Quasi-contractive Nonlinear Calculus of Variations and Smoothness of Discontinuous Semigroups, generated by Non-Lipschitz Stochastic Differential Equations.*

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ABSTRACT

In this paper we propose an approach to the investigation of smooth properties of not strongly continuous semigroups, based on observation about the nature of corresponding system in variations.

Quasi-contractive nonlinear estimates permit us to study the action of semigroups in spaces of continuously differentiable functions and to achieve the smoothness of flows, generated by non-Lipschitz (even locally) stochastic differential equations.

1 Introduction.

The appearance of different type nonlinearities in coefficients of equation can to a great extent influence the behaviour of solutions and usually requires relevant techniques, adopted to the description of arising phenomenons.

There is a large class of semigroups, associated with operator parabolic Cauchy problems, which does not permit the direct investigation by methods of classical semigroup theory, because the property of strong continuity fails to hold. They correspond to the second order differential operators of finite and infinite number of variables and have numerous applications as associated with nonlinear stochastic differential equations with unbounded coefficients. Such semigroups are known as Feller and the investigation of their regular properties in spaces of continuously differentiable functions leads to the problem of smooth dependence with respect to the initial data for solutions to stochastic differential equations.

In the case of globally Lipschitz coefficients with bounded derivatives the question of C^∞ - smoothness of the stochastic flow and corresponding semigroup has been already solved due to the elaboration of keen techniques, connected with direct treatment of system in variations and fixed point arguments [10, 11, 16, 17, 19], stochastic analysis on Wiener space and subtle properties of Ornstein-Uhlenbeck process [18], [31]-[34], [36]-[38], [40].

This paper is dedicated to the study of the smooth properties of semigroups, associated with nonlinear SDE with coefficients, for which the Lipschitz property breaks even locally on balls. Such semigroups has already formed an topic of steady interest in the theory of stochastic differential equations and have a large domain of applications to the rigorous description of lattice models in Statistical physics, see [9], [12]-[15], [20, 30, 35, 39] and references therein.

We describe a nonlinear technique, adopted to the investigation of C^∞ smoothness of semigroups and associated stochastic flows with essentially non-Lipschitz coefficients. We aim to show that a simple observation on the nature of system in variations enables to predict the simultaneous behaviour of all variations in a quasi-contractive way.

Let's explain the key idea in one-dimensional case. Consider semigroup

$$(P_t f)(x^0) = \mathbf{E} f(\xi^0(t, x^0))$$

associated with non-Lipschitz SDE

$$\xi^0(t, x^0) = x^0 + \int_0^t dW_\tau - \int_0^t F(\xi^0(\tau, x^0)) d\tau$$

To obtain the smoothness of semigroup, i.e. to study its derivatives

$$\frac{\partial^{(k)}}{(\partial x^0)^k}(P_t f)(x^0) = \sum_{j_1+\ldots+j_s=k,\, s\geq 1} \mathbf{E}\{f^{(s)}(\xi^0(t, x^0))\, \xi^{j_1}(t, x^0)\ldots\xi^{j_s}(t, x^0)\}$$

we are to consider variations $\xi^j(t, x) = \dfrac{\partial^j}{\partial x^j}\xi^0(t, x)$, which solve associated nonlinear equations

$$\xi^i(t, x^0) = x^i - \int_0^t d\sigma \sum_{j_1+\ldots+j_s=i,\, s\geq 1} F^{(s)}(\xi^0(\sigma, x^0))\xi^{j_1}(\sigma, x^0)\ldots\xi^{j_s}(\sigma, x^0) \tag{1.1}$$

with $x^1 = 1$ and $x^i = 0,\ i \geq 2$.

The key idea is that *the behaviour of i^{th} variation ξ^i is comparable with the behaviour of the first variation ξ^1 in the i^{th} degree.* This is connected with representation (1.1), where under summation we find simultaneously the terms $F'(\xi^0)\xi^i$ and $F^{(i)}(\xi^0)\,[\,\xi^1\,]^i$ for $s = 1$ and $s = i$. Taking into account this observation we introduce a special nonlinear expression

$$\rho(\xi; t) = \mathbf{E} \sum_{i=1}^n p_i(\xi^0(t))\, |\xi^i(t)\,|^{m_1/i}$$

which in essentially nonlinear way interlaces different variations $\xi^{(i)}$ and is controlled by weights, depending on initial solution ξ^0. We note that $\rho(\xi;t)$ does not permit the interpretation of norm in some Banach space.

Under assumption that monotone function F has no more than polynomial growth

$$\exists\, \mathbf{k} \geq -1\ \forall\, i \geq 1 \quad |F^{(i)}(x) - F^{(i)}(y)| \leq C_i |x-y|(1+|x|+|y|)^{\mathbf{k}}$$

we obtain a quasi-contractive nonlinear estimate on variations

$$\exists\, M\ \forall\, t \geq 0 \quad \rho(\xi;t) \leq e^{Mt}\rho(\xi;0)$$

Quasi-contractive estimates and observation above permit us to investigate the smooth properties of semigroup P_t in the special scales of continuously differentiable functions. The topologies of these spaces depend on order of nonlinearity $\mathbf{k}$ and require the smaller growth for lower derivatives of function, reflected in special hierarchy of weights in seminorms.

A more simple finite dimensional case of $I\!\!R^d$ is already studied in [5], the peculiarities of regularity problems for diffusion flows on manifolds are discussed in [6]. Similar estimates are also valid for stochastic derivatives, they give possibility to obtain Malliavin calculus applications for non-Lipschitz diffusion on non compact manifold [7] via a scheme of [4].

Paper consists of three parts. In second part we describe the problem, outline the general scheme of paper and give main results: nonlinear estimate on variations and theorem about preservance of spaces of continuously differentiable functions under the action of semigroup.

The third part is completely devoted to the proof of theorem on smooth properties of semigroup. In Section 3.1 and 3.2 we study the variations of stochastic differential equations with non-Lipschitz coefficients and prove their solvability and smooth dependence with respect to the initial data. Section 3.3 provides an integral representation for derivatives of semigroup along absolutely continuous paths. In Section 3.4 we show the quasi-contractive property of semigroup in scales of continuously differentiable functions and end the proof of main theorem. We also obtain ergodicity in derivatives for semigroup.

2 Main results: nonlinear quasi-contractive estimate and smooth properties of discontinuous semigroups.

The investigation of properties of infinite-dimensional semigroups, which do not fulfill the standard assumption of strong continuity, makes a prevailing topic of this article. These semigroups usually appear at consideration of the stochastic differential equations with unbounded coefficients and we apply nonlinear quasi-contractive estimates [3] to study the smooth properties of such semigroup in spaces of continuously differentiable functions.

