solution to the problem (2.1-2.4) in the space* $\ell_{m_0}(a)$. *Fix* $m_1>1,\ m_\gamma=m_1/|\gamma|$ *and the vector weight* $\{c_\tau\}$.

Suppose the weight $d=\{d_k\}_{k\in\mathbb{Z}^d}\in I\!P$ *satisfies*

$$d_k\ge a_k^{-\frac{(\mathbf{k}+1)}{m_0}m_1}\ge 1\boxed{trtr}\qquad(4.9)$$

Then for any $|\tau|\le m_1$ *the family of operators*

$$A(t)=F'(y^{(0)}(t))+B$$

fulfill conditions 1,2 of Theorem 4.1 in the spaces

$$X = X_\tau = \ell_{m_\tau}(c_\tau) \qquad Y = Y_\tau = \ell_{m_\tau}(d \cdot c_\tau) \boxed{d40} \tag{4.10}$$

Proof. <u>Condition 1</u> follows from the fact that $\forall x \in I\!R^{\mathbb{Z}^d}$ the linear operator $F'(x) + B$ is quasi m - monotone in any space $\ell_p(c),\ c \in I\!P,\ p \in (1, \infty)$.

Indeed, consider $x \in I\!R^{\mathbb{Z}^d}$. Then from monotonicity of function $F : I\!R^1 \to I\!R^1$ it follows that for any $h \in \mathcal{D}_{\ell_p(c)}(F'(x))$ we have

$$< F'(x)h, \mathcal{F}h >_{\ell_p(c)} = \sum_{k\in\mathbb{Z}^d} c_k \frac{F'(x_k)|h_k|^p}{\|h\|^{p-2}_{\ell_p(c)}} \geq 0$$

i.e. $F'(x)$ is monotone map in $\ell_p(c)$. Remark that

$$\forall x \in I\!R^{\mathbb{Z}^d} \quad \mathcal{D}_{\ell_p(c)}(F'(x) + B) \neq \emptyset$$

and contains any vectors $h = \{h_k\}_{k\in\mathbb{Z}^d}$ with finite number of non-zero components h_k. To obtain m - monotonicity of $F'(x)$ we to need prove $\forall \lambda > 0\ \ \mathcal{R}(1+\lambda F'(x)) = \ell_p(c)$. First note that for any z_k there is y_k such that

$$y_k + \lambda F'(x_k) y_k = z_k$$

which is $y_k = (1 + \lambda F'(x_k))^{-1} z_k$. Inequality $F'(x_k) \geq 0$ implies $\|y\|_{\ell_p(c)} \leq \|z\|_{\ell_p(c)}$ and

$$\|F'(x)y\|_{\ell_p(c)} \leq \frac{1}{\lambda}\|z\|_{\ell_p(c)}$$

Therefore $\forall z \in \ell_p(c)\ \exists y \in \mathcal{D}_{\ell_p(c)}(F'(x))$ and thus the map $F'(x)$ is m-monotone.

The boundedness of operator B in any space $\ell_p(c),\ c \in I\!P,\ p \in (1, \infty)$ (7.7) and the criterion on quasi m - monotonicity of sum of m - monotone and bounded linear operator [2, p.158] give condition 1.

<u>Condition 2</u>. The continuity of constant map

$$[0, T] \ni t \to B \in \mathcal{L}(\ell_{m_\tau}(dc_\tau), \ell_{m_\tau}(c_\tau))$$

follows from the embedding $\ell_{m_\tau}(dc_\tau) \subset \ell_{m_\tau}(c_\tau)$ (as $d_k \geq 1$) and from the boundedness of B in $\ell_{m_\tau}(dc_\tau)$, see (7.7).

From (4.9) at $p = m_\tau = m_1/|\tau|$ we have

$$d_k \geq a_k^{-\frac{(\mathbf{k}+1)}{m_0} m_1} \geq a_k^{-\frac{(\mathbf{k}+1)}{m_0} p}$$

for any $m_1 \geq |\tau| \geq 1$. This by Theorem 4.2 gives

$$\|F'(y^{(0)}(t)) - F'(y^{(0)}(s))\|_{\mathcal{L}(\ell_{m_\tau}(dc_\tau), \ell_{m_\tau}(c_\tau))} \leq$$

$$\leq const\, \|y^{(0)}(t) - y^{(0)}(s)\|_{\ell_{m_0}(a)} (1 + \|y^{(0)}(t)\|_{\ell_{m_0}(a)} + \|y^{(0)}(s)\|_{\ell_{m_0}(a)})^{\mathbf{k}}$$

By the continuity of the strong solution $y^{(0)}$ the expressions $\|y^{(0)}(t)\|_{\ell_{m_0}(a)}$ and $\|y^{(0)}(s)\|_{\ell_{m_0}(a)}$ are bounded. Therefore the map

$$[0,T] \ni t \to \{F'(y^{(0)}(t)) + B\} \in \mathcal{L}(Y_\tau, X_\tau) \boxed{\text{d41}} \tag{4.11}$$

is continuous for any τ, $|\tau| \le m_1$. Remark that (4.11) also leads to

$$\forall\, t \in [0,T] \quad Y_\tau \subset \mathcal{D}_{X_\tau}(F'(y^{(0)}(t)) + B) = \mathcal{D}_{X_\tau}(F'(y^{(0)}(t))) \quad \blacksquare$$

The following Proposition we use in Theorem 4.5 to control condition 3 of Theorem 4.1

$$f_\tau \in C([0,T], X_\tau) \cap L^\infty([0,T], Y_\tau)$$

In fact we obtain below the continuity of the multiplicative map

$$\ell_{m_0}(a) \times \mathop{\times}_{\gamma \subset \tau} \ell_{m_\gamma}(c_\gamma) \to \ell_{m_\tau}(c_\tau)$$

$$(y^{(0)},\ y_\gamma) \to f_\tau = \sum_{\gamma_1 \cup .. \cup \gamma_s = \tau,\ s \ge 2} F^{(s)}(y_k^{(0)}) y_{k,\gamma_1} ... y_{k,\gamma_s}$$

which in the essentially nonlinear manner depends on the first coordinate $y^{(0)} \in \ell_{m_0}(a)$. The continuity of this map is actually guaranteed by the special hierarchy of $\{c_\gamma\}$ reflected in the notion of vector weight.

Later we use this Proposition to justify part $f_\tau \in \ell_{m_\tau}(c_\tau)$ in Definition 2.2 of strong solution. We also apply it to the investigation of the C^∞ – properties of the flow $x^0 \to y^{(0)}(t, x^0)$.

Proposition 4.4. *Fix $m_1 > 1$ and vector weight $\{c_\tau\}$. Consider function $Q : \mathbb{R} \to \mathbb{R}$, which satisfy*

$$\exists\, K \quad \forall\, x, y \in \mathbb{R} \qquad |Q(x) - Q(y)| \le const\, |x - y|(1 + |x| + |y|)^{\mathbf{k}}$$

Let $n \ge 2$. Fix $\gamma_1, .., \gamma_n$ to be finite arrays of points from $\mathbb{Z}^d$ and denote $m_\gamma = m_1/|\gamma|$, $\tau = \gamma_1 \cup .. \cup \gamma_n$.

Suppose that $|\tau| = |\gamma_1| + .. + |\gamma_n| \le m_1$. Then:

1. *The map*

$$\ell_{m_0}(a) \times \mathop{\times}_{i=1}^{n} \ell_{m_{\gamma_i}}(c_{\gamma_i}) \ni (y^0, y_{\gamma_1}, .., y_{\gamma_n}) \to Q(y^0) y_{\gamma_1} ... y_{\gamma_n} \in \ell_{m_\tau}(c_\tau)$$

is continuous, i.e. the estimate holds

$$\|Q(y^0) y_{\gamma_1} ... y_{\gamma_n} - Q(z^0) z_{\gamma_1} ... z_{\gamma_n}\|_{\ell_{m_\tau}(c_\tau)} \le$$

$$\le K_1 \{1 + \|y^0\|_{\ell_{m_0}(a)} + \|z^0\|_{\ell_{m_0}(a)}\}^{\mathbf{k}+1} \prod_{i=1}^{n} [1 + \|y_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})} + \|z_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}] \times$$

$$\times \{\|y^0 - z^0\|_{\ell_{m_0}(a)} + \sum_{i=1}^{n} \|y_{\gamma_i} - z_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}\} \boxed{d43} \tag{4.12}$$

2. For any fixed $s = 1, .., n-1$ the map

$$\ell_{m_0}(a) \times \underset{i=1}{\overset{s}{\times}} \ell_{m_{\gamma_i}}(c_{\gamma_i}) \ni (y^0, y_{\gamma_1}, .., y_{\gamma_s}) \to$$

$$\to Q(y^0) y_{\gamma_1} ... y_{\gamma_s} \in \mathcal{L}(\bigotimes_{i=s+1}^{n} \ell_{m_{\gamma_i}}(c_{m_{\gamma_i}}), \ell_{m_\tau}(c_\tau)), \quad n = |\tau|$$

is continuous, i.e. the estimate holds

$$\|Q(y^0) y_{\gamma_1} ... y_{\gamma_s} - Q(z^0) z_{\gamma_1} ... z_{\gamma_s}\|_{\mathcal{L}(\bigotimes_{i=s+1}^{n} \ell_{m_{\gamma_i}}(c_{\gamma_i}), \ell_{m_\tau}(c_\tau))} \leq$$

$$\leq K_2 \{1 + \|y^0\|_{\ell_{m_0}(a)} + \|z^0\|_{\ell_{m_0}(a)}\}^{\mathbf{k}+1} \prod_{i=1}^{s} [1 + \|y_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})} + \|z_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}] \times$$

$$\times \{\|y^0 - z^0\|_{\ell_{m_0}(a)} + \sum_{i=1}^{s} \|y_{\gamma_i} - z_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}\} \boxed{d44} \tag{4.13}$$

Above in 1,2 we understand under $Q(y^{(0)}) y_{\gamma_1} ... y_{\gamma_n}$ the map

$$\{Q(y^{(0)}) y_{\gamma_1} \cdots y_{\gamma_n}\}_k = Q(y_k^{(0)}) y_{k,\gamma_1} \cdots y_{k,\gamma_n}, \quad k \in \mathbb{Z}^d$$

with the corresponding action of $Q(y^{(0)}) y_{\gamma_1} \cdots y_{\gamma_s}$ in the space $\bigotimes_{i=s+1}^{n} \ell_{m_{\gamma_i}}(c_{\gamma_i})$.

Proof. The proof of the both parts of Proposition simply follows from estimate

$$\|Q(y^0) y_{\gamma_1} \cdots y_{\gamma_n} - Q(z^0) z_{\gamma_1} ... z_{\gamma_n}\|_{\ell_{m_\tau}(c_\tau)} \leq$$

$$\leq \textit{const}\,(1 + |Q(0)|)(1 + \|y^0\|_{\ell_{m_0}(a)} + \|z^0\|_{\ell_{m_0}(a)})^{\mathbf{k}}$$

$$\Bigg\{ \|y^0 - z^0\|_{\ell_{m_0}(a)} \prod_{i=1}^{n} (\|y_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})} + \|z_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}) + \boxed{\text{d45}} \tag{4.14}$$

$$+ (1 + \|y^0\|_{\ell_{m_0}(a)} + \|z^0\|_{\ell_{m_0}(a)}) \sum_{s=1}^{n} \|y_{\gamma_s} - z_{\gamma_s}\|_{\ell_{m_{\gamma_s}}(c_{\gamma_s})} \cdot \boxed{\text{d46}} \tag{4.15}$$

$$\cdot \prod_{i=1, i \neq s}^{n} (\|y_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})} + \|z_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}) \Bigg\}$$

To obtain (4.12) from (4.14-4.15) we enlarge (where necessary) the multiplicators and powers and put $K_1 = const(1 + |Q(0)|)$.

To obtain (4.13) we set $y_{\gamma_i} = z_{\gamma_i}$ for $i = s+1, .., n$ in (4.14-4.15), extract the product $\prod_{i=s+1}^{n} (2\|y_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})})$ and put $K_2 = 2^{n-s} const(1 + |Q(0)|)$.