Consider the stochastic differential equation

$$\begin{cases} d\xi_k^0(t,x^0) = dW_k(t) - \{F(\xi_k^0(t,x^0)) + (B\xi^0(t,x^0))_k\}dt \\ \xi_k^0(0,x^0) = x_k^0, \quad k \in Z\!\!\!Z^d \end{cases} \tag{2.1}$$

where, for probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathcal{F}_t$, the process $W(t) = \{W_k(t)\}_{k\in Z\!\!\!Z^d}$, $t \geq 0$ is $\mathcal{F}_t$ – adapted Wiener process defined on Ω with values in $\ell_2(a) = \ell_2(a, Z\!\!\!Z^d)$, and identity covariance operator. The linear finite-diagonal map $B : I\!\!R^{Z\!\!\!Z^d} \to I\!\!R^{Z\!\!\!Z^d}$ is defined by

$$Bx = \{\sum_{j:\,|j-k|\leq r_0} b(k-j)x_j\}_{k\in Z\!\!\!Z^d}$$

and the nonlinear map $F:\ I\!\!R^{Z\!\!\!Z^d} \ni x \longrightarrow F(x) = \{F(x_k)\}_{k\in Z\!\!\!Z^d} \in I\!\!R^{Z\!\!\!Z^d}$ is generated by the C^∞ monotone function F, $F(0)=0$, which satisfies condition of polynomial growth on the infinity

$$\exists\, \mathbf{k} \geq -1\ \forall\, i \geq 1 \quad |F^{(i)}(x) - F^{(i)}(y)| \leq C_i |x-y|(1+|x|+|y|)^{\mathbf{k}} \tag{2.2}$$

Note that for $\mathbf{k} > -1$ the map $F : I\!R^{\mathbb{Z}^d} \to I\!R^{\mathbb{Z}^d}$ is *non-Lipschitz* even on balls in any space $\ell_p(c) = \ell_p(c, \mathbb{Z}^d)$.

The questions on solvability of equations like (2.1) are profoundly studied [12]-[14], [30, 35, 39]. For example, for initial data $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$ there is a unique strong solution, i.e. $\ell_2(a)$-continuous $\mathcal{F}_t$ adapted process $\xi^0(t, x^0)$, which a.e. fulfils the integral form of equation (2.1), $\xi^0 \in \mathcal{D}_F(\ell_2(a))$ and

$$\forall\, r \geq 1 \qquad \mathbf{E} \sup_{t\in[0,T]} \|\xi^0(t, x^0)\|^r_{\ell_2(a)} < \infty \tag{2.3}$$

Furthermore, for $x^0 \in \ell_2(a)$ there is a generalized solution, obtained as uniform on $[0, T]$ $\mathbf{P}$ a.e. limit of strong solutions [13, 14]. Moreover, the associated not strongly continuous (discontinuous) semigroup

$$(P_t f)(x^0) = \mathbf{E}(f(\xi^0(t, x^0))) \tag{2.4}$$

corresponding to formal generator, given on cylinder test functions by

$$[H_\mu f](\cdot) = \frac{1}{2} \sum_{k\in\mathbb{Z}^d} \{-\partial_k^2 + v_k\partial_k\} f(\cdot) \tag{2.5}$$

where $\partial_k = \dfrac{\partial}{\partial x_k}$ and $v_k = F(x_k) + \sum_{j\in\mathbb{Z}^d} b(k-j)x_j$, is Feller in the space of bounded continuous functions $C_b(\ell_2(a))$.

In appendix we give a simple extension of some results [13, 14] to the case of pathwise estimates on solutions ξ^0 in spaces $\ell_m(a) \subset \ell_2(a)$. These facts we use to prove the smoothness of variations. Moreover, it is shown that the Banach space $Lip_r(\ell_2(a))$, $r \geq 0$, equipped with norm

$$\|f\|_{Lip_r} = \sup_{x\in\ell_2(a)} \frac{|f(x)|}{(1 + \|x\|_{\ell_2(a)})^{r+1}} + \tag{2.6}$$
$$+ \sup_{x,y\in\ell_2(a)} \frac{|f(x) - f(y)|}{\|x - y\|_{\ell_2(a)}(1 + \|x\|_{\ell_2(a)} + \|y\|_{\ell_2(a)})^r} < \infty$$

is preserved under the action of semigroup P_t.

The aim of this paper lies in the study of semigroup (2.4) in the subspaces of space $Lip_r(\ell_2(a))$, formed by functions with continuous derivatives of first and higher orders. We need to write the representation for derivatives of semigroup $\partial_\tau P_t f$, where $\tau = \{k_1, ..., k_m\}$ and $\partial_\tau = \partial^{|\tau|}/\partial x_{k_1} ... \partial x_{k_m}$. Formula (2.4) leads to

$$\partial_\tau(P_t f)(x^0) = \sum_{s=1}^{m} \sum_{\gamma_1\cup...\cup\gamma_s=\tau} \mathbf{E} < \partial^{(s)} f(\xi^0), \xi_{\gamma_1} \otimes ... \otimes \xi_{\gamma_s} > \tag{2.7}$$

where $\partial^{(s)} f = \{\partial_\gamma f\}_{|\gamma|=s}$ denotes the set of s^{th} order partial derivatives of function and we used notation

$$< \partial^{(s)} f(\xi^0), \xi_{\gamma_1} \otimes ... \otimes \xi_{\gamma_s} >= \sum_{j_1,..,j_s\in\mathbb{Z}^d} (\partial_{\{j_1,...,j_s\}} f)(\xi^0)\, \xi_{j_1,\gamma_1} ... \xi_{j_s,\gamma_s}$$

Vector $\xi_\tau = \{\xi_{k,\tau}\}_{k\in\mathbb{Z}^d}$ is interpretated as a derivative of ξ^0 with respect to the initial data $x^0 = \{x^0_k\}_{k\in\mathbb{Z}^d}$

$$\xi_{k,\tau} = \frac{\partial^{|\tau|}\xi^0_k(t, x^0)}{\partial x^0_{j_n} ... \partial x^0_{j_1}} \tag{2.8}$$

and is called below a τ^{th} *variation* of ξ^0. The equation on ξ_τ is derived by the formal successive differentiation of (2.1) with respect to x^0:

$$\begin{cases} \dfrac{d\xi_{k,\tau}}{dt} = -F'(\xi^0_k)\xi_{k,\tau} - \sum\limits_{j:\,|j-k|\leq r_0} b(k-j)\xi_{j,\tau} - \varphi_{k,\tau} \\ \xi_{k,\tau}(0) = x_{k,\tau} \end{cases} \tag{2.9}$$

where $\varphi_{k,\tau} = \varphi_{k,\tau}(\xi^0, \xi_{\cdot,\gamma}, \gamma \subset \tau, \gamma \neq \tau)$

$$\varphi_{k,\tau} = \sum_{\gamma_1\cup...\cup\gamma_s=\tau,\ s\geq 2} F^{(s)}(\xi^0)\xi_{k,\gamma_1} ... \xi_{k,\gamma_s} \tag{2.10}$$

In (2.10) the summation $\sum_{\gamma_1\cup..\cup\gamma_s=\tau,\, s\geq 2}$ runs on all possible subdivisions of the set $\tau = \{j_1,..,j_n\}$ $j_i \in \mathbb{Z}^d$ on the nonintersecting subsets $\gamma_1,..,\gamma_s \subset \tau$, with $|\gamma_1| + .. + |\gamma_s| = |\tau|$, $s \geq 2$, $|\gamma_i| \geq 1$. The precise sense to expression (2.8) as a solution to (2.9) can be given only under the special choice of initial data

$$\widetilde{x}_{k,\tau} = \begin{cases} \delta_{k,j}, & |\tau| = 1,\ \tau = \{j\} \subset \mathbb{Z}^d \\ 0, & |\tau| > 1 \end{cases} \tag{2.11}$$

We see, that the representation (2.7) gives the connection between the partial derivatives of semigroup (2.4) and the behaviour of variations with respect to the initial data. To show the preservance of spaces of continuously differentiable functions under the action of Feller semigroup P_t we need to investigate the differentiability of $\xi^0(t, x^0)$ with respect to x^0 or study its variations.