Let us prove (4.14-4.15). First we add and subtract additional terms and obtain

$$\|Q(y^0) y_{\gamma_1} ... y_{\gamma_n} - Q(z^0) z_{\gamma_1} ... z_{\gamma_n}\|_{\ell_{m_\tau}(c_\tau)} \leq$$

$$\leq \|(Q(y^0) - Q(z^0)) y_{\gamma_1} ... y_{\gamma_n}\|_{\ell_{m_\tau}(c_\tau)} + \boxed{\text{d47}} \tag{4.16}$$

$$+\sum_{s=1}^{n}\|Q(z^0)z_{\gamma_1}...z_{\gamma_{s-1}}(y_{\gamma_s}-z_{\gamma_s})y_{\gamma_{s+1}}...y_{\gamma_n}\|_{\ell_{m_\tau}(c_\tau)}\boxed{\text{d47-2}} \tag{4.17}$$

Estimate term (4.16) in power m_τ

$$(4.16)^{m_\tau}=\sum_{k\in\mathbb{Z}^d}c_{k,\tau}|Q(y_k^0)-Q(z_k^0)|^{m_\tau}|y_{k,\gamma_1}...y_{k,\gamma_n}|^{m_\tau}\le \boxed{\text{d48}} \tag{4.18}$$

$$\le const\sum_{k\in\mathbb{Z}^d}c_{k,\tau}|y_k^0-z_k^0|^{m_\tau}(1+|y_k^0|+|z_k^0|)^{m_\tau\mathbf{k}}|y_{k,\gamma_1}...y_{k,\gamma_n}|^{m_\tau}=$$

$$=const\sum_{k\in\mathbb{Z}^d}\frac{c_{k,\tau}}{a_k^{m_\tau/m_0}a_k^{m_\tau\mathbf{k}/m_0}}(a_k|y_k^0-z_k^0|^{m_0})^{m_\tau/m_0}\cdot$$

$$\cdot(a_k^{1/m_0}+(a_k|y_k^0|^{m_0})^{1/m_0}+(a_k|z_k^0|^{m_0})^{1/m_0})^{m_\tau\mathbf{k}}|y_{k,\gamma_1}...y_{k,\gamma_n}|^{m_\tau}\le$$

$$\le const\,\|y^0-z^0\|^{m_\tau}_{\ell_{m_0}(a)}(1+\|y^0\|_{\ell_{m_0}(a)}+\|z^0\|_{\ell_{m_0}(a)})^{m_\tau\mathbf{k}}\cdot$$

$$\cdot\sum_{k\in\mathbb{Z}^d}c_{k,\tau}a_k^{-\frac{\mathbf{k}+1}{m_0}m_1}|y_{k,\gamma_1}...y_{k,\gamma_n}|^{m_\tau}$$

From (3.2) and the Hölder inequality with $q_i=|\tau|/|\gamma_i|$, $\sum 1/q_i=1$ we have

$$\sum_{k\in\mathbb{Z}^d}c_{k,\tau}a_k^{-\frac{\mathbf{k}+1}{m_0}m_1}|y_{k,\gamma_1}...y_{k,\gamma_n}|^{m_\tau}\le$$

$$\le\sum_{k\in\mathbb{Z}^d}[c_{k,\gamma_1}]^{|\gamma_1|/|\tau|}...[c_{k,\gamma_n}]^{|\gamma_n|/|\tau|}|y_{k,\gamma_1}...y_{k,\gamma_n}|^{m_\tau}=$$

$$=\sum_{k\in\mathbb{Z}^d}[c_{k,\gamma_1}|y_{k,\gamma_1}|^{m_{\gamma_1}}]^{|\gamma_1|/|\tau|}...[c_{k,\gamma_n}|y_{k,\gamma_n}|^{m_{\gamma_n}}]^{|\gamma_n|/|\tau|}\le$$

$$\le\prod_{i=1}^{n}(\sum_{k\in\mathbb{Z}^d}c_{k,\gamma_i}|y_{k,\gamma_i}|^{m_{\gamma_i}})^{|\gamma_i|/|\tau|}\boxed{\text{d49-1}} \tag{4.19}$$

Taking the m_τ root in (4.18)-(4.19) we obtain

$$(4.16)=\|(Q(y^0)-Q(z^0))y_{\gamma_1}...y_{\gamma_n}\|_{\ell_{m_\tau}(c_\tau)}\le$$

$$\le const\,\|y^0-z^0\|_{\ell_{m_0}(a)}(1+\|y^0\|_{\ell_{m_0}(a)}+\|z^0\|_{\ell_{m_0}(a)})^{\mathbf{k}}\cdot$$

$$\cdot\|y_{\gamma_1}\|_{\ell_{m_{\gamma_1}}(c_{\gamma_1})}...\|y_{\gamma_n}\|_{\ell_{m_{\gamma_n}}(c_{\gamma_n})}\le$$

$$\le const\,\|y^0-z^0\|_{\ell_{m_0}(a)}(1+\|y^0\|_{\ell_{m_0}(a)}+\|z^0\|_{\ell_{m_0}(a)})^{\mathbf{k}}\cdot$$

$$\cdot\prod_{i=1}^{n}(\|y_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}+\|z_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})})\boxed{\text{d50}} \tag{4.20}$$

which gives (4.14).

Note that each term in (4.17) has analogous to (4.16) structure, with $Q(y^0)-Q(z^0)$ replaced by $Q(z^0)$, y_{γ_i} for $i=1,..,s-1$ replaced by z_{γ_i} and y_{γ_s} by $y_{\gamma_s}-z_{\gamma_s}$.

Therefore, the reasoning (4.18-4.20) can be applied in this case and we need only to estimate $Q(z_k^0)$

$$|Q(z_k^0)| \leq |Q(0)| + |Q(z_k^0) - Q(0)| \leq$$
$$\leq |Q(0)| + const\,|z_k^0|(1+|z_k^0|)^{\mathbf{k}} \leq$$
$$\leq const\,(1+|Q(0)|)(1+|z_k^0|)^{\mathbf{k}+1}$$

in step (4.18).

Finally for any fixed $s = 1,..,n$

$$(4.17)_s = \|Q(z^0)z_{\gamma_1}...z_{\gamma_{s-1}}(y_{\gamma_s} - z_{\gamma_s})y_{\gamma_{s+1}}...y_{\gamma_n}\|_{\ell_{m_\tau}(c_\tau)} \leq$$

$$\leq const(1+|Q(0)|)(1+\|z^0\|_{\ell_{m_0}(a)})^{\mathbf{k}+1} \cdot (\prod_{i=1}^{s-1} \|z_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}\|)\cdot$$

$$\cdot\|y_{\gamma_s} - z_{\gamma_s}\|_{\ell_{m_{\gamma_s}}(c_{\gamma_s})} \cdot (\prod_{i=s+1}^{n} \|y_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})})$$

which imply (4.15). ■

Now we apply Theorem 4.1 and construct the strong solutions $\{y_\tau\}_\tau$ to the system in variations (2.8). This gives that the non-linear quasi-contractive estimate (3.6) holds for arbitrary initial data $x_\tau \in Y_\tau$.

Proof. We prove Theorem 4.5 by induction on $|\alpha|,\ \alpha \subset \tau$ as an application of Theorem 4.1.

The conditions 1,2 on the map $A(t) = F'(y^{(0)}(t)) + B$ are already checked in Theorem 4.3. Moreover, the closed ball property is satisfied by the reflexivity of $\ell_p(c),\ 1 < p < \infty$.

In fact it remains to control reccurently condition 3.

1. The base of induction. For $|\alpha| = 1$ the condition 3 is obvious and we have for $\alpha \subset \tau,\ |\alpha| = 1$ a unique strong solution y_α.

2. The inductive step. Suppose that for all $\alpha \subset \tau,\ |\alpha| \leq n_0 - 1$ with $n_0 \leq |\tau|$ the function y_α is a strong solution to problem (2.8).

We only have to prove that

$$\forall\,\alpha \subset \tau\ |\alpha| = n_0 \quad f_\alpha \in C([0,T], X_\alpha) \cap L^\infty([0,T], Y_\alpha)$$

with

$$f_{k,\alpha} = \sum_{\gamma_1\cup..\cup\gamma_s=\alpha,\ s\geq 2} F^{(s)}(y_k^{(0)})y_{k,\gamma_1}...y_{k,\gamma_s}\boxed{\text{d53}} \tag{4.21}$$

a. First we check that $\forall\,\alpha \subset \tau,\ |\alpha| = n_0$: $f_\alpha \in L^\infty([0,T], Y_\alpha)$

$$\|f_\alpha\|_{Y_\alpha}^{m_\alpha} = \|f_\alpha\|_{\ell_{m_\alpha}(d\cdot c_\alpha)}^{m_\alpha} =$$

$$= \sum_{k\in\mathbb{Z}^d} d_k c_{k,\alpha}|\sum_{\gamma_1\cup..\cup\gamma_s=\alpha,\ s\geq 2} F^{(s)}(y_k^{(0)})y_{k,\gamma_1}...y_{k,\gamma_s}|^{m_\alpha} \leq$$

$$\leq K \sum_{\gamma_1\cup...\cup\gamma_s=\alpha, s\geq 2} \|F^{(s)}(y^{(0)})y_{k,\gamma_1}...y_{k,\gamma_s}\|_{\ell_{m_\alpha}(dc_\alpha)}^{m_\alpha}\boxed{\text{d54}} \tag{4.22}$$

From (2.2) and Proposition 4.4, we have

$$\|F^{(s)}(y^{(0)})y_{\gamma_1}...y_{\gamma_s}\|_{\ell_{m_\alpha}(dc_\alpha)} \leq$$

$$\leq const\,(1+\|y^{(0)}\|_{\ell_{m_0}(a)})^{\mathbf{k}+1}\prod_{i=1}^{s}(1+\|y_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(dc_{\gamma_i})})\times$$

$$\times(\|y^{(0)}\|_{\ell_{m_0}(a)}+\sum_{i=1}^{s}\|y_{\gamma_i}\|_{\ell_{m_{\gamma_i}}(dc_{\gamma_i})})\boxed{\text{d55}} \tag{4.23}$$

with $m_{\gamma_i} = m_1/|\gamma_i|$.

Inductive assumption implies that $\forall\,\gamma \subset \tau,\ |\gamma| \leq n_0-1\ y_\gamma \in L^\infty([0,T],Y_\gamma)$. This gives the boundedness of terms

$$\|y^{(0)}\|_{\ell_{m_0}(a)},\ \ \|y_{\gamma_1}\|_{\ell_{m_{\gamma_1}}(dc_{\gamma_1})},..,\|y_{\gamma_s}\|_{\ell_{m_{\gamma_s}}(dc_{\gamma_s})}$$

Therefore (4.22) is uniformly on $t \in [0,T]$ bounded and $f_\alpha \in L^\infty([0,T],Y_\alpha)$ for $\alpha \subset \tau,\ |\alpha| \leq n_0$.

b. We prove that $\forall\,\alpha \subset \tau,\ |\alpha| = n_0$: $f_\alpha \in C([0,T],X_\alpha)$

Indeed

$$\|f_\alpha(t_1)-f_\alpha(t_2)\|_{X_\alpha}^{m_\alpha} = \|f_\alpha(t_1)-f_\alpha(t_2)\|_{\ell_{m_\alpha}(c_\alpha)}^{m_\alpha} =$$

$$= \sum_{k\in\mathbb{Z}^d} c_{k,\alpha}|\sum_{\gamma_1\cup..\cup\gamma_s=\alpha,s\geq 2}\{F^{(s)}(y_k^{(0)}(t_1))y_{k,\gamma_1}(t_1)...y_{k,\gamma_s}(t_1)-$$

$$-F^{(s)}(y_k^{(0)}(t_2))y_{k,\gamma_1}(t_2)...y_{k,\gamma_s}(t_2)\}|^{m_\alpha} \leq$$

$$\leq K\sum_{\gamma_1\cup..\cup\gamma_s=\alpha,s\geq 2}\|F^{(s)}(y^{(0)}(t_1))y_{\cdot,\gamma_1}(t_1)...y_{\cdot,\gamma_s}(t_1)-$$

$$-F^{(s)}(y_\cdot^{(0)}(t_2))y_{\cdot,\gamma_1}(t_2)...y_{\cdot,\gamma_s}(t_2)\|_{\ell_{m_\alpha}(c_\alpha)}^{m_\alpha}\boxed{\text{d56}} \tag{4.24}$$

Proposition 4.4 implies

$$\|F^{(s)}(y^{(0)}(t_1))y_{\cdot,\gamma_1}(t_1)...y_{\cdot,\gamma_s}(t_1)-$$

$$-F^{(s)}(y^{(0)}(t_2))y_{\cdot,\gamma_1}(t_2)...y_{\cdot,\gamma_s}(t_2)\|_{\ell_{m_\alpha}(c_\alpha)} \leq$$

$$\leq const\,(1+\|y^{(0)}(t_1)\|_{\ell_{m_0}(a)}+\|y^{(0)}(t_2)\|_{\ell_{m_0}(a)})^{\mathbf{k}+1}\times$$

$$\times\prod_{i=1}^{s}(1+\|y_{\cdot,\gamma_i}(t_1)\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}+\|y_{\cdot,\gamma_i}(t_2)\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})})\times$$

$$\times\{\|y^{(0)}(t_1)-y^{(0)}(t_2)\|_{\ell_{m_0}(a)}+\sum_{i=1}^{s}\|y_{\cdot,\gamma_i}(t_1)-y_{\cdot,\gamma_i}(t_2)\|_{\ell_{m_{\gamma_i}}(c_{\gamma_i})}\}\boxed{\text{d57}} \tag{4.25}$$

By the inductive assumption and Theorem 2.1

$$\forall\,\gamma \subset \tau,\ |\gamma| \leq n_0-1\ \ y_\gamma \in C([0,T],X_\gamma)\ \ \&\ \ y^{(0)} \in C([0,T],\ell_{m_0}(a))$$

Therefore we have the required continuity

$$f_\alpha \in C([0,T],X_\alpha)$$

for $\alpha \subset \tau,\ |\alpha| = n_0$.