However

1. The coefficients of system (2.9) are controlled in essentially nonlinear manner by the solution ξ^0 of initial equation.
2. With respect to ξ_τ system (2.9) is non-autonomous one with unbounded operator coefficients.
3. Due to (2.10) each variation is interlaced with lower rank variations in nonlinear inherently multiplicative way.
4. To reconstruct the derivatives of semigroup it is necessary to guess the natural consent topologies of solvability for equations in variations.

The main tool to treat the system in variations is provided by next Theorem, which generalizes to stochastic case the results of [3]. As the evolution in time of ξ_τ shifts from zero-one data $\widetilde{x}_\tau$, we are to consider the case of arbitrary initial data. First we give the definition of strong solution to system (2.9). In Theorem 3.1 we show the strong solvability of system (2.9) in the sense of Definition 2.1 for $x_\gamma \in \ell_{m_\gamma}(dc_\gamma),\ \gamma \subset \tau,\ d \geq a^{-\frac{\mathbf{k}+1}{2}m_1}$.

Definition 2.1. The $\mathcal{F}_t$ adapted processes $\xi_\gamma(t, x^0),\ \gamma \subset \tau$ form *strong solutions to the system* (2.9) in spaces $\ell_{m_\gamma}(c_\gamma),\ \gamma \subset \tau$ iff $\forall\, \gamma \subset \tau$ for $\mathbf{P}$ a.e. $\omega \in \Omega$ the map $[0, T] \ni t \to \xi_\gamma(t, x^0) \in \ell_{m_\gamma}(c_\gamma)$ is Lipschitz continuous, $\xi_{k,\gamma}(0, x^0) = x_{k,\gamma},\ k \in \mathbb{Z}^d$ and for a.e. $t \in [0, T]$ $\xi_\gamma(t, x^0) \in \mathcal{D}_{\ell_{m_\gamma}(c_\gamma)}(F'(\xi^0(t, x^0)) + B)$. Moreover, there is a strong $\ell_{m_\gamma}(c_\gamma)$ derivative $d\xi_\gamma(t, x^0)/dt$ a.e. on $[0, T]$, equation (2.9) is satisfied in $\ell_{m_\gamma}(c_\gamma)$ a.e on $[0, T]$ and $\forall\, x^0 \in \ell_2(a)\ \ \forall\, q \geq 1\ \ \forall T > 0\ \ \forall\, \gamma \subset \tau$

$$\mathbf{E} \sup_{t\in[0,T]} \|\xi_\gamma(t)\|^q_{\ell_{m_\gamma}(c_\gamma)} < \infty \tag{2.12}$$

$$\mathbf{E} \sup_{t\in[0,T]} \|d\xi_\gamma(t, x^0)/dt\|^q_{\ell_{m_\gamma}(c_\gamma)} < \infty \tag{2.13}$$

Let us introduce the nonlinear expression:

$$\rho_\tau(\xi; t) = \mathbf{E} \sum_{s=1}^{n} \{p_s(z_t) \sum_{\gamma\subset\tau,\ |\gamma|=s} \|\xi_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}\} \tag{2.14}$$

where $\tau = \{j_1, ..., j_n\},\ j_i \in \mathbb{Z}^d$, p_s are polynomial functions depending on $z_t = \|\xi^0(t, x^0)\|^2_{\ell_2(a)}$ and $m_\gamma = m_1/|\gamma|$, $|\gamma|$ is a number of points in set $\gamma \subset \mathbb{Z}^d$. The set of all vectors $c = \{c_k\}_{k\in\mathbb{Z}^d}$, such that $\delta_c = \sup_{|k-j|=1} |c_k/c_j| < \infty$ we denote by $\mathbb{P}$.

Theorem 2.2. *Let F satisfy (2.2), $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$, $tr\, a = 1,\ a \in \mathbb{P}$, $x_\gamma \in \ell_{m_\gamma}(dc_\gamma),\ \gamma \subset \tau$, $d \geq a^{-\frac{\mathbf{k}+1}{2}m_1}$, $m_\gamma = m_1/|\gamma|$, $m_1 \geq |\tau|$ and $\xi^0,\ \{\xi_\tau\}_{\gamma\subset\tau}$ form the strong solutions to systems (2.1), (2.9). Suppose that functions $p_i(z),\ i = 1,..,n$ and vectors $\{c_\gamma\}_{\gamma\subset\tau} \subset \mathbb{P}$ in (2.14) satisfy:*

(1) $\exists\, \varepsilon > 0,\ \exists\, K > 0$, such that $\forall\, z \in \mathbb{R}_+$

$$p_i(z) \geq \varepsilon \quad \& \quad (1 + z)(|p_i'(z)| + |p_i''(z)|) \leq K p_i(z) \tag{2.15}$$

(2) $\exists K_p \ \ \forall j = 2,..,n \ \forall i_1,..,i_s: \ i_1 + ... + i_s = j, \ s \geq 2$

$$p_j^j(z)(1+z)^{\frac{\mathbf{k}_{+1}}{2}m_1} \leq K_p \cdot p_{i_1}^{i_1}(z)...p_{i_s}^{i_s}(z) \tag{2.16}$$

(3) for any subdivision of the set $\gamma = \alpha_1 \cup .. \cup \alpha_s, \ \gamma \subset \tau$ *on nonempty nonintersecting subsets* $\gamma_1,..,\gamma_s, \ s \geq 2$ *there is* $R_{\gamma,\alpha_1,..,\alpha_s}$ *such that* $\forall\, k \in \mathbb{Z}^d$

$$[c_{k,\gamma}]^{|\gamma|} a_k^{-\frac{\mathbf{k}_{+1}}{2}m_1} \leq R_{\gamma,\alpha_1,..,\alpha_s}[c_{k,\alpha_1}]^{|\alpha_1|}...[c_{k,\alpha_s}]^{|\alpha_s|} \tag{2.17}$$

Then there is a constant $M \in \mathbb{R}^1$*, such that the* **nonlinear estimate of quasi-contractive type** *holds*

$$\rho_\tau(\xi;t) \leq e^{Mt}\rho_\tau(\xi;0) \tag{2.18}$$

Upper indexes mean the powers and the parameter $\mathbf{k}$ above is introduced in (2.2) and reflects the order of nonlinearity of map F. In [3] it was shown that the set of vectors, which satisfy condition (2.17), is sufficiently wide. In particular, for the weight $\{c_\tau\}$ and arbitrary vector $d \in \mathbb{P}$ vector $\{d \cdot c_\tau\}$ also form a weight.

Proof. Introduce $h_\tau^i(\xi;t) = \sum\limits_{s=1}^{i} \mathbf{E}\{p_s(z_t) \sum\limits_{\gamma\subset\tau, \ |\gamma|=s} \|\xi_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}\}, \ \ i \geq 1$ and zero at $i = 0$, where $z_t = \|\xi^0(t,x^0)\|^2_{\ell_2(a)}$. Then at $i = n$ $h_\tau^n(\xi;t) = \rho_\tau(\xi;t)$ and

$$h_\tau^i(\xi;t) = h_\tau^{i-1}(\xi;t) + \sum_{\gamma\subset\tau, \ |\gamma|=i} g_\gamma(t) \tag{2.19}$$

with $g_\gamma(t) = \mathbf{E}\left[\, p_i(z)\|\xi_\gamma(t)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}\right]$. We prove inductively that

$$\forall\, i = 0,..,n \ \ \exists\, M_i \in \mathbb{R} \quad h_\tau^i(\xi;t) \leq e^{M_i t}h_\tau^i(\xi;0) \tag{2.20}$$

which at $i = n$ gives the statement of theorem. Base of induction at $i = 0$ is obvious. If for any $\gamma \subset \tau, \ |\gamma| = i$ we prove

$$\frac{dg_\gamma(t)}{dt} \leq K_1 g_\gamma(t) + K_2 h_\tau^{i-1}(\xi;t) \tag{2.21}$$

then due to the inductive assumption and representation (2.19) we obtain

$$\begin{gathered} h_\tau^i(\xi;t) \leq e^{M_{i-1}t}h_\tau^{i-1}(\xi;0)+ \\ + \sum_{\gamma\subset\tau, \ |\gamma|=i}\{e^{K_1 t}g_\gamma(0) + K_2\int_0^t e^{K_1(t-s)}e^{M_{i-1}s}h_\tau^{i-1}(\xi;0)ds\} \leq \\ \leq e^{(M_{i-1}+K_1)t}(1+2^{|\tau|}K_2 t)h_\tau^i(\xi;0) \leq e^{(M_{i-1}+K_1+2^{|\tau|}K_2)t}h_\tau^i(\xi;0) \end{gathered}$$

which gives (2.20). To prove (2.21) first note that Ito formula gives

$$p_i(z_t)\|\xi_\gamma(t)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} = p_i(z_0)\|\xi_\gamma(0)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} + 2\int_0^t p_i'(z_s)\|\xi_\gamma(s)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}(\xi^0(s), dW(s))+$$

$$+\int_0^t\{m_\gamma p_i(z_s) < \frac{d\xi_\gamma(s)}{ds}, [\xi_\gamma(s)]^\# >_{\ell_{m_\gamma}(c_\gamma)} - \|\xi_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}(H_\mu p)(z_s)\}ds \tag{2.22}$$

Here $(x,y) = \sum_{k\in\mathbb{Z}^d} a_k x_k y_k, \ < u, v^\# >_{\ell_m(c)} = \sum_{k\in\mathbb{Z}^d} c_k u_k v_k \cdot |v_k|^{m-2}$ for $v^\# = \|v\|^{m-2}_{\ell_m(c)}\mathcal{F}v$ with duality map $\mathcal{F}$ in space $\ell_m(c)$ and operator H_μ is introduced in (2.5). Estimates (2.3), (3.61), (2.12), (2.13) and inequality $|H_\mu p|(z_t) \leq Cp(z_t)[M + K\|\xi^0(t)\|^{2(\mathbf{k}+1)}_{\ell_{2(\mathbf{k}+1)}(a)}]$ guarantees the integrability on $[0,T]\times\Omega$ of all expressions in (2.22). Therefore function $g_\gamma(t)$ is a.e. on $[0,T]$ differentiable with derivative

$$\frac{dg_\gamma(t)}{dt} = m_\gamma \mathbf{E}\, p_i(z_t) < \frac{d\xi_\gamma(t)}{dt}, [\xi_\gamma(t)]^\# > -\mathbf{E}\{\|\xi_\gamma(t)\|^{m_\gamma \cdot}_{\ell_{m_\gamma}(c_\gamma)} \cdot [H_\mu p_i](z_t)\} \tag{2.23}$$

Due to the inequality $H_\mu p_i(z_t) \geq -M_{p_i} p_i(z_t)$ (see [2],Hint 9) the second term in (2.23) is estimated by $M_{p_i} \cdot g_\gamma(t)$. To estimate the first term we use Definition 2.1 of strong solution to (2.9), property $F'(x) \geq 0,\ x \in I\!R$ and boundedness of map B in any space $\ell_p(c)$, $p \geq 1,\ c \in I\!P$. Finally we apply inequality $| < f, \xi^\# > | \leq \frac{1}{m} \|f\|^m_{\ell_m(c)} + \frac{m-1}{m} \|\xi\|^m_{\ell_m(c)}$ and obtain

$$m_\gamma < \frac{d}{dt}\xi_\gamma(t), [\xi_\gamma]^\# >= -m_\gamma < (F'(\xi^0) + B)\xi_\gamma + \varphi_\gamma, [\xi_\gamma]^\# > \leq \tag{2.24}$$

$$\leq m_\gamma \|B\| \, \|\xi_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} + m_\gamma \sum_{\alpha_1..\alpha_s} | < F^{(s)}(\xi^0)\xi_{\alpha_1}...\xi_{\alpha_s}, [\xi_\gamma]^\# > | \leq$$

$$\leq (m_\gamma \|B\| + (m_\gamma - 1)2^{|\gamma|^2}) \|\xi_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} + \sum_{\alpha_1..\alpha_s} \|F^{(s)}(\xi^0)\xi_{\alpha_1}...\xi_{\alpha_s}\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}$$

Therefore

$$\frac{dg_\gamma(t)}{dt} \leq (m_\gamma \|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))} + (m_\gamma - 1)2^{|\gamma|^2} + M_{p_i}) g_\gamma(t) +$$

$$+ \sum_{\alpha_1 \cup ... \cup \alpha_s = \gamma,\ s \geq 2} \mathbf{E} p_i(\|\xi^0\|^2_{\ell_2(a)}) \ \|F^{(s)}(\xi^0)\xi_{\alpha_1}...\xi_{\alpha_s}\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} \tag{2.25}$$

Condition (2.2) on the function F gives

$$|F^{(s)}(\xi^0_k)| \leq C(1 + |\xi^0_k|)^{\mathbf{k}+1} \leq C \cdot a_k^{-\frac{\mathbf{k}+1}{2}} (1 + \|\xi^0(t)\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}}$$

Thus each term in (2.25) is estimated from above by

$$C\mathbf{E} \sum_{k \in Z\!\!\!Z^d} c_{k,\gamma} a_k^{-\frac{\mathbf{k}+1}{2} m_\gamma} \ p_i(\|\xi^0\|^2_{\ell_2(a)})(1 + \|\xi^0\|^2_{\ell_2(a)})^{-\frac{\mathbf{k}+1}{2} m_\gamma} |\xi_{k,\alpha_1}|^{m_\gamma} ... |\xi_{k,\alpha_s}|^{m_\gamma} \tag{2.26}$$

Special choice of $m_\gamma = m_\alpha \cdot |\alpha|/|\gamma|$ gives the representation

$$|\xi_{k,\alpha_1}|^{m_\gamma} ... |\xi_{k,\alpha_s}|^{m_\gamma} = [|\xi_{k,\alpha_1}|^{m_{\alpha_1}}]^{|\alpha_1|/|\gamma|} ... [|\xi_{k,\alpha_s}|^{m_{\alpha_s}}]^{|\alpha_s|/|\gamma|}.$$

Properties (2.16) and (2.17) imply

$$(2.26) \leq C\, K_p\, R_{\gamma,\alpha_1..\alpha_s} \mathbf{E} \sum_{k \in Z\!\!\!Z^d} \prod_{i=1}^{s} \{p_{|\alpha_i|}(\|\xi^0\|^2_{\ell_2(a)}) c_{k,\alpha_i} |\xi_{k,\alpha_i}|^{m_{\alpha_i}}\}^{|\alpha_i|/|\gamma|} \tag{2.27}$$