Applying at last Theorem 4.1 we have that for any $\alpha \subset \tau,\ |\alpha| \le n_0$ there is a unique strong solution

$$y_\alpha \in C([0,T], X_\alpha) \cap L^\infty([0,T], Y_\alpha) \qquad \blacksquare$$

Corollary 4.6. *In the case of zero-one initial data (2.11) the statement of Theorem 4.5 reads as follows:*

Let $x_0 \in \ell_{m_0(\mathbf{k}+1)}(a),\ m_1 > 1$ and the vector weights $\{c_\tau\}$ be fixed. Let vector $d = \{d_k\}_{k\in\mathbb{Z}^d} \in \mathbb{P}$ satisfy

$$d_k \ge a_k^{-\frac{(\mathbf{k}+1)}{m_0}m_1} \boxed{d58} \tag{4.26}$$

and put by definition $m_\tau \stackrel{def}{=} m_1/|\tau|$.

Then for zero-one initial data (2.11) we have that there are functions

$$y_\tau \in \bigcap_{i\ge 0} C([0,T], \ell_{m_\tau}(d^i c_\tau))$$

satisfying the Cauchy problems (2.8) in any space $\ell_{m_\tau}(d^i c_\tau)$

$$\begin{cases} \dfrac{dy_{k,\tau}}{dt} = -F'(y_k^{(0)})y_{k,\tau} - \sum\limits_{j\in\mathbb{Z}^d} b(k-j)y_{j,\tau} - \\ \qquad - \sum\limits_{\gamma_1\cup..\cup\gamma_s=\tau,\ s\ge 2} F^{(s)}(y_k^{(0)})y_{k,\gamma_1}...y_{k,\gamma_s} \boxed{d59} \\ y_{k,\tau}(0) = \begin{cases} \delta_{kj}, & \tau = \{j\},\ |\tau| = 1 \\ 0, & |\tau| \ge 2 \end{cases} \end{cases} \tag{4.27}$$

Moreover, for $\tau = \{j_1,..,j_n\},\ j_s \in \mathbb{Z}^d,\ |\tau| \le m_1/2$, we have the reduced form of the nonlinear quasi-contractive estimate (3.6)

$$\sum_{s=1}^n \{p_s(z) \sum_{\gamma\subset\tau,\ |\gamma|=s} \|y_\gamma(t)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}\} \le De^{Mt}p_1(\|x^0\|^{m_0}_{\ell_{m_0}(a)}) \boxed{d59\text{-}3} \tag{4.28}$$

with $D = \sum\limits_{i=1}^n c_{j_i,\{j_i\}}$ and $z = \|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)}$. Above functions $\{p_i\}$ satisfy (a-b) in (3.5).

Proof. The zero-one initial data x_τ (2.11) belong actually to $\bigcap\limits_{i\ge 1} \ell_{m_\tau}(d^i c_\tau)$, so it is only to apply Theorem 4.5 succesively on $i \in \mathbb{N}$ with $X_\tau^i = \ell_{m_\tau}(d^i c_\tau)$ and $Y_\tau^i = \ell_{m_\tau}(d^{i+1} c_\tau)$, $Y_\tau^i = X_\tau^{i+1}$. Estimate (4.28) follows from Theorem 3.3. $\blacksquare$

5 C^∞ differentiability on the initial data.

In this part we investigate how the solution $y^{(0)}(t,x^0) \in \ell_{m_0}(a)$ of the initial Cauchy problem (2.1) depends on x^0. It's shown that for the special zero-one initial data

(2.11) we can consider the solutions $\{y_\tau\}$ of the system in variation (2.8) as a set of partial derivatives with respect to the initial data in strong $\ell_p(c)$ - sense, i.e.

$$y_{\{i\}} = \ell_{m_1}(c_{\{i\}}) - \frac{\partial}{\partial x_i^0} y^{(0)}, \quad \ldots\ldots, \quad y_{\tau\cup\{i\}} = \ell_{m_{\tau\cup\{i\}}}(c_{\tau\cup i}) - \frac{\partial}{\partial x_i^0} y_\tau \boxed{\text{a1}} \tag{5.1}$$

To achieve the interpretation (5.1), we need to impose some additional conditions on the vector weights $\{c_\tau\}$. Without doubts, the justification of (5.1) also requires the special relation between the space $\ell_{m_0}(a)$ of solvability for the initial system (2.1) and the spaces $\{\ell_{m_\tau}(c_\tau)\}$ of solvability for the τ^{th} Cauchy problem (2.8).

In theorem below we show that the strong solution $y^{(0)}(t, x^0)$ possesses the first derivatives on the initial data x^0.

Theorem 5.1. *Let $m_0 \geq m_1 > 1$ and the vector $c_{\{j\}} \in I\!\!P$ be such that*

$$\forall\, k \in Z\!\!\!Z^d \quad d_k c_{k,\{j\}} \leq a_k \boxed{rzdk} \tag{5.2}$$

with $d_k \geq a_k^{-\frac{\mathbf{k}+1}{m_0} m_1}$, $k \in Z\!\!\!Z^d$.

Suppose that $x^0 \in \ell_{m_0(\mathbf{k}+1)}(a)$ and the maps F and B fulfill conditions (2.2-2.3).

Then the strong solution $y^{(0)}(t, x^0)$ to problem (2.1) possesses the strong partial derivatives of the first order, i.e.

$$\ell_{m_1}(c_{\{j\}}) - \frac{\partial}{\partial x_j^0} y_k^{(0)}(t, x^0)$$

These derivatives form a set of the strong solutions $y_{k,\{j\}}(t, x^0)$ at $\tau = \{j\}$, $j \in Z\!\!\!Z^d$ to the system in variations (2.8) with zero-one initial data (2.11) and are understood in the sense of convergence

$$\sup_{\alpha\in(0,\alpha_0]} \sup_{t\in[0,T]} \| \frac{y^{(0)}(t, x^0 + \alpha\ell_j) - y^{(0)}(t, x^0)}{\alpha} - y_{\{j\}}(t, x^0)\|_{\ell_{m_1}(c_{\{j\}})} \to 0 \boxed{a3} \tag{5.3}$$

when $\alpha_0 \to 0$. Above ℓ_j denotes the j^{th} coordinate vector in $I\!\!R^{Z\!\!\!Z^d}$.

Proof. Fix $j \in Z\!\!\!Z^d$. Let $y_{\{j\}}(t, x^0)$ be the strong solution to the problem (2.8) in the space $\ell_{m_1}(c_{\{j\}})$, constructed by Corollary 4.6. Due to the imbeddings (7.1)

$$\ell_{m_0}(a) \subset \ell_{m_1}(a) \subset \ell_{m_1}(c_{\{j\}})$$

the following expression is element of the space $\ell_{m_1}(c_{\{j\}})$

$$\theta_{\cdot,j}^{(0)}(\alpha, t) = \frac{y^{(0)}(t, x^0 + \alpha\ell_j) - y^{(0)}(t, x^0)}{\alpha} - y_{\{j\}}(t, x^0)$$

We used that $\forall\, \alpha > 0 \;\; x^0 + \alpha\ell_j \in \ell_{m_0(\mathbf{k}+1)}(a)$, therefore $y^{(0)}(t, x^0 + \alpha\ell_j)$ is a strong solution to (2.1) in $\ell_{m_0}(a)$.

To obtain convergence (5.3) it is sufficient to prove the estimate

$$\frac{d}{dt} \|\theta_{\cdot,j}^{(0)}(\alpha, t)\|_{\ell_{m_1}(c_{\{j\}})}^{m_1} \leq K \|\theta_{\cdot,j}^{(0)}(\alpha, t)\|_{\ell_{m_1}(c_{\{j\}})}^{m_1} + \varepsilon(\alpha) \boxed{\text{k1}} \tag{5.4}$$

with $\varepsilon(\alpha) \to 0,\ \alpha \to 0$.

Inequality (5.4) leads to

$$\|\theta^{(0)}_{\cdot,j}(\alpha,t)\|^{m_1}_{\ell_{m_1}(c_{\{j\}})} \le e^{Kt}\|\theta^{(0)}_{\cdot,j}(\alpha,0)\|^{m_1}_{\ell_{m_1}(c_{\{j\}})} + \int_0^t e^{K(t-s)}\varepsilon(\alpha)ds$$

Using $\theta^{(0)}_{\cdot,j}(\alpha,0) = \dfrac{x^0+\alpha\ell_j - x^0}{\alpha} - \ell_j \equiv 0$ we have that

$$\sup_{\alpha\in(0,\alpha_0]}\ \sup_{t\in[0,T]} \|\frac{y^{(0)}(x^0+\alpha\ell_j)-y^{(0)}(x^0)}{\alpha} - y_{\{j\}}\|_{\ell_{m_1}(c_{\{j\}})} \to 0,$$

when $\alpha_0 \to 0$.

Let's prove (5.4).

$$\frac{d}{dt}\|\theta^{(0)}_{\cdot,j}(\alpha,t)\|^{m_1}_{\ell_{m_1}(c_{\{j\}})} = \frac{d}{dt}\|\frac{y^{(0)}(x^0+\alpha\ell_j)-y^{(0)}(x^0)}{\alpha} - y_{\{j\}}\|^{m_1}_{\ell_{m_1}(c_{\{j\}})} =$$

$$= -m_1 < \frac{F(y^{(0)}(x^0+\alpha\ell_j)) - F(y^{(0)}(x^0))}{\alpha} - F'(y^{(0)}(x^0))y_{\{j\}}(x^0) + \boxed{\text{a5}} \tag{5.5}$$

$$+B\theta^{(0)}_{\cdot,j}(\alpha,t), [\theta^{(0)}_{\cdot,j}(\alpha,t)]^{\#} >$$

For any continuously differentiable function the following representation

$$F(y) - F(x) = F'(x)(y-x) +$$

$$+\int_0^1 \{F'(x+\eta(y-x)) - F'(x)\}(y-x)d\eta$$

leads to

$$\frac{F(y^{(0)}_k(x^0+\alpha\ell_j)) - F(y^{(0)}_k(x^0))}{\alpha} - F'(y^{(0)}_k(x^0))y_{k,j}(x^0) =$$

$$= F'(y^{(0)}_k(x^0))\theta^{(0)}_{k,j}(\alpha,t) + \int_0^1 \{F'(y^{(0)}_{k;\alpha,\eta}) - F'(y^{(0)}_k(x^0))\}\Delta^\alpha_j y^{(0)}_k(x^0)d\eta$$

with

$$\Delta^\alpha_j y^{(0)}(x^0) = \frac{y^{(0)}(x^0+\alpha\ell_j)-y^{(0)}(x^0)}{\alpha}$$

and

$$y^{(0)}_{\alpha,\eta} = y^{(0)}(x^0) + \eta[y^{(0)}(x^0+\alpha\ell_j) - y^{(0)}(x^0)]$$

Substituting the above expression into (5.5) and using the boundedness of B in the space $\ell_{m_1}(c_{\{j\}})$ and the monotonicity of the map F we have

$$\frac{d}{dt}\|\theta^{(0)}_{\cdot,j}(\alpha,t)\|^{m_1}_{\ell_{m_1}(c_{\{j\}})} \le m_1(\|B\|_{\mathcal{L}(\ell_{m_1}(c_{\{j\}}))} + 0)\|\theta^{(0)}_{\cdot,j}(\alpha,t)\|^{m_1}_{\ell_{m_1}(c_{\{j\}})} +$$

$$+m_1| < \int_0^1 \{F'(y^{(0)}_{\alpha,\eta}) - F'(y^{(0)}(x^0))\}\Delta^\alpha_j y^{(0)}(x^0)d\eta, [\theta^{(0)}_{\cdot,j}(\alpha)]^{\#} > |\boxed{\text{a6}} \tag{5.6}$$

Consider the last expression. Using (7.3) we have

$$(5.6) \le m_1 \|\theta^{(0)}_{\cdot,j}\|^{m_1-1}_{\ell_{m_1}(c_{\{j\}})} \times$$

$$\times \int_0^1 \|\{F'(y^{(0)}_{\alpha,\eta}) - F'(y^{(0)}(x^0))\}\Delta^\alpha_j y^{(0)}(x^0)\|_{\ell_{m_1}(c_{\{j\}})} d\eta \boxed{\text{a7}} \tag{5.7}$$