Applying inequality $|x_1 ... x_s| \leq \frac{|x_1|^{q_1}}{q_1} + ... + \frac{|x_s|^{q_s}}{q_s}$ with $q_j = |\gamma|/|\alpha_j|$ we obtain

$$(2.27) \leq C\, K_p\, R_{\gamma,\alpha_1..\alpha_s} \mathbf{E} \sum_{j=1}^{s} \frac{|\alpha_j|}{|\gamma|} p_{|\alpha_j|}(\|\xi^0\|^2_{\ell_2(a)}) \ \|\xi_{\alpha_j}\|^{m_{\alpha_j}}_{\ell_{m_{\alpha_j}}(c_{\alpha_j})} \leq$$

$$\leq C\, K_p\, R_{\gamma,\alpha_1,..,\alpha_s} h^{i-1}_\tau(\xi; t) \tag{2.28}$$

In the last inequality we used that for subdivision $\alpha_1 \cup ... \cup \alpha_s = \gamma$, $|\gamma| = i$ at $s \geq 2$ we have $|\alpha_j| \leq i - 1$. Therefore (2.21) is proved with

$$K_1 = 2M_{p_i} + \sup_{\gamma \subset \tau} (m_\gamma \|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))} + (m_\gamma - 1)2^{|\gamma|^2}), \tag{2.29}$$

$$K_2 = K_p\, C\, 2^{|\tau|^2} \max_{\alpha_1 \cup .. \cup \alpha_s = \gamma,\ \gamma \subset \tau} R_{\gamma,\alpha_1...\alpha_s} \tag{2.30}$$

see (2.25), (2.26) and (2.28). ■

Now we can discuss the structure of topologies in spaces, preserved under the action of semigroup P_t. It is intrinsically determined by the structure of nonlinear estimate just proved and essentially depends on the order of nonlinearity $\mathbf{k}$ of map F. The similar hierarchy, connected with the special reduction of weights in seminorm, was already noticed in [2], where we've shown, that this hierarchy *can not be avoided*, if one wishes to have the quasi-accreative property for semigroup generator.

Denote, for some $m \in I\!N$, by Θ^m the array of pairs $\{(p,\mathcal{G}) : (p,\mathcal{G}) \in \Theta^m\}$ where $\mathcal{G} = G^1 \otimes ... \otimes G^m$ is m-tensor constructed by vectors $G^i \in I\!P$, $i = 1,...,m$ and p is a smooth function of polynomial behavior (2.15).

Definition 2.3. The array $\Theta = \Theta^1 \cup ... \cup \Theta^n$, $n \in I\!N$, is a *quasi-contractive with parameter* $\mathbf{k}$ iff $\forall\, m = 2,...,n\ \forall\,(p,\mathcal{G}) \in \Theta^m$ and $\forall\, i,j \in \{1,...,m\}$, $i \neq j$, there is a pair $(\widetilde{p},\widetilde{\mathcal{G}}) \in \Theta^{m-1}$ such that it majorizes a pair
$(\,(1+z)^{\frac{\mathbf{k}+1}{2}} p(z), \widehat{\mathcal{G}}^{\{i,j\}})$, i.e.

$$\exists\, K:\quad \forall\, z \in I\!R^1_+ \quad (1+z)^{\frac{\mathbf{k}+1}{2}}\widetilde{p}(z) \leq K p(z)$$

$$(\widehat{\mathcal{G}}^{\{i,j\}})^\ell \leq K\, \widetilde{G}^\ell, \quad \ell = 1,..,m-1 \tag{2.31}$$

Above $(m-1)$-tensor $\widehat{\mathcal{G}}^{\{i,j\}}$ is constructed from m-tensor $\mathcal{G}$ by the rule

$$\widehat{\mathcal{G}}^{\{i,j\}} = G^1 \otimes \underset{\widehat{i}}{....} \otimes (A)^{-(\mathbf{k}+1)} \underset{\uparrow j}{G^i G^j} \otimes .. \otimes G^m$$

Notation $G^1 \otimes \underset{\widehat{i}}{....} \otimes G^s$ means that in tensor product the i^{th} - vector is omitted and $G^1 \otimes .. \otimes \underset{\uparrow j}{B} \otimes .. \otimes G^s$ means that on j^{th} - place in tensor product it is inserted vector B. Inequality (2.31) is understood as a coordinate inequality between two vectors.

For multifunction of m^{th} order $u^{(m)}(x) = \{u_\tau(x),\ \tau = \{k_1,...,k_m\},\ k_i \in Z\!\!\!Z^d\}$, $x \in \ell_2(a)$ we introduce the seminorm

$$\|u^{(m)}\|_{\Theta^m} = \sup_{x\in\ell_2(a)} \max_{(p_m,\mathcal{G}^m)\in\Theta^m} \frac{|u^{(m)}(x)|_{\mathcal{G}^m}}{p_m(\|x\|^2_{\ell_2(a)})} \tag{2.32}$$

with $|u^{(m)}(x)|^2_{\mathcal{G}^m} = \sum\limits_{\tau=\{j_1...j_m\}\subset Z\!\!\!Z^d} G^1_{j_1}...G^m_{j_m}|u_\tau(x)|^2$ for $\mathcal{G}^m = G^1 \otimes ... \otimes G^m$.

Let $r \geq 0$, $n \geq 1$ and $\Theta = \Theta^1 \cup ... \cup \Theta^n$ be a quasi-contractive array with parameter $\mathbf{k}$. We say that $f \in C_{\Theta,r}(\ell_2(a))$ iff $f \in Lip_r(\ell_2(a))$ and

1) There is a set of partial derivatives $\{\partial^{(1)}f,...,\partial^{(n)}f\}$ such that for any $m \in \{1,...,n\}$ the coordinates of multifunctions $\{\partial^{(m)}f(x)\}_{j_1...j_m} = \partial_\tau f(x)$, $\tau = \{j_1,...,j_m\}$ are continuous $\partial_\tau f \in C(\ell_2(a), I\!R^1)$ and $\forall\, x^0 \in \ell_2(a)$, $\forall\, h \in \mathbf{X}_\infty([a,b])$ it holds

$$f(x^0 + h(\cdot))\Big|_a^b = \int_a^b ds \sum_{k\in Z\!\!\!Z^d} \partial_k f(x^0 + h(s))h'_k(s) \tag{2.33}$$

and $\forall\, \tau = \{j_1,...,j_\ell\}$, $|\tau| = \ell \leq n-1$

$$\partial_\tau f(x^0 + h(\cdot))\Big|_a^b = \int_a^b ds \sum_{k\in Z\!\!\!Z^d} \partial_{\tau\cup\{k\}} f(x^0 + h(s))h'_k(s) \tag{2.34}$$

2) The norm is finite

$$\|f\|_{C_{\Theta,r}} = \|f\|_{Lip_r} + \max_{m=1,n} \|\partial^{(m)}f\|_{\Theta^m} < \infty \tag{2.35}$$

Above

$$\mathbf{X}_\infty([a,b]) = \bigcap_{p\geq 1, c\in I\!P} AC_\infty([a,b], \ell_p(c)) \tag{2.36}$$

and $AC_\infty([a,b], X) = \{h \in C([a,b], X) :\ \exists\, h' \in L_\infty([a,b], X\}$ for Banach space X.