Applying Proposition 4.2 we estimate the expression under integral the following way

$$\|\{F'(y^{(0)}_{\alpha,\eta}) - F'(y^{(0)}(x^0))\}\Delta^\alpha_j y^{(0)}(x^0)\|_{\ell_{m_1}(c_{\{j\}})} \le$$

$$\le const \|\Delta^\alpha_j y^{(0)}(x^0)\|_{\ell_{m_1}(dc_{\{j\}})}(1 + \|y^{(0)}_{\alpha,\eta}\|_{\ell_{m_0}(a)} + \|y^{(0)}(x^0)\|_{\ell_{m_0}(a)})^{\mathbf{k}} \times$$

$$\times \|y^{(0)}_{\alpha,\eta} - y^{(0)}(x^0)\|_{\ell_{m_0}(a)} \boxed{\text{a8}} \tag{5.8}$$

By (5.2) and (2.6)

$$\|\Delta^\alpha_j y^{(0)}(x^0)\|_{\ell_{m_1}(dc_{\{j\}})} \le \|\Delta^\alpha_j y^{(0)}(t, x^0)\|_{\ell_{m_0}(a)} \le$$

$$\le e^{\omega t} \|\frac{x^0 + \alpha \ell_j - x^0}{\alpha}\|_{\ell_{m_0}(a)} \le e^{\omega t}\|\ell_j\|_{\ell_{m_0}(a)} < \infty$$

Moreover the above estimate gives that

$$\sup_{\eta \in [0,1]} \|y^{(0)}_{\alpha,\eta}(t) - y^{(0)}(t, x^0)\|_{\ell_{m_0}(a)} =$$

$$= \sup_{\eta \in [0,1]} \eta \|y^{(0)}(t, x^0 + \alpha \ell_j) - y^{(0)}(t, x^0)\|_{\ell_{m_0}(a)} \le e^{\omega t}\alpha\|\ell_j\|_{\ell_{m_0}(a)} \to 0, \quad \alpha \to 0$$

So all expressions in (5.8) are uniformly on $(\alpha, t) \in (0, \alpha_0] \times [0, T]$ bounded
and the last one tends to zero at $\alpha_0 \to 0$. Applying to (5.7) inequality

$$|xy^{m_1-1}| \le |x|^{m_1}/m_1 + (m_1 - 1)|y|^{m_1}/m_1 \boxed{\text{a9}} \tag{5.9}$$

we finally have the statement (5.4)

$$\frac{d}{dt}\|\theta^{(0)}_{\cdot,j}(\alpha, t)\|^{m_1}_{\ell_{m_1}(c_{\{j\}})} \le$$

$$\le (m_1\|B\| + (m_1 - 1))\|\theta^{(0)}_{\cdot,j}(\alpha, t)\|^{m_1}_{\ell_{m_1}(c_{\{j\}})} + \varepsilon(\alpha) \quad \blacksquare$$

The Lemma below we use as the important intermediate step in the proof of the second and higher order differentiability on the initial data in Theorem 5.3.

Lemma 5.2. *Under conditions of Corollary 4.6 we have*

$$\forall i \ge 0 \ \ \forall \tau \ \ |\tau| \le [m_1] \ \forall j \in \mathbb{Z}^d \ \exists \alpha_0 > 0$$

$$\sup_{\alpha \in (0, \alpha_0]} \sup_{t \in [0,T]} \|\frac{y_\tau(x^0 + \alpha \ell_j) - y_\tau(x^0)}{\alpha}\|_{\ell_{m_\tau}(d^i c_\tau)} < \infty \boxed{\textit{a2}} \tag{5.10}$$

Proof. First note that due to the Corollary 4.6

$$y_\tau \in \bigcap_{i\geq 0} C([0,T], \ell_{m_\tau}(d^i c_\tau))$$

We prove the statement of Lemma inductively on $|\tau|$. At $|\tau| = 1$ the inhomogeneous part f_τ of equation (2.8) is zero and thus the proof of inductive base can be considered as a particular case of the inductive step. Therefore we give only the proof of inductive step.

Inductive step and base. Introduce notation

$$\Delta_j^\alpha y_{k,\tau}(x^0) = \frac{y_{k,\tau}(x^0+\alpha\ell_j) - y_{k,\tau}(x^0)}{\alpha} \boxed{\text{aa1*}} \tag{5.11}$$

The following differentiation is justified by (7.2) because $\Delta_j^\alpha y_{k,\tau}(x^0)$ is a difference of strong solutions to (2.8)

$$\frac{d}{dt}\|\Delta_j^\alpha y_\tau(x^0)\|^{m_\tau}_{\ell_{m_\tau}(d^i c_\tau)} =$$

$$= -m_\tau < \frac{F'(y^{(0)}(x^0+\alpha\ell_j))y_\tau(x^0+\alpha\ell_j) - F'(y^{(0)}(x^0))y_\tau(x^0)}{\alpha} +$$

$$+B\Delta_j^\alpha y_\tau(x^0) + \frac{f_\tau(x^0+\alpha\ell_j) - f_\tau(x^0)}{\alpha}, [\Delta_j^\alpha y_\tau(x^0)]^\# > \boxed{\text{c1}} \tag{5.12}$$

Adding and subtracting $F'(y^{(0)}(x^0))y_\tau(x^0+\alpha\ell_j)$ in (5.12) we have

$$\frac{F'(y^{(0)}(x^0+\alpha\ell_j))y_\tau(x^0+\alpha\ell_j) - F'(y^{(0)}(x^0))y_\tau(x^0)}{\alpha} =$$

$$= F'(y^{(0)}(x^0))\Delta_j^\alpha y_\tau(x^0) + \frac{F'(y^{(0)}(x^0+\alpha\ell_j)) - F'(y^{(0)}(x^0))}{\alpha} y_\tau(x^0+\alpha\ell_j)$$

Using (7.3), (5.9), the boundedness of B in space $\ell_{m_\tau}(dc_\tau)$ (7.7) and $F'(x) \geq 0$, $x \in I\!R$, we estimate (5.12) by

$$\frac{d}{dt}\|\Delta_j^\alpha y_\tau(x^0)\|^{m_\tau}_{\ell_{m_\tau}(d^i c_\tau)} \leq m_\tau(\|B\|_{\mathcal{L}(\ell_{m_\tau}(d^i c_\tau))} + 0)\|\Delta_j^\alpha y_\tau(x^0)\|^{m_\tau}_{\ell_{m_\tau}(d^i c_\tau)} +$$

$$+m_\tau\|A\|_{\ell_{m_\tau}(d^i c_\tau)} \cdot \|\Delta_j^\alpha y_\tau(x^0)\|^{m_\tau - 1}_{\ell_{m_\tau}(d^i c_\tau)} +$$

$$+m_\tau| < \frac{f_\tau(x^0+\alpha\ell_j) - f_\tau(x^0)}{\alpha}, [\Delta_j^\alpha y_\tau(x^0)]^\# > | \leq$$

$$\leq (m_\tau\|B\| + 2(m_\tau - 1))\|\Delta_j^\alpha y_\tau(x^0)\|^{m_\tau}_{\ell_{m_\tau}(d^i c_\tau)} +$$

$$+\|A\|^{m_\tau}_{\ell_{m_\tau}(d^i c_\tau)} + \|\frac{f_\tau(x^0+\alpha\ell_j) - f_\tau(x^0)}{\alpha}\|^{m_\tau}_{\ell_{m_\tau}(d^i c_\tau)}$$

where

$$\|A\|_{\ell_{m_\tau}(d^i c_\tau)} = \|\frac{F'(y^{(0)}(x^0+\alpha\ell_j)) - F'(y^{(0)}(x^0))}{\alpha} y_\tau(x^0+\alpha\ell_j)\|_{\ell_{m_\tau}(d^i c_\tau)} \leq$$

$$\leq const(1+\|y^{(0)}(x^0+\alpha\ell_j)\|_{\ell_{m_0}(a)}+\|y^{(0)}(x^0)\|_{\ell_{m_0}(a)})^{\mathbf{k}}\times$$

$$\times\|\frac{y^{(0)}(x^0+\alpha\ell_j)-y^{(0)}(x^0)}{\alpha}\|_{\ell_{m_0}(a)}\,\|y_\tau(x^0+\alpha\ell_j)\|_{\ell_{m_\tau}(d^{i+1}c_\tau)}\boxed{\text{c2}} \tag{5.13}$$

by Proposition 4.2. In (5.13) all multiplies are bounded uniformly on $t\in[0,T]$ and $\alpha\in(0,\alpha_0]$, for some α_0. To finish we only need to estimate the expression

$$\|\frac{f_\tau(x^0+\alpha\ell_j)-f_\tau(x^0)}{\alpha}\|_{\ell_{m_\tau}(d^ic_\tau)}\leq$$

$$\leq\sum_{\gamma_1\cup..\cup\gamma_s=\tau}\frac{1}{\alpha}\|F^{(s)}(y^{(0)}(x^0+\alpha\ell_j))y_{\gamma_1}(x^0+\alpha\ell_j)...y_{\gamma_s}(x^0+\alpha\ell_j)-$$

$$-F^{(s)}(y^{(0)}(x^0))y_{\gamma_1}(x^0)...y_{\gamma_s}(x^0)\|_{\ell_{m_\tau}(d^ic_\tau)}\boxed{\text{c3}} \tag{5.14}$$

Due to the Proposition 4.4, each term in the summation above is estimated the following way

$$\frac{1}{\alpha}\|F^{(s)}(y^{(0)}(x^0+\alpha\ell_j)y_{\gamma_1}(x^0+\alpha\ell_j)...y_{\gamma_s}(x^0+\alpha\ell_j)-$$

$$-F^{(s)}(y^{(0)}(x^0))y_{\gamma_1}(x^0)...y_{\gamma_s}(x^0)\|_{\ell_{m_\tau}(d^ic_\tau)}\leq$$

$$\leq const(1+\|y^{(0)}(x^0+\alpha\ell_j)\|_{\ell_{m_0}(a)}+\|y^{(0)}(x^0)\|_{\ell_{m_0}(a)})^{\mathbf{k}+1}\cdot$$

$$\cdot\prod_{q=1}^{s}(1+\|y_{\gamma_q}(x^0+\alpha\ell_j)\|_{\ell_{m_{\gamma_q}}(d^ic_{\gamma_q})}+\|y_{\gamma_q}(x^0)\|_{\ell_{m_{\gamma_q}}(d^ic_{\gamma_q})})\cdot$$

$$\cdot\{\|\frac{y^{(0)}(x^0+\alpha\ell_j)-y^{(0)}(x^0)}{\alpha}\|_{\ell_{m_0}(a)}+\sum_{q=1}^{s}\|\frac{y_{\gamma_q}(x^0+\alpha\ell_j)-y_{\gamma_q}(x^0)}{\alpha}\|_{\ell_{m_{\gamma_q}}(d^ic_{\gamma_q})}\}\leq$$

$$\leq K'\{\|\Delta_j^\alpha y^{(0)}(x^0)\|_{\ell_{m_0}(a)}+\sum_{q=1}^{s}\|\Delta_j^\alpha y_{\gamma_q}(x^0)\|_{\ell_{m_{\gamma_q}}(d^ic_{\gamma_q})}\}\boxed{\text{rere}} \tag{5.15}$$

with

$$K'=const(1+2\|y^{(0)}(x^0)\|_{\ell_{m_0}(a)}+\alpha\|\Delta_j^\alpha y^{(0)}(x^0)\|_{\ell_{m_0}(a)})^{\mathbf{k}+1}\cdot$$

$$\cdot\prod_{q=1}^{s}(1+2\|y_{\gamma_q}(x^0)\|_{\ell_{m_{\gamma_q}}(d^ic_{\gamma_q})}+\alpha\|\Delta_j^\alpha y_{\gamma_q}(x^0)\|_{\ell_{m_{\gamma_q}}(d^ic_{\gamma_q})})\boxed{\text{rara}} \tag{5.16}$$

By the inductive assumption all expressions in (5.15) and (5.16) are finite. Therefore (5.14) is also uniformly bounded on α, t. This gives

$$\frac{d}{dt}\|\Delta_j^\alpha y_\tau(x^0)\|^{m_\tau}_{\ell_{m_\tau}(d^ic_\tau)}\leq$$

$$\leq(m_\tau\|B\|+2(m_\tau-1))\|\Delta_j^\alpha y_\tau(x^0)\|^{m_\tau}_{\ell_{m_\tau}(d^ic_\tau)}+const$$

and proves the inductive step.

The considerations above, applied to the function $f_\tau\equiv 0$ at $|\tau|=1$ also give the inductive base. ■.

The following theorem states the second and higher order strong differentiability of the solution $y^{(0)}(t,x^0)$ to the nonlinear non-Lipschitz's equation (2.1).

Theorem 5.3. *Let $m_0 \geq m_1 > 1$ and the vector weight $\{c_\tau\}$ be such that*

$$\forall\, k \in \mathbb{Z}^d \quad \begin{cases} d_k c_{k,\{j\}} \leq a_k, & \forall\, j \in \mathbb{Z}^d \\ d_k c_{k,\tau\cup j} \leq c_{k,\tau}, & \forall \tau\ \forall\, j \in \mathbb{Z}^d \end{cases} \quad \boxed{przr} \tag{5.17}$$

with $d_k \geq a_k^{-\frac{\mathbf{k}+1}{m_0}m_1}$, $k \in \mathbb{Z}^d$.

Suppose that $x^0 \in \ell_{m_0(\mathbf{k}+1)}(a)$ and the maps F and B fulfill conditions (2.2-2.3).

Then the strong solution $y^{(0)}(t,x^0)$ of the Cauchy problem (2.1) is $[m_1]$ - times strongly differentiable on the initial data $x^0 = \{x_k^0\}_{k\in\mathbb{Z}^d}$ and possesses the partial derivatives on the variables $\{x_k^0\}$ up to the $[m_1]$ - order

$$\ell_{m_{\tau\cup j}}(c_{\tau\cup j}) - \frac{\partial}{\partial x_j^0} y_{k,\tau}(t,x^0) = y_{k,\tau\cup j} \boxed{a1\text{-}2} \tag{5.18}$$

These derivatives form a set of strong solutions for the system in variations (2.8) in the corresponding spaces $\{\ell_{m_\tau}(c_\tau)\}$ with zero-one initial data (2.11) and are understood in the sense of convergence

$$\sup_{\varepsilon\in(0,\alpha_0]} \sup_{t\in[0,T]} \|\frac{y_\tau(t;x^0+\varepsilon\ell_j) - y_\tau(t;x^0)}{\varepsilon} - y_{\tau\cup j}(t;x^0)\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} \to 0,$$

when $\alpha_0 \to 0$.

Moreover, by Corollary 4.6, the following non-linear estimate on partial derivatives of $y^{(0)}(t,x^0)$ holds:

$$for\ any\ \tau = \{j_1,..,j_n\},\ j_s \in \mathbb{Z}^d,\ |\tau| \leq [m_1/2]\ we\ have$$

$$\sum_{s=1}^{n} \{p_s(z) \sum_{\gamma\subset\tau,\ |\gamma|=s} \|\frac{\partial^s y^{(0)}(t,x^0)}{\partial x^0_{j_{r(s)}}...\partial x^0_{j_{r(1)}}}\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}\} \leq e^{Mt} p_1(\|x^0\|^{m_0}_{\ell_{m_0}(a)}) \sum_{i=1}^{n} c_{j_i,\{j_i\}} \boxed{a1\text{-}3} \tag{5.19}$$

with $\gamma = \{j_{r(1)},..,j_{r(s)}\}$, $j_r \in \mathbb{Z}^d$ and $z = \|y^{(0)}(t,x^0)\|^{m_0}_{\ell_{m_0}(a)}$.

Remark. The set of vector weights $\{c_\tau\}$, required in Theorem 5.3, is non-empty. For example one can use the vectors

$$c_{k,\tau} = e_k \cdot a_k^{\frac{\mathbf{k}+1}{m_0}m_1|\tau|}, \quad |\tau| > 1$$

with $e \in \mathbb{P}$, $e \leq a$.

Proof. We prove the statement of Theorem inductively on $|\tau| \geq 1$.

First of all note that for $|\tau| = 1$ the inhomogeneous part $f_\tau = 0$ in (2.8). This property permits us to avoid the estimation of (II), (III) terms in (5.34) for the proof of inductive base.

Therefore we are able to prove the inductive base directly in the proof of the inductive step. For convenience we formulate both statements.

Inductive step ($n \geq 2$).

Suppose that $\forall \gamma : \ |\gamma| \leq n-1 \ \forall i \in \mathbb{Z}^d$ the following convergence holds

$$\sup_{\alpha \in (0,\alpha_0]} \sup_{t \in [0,T]} \| \frac{y_\gamma(x^0 + \alpha \ell_i) - y_\gamma(x^0)}{\alpha} - y_{\gamma \cup i}(x^0) \|_{\ell_{m_{\gamma \cup i}}(c_{\gamma \cup i})} \to 0, \quad \alpha_0 \to 0$$

Then $\forall \tau : \ |\tau| = n, \ \forall j \in \mathbb{Z}^d$

$$\sup_{\alpha \in (0,\alpha_0]} \sup_{t \in [0,T]} \| \frac{y_\tau(x^0 + \alpha \ell_j) - y_\tau(x^0)}{\alpha} - y_{\tau \cup j}(x^0) \|_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} \to 0, \quad \alpha_0 \to 0 \boxed{\text{s1}} \tag{5.20}$$

Inductive base ($n = 1$). $\forall \tau : \ |\tau| = 1, \ \forall j \in \mathbb{Z}^d$

$$\sup_{\alpha \in (0,\alpha_0]} \sup_{t \in [0,T]} \| \frac{y_\tau(x^0 + \alpha \ell_j) - y_\tau(x^0)}{\alpha} - y_{\tau \cup j}(x^0) \|_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} \to 0, \quad \alpha_0 \to 0$$

To prove (5.20) it is sufficient to obtain estimate

$$\frac{d}{dt} \|\theta^\tau_{\cdot,j}(\alpha,t)\|^{m_{\tau \cup j}}_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} \leq K_1 \|\theta^\tau_{\cdot,j}\|^{m_{\tau \cup j}}_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} + const \cdot \varepsilon(\alpha_0) \boxed{\text{**18}} \tag{5.21}$$

where $\varepsilon(\alpha_0) \to 0$ when $\alpha_0 \to 0$ and

$$\theta^\tau_{\cdot,j}(\alpha,t) = \frac{y_\tau(x^0 + \alpha \ell_j) - y_\tau(x^0)}{\alpha} - y_{\tau \cup j}(x^0) \boxed{\text{bb1*}} \tag{5.22}$$

From (5.21) it follows that

$$\|\theta^\tau_{\cdot,j}(\alpha,t)\|^{m_{\tau \cup j}}_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} \leq e^{K_1 t} \|\theta^\tau_{\cdot,\tau \cup j}(\alpha,0)\|^{m_{\tau \cup j}}_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} +$$

$$+ const \int_0^t e^{K_1(t-s)} \varepsilon(\alpha_0) ds$$

For the zero-one initial data (2.11) we have $\theta^\tau_{\cdot,j}(\alpha,0) \equiv 0, \quad |\tau| \geq 1$. This gives the required convergence (5.20).

Let us prove (5.21). From (5.2) and (7.1) we have

$$\ell_{m_0}(a) \subset \ell_{m_1}(c_{\{j\}}) \subset \ \dots \ \subset \ell_{m_\tau}(c_\tau) \subset \ell_{m_{\tau \cup j}}(c_{\tau \cup j}) \subset \ \dots$$

Fix $\tau, \ |\tau| = n$ and $j \in \mathbb{Z}^d$. Due to the Corollary 4.6 and (7.2) the following differentiation a.e. on $t \in [0,T]$ is justified

$$\frac{d}{dt} \|\theta^\tau_{\cdot,j}(\alpha,t)\|^{m_{\tau \cup j}}_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} = \frac{d}{dt} \| \frac{y_\tau(x^0 + \alpha \ell_j) - y_\tau(x^0)}{\alpha} - y_{\tau \cup j}(x^0) \|^{m_{\tau \cup j}}_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} =$$

$$= -m_{\tau \cup j} < \frac{F'(y^{(0)}(x^0_\alpha)) y_\tau(x^0_\alpha) - F'(y^{(0)}(x^0)) y_\tau(x^0)}{\alpha} +$$

$$+B\theta^{\tau}_{\cdot,j}(\alpha,t) - F'(y^{(0)}(x^0))y_{\tau\cup j}(x^0) + \boxed{**4} \tag{5.23}$$
$$+\frac{f_\tau(x^0_\alpha) - f_\tau(x^0)}{\alpha} - f_{\tau\cup j}(x^0), [\theta^{\tau}_{\cdot,j}]^{\#} >$$

where

$$x^0_\alpha = x^0 + \alpha\ell_j$$

In the proof of the inductive base above $\frac{f_\tau(x^0_\alpha) - f_\tau(x^0)}{\alpha} = 0$ at $|\tau| = 1$.

Add and subtract in the expression (5.23) the terms of the form

$$\frac{1}{\alpha}F'(y^{(0)}(x^0_\alpha))y_\tau(x^0) \quad and \quad F'(y^{(0)}(x^0_\alpha))y_{\tau\cup j}(x^0)$$

Then

$$\frac{F'(y^{(0)}(x^0_\alpha))y_\tau(x^0_\alpha) - F'(y^{(0)}(x^0))y_\tau(x^0)}{\alpha} - F'(y^{(0)}(x^0))y_{\tau\cup j}(x^0) =$$
$$= F'(y^{(0)}(x^0_\alpha))\{\frac{y_\tau(x^0_\alpha) - y_\tau(x^0)}{\alpha} - y_{\tau\cup j}(x^0)\} +$$
$$+\frac{F'(y^{(0)}(x^0_\alpha)) - F'(y^{(0)}(x^0))}{\alpha}y_\tau(x^0) + (F'(y^{(0)}(x^0_\alpha)) - F'(y^{(0)}(x^0)))y_{\tau\cup j}(x^0) =$$
$$= F'(y^{(0)}(x^0_\alpha))\theta^{\tau}_{\cdot,j}(\alpha,t) + \frac{F'_\alpha - F'}{\alpha}y_\tau(x^0) + (F'_\alpha - F')y_{\tau\cup j}(x^0)\boxed{\text{reka}} \tag{5.24}$$

where we used notations

$$F'_\alpha = F'(y^{(0)}(x^0_\alpha)), \qquad F' = F'(y^{(0)}(x^0))$$

Substituting (5.24) in (5.23), using $F' \geq 0$ and the boundedness of operator B in $\ell_p(c)$, $c \in I\!P$ (7.7), we transform estimate (5.23) to

$$\frac{d}{dt}\|\theta^{\tau}_{\cdot,j}(\alpha,t)\|^{m_{\tau\cup j}}_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} \leq m_{\tau\cup j}(\|B\| + 0)\|\theta^{\tau}_{\cdot,j}(\alpha,t)\|^{m_{\tau\cup j}}_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} +$$
$$+m_{\tau\cup j}| < (F'_\alpha - F')y_{\tau\cup j}(x^0), [\theta^{\tau}_{\cdot,j}]^{\#} > | + \boxed{**6} \tag{5.25}$$
$$+m_{\tau\cup j}| < \frac{F'_\alpha - F'}{\alpha}y_\tau(x^0) + \frac{f_\tau(x^0_\alpha) - f_\tau(x^0)}{\alpha} - f_{\tau\cup j}(x^0), [\theta^{\tau}_{\cdot,j}]^{\#} > |\boxed{**5} \tag{5.26}$$

Applying inequality (3.16) we estimate the term (5.25) from above by

$$(5.25) \leq \|(F'_\alpha - F')y_{\tau\cup j}(x^0)\|^{m_{\tau\cup j}}_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} + (m_{\tau\cup j} - 1)\|\theta^{\tau}_j\|^{m_{\tau\cup j}}_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})}\boxed{**7} \tag{5.27}$$

Theorem 4.2 gives

$$\|(F'_\alpha - F')y_{\tau\cup j}(x^0)\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} =$$
$$= \|\{F'(y^{(0)}(x^0_\alpha)) - F'(y^{(0)}(x^0))\}y_{\tau\cup j}(x^0)\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} \leq$$
$$\leq const\,\|y^{(0)}(x^0_\alpha) - y^{(0)}(x^0)\|_{\ell_{m_0}(a)} \times$$

$$\times(1+\|y^{(0)}(x^0_\alpha)\|_{\ell_{m_0}(a)}+\|y^{(0)}(x^0)\|_{\ell_{m_0}(a)})^{\mathbf{k}}\cdot\|y_{\tau\cup j}(x^0)\|_{\ell_{m_{\tau\cup j}}(dc_{\tau\cup j})}\boxed{***7} \tag{5.28}$$

As $y_{\tau\cup j}$ is a solution of the Cauchy problem of $\tau\cup j^{th}$ order, then by Theorem 4.6 $\|y_{\tau\cup j}(x^0)\|_{\ell_{m_{\tau\cup j}}(dc_{\tau\cup j})}$ is bounded uniformly on $t\in[0,T]$. Moreover, from Theorem 2.1,

we have that the first expression in (5.28) tends to zero when $\alpha\to 0$ and finally

$$(5.25)\le(m_{\tau\cup j}-1)\|\theta^\tau_{\cdot,\tau\cup j}\|^{m_{\tau\cup j}}_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})}+\varepsilon(\alpha)\boxed{**8} \tag{5.29}$$

with $\varepsilon(\alpha)\to 0$, when $\alpha\to 0$.