Remark. Definition of $C_{\Theta,r}$ is not transparent at the first look and we would like to give some comments. The structure of seminorm $\|\cdot\|_{\Theta^m}$ and condition of quasi-contractivity of array Θ

is actually *dictated* by the structure of nonlinear estimate and, as it will be seen, guarantees the quasi-contractive property of semigroup P_t is scale $C_{\Theta,r}$.

For fixed $\omega \in \Omega,\ t \in [0,T]$ the map $\ell_2(a) \ni x^0 \to \xi^0(\omega, t, x^0) \in \ell_2(a)$ and its variations $\{\xi_\tau\}$ have *nonlinear responses* with respect to initial data in representation of $\partial_\tau P_t f$ (2.7). This circumstance motivated us to give a sense (2.33), (2.34) to derivatives of function $f \in C_{\Theta,r}$.

Properties (2.35) and (2.33), (2.34) give in particular that for function $f \in C_{\Theta,r}$ there exist continuous partial derivatives up to order n $\partial_{j_1\dots j_m} f(x) = \partial^{|\tau|} f(x)/\partial x_{j_1}\dots\partial x_{j_m},\quad x = \{x_k\}_{k\in\mathbb{Z}^d}$ To show this one should take $h(t) = te_k$, $t \in [0,1]$ in (2.33) and (2.34). Due to the finiteness of norm $\|f\|_{C_{\Theta,r}}$ the r.h.s. of (2.33) and (2.34) are well-defined for any $h \in \mathbf{X}_\infty([a,b])$.

Theorem below give the main result of paper about the preservance of $C_{\Theta,r}$ under the action of discontinuous Feller semigroup, defined by (2.4).

Theorem 2.4. *Let F fulfill (2.2) and Θ be a quasi-contractive array with parameter $\mathbf{k}$. Then $\forall\, t \ge 0: \ \ P_t : C_{\Theta,r} \to C_{\Theta,r}$ and $\exists\, K_{\Theta,r},\ M_{\Theta,r}$*

$$\forall\, f \in C_{\Theta,r} \quad \|P_t f\|_{C_{\Theta,r}} \le K_{\Theta,r} e^{M_{\Theta,r} t} \|f\|_{C_{\Theta,r}}$$

Proof. The remaining part of paper is dedicated to the proof of this theorem. In Sections 3.1 and 3.2 we conduct all necessary preparations about the equation in variations, including their solvability, continuous and smooth dependence on initial data, and also obtain the integral representations of variations with respect to nonlinear increments of initial data.

In Sections 3.3 and 3.4 we check the conditions of Definition 2.3 for semigroup. Moreover, Theorem 3.10 gives the simple consequence of nonlinear estimate — ergodicity in derivatives. Necessary facts on properties of initial SDE and the preservance of space $Lip_r(\ell_2(a))$ are given in Appendix. ■

3 Proof of theorem on smooth properties of semigroup.

3.1 Strong solvability of the system in variations and continuity of variations with respect to the initial data.

Here we construct the variations ξ_τ to give a rigorous sense to partial derivatives $\partial_\tau P_t f$ (2.7). The theorem below states the strong solvability of system in variations (2.9) and justifies a priori nonlinear estimate (2.18).

Theorem 3.1. *Let $a \in \mathbb{P}$, $tr\, a = 1$, vectors $\{c_\tau\}$ satisfy (2.17), $|\tau| < m_1$, and $d \in \mathbb{P}$ be such that $d_k \ge a_k^{-\frac{(\mathbf{k}+1)}{2} m_1} \ge 1$. Denote $X_\gamma = \ell_{m_\gamma}(c_\gamma)$, $Y_\gamma = \ell_{m_\gamma}(dc_\gamma)$, $m_\gamma = m_1/|\gamma|$, $\gamma \subset \tau$. Then $\forall\, x^0 \in \ell_2(a)$ and $\forall\, x_\gamma \in Y_\gamma$, $\gamma \subset \tau$ we have that $\forall\, \gamma \subset \tau$ there is a unique $\mathcal{F}_t$ — adapted process*

$$\xi_\gamma(t) \in C([0,T], X_\gamma) \cap L^\infty([0,T], Y_\gamma), \qquad \mathbf{P} - a.e.\, \omega \in \Omega,$$

which gives a strong solution to the γ^{th} Cauchy problem (2.9) in the space X_γ. Moreover, $\exists\, K(\cdot, \tau, R) \in L_p(\Omega, \mathbf{P})$, $p \ge 1$, such that

$$\sup_{t\in[0,T]} \|\xi_\gamma(t, x^0)\|_{Y_\gamma} \le K(\omega, \tau, R) \tag{3.1}$$

with $R = \max(\|x^0\|_{\ell_2(a)},\ \|x_\gamma\|_{Y_\gamma},\ \gamma \subset \tau)$.

The proof essentially uses the following propositions, first of which states the solvability of non-autonomous equations [21]-[26] and the second one enables to work with multiplicative structure of system in variations.

Proposition 3.2 [21],Th.2.1-2.3. Let $Y \subset X$ be continuously imbedded reflexive Banach spaces, operators $A(t),\ A(t) \upharpoonright_Y,\ t \in [0,T]$ generate the strongly continuous semigroups in X

and Y, $\forall\, t \in [0,T]$ $\; Y \subset \mathcal{D}_X(A(t))$, and $\exists\, \lambda\; \forall\, t \in [0,T]$: $\|e^{-sA(t)}\|_{\mathcal{L}(X)} \le e^{\lambda s}$, $\|e^{-sA(t)} \upharpoonright_Y \|_{\mathcal{L}(Y)} \le e^{\lambda s}$, $\; s \ge 0$. Suppose that

$$A(\cdot) \upharpoonright_Y \in C([0,T], \mathcal{L}(Y,X)) \tag{3.2}$$

$$\varphi \in C([0,T],X) \cap L^\infty([0,T],Y) \tag{3.3}$$

Then the Cauchy problem

$$\begin{cases} y'(t) = -A(t)y(t) + \varphi(t) \\ y(0) = y_0, \quad y_0 \in Y \end{cases} \tag{3.4}$$

has a unique strong solution $y \in C([0,T],X) \cap L^\infty([0,T],Y)$ which possesses the strong X - derivative and fulfills (3.4) a.e. on $t \in [0,T]$. It is represented in the form

$$y(t) = U(t,0)y_0 + \int_0^t U(t,s)\varphi(s)ds$$

with unique strongly continuous in X evolution system $\{U(t,s),\; 0 \le s \le t\}$, generated by $\{A(t)\}_{t\in[0,T]}$, such that $\|U(t,s)\|_{\mathcal{L}(X)} \le e^{\lambda(t-s)}$, $\|U(t,s)\|_{\mathcal{L}(Y)} \le e^{\lambda(t-s)}$. Moreover, if $\{A(t)\}_{t\in[0,T]}$ and $\{\widetilde{A}(t)\}_{t\in[0,T]}$ are two families, which fulfill the assumptions above and generate evolution systems $U, \widetilde{U}$ respectively, then $\forall\, y \in Y$

$$\|U(t,s)y - \widetilde{U}(t,s)y\|_X \le |t-s| e^{2T(\lambda+\widetilde{\lambda})} \|y\|_Y \sup_{\tau\in[s,t]} \|A(\tau) - \widetilde{A}(\tau)\|_{\mathcal{L}(Y,X)} \tag{3.5}$$

Proposition 3.3 [3], Prop.4.2 and 4.4. Let $a \in I\!\!P$, $\; tr\, a = 1$, $\; |\tau| < m_1$, vectors $\{c_\gamma\}_{\gamma\subset\tau}$ satisfy (2.17) and $d \in I\!\!P$ such that $d_k \ge a_k^{-\frac{\mathbf{k}+1}{2}m_1}$. Denote $X_\gamma = \ell_{m_\gamma}(c_\gamma)$, $Y_\gamma = \ell_{m_\gamma}(dc_\gamma)$, $m_\gamma = m_1/|\gamma|$, $\gamma \subset \tau$. Then for function Q, which satisfies $\exists\, K \quad |Q(x) - Q(y)| \le K|x-y|(1+|x|+|y|)^{\mathbf{k}}$, $\forall\, x,y \in I\!\!R^1$ the map $\ell_2(a) \ni \xi \to Q(\xi) \in \mathcal{L}(Y_\tau, X_\tau)$, $[Q(\xi)u]_k = Q(\xi_k)u_k$ is continuous and

$$\exists\, C\; \forall\, \xi,\zeta \qquad \|Q(\xi)\|_{\mathcal{L}(Y_\tau,X_\tau)} \le C(1+|Q(0)|)(1+\|\xi\|_{\ell_2(a)})^{\mathbf{k}+1} \tag{3.6}$$

$$\|\, [\, Q(\xi) - Q(\zeta)\,]\, u\|_{X_\tau} \le C\|\xi - \zeta\|_{\ell_2(a)}(1+\|\xi\|_{\ell_2(a)} + \|\zeta\|_{\ell_2(a)})^{\mathbf{k}}\|u\|_{Y_\tau} \tag{3.7}$$

Moreover, introduce $\{Q(\xi^0)\xi_{\gamma_1}\ldots\xi_{\gamma_n}\}_k = Q(\xi^0_k)\xi_{k,\gamma_1}\ldots\xi_{k,\gamma_n}$, $\; k \in Z\!\!\!Z^d$ for $n \ge 2$, $\; \gamma_1 \cup \ldots \cup \gamma_n = \tau$ with the corresponding action of $Q(\xi^0)\xi_{\gamma_1}\ldots\xi_{\gamma_\ell}$ in the space $\bigotimes_{i=\ell+1}^{n} X_{\gamma_i}$. Then for any fixed $\ell = 1,..,n$ the map

$$\ell_2(a) \times \mathop{\times}_{i=1}^{\ell} X_{\gamma_i} \ni (\xi^0, \xi_{\gamma_1}, .., \xi_{\gamma_\ell}) \to Q(\xi^0)\xi_{\gamma_1}\ldots\xi_{\gamma_\ell} \in \mathcal{L}(\bigotimes_{i=\ell+1}^{n} X_{\gamma_i}, X_\tau)$$

is continuous and the estimate holds

$$\begin{aligned} \exists\, K \qquad & \|\, [\, Q(\xi^0)\xi_{\gamma_1}\ldots\xi_{\gamma_\ell} - Q(\zeta^0)\zeta_{\gamma_1}\ldots\zeta_{\gamma_\ell}\,]\, u_{\gamma_{\ell+1}}\ldots u_{\gamma_n}\|_{X_\tau} \le \\ & \le K\{1 + \|\xi^0\|_{\ell_2(a)} + \|\zeta^0\|_{\ell_2(a)}\}^{\mathbf{k}+1} \prod_{i=1}^{\ell} [1 + \|\xi_{\gamma_i}\|_{X_{\gamma_i}} + \|\zeta_{\gamma_i}\|_{X_{\gamma_i}}] \times \\ & \times \{\|\xi^0 - \zeta^0\|_{\ell_2(a)} + \sum_{i=1}^{\ell} \|\xi_{\gamma_i} - \zeta_{\gamma_i}\|_{X_{\gamma_i}}\} \prod_{i=\ell+1}^{n} \|u_{\gamma_i}\|_{X_{\gamma_i}} \end{aligned} \tag{3.8}$$

Proof of Theorem 3.1. We can view the system (2.9) as a family of nonautonomous inhomogeneous equations, parameterized by $\omega \in \Omega$

$$\begin{cases} \dfrac{d\xi_\tau}{dt} = -A(t)\xi_\tau - \varphi_\tau, \\ \xi_\tau(0) = x_\tau \end{cases} \tag{3.9}$$

with operator $A(t) = A(\omega,t,x^0) = F'(\xi^0(\omega,t,x^0)) + B$ and function $\varphi_\tau = \varphi_\tau(\omega,t,x^0)$ introduced in (2.10). For any fixed $\omega \in \Omega$ $\forall\, x^0 \in \ell_2(a)$ $\forall\, t \in [0,T]$ the operator $A(\omega,t,x^0)$ generates the strongly continuous semigroup in any space $\ell_p(c)$, $\; c \in I\!\!P$, $\; p \ge 1$ with uniform on $\omega \in \Omega$, $\; x^0 \in$

$\ell_2(a)$ constant $\lambda = \inf_{t\in I\!R^1} F'(t) + \|B\|_{\mathcal{L}(\ell_p(c))} = \|B\|$. This follows from m-monotonicity of linear diagonal operator $F'(\xi)y = \{F'(\xi_k)y_k\}_{k\in Z\!\!Z^d}$ for any $\xi \in I\!R^{Z\!\!Z^d}$ and boundedness of map B in any space $\ell_p(c),\ c \in I\!P,\ p \geq 1$ [8],p.158.

Pathwise continuity of ξ^0 and estimate (3.7) with $Q(\cdot) = F'(\cdot)+B$, $\xi = \xi^0(t,x^0)$, $\zeta = \xi^0(s,x^0)$ imply property (3.2), i.e. $A(\omega,\cdot,x^0) \in C([0,T],\mathcal{L}(Y_\tau,X_\tau))$ and generates the evolution system $U^\omega_{x^0}(t,s)$.

To construct the strong solutions to (3.9) we check condition (3.3) for φ_τ inductively on $|\tau|$. For $|\alpha| = 1,\ \varphi_\alpha \equiv 0$ and by Proposition 3.2 there is a unique finite variation process

$$\xi_\alpha(\omega,t,x^0;x_\alpha) \in C([0,T],X_\alpha) \cap L^\infty([0,T],Y_\alpha), \quad |\alpha| = 1 \tag{3.10}$$

which solves (3.9) and possesses representation $\xi_\alpha(t,x^0;x_\alpha) = U^\omega_{x^0}(t,0)x_\alpha$, $|\alpha| = 1$.