It remains to estimate the expression (5.26). To do this it is sufficient to show that

$$\sup_{\alpha\in(0,\alpha_0]}\sup_{t\in[0,T]}\|\frac{F'_\alpha-F'}{\alpha}y_\tau(x^0)+\frac{f_\tau(x^0_\alpha)-f_\tau(x^0)}{\alpha}-f_{\tau\cup j}(x^0)\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})}\le\varepsilon(\alpha_0)\boxed{**10} \tag{5.30}$$

with $\varepsilon(\alpha_0)\to 0$ at $\alpha_0\to 0$.

First of all we recall the recurrent form of functions f_τ (2.9)

$$\begin{cases} f_\tau\equiv 0, \quad |\tau|=1 \\ f_\tau(x^0)=\sum\limits_{\gamma\subset\tau,\ |\gamma|\ge 0}\dfrac{\partial f_\tau(y_\gamma(x^0),\ \gamma\subset\tau)}{\partial y_\gamma}y_{\gamma\cup\{j\}}(x^0)+ \qquad \boxed{**9} \\ \qquad +F''(y^{(0)}(x^0))y_j(x^0)y_\tau(x^0), \qquad |\tau|\ge 2\end{cases} \tag{5.31}$$

which we have shorthanded by the usage of notation that $y_\emptyset=y^{(0)}$ at $\gamma=\emptyset$.

Note that for $\tau=\{j,k\},\ |\tau|=2$

$$f_\tau=F''(y^{(0)}(x^0))y_{\{j\}}(x^0)y_{\{k\}}$$

Now we apply the formula

$$f(x_1,..,x_s)-f(y_1,..,y_s)=\sum_{i=1}^s\frac{\partial f}{\partial i}(\vec{x})(x_i-y_i)+$$

$$+\sum_{i=1}^s\int_0^1\{\frac{\partial f}{\partial i}(\vec{y}+\eta(\vec{x}-\vec{y}))-\frac{\partial f}{\partial i}(\vec{x})\}(x_i-y_i)d\eta,$$

$$\vec{y}=(y_1,..,y_s),\ \vec{x}=(x_1,..,x_s)$$

to the function $f_\tau=f_\tau(y_\gamma,\ \gamma\subset\tau)$. We understand under the vector $\vec{y}(x^0)$ the set of the strong solutions $\{y_\gamma,\ \gamma\subset\tau\}$ to the system in variations on which depends the non-autonomous part f_τ.

Using (5.31) and notation

$$\vec{y}_{\eta,\alpha}=\vec{y}(x^0)+\eta[\vec{y}(x^0_\alpha)-\vec{y}(x^0)]$$

we have

$$\frac{F'_\alpha-F'}{\alpha}y_\tau(x^0)+\frac{f_\tau(x^0_\alpha)-f_\tau(x^0)}{\alpha}-f_{\tau\cup j}(x^0)=$$

$$= [\frac{F'(y^{(0)}(x^0_\alpha)) - F'(y^{(0)}(x^0))}{\alpha} - F''(y^{(0)}(x^0))y_j(x^0)]y_\tau(x^0) +$$

$$+ \sum_{\gamma\subset\tau,\ |\gamma|\geq 0} \frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}(x^0))\frac{y_\gamma(x^0_\alpha) - y_\gamma(x^0)}{\alpha} + \boxed{**11} \tag{5.32}$$

$$+ \sum_{\gamma\subset\tau,|\gamma|\geq 0} \int_0^1 \{\frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}_{\eta,\alpha}) - \frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}(x^0))\}\frac{y_\gamma(x^0_\alpha) - y_\gamma(x^0)}{\alpha}d\eta -$$

$$- \sum_{\gamma\subset\tau,|\gamma|\geq 0} \frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}_{\eta,\alpha}(x^0))y_{\gamma\cup j}(x^0)$$

We remark immediately, that the "coordinates" $\{\vec{y}_{\eta,\alpha}\}_\gamma$ of vector $\vec{y}_{\eta,\alpha}$ have the next property

$$\sup_{\eta\in[0,1]} \sup_{t\in[0,T]} \|\{\vec{y}_{\eta,\alpha}\}_\gamma - y_\gamma(x^0)\|_{\ell_{m_\gamma}(c_\gamma)} \to 0, \quad \alpha \to 0 \boxed{aaa*} \tag{5.33}$$

Indeed, y_γ form the solutions of the associated Cauchy problems of orders $|\gamma| < |\tau|$ and

$$\{\vec{y}_{\eta,\alpha}\}_\gamma - y_\gamma(x^0) = y_\gamma(x^0) + \eta[y_\gamma(x^0_\alpha) - y_\gamma(x^0)] - y_\gamma(x^0) =$$
$$= \eta[y_\gamma(x^0_\alpha) - y_\gamma(x^0)]$$

and due to (5.10) the last expression tends to zero

$$\|y_\gamma(x^0_\alpha) - y_\gamma(x^0)\|_{\ell_{m_\gamma}(c_\gamma)} \leq$$

$$\leq \|y_\gamma(x^0_\alpha) - y_\gamma(x^0)\|_{\ell_{m_\gamma}(dc_\gamma)} \leq const \cdot \alpha \to 0, \quad \alpha \to 0$$

The expression (5.32) consists of three terms

$$I = [\frac{F'(y^{(0)}(x^0_\alpha)) - F'(y^{(0)}(x^0))}{\alpha} - F''(y^{(0)}(x^0))y_j(x^0)]y_\tau(x^0)$$

$$II = \sum_{\gamma\subset\tau,\ |\gamma|\geq 0} \frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}(x^0))\theta^{\gamma}_{\cdot,j}(\alpha, t) \qquad \boxed{acdc} \tag{5.34}$$

$$III = \sum_{\gamma\subset\tau,|\gamma|\geq 0} \int_0^1 [\frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}_{\eta,\alpha}) - \frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}(x^0))]\Delta^\alpha_j y_\gamma(x^0)d\eta$$

with $\Delta^\alpha_j y_\gamma(x^0)$ introduced in (5.11) and $\theta^{\gamma}_{\cdot,j}$ introduced in (5.22).
Therefore

$$(5.30) \leq \|I\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} + \|II\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} + \|III\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})}$$

(I) We begin by estimate of I

$$\sup_{\alpha\in(0,\alpha_0]} \sup_{t\in[0,T]} \|I\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} \leq \varepsilon(\alpha_0)$$

where $\varepsilon(\alpha_0) \to 0$ at $\alpha_0 \to 0$. The next representation of function $F'(x)$

$$F'(y^{(0)}(x^0_\alpha)) - F'(y^{(0)}(x^0)) = F''(y^{(0)}(x^0))[y^{(0)}(x^0_\alpha) - y^{(0)}(x^0)] +$$
$$+ \int_0^1 \{F''(y_{\alpha,\eta}) - F''(y^{(0)}(x^0))\}(y^{(0)}(x^0_\alpha) - y^{(0)}(x^0)) d\eta$$

with $y_{\alpha,\eta} = y^{(0)}(x^0) + \eta[y^{(0)}(x^0_\alpha) - y^{(0)}(x^0)]$ implies

$$\frac{F'(y^{(0)}(x^0_\alpha)) - F'(y^{(0)}(x^0))}{\alpha} - F''(y^{(0)}(x^0))y_j(x^0) =$$
$$= F''(y^{(0)}(x^0))[\frac{y^{(0)}(x^0_\alpha) - y^{(0)}(x^0)}{\alpha} - y_j(x^0)] +$$
$$+ \int_0^1 [F''(y_{\alpha,\eta}) - F''(y^{(0)})]\frac{y^{(0)}(x^0_\alpha) - y^{(0)}(x^0)}{\alpha} d\eta$$

By Proposition 4.4, we have

$$\|I\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} \le \|F''(y^{(0)}(x^0))\theta^{\emptyset}_{\cdot,j}(\alpha,t)y_\tau(x^0)\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} +$$
$$+ \int_0^1 \|(F''(y_{\alpha,\eta}) - F''(y^{(0)}(x^0)))\Delta^\alpha_j y^{(0)}(x^0) y_\tau(x^0)\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})} d\eta \le$$
$$\le K_1(1 + \|y^{(0)}\|_{\ell_{m_0}(a)})^{\mathbf{k}+1}\|\theta^{\emptyset}_{\cdot,j}(\alpha,t)\|_{\ell_{m_1}(c_{\{j\}})}\|y_\tau(x^0)\|_{\ell_{m_\tau}(c_\tau)} + \boxed{**17} \tag{5.35}$$
$$+ K_2 \sup_{\eta\in[0,1]} \|y_{\alpha,\eta} - y^{(0)}(x^0)\|_{\ell_{m_0}(a)} \cdot$$
$$\cdot (1 + \|y_{\alpha,\eta}\|_{\ell_{m_0}(a)} + \|y^{(0)}(x^0)\|_{\ell_{m_0}(a)})^{\mathbf{k}+1}\|\Delta^\alpha_j y^{(0)}(x^0)\|_{\ell_{m_1}(c_{\{j\}})}\|y_\tau\|_{\ell_{m_\tau}(c_\tau)}$$

It is obvious that the expressions in (5.35) can be estimated by some number $\varepsilon(\alpha_0) \to 0$ when $\alpha_0 \to 0$. This follows from Theorem 5.1

$$\sup_{\alpha\in(0,\alpha_0]} \sup_{t\in[0,T]} \|\theta^{\emptyset}_{\cdot,j}(\alpha,t)\|_{\ell_{m_1}(c_{\{j\}})} \to 0, \quad \textit{when } \alpha_0 \to 0$$

from the estimate

$$\sup_{\eta\in[0,1]} \|y_{\eta,\alpha} - y^{(0)}(x^0)\|_{\ell_{m_0}(a)} \le$$
$$\le \|y^{(0)}(x^0_\alpha) - y^{(0)}(x^0)\|_{\ell_{m_0}(a)} \le e^{\omega t}\alpha\|\ell_j\|_{\ell_{m_0}(a)} \to 0, \quad \alpha \to 0$$

and from the uniform boundedness of the other terms.

As for $|\tau| = 1$ $f_\tau \equiv 0$ we see that the proof of the inductive base is completed, because we do not need to estimate the parts (II), (III) in (5.34).

(II) To estimate $\|II\|$ we clarify the structure of the $\dfrac{\partial f_\tau}{\partial y_\gamma}(\vec{y}(x^0))\theta^\tau_{\cdot,j}(\alpha,t)$ term.

Consider first the case when $|\gamma| \ne 0$. From the representation (2.10) of function f_τ

$$f_\tau = \sum_{\beta_1\cup..\cup\beta_s=\tau,\ s\ge 2} F^{(s)}(y^{(0)})y_{\beta_1}...y_{\beta_s}$$

we see that $\dfrac{\partial f_\tau}{\partial y_\gamma} \neq 0$ only in terms when at least one of the sets $\{\beta_1, .., \beta_s\}$ is equal to γ. Without loss of generality we can suppose that $\beta_s = \gamma$ and we have

$$\frac{\partial f_\tau}{\partial y_\gamma} = \sum_{\beta_1 \cup .. \cup \beta_{s-1} = \tau \setminus \gamma,\, s \geq 2} K_{\beta_1,..,\beta_{s-1},\gamma;\tau}\, F^{(s)}(y^{(0)}) y_{\beta_1} ... y_{\beta_{s-1}}$$

with some combinatorial constants $K_{\beta_1,..,\beta_{s-1},\gamma;\tau}$.

Changing in the summation above $(s-1)$ to s we have the following representation for the derivative of function f_τ on variable y_γ at $|\gamma| \neq 0$

$$\frac{\partial f_\tau}{\partial y_\gamma} = \sum_{\beta_1 \cup .. \cup \beta_s = \tau \setminus \gamma,\, s \geq 1} K_{\beta_1,..,\beta_s,\gamma;\tau}\, F^{(s+1)}(y^{(0)}) y_{\beta_1} ... y_{\beta_s} \boxed{***} \tag{5.36}$$

In the case $|\gamma| = 0$, i.e. $y_\emptyset = y^{(0)}$, from the representation (2.10) we obtain

$$\frac{\partial f_\tau}{\partial y_\emptyset} = \sum_{\beta_1 \cup .. \cup \beta_s = \tau \setminus \emptyset,\, s \geq 2} F^{(s+1)}(y^{(0)}) y_{\beta_1} ... y_{\beta_s}$$

Therefore we have reduced the both cases at $|\gamma| = 0$ and $|\gamma| = 1$ to the common form

$$\frac{\partial f_\tau}{\partial y_\gamma} \theta^\gamma_{\cdot,j}(\alpha, t) = \sum_{\beta_1 \cup .. \cup \beta_s = \tau \setminus \gamma} K_{\beta_1,..,\beta_s,\gamma;\tau}\, F^{(s+1)}(y^{(0)}) y_{\beta_1} ... y_{\beta_s} \theta^\gamma_{\cdot,j}(\alpha, t) \boxed{**12} \tag{5.37}$$

where $s \geq 2$ at $|\gamma| = 0$ and $s \geq 1$ at $|\gamma| \geq 1$. We understand the summation above as one running over all representations of the set τ on the non-intersecting subsets $\beta_1, .., \beta_s$: $\beta_1 \cup .. \cup \beta_s \cup \gamma = \tau$ with $|\gamma| \geq 0$.

Because of $\beta_1 \cup .. \cup \beta_s \cup \gamma \cup \{j\} = \tau \cup j$ each term in the summation (5.37) satisfies the conditions of Proposition 4.4, part 2.

$$\|II\|_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} \leq \sum_{\gamma \subset \tau, |\gamma| \geq 0} \| \frac{\partial f_\tau}{\partial y_\gamma} (\vec{y}(x^0)) \theta^\gamma_{\cdot,j}(\alpha, t) \|_{\ell_{m_{\tau \cup j}}(c_{\tau \cup j})} \leq$$

$$\leq const \sum_{\gamma \subset \tau, |\gamma| \geq 0} \|\theta^\gamma_{\cdot,j}\|_{\ell_{m_{\gamma \cup j}}(c_{\gamma \cup j})} \cdot$$

$$\cdot \sum_{\beta_1 \cup .. \cup \beta_s = \tau \setminus \gamma} \{ \prod_{i=0}^s (1 + \|y_{\beta_i}\|_{\ell_{m_{\beta_i}}(c_{\beta_i})})^{\mathbf{k}_i} \cdot (\sum_{i=0}^s \|y_{\beta_i}\|_{\ell_{m_{\beta_i}}(c_{\beta_i})}) \} \boxed{**14} \tag{5.38}$$

Here under y_{β_0} it is understood $y^{(0)}(x^0)$, $\ell_{m_{\beta_0}}(c_{\beta_0}) = \ell_{m_0}(a)$ and $\mathbf{k}_i = 1$, $i = 1, .., s$, $\mathbf{k}_0 = \mathbf{k} + 1$.

Functions y_β are solutions of the Cauchy problem of β^{th} order, thus all corresponding norms in the expression (5.38) are uniformly bounded on $t \in [0, T]$. By the inductive assumption

$$\sup_{\alpha \in (0, \alpha_0]} \sup_{t \in [0,T]} \|\theta^\gamma_{\cdot,j}(\alpha, t)\|_{\ell_{m_{\gamma \cup j}}(c_{\gamma \cup j})} \to 0, \quad \alpha_0 \to 0$$

and therefore

$$\sup_{\alpha\in(0,\alpha_0]}\sup_{t\in[0,T]}\|II\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})}\le\varepsilon(\alpha_0)\boxed{**16} \tag{5.39}$$

with $\varepsilon(\alpha_0)\to 0,\ \alpha_0\to 0$.

(III) Now we estimate $\|III\|$ in (5.34). Due to (5.37) we can write

$$[\frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}_{\eta,\alpha})-\frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}(x^0))]\Delta_j^\alpha y_\gamma(x^0)=$$

$$=\sum_{\beta_1\cup..\cup\beta_s=\tau\setminus\gamma}K_{\beta_1,..,\beta_s,\gamma;\tau}\ \{\ F^{(s+1)}(\{\vec{y}_{\eta,\alpha}\}_0)\{\vec{y}_{\eta,\alpha}\}_{\beta_1}...\{\vec{y}_{\eta,\alpha}\}_{\beta_s}-\boxed{\text{addd}} \tag{5.40}$$

$$-F^{(s+1)}(y^{(0)}(x^0))y_{\beta_1}(x^0)...y_{\beta_s}(x^0)\ \}\ \Delta_j^\alpha y_\gamma(x^0)$$

By Proposition 4.4 (part 2) like in (5.38) we have

$$\|[\frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}_{\eta,\alpha})-\frac{\partial f_\tau}{\partial y_\gamma}(\vec{y}(x^0))]\Delta_j^\alpha y_\gamma(x^0)\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})}\le$$

$$\le const\sum_{\beta_1\cup...\cup\beta_s=\tau\setminus\gamma}\prod_{i=0}^{s}(1+\|y_{\beta_i}(x^0)\|_{\ell_{m_{\beta_i}}(c_{\beta_i})}+\|\{\vec{y}_{\eta,\alpha}\}_{\beta_i}\|_{\ell_{m_{\beta_i}}(c_{\beta_i})})^{\mathbf{k}_i}\cdot\boxed{\text{addb}} \tag{5.41}$$

$$\cdot(\sum_{i=0}^{s}\|\{\vec{y}_{\eta,\alpha}\}_{\beta_i}-y_{\beta_i}(x^0)\|_{\ell_{m_{\beta_i}}(c_{\beta_i})})\cdot\|\Delta_j^\alpha y_\gamma(x^0)\|_{\ell_{m_{\gamma\cup j}}(c_{\gamma\cup j})}$$

with $y_{\beta_0}=y^{(0)},\ \ell_{m_{\beta_0}}(c_{\beta_0})=\ell_{m_0}(a)$ and $\mathbf{k}_0=\mathbf{k}+1,\ \mathbf{k}_i=1,\ i=1,...,s$.

Due to $m_{\gamma\cup j}\le m_\gamma$, assumption $d_k\ge a_k^{-\frac{\mathbf{k}+1}{m_0}m_1}$ and (5.17) we have

$$\|\cdot\|_{\ell_{m_{\gamma\cup j}}(c_{\gamma\cup j})}\le const\|\cdot\|_{\ell_{m_\gamma}(c_{\gamma\cup j})}\le const\|\cdot\|_{\ell_{m_\gamma}(dc_{\gamma\cup j})}\le$$

$$\le const\|\cdot\|_{\ell_{m_\gamma}(c_\gamma)}\le const\|\cdot\|_{\ell_{m_\gamma}(dc_\gamma)}$$

This and Lemma 5.2 give that $\exists\,\alpha_0>0$

$$\sup_{\alpha\in(0,\alpha_0]}\sup_{t\in[0,T]}\|\Delta_j^\alpha y_\gamma(x^0)\|_{\ell_{m_{\gamma\cup j}}(c_{\gamma\cup j})}<\infty$$

Together with convergence (5.33) this shows that (5.41) tends to the zero when $\alpha_0\to 0$ uniformly on $\alpha\in(0,\alpha_0],\ t\in[0,T]$. Therefore in (5.34)

$$\sup_{\alpha\in(0,\alpha_0]}\sup_{t\in[0,T]}\|III\|_{\ell_{m_{\tau\cup j}}(c_{\tau\cup j})}\le\varepsilon(\alpha_0)$$

with $\varepsilon(\alpha_0)\to 0,\ \alpha_0\to 0$.

Steps (I)-(III) together with (5.29) give (5.21) with $K_1=m_{\tau\cup j}\|B\|+(m_{\tau\cup j}-1)$. Thus the inductive step (& base) are proved. ■

6 Exponential ergodicity in variations via nonlinear quasi-contractive estimate.

In this section we apply the non-linear estimate of quasi-contractive type, obtained in Theorem 3.3, to the investigation of the asymptotic behaviour of solution of the equation (2.1) when $t \to \infty$.

It is known that the strictly contractive estimates on nonlinear semigroups in Banach spaces lead to the exponential ergodicity of corresponding system.

More concretely, let the nonlinear semigroup $\{S_t \in Lip(X,X),\ t \geq 0\}$ in the Banach space X be associated with strictly m-monotone generator G, i.e. $\forall\, x^0 \in \mathcal{D}_X(G)$ the function $y(t) = S_t x^0$ solves equation

$$\frac{dy(t)}{dt} = -G(y(t)), \quad y(0) = x^0$$

where the nonlinear map G fulfills

$$\exists\, \varepsilon > 0\ \forall\, x, y \in \mathcal{D}_X(G) \quad < G(x) - G(y), \mathcal{F}(x-y) >_X \geq \varepsilon \|x-y\|_X^2$$

and $\forall\, \lambda > 0 \ \ Ran(1+\lambda G) = X$. Above $\mathcal{F}$ is the duality map in the space X.

Then

$$\exists\, !x_* \in \mathcal{D}_X(G): \quad S_t x_* = x_*,\ t \geq 0$$

and the exponential ergodic property holds

$$\forall\, x^0 \in X: \quad \|S_t x^0 - x_*\|_X \leq e^{-\varepsilon t}\|x^0 - x_*\|_X$$

In fact the ergodic property follows from fixed points arguments applied to the inequalities of (2.6)-type with $\omega = -\varepsilon$.

For example, for the system (2.1) at choice of parameter

$$\varepsilon = -\omega = \inf_{x \in I\!R} F'(x) - \|B\|_{\ell_{m_0}(a)} > 0$$

we have exponential ergodicity of solution $y^{(0)}(t, x^0)$ in the space $\ell_{m_0}(a)$ with $x_* = 0$

$$\|y^{(0)}(t, x^0) - 0\|_{\ell_{m_0}(a)} \leq e^{-\varepsilon t}\|x^0 - 0\|_{\ell_{m_0}(a)} \boxed{\text{p2}} \tag{6.1}$$

The above inequality can be obtained by the scheme of Theorem 2.1 proof with usage at point (7.6) the mean value theorem

$$(F(x) - F(y))(x-y) = [F'(\theta)(x-y)](x-y) \geq \inf_{z \in I\!R} F'(z)|x-y|^2$$

The choice $x_* = 0$ is obvious from $F(0) = 0$ and linearity of the map B (2.2)-(2.3).

Below we are going to prove that at fixed matrix B the parameter $\inf\limits_{x \in I\!R} F'(x)$ controls the asymptotic behaviour on infinity for the solutions of the system in variations (2.8). We show that the more monotone is function F, the more variations (derivatives on the initial data) of solution $y^{(0)}(t,x^0)$ converge exponentially to zero at $t\to\infty$.

Theorem 6.1. *Let conditions of Theorem 5.3 hold with additional restriction that the constant $const_{\tau;\gamma_1,..,\gamma_s}$ in the Definition 3.1 is uniform on $\tau,\gamma_1,..,\gamma_s$.*

Then there is positive increasing numbers $a_n\uparrow,\ n=1,..,[m_1]$ such that if

$$a_{s+1}\ge\inf_{x\in I\!R}F'(x)>a_s>0\quad for\ some\ s\in\{1,..,[m_1]\}$$

then $\exists\,\varepsilon_s>0\ \forall\,\tau=\{j_1,..,j_s\},\ |\tau|=s,\ j_i\in Z\!\!Z^d$ we have the exponential ergodicity in variations up to the s^{th} order, i.e. the following estimate

$$\sum_{i=1}^{s}\sum_{\gamma\subset\tau,|\gamma|=i}\|\frac{\partial^{|\gamma|}y^{(0)}(t,x^0)}{\partial x^0_{j_{r(1)}}...\partial x^0_{j_{r(i)}}}\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}\le C_{x_0}e^{-\varepsilon t}\boxed{p3}\tag{6.2}$$

In the summation above $\gamma=\{j_{r(1)},..,j_{r(i)}\}$ and the constant C_{x_0} is uniformly bounded on balls in space $\ell_{m_0}(a)$.

Proof. From Theorem 5.3 we have the existence of partial derivatives and the nonlinear quasi-contractive estimate (5.19). We only need to ensure that the constant M in (5.19) is negative.

Function $F_a(x)=F(x)+ax,\ a\ge0$ satisfies the requirement (2.2) with the same constant $\mathbf{k}$, because $F'_a=F'+a$ and $F_a^{(i)}=F^{(i)},\ i\ge2$. Therefore Theorem 3.3 is true for system (2.1) with F replaced by F_a.

Due to $\inf_{x\in I\!R}F'_a(x)\ge a$ the steps (3.13-3.14) in the proof of Theorem 3.3 transform to

$$\begin{gathered}\frac{dg_\gamma(t)}{dt}=\{\frac{d}{dt}p_i(\|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)})\}\|y_\gamma(t)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}+\\ +p_i(\|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)})\frac{d}{dt}\|y_\gamma(t)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}\le\\ \le m_0K\omega g_\gamma(t)+m_\gamma(\|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))}-a)g_\gamma(t)+\boxed{\text{bzbz}}\\ +m_\gamma p_i(\|y^{(0)}\|^{m_0}_{\ell_{m_0}(a)})|<f_\gamma,[y_\gamma]^{\#}>|\end{gathered}\tag{6.3}$$

i.e. in (6.3) the coefficient $m_\gamma(\|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))}-0)$ is replaced by

$$m_\gamma(\|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))}-a)<m_\gamma\|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))}-a$$

as $m_\gamma>1$. This changes estimate (3.12) to

$$h^i_\tau(y;t)\le e^{(M_{i-1}+K_1-a+2^{|\tau|}K_2)t}h^i_\tau(y;0)$$

and gives in (3.9) the relation

$$M_i=M_{i-1}+K(i)-a$$

with the increasing on i constants $K(i)=K_1+2^{|\tau|}K_2$.

Therefore we have the nonlinear estimate (3.6) in form

$$\rho_\tau(y;t) \le e^{(M_s - sa)t}\rho_\tau(y;0), \quad |\tau| = s$$

and so can obviously choose numbers

$$a_s = \frac{M_s}{s}, \quad s \in \{1, .., m_1\}$$

The monotonicity of a_s follows from the general fact that

$$\text{if } M_s = \sum_{i=1}^{s} K_i \text{ with increasing } K_i > 0 \text{ then } \frac{M_s}{s} > \frac{M_{s-1}}{s-1}.$$

As the functions p_i in (??) are strictly positive, $p_i \ge \varepsilon$, we can transform the nonlinear estimate at zero-one initial data (4.28) to the required form (6.2) with

$$C_{x^0} = \frac{1}{\varepsilon} p_1(\|x^0\|^{m_0}_{\ell_{m_0}(a)}) \sum_{j=1}^{n} c_{j,\{j\}}$$

where $\{j_1, .., j_n\} = \tau$. ■

Remark. The statement of the Theorem 6.1 actually holds also for the system in variations (2.8) at arbitrary initial data.

One should proceed like in the proof above and use the Theorem 4.5 instead of Theorem 5.3. This will give the nonlinear quasi-contractive estimate (3.6) with negative constant M in the exponent.

Thus we see that the nevertheless of the nonlinear and multiplicative structure of the system (2.8) the nonlinear non-autonomous flow

$$\bigotimes_{\gamma \subset \tau} \ell_{m_\gamma}(c_\gamma) \ni \{x_\gamma\}_{\gamma \subset \tau} \longrightarrow \{y_\gamma(t, x_\alpha, \alpha \subset \tau)\}_{\gamma \subset \tau} \in \bigotimes_{\gamma \subset \tau} \ell_{m_\gamma}(c_\gamma)$$

parameterized by $y^{(0)}$ fulfills the exponential ergodic property at sufficiently monotone function F.

7 Appendix.

We briefly list the necessary facts and notations. We also sketch the scheme of the Theorem 2.1 proof.

Let $I\!P$ denote vectors $c = \{c_k\}_{k \in \mathbb{Z}^d} \in (I\!R_+)^{\mathbb{Z}^d}$ satisfying condition

$$\gamma_c = \sup_{|k-j|=1} |c_k / c_j| < \infty$$

For $c \in I\!P,\ 1 < p < \infty$ introduce the space $\ell_p(c, \mathbb{Z}^d) = \ell_p(c)$ by the next way

$$\ell_p(c) = \{x \in I\!R^{\mathbb{Z}^d} :\ \|x\|_{\ell_p(c)} = (\sum_{k \in \mathbb{Z}^d} c_k |x_k|^p)^{1/p} < \infty\}$$

Note immediately that for the $\ell_p(c)$ spaces with weight $\sum_{k \in \mathbb{Z}^d} c_k < \infty$ the following inclusion holds

$$\ell_1(c) \supset \ell_2(c) \supset ... \supset \ell_p(c) \supset \ell_\infty \boxed{\text{d6}} \tag{7.1}$$

Indeed for $q > p$

$$\sum_{k \in \mathbb{Z}^d} c_k |x_k|^p = \sum_{k \in \mathbb{Z}^d} c_k^{(q-p)/q} c_k^{p/q} |x_k|^p \le (\sum_{k \in \mathbb{Z}^d} c_k)^{(q-p)/q} (\sum_{k \in \mathbb{Z}^d} c_k |x_k|^q)^{p/q}$$

which gives

$$\|x\|_{\ell_p(c)} \le (\sum_{k \in \mathbb{Z}^d} c_k)^{(q-p)/qp} \|x\|_{\ell_q(c)}$$

Let X be a Banach space. The multi-valued operator

$$\mathcal{F} : X \to X^*$$

given by the formula

$$\mathcal{F}(x) = \{x^* \in X^* \ : < x, x^* >= \|x\|^2 = \|x^*\|^2\}$$

is called a *duality map* . Here $< x, x^* >$ denotes the value of $x^* \in X^*$ at point $x \in X$.

The dual space to the $\ell_p(c),\ 1 < p < \infty$ can be identified with $\ell_q(c)$ with $1/p + 1/q = 1$. Moreover in the space $\ell_p(c),\ 1 < p < \infty$ the duality map is uniquely defined by

$$(\mathcal{F}x)_k = \frac{x_k |x_k|^{p-2}}{\|x\|^{p-2}_{\ell_p(c)}}$$

Let function $u \in C([0,T], \ell_p(c)),\ 1 < p < \infty$ be a.e. strongly differentiable and suppose that $\|u(t)\|$ is differentiable at $t = s$. Then by the reflexivity of space $\ell_p(c),\ 1 < p < \infty$ we have that [2, Ch.3,§1,Lemma 2.1]

$$\frac{d}{ds} \|u(s)\|^p_{\ell_p(c)} = p < \frac{du(s)}{ds}, u^\#(s) > \boxed{\text{d7}} \tag{7.2}$$

where for $x \in \ell_p(c)$ we use notation $x^\# = \|x\|^{p-2}_{\ell_p(c)} \mathcal{F}x$ and

$$\|x^\#\|_{\ell_p^*(c)} = \|x\|^{p-1}_{\ell_p(c)} \boxed{\text{d8}} \tag{7.3}$$

Proof of Theorem 2.1. First we prove that the map F is m - monotone, i.e.

$$\forall\, x, y \in \mathcal{D}_{\ell_{m_0}(a)}(F) : \quad < F(x) - F(y), \mathcal{F}(x - y) >\ge 0 \boxed{\text{d10}} \tag{7.4}$$

and

$$\forall\, \lambda \ge 0 : \quad \mathcal{R}(1 + \lambda F) = \ell_{m_0}(a) \boxed{\text{d11}} \tag{7.5}$$

where $\mathcal{F}(x) = \{\mathcal{F}(x)\}_{k\in \mathbb{Z}^d} = x_k|x_k|^{m_0-2}/\|x\|^{m_0-2}_{\ell_{m_0}(a)}$ denotes the duality map in the space $\ell_{m_0}(a)$.

Property (7.4) follows from estimate

$$< F(x) - F(y), \mathcal{F}(x-y) >=$$

$$= \sum_{k\in \mathbb{Z}^d} a_k \frac{(F(x_k) - F(y_k))(x_k - y_k)|x_k - y_k|^{m_0-2}}{\|x-y\|^{m_0-2}_{\ell_{m_0}(a)}} \geq 0 \boxed{\text{d10-1}} \tag{7.6}$$

due to the coordinate monotonicity of function F.

Moreover for any $\lambda \geq 0$ and $y \in \ell_{m_0}(a)$ due to $F(0) = 0$ the problem

$$x_k + \lambda F(x_k) = y_k$$

has a unique solution $x = \{x_k\}_{k\in \mathbb{Z}^d}$ [1, Thm.8,p.383] which fulfills

$$|x_k| \leq |y_k|$$

So vector $x = \{x_k\}_{k\in \mathbb{Z}^d} \in \ell_{m_0}(a)$ and (7.5) holds.

For every $a = \{a_k\}_{k\in \mathbb{Z}^d}$, $\sum_{k\in \mathbb{Z}^d} a_k < \infty$, $\sup\limits_{|k-j|=1} |a_k/a_j| < \infty$, $m_0 \in (1, \infty)$ we have that the linear map B is bounded in $\ell_{m_0}(a)$

$$\|Bx\|_{\ell_{m_0}(a)} = (\sum_{k\in \mathbb{Z}^d} | \sum_{|k-j|\leq r_0} b(k-j)x_j|^{m_0})^{1/m_0} \leq$$

$$\leq \max_{|j|\leq r_0} |b(j)| \sum_{|i|\leq r_0} (\sum_{k\in \mathbb{Z}^d} a_k |x_{k+i}|^{m_0})^{1/m_0} \leq$$

$$\leq \max_{|i|\leq r_0} |b(i)|(2r_0)^d \gamma_a^{r_0/m_0} \|x\|_{\ell_{m_0}(a)} \boxed{\text{d11-2}} \tag{7.7}$$

From Thm.3.2 in [1, p.158] it follows that $(F+B)$ is a quasi m - monotone map in $\ell_{m_0}(a)$, i.e. $\exists\, \omega = \max |b(i)|(2r_0)^d \gamma_a^{r_0/m_0} > 0 \;\; \forall\, x, y \in \mathcal{D}_{\ell_{m_0}(a)}(F+B)$

$$< (F+B)(x) - (F+B)(y), \mathcal{F}(x-y) > \geq -\omega \|x-y\|^2_{\ell_{m_0}(a)} \boxed{\text{d11-3}} \tag{7.8}$$

and $\exists\, \lambda_0 > 0 \;\; \forall\, \lambda \in [0, \lambda_0)$

$$\mathcal{R}(1 + \lambda(F+B)) = \ell_{m_0}(a)$$

By Theorem I in [6] we can define the function

$$y^{(0)}(t) = \ell_{m_0}(a) - \lim_{n\to\infty} ([1 + \frac{t}{n}(F+B)]^{inv})^n x^0$$

as a strong limit in $\ell_{m_0}(a)$ for any $x^0 \in \mathcal{D}_{\ell_{m_0}(a)}(F+B)$. Moreover by [6, (1.11)] function $y^{(0)}(t)$ is Lipschitz continuous on $t \in [0, T]$. By reflexivity of the space $\ell_{m_0}(a)$ we have that $y^{(0)}(t)$ is a.e. on $[0, T]$ strongly differentiable in $\ell_{m_0}(a)$ [2, Ch.1,§1,Thm.2.1].

As an application of Theorem II(ii⇒i) and Lemma 2.3 in [6] we have that for $x^0 \in \mathcal{D}_{\ell_{m_0}(a)}(F+B)$ function $y^{(0)}(t)$ is a strong solution of problem (2.1), i.e. satisfies properties 1-3 stated in Theorem.

It remains to show that

$$\ell_{m_0(\mathbf{k}+1)}(a) \subset \mathcal{D}_{\ell_{m_0}(a)}(F+B)$$

But this is obvious due to the continuity of B in $\ell_{m_0(\mathbf{k}+1)}(a)$, the continuity of embedding $\ell_{m_0(\mathbf{k}+1)}(a) \subset \ell_{m_0}(a)$ and estimate: for $x \in \ell_{m_0(\mathbf{k}+1)}(a)$

$$\|F(x)\|_{\ell_{m_0}(a)} = (\sum_{k\in\mathbb{Z}^d} a_k |F(x_k)|^{m_0})^{1/m_0} \leq$$

$$\leq const(\sum_{k\in\mathbb{Z}^d} a_k (1+|x_k|)^{m_0(\mathbf{k}+1)})^{1/m_0} \leq$$

$$\leq const[(\sum_{k\in\mathbb{Z}^d} a_k)^{1/m_0(\mathbf{k}+1)} + \|x\|_{\ell_{m_0(\mathbf{k}+1)}(a)}]^{\mathbf{k}+1} < \infty$$

which follows from (2.2) and $F(0) = 0$.

Estimate (2.6) is a consequence of Theorem I in [6]. This estimate also enables us to construct the generalized solution by choosing any $\ell_{m_0(\mathbf{k}+1)}(a) \ni x_n \to x_0 \in \ell_{m_0}(a)$ and tending to the limit $\lim_{n\to\infty} y^{(0)}(t, x_n)$

By above we have that $y^{(0)}(t)$ is Lipschitz continuous on $t \in [0, T]$. This leads to the Lipschitz continuity of $\|y^{(0)}(t)\|_{\ell_{m_0}(a)}$ on $t \in [0, T]$, which gives its differentiability a.e. on $[0, T]$.

So we can apply (7.8),(7.2) and obtain that for almost all $t \in [0, T]$

$$\frac{d}{dt}\|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)} = m_0 < \frac{dy^{(0)}(t)}{dt}, [y^{(0)}(t)]^{\#} >=$$

$$= -m_0 < (F+B)(y^{(0)}(t)), [y^{(0)}(t)]^{\#} > \leq m_0\omega\|y^{(0)}(t)\|^{m_0}_{\ell_{m_0}(a)} \quad \blacksquare$$

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