Suppose that (3.10) is true for all $\alpha \subset \tau,\ |\alpha| \leq n_0 - 1$. Then $\forall\, \alpha \subset \tau,\ |\alpha| = n_0$ we have from inductive assumption, (3.61) and (3.62) that $\varphi_\alpha \in C([0,T],X_\alpha)$ because by (3.8)

$$\|\varphi_\alpha(t_1) - \varphi_\alpha(t_2)\|_{X_\alpha} \leq \sum_{\gamma_1\cup..\cup\gamma_s=\alpha, s\geq 2} \|F^{(s)}(\xi^0(t_1))\xi_{\gamma_1}(t_1)...\xi_{\gamma_s}(t_1) -$$

$$-F^{(s)}(\xi^0(t_2))\xi_{\gamma_1}(t_2)...\xi_{\gamma_s}(t_2)\|_{X_\alpha} \leq K \sum_{\gamma_1\cup..\cup\gamma_s=\alpha, s\geq 2}$$

$$(1 + \|\xi^0(t_1)\|_{\ell_2(a)} + \|\xi^0(t_2)\|_{\ell_2(a)})^{\mathbf{k}+1} \prod_{i=1}^{s}(1 + \|\xi_{\gamma_i}(t_1)\|_{X_{\gamma_i}} + \|\xi_{\gamma_i}(t_2)\|_{X_{\gamma_i}})\times$$

$$\times \{\|\xi^0(t_1) - \xi^0(t_2)\|_{\ell_2(a)} + \sum_{i=1}^{s} \|\xi_{\gamma_i}(t_1) - \xi_{\gamma_i}(t_2)\|_{X_{\gamma_i}}\}$$

Similarly, estimate (3.8) with $n = \ell = s$ and $\zeta_{\gamma_1} = ... = \zeta_{\gamma_s} = 0$ implies $\varphi_\alpha \in L^\infty([0,T],Y_\alpha)$

$$\|\varphi_\alpha(\omega,x^0,t)\|_{Y_\alpha} \leq K \sum_{\gamma_i\cup...\cup\gamma_s=\alpha, s\geq 2} \sup_{t\in[0,T]} \{1 + \|\xi^0(t,x^0)\|_{\ell_2(a)}\}^{\mathbf{k}+1}\cdot \tag{3.11}$$

$$\cdot \prod_{i=1}^{s}(1 + \|\xi_{\gamma_i}(t,x^0)\|_{Y_{\gamma_i}})\{\|\xi^0(t,x^0)\|_{\ell_2(a)} + \sum_{i=1}^{s} \|\xi_{\gamma_i}(t,x^0)\|_{Y_{\gamma_i}}\}$$

Therefore at $|\alpha| = n_0$ Proposition 3.2 gives unique strong solution ξ_α to (3.9), which fulfills (3.10) and has representation

$$\xi_\alpha(\omega,t,x^0;x_\gamma,\gamma \subset \alpha) = U^\omega_{x^0}(t,0)x_\alpha + \int_0^t U^\omega_{x^0}(t,s)\varphi_\alpha(s)ds \tag{3.12}$$

The $\mathcal{F}_t$ measurability of $\xi_\alpha(t)$ follows from the representation (3.12) and $\mathcal{F}_t$ measurability of $U^\omega_{x^0}(t,s)x_\gamma,\ x_\gamma \in Y_\gamma$, which is obtained from (3.5) and (3.7) with $Q(\cdot) = F'(\cdot) + B$, $\xi = \xi^0(\omega,\sigma,x^0)$, $\zeta = \xi^0(\widetilde{\omega},\sigma,x^0)$:

$$\|U^\omega_{x^0}(t,s)x_\gamma - U^{\widetilde{\omega}}_{x^0}(t,s)x_\gamma\|_{X_\gamma} \leq C\,|t-s|\,\|x_\gamma\|_{Y_\gamma}\cdot$$

$$\cdot \sup_{\sigma\in[0,t]} \|\xi^0(\omega,\sigma) - \xi^0(\widetilde{\omega},\sigma)\|_{\ell_2(a)} \cdot (1 + \|\xi^0(\omega,\sigma)\|_{\ell_2(a)} + \|\xi^0(\widetilde{\omega},\sigma)\|_{\ell_2(a)})^{\mathbf{k}}$$

Estimate (3.1) simply follows from representation (3.12), iteration of (3.11) with usage of (3.61), (3.62) and estimate below

$$\|\xi_\alpha(\omega,t,x^0)\|_{Y_\alpha} \leq e^{\lambda T}\|x_\alpha\|_{Y_\alpha} + Te^{\lambda T} \sup_{t\in[0,T]} \|\varphi_\alpha(x^0,t)\|_{Y_\alpha} \tag{3.13}$$

Finally, properties (2.12), (2.13) in the definition of strong solution follow from (3.6), inequality

$$\|\frac{d\xi_\gamma(t,x^0)}{dt}\|_{X_\gamma} = \| - \{F'(\xi^0) + B\}\xi_\gamma - \varphi_\gamma\|_{X_\gamma} \leq$$

$$\leq C(1 + \|\xi^0\|_{\ell_2(a)})^{\mathbf{k}+1}\|\xi_\gamma\|_{Y_\gamma} + \|\varphi_\gamma\|_{X_\gamma}$$

estimates (3.61), (3.1) and (3.11). ■

Next theorem gives the continuity of variation $\xi_\tau(t, x^0, x_\gamma)$ with respect to the initial data x^0. Later we apply this result to prove C^∞ differentiability of $\xi^0(t, x^0)$ with respect to x^0 and to extend the nonlinear quasi-contractive estimate (2.18) from $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$ to $x^0 \in \ell_2(a)$.

Theorem 3.4. *Under conditions of Theorem 3.1* $\forall\, x^0, y^0 \in \ell_2(a)$ $\forall\, x_\gamma \in Y_\gamma, \gamma \subset \tau$ *there is* $K(\cdot, \tau, R) \in L_p(\Omega, \mathbf{P}), p \geq 1,$ *such that* $\forall\, \gamma \subset \tau,\ |\gamma| \leq m_1$

$$\sup_{t\in[0,T]} \|\xi_\tau(t, x^0; x_\gamma) - \xi_\tau(t, y^0; x_\gamma)\|_{X_\tau} \leq K(\omega, \tau, R)\|x^0 - y^0\|_{\ell_2(a)} \tag{3.14}$$

with $R = \max(\|x^0\|_{\ell_2(a)}, \|y^0\|_{\ell_2(a)}, \|x_\gamma\|_{Y_\gamma}, \gamma \subset \tau)$.

Proof. Application of (3.5), (3.7) gives

$$\begin{aligned} &\sup_{t,s\in[0,T]} \|U^\omega_{x^0}(t,s) - U^\omega_{y^0}(t,s)\|_{\mathcal{L}(Y_\tau, X_\tau)} \leq \\ &\leq Te^{2\lambda T} \sup_{s\in[0,T]} \|F'(\xi^0(s, x^0)) - F'(\xi^0(s, y^0))\|_{\mathcal{L}(Y_\tau, X_\tau)} \leq \\ &\leq C \sup_{s\in[0,T]} \|\xi^0(s, x^0) - \xi^0(s, y^0)\|_{\ell_2(a)}\{1 + \|\xi^0(s, x^0)\|_{\ell_2(a)} + \|\xi^0(s, y^0)\|_{\ell_2(a)}\}^{\mathbf{k}} \leq \\ &\leq C'\|x^0 - y^0\|_{\ell_2(a)}(1 + \|x^0\|_{\ell_2(a)} + \|y^0\|_{\ell_2(a)} + 2K(\omega))^{\mathbf{k}} \end{aligned} \tag{3.15}$$

where $K(\cdot) \in L_p(\Omega, \mathbf{P}),\ p \geq 1$ appear due to (3.61) and (3.62).

At $|\tau| = 1$ $\varphi_\tau \equiv 0$ and due to representation (3.12) and estimate (3.15) we have the statement of theorem

$$\|\xi_\tau(\omega, t, x^0) - \xi_\tau(\omega, t, y^0)\|_{X_\tau} \leq K'(\omega, R)\|x_\tau\|_{Y_\tau} \cdot \|x^0 - y^0\|_{\ell_2(a)}$$

Suppose that $\forall\, \tau$ $|\tau| \leq n-1$ the statement is proved. From (3.12) we have

$$\|\xi_\tau(\omega, t, x^0) - \xi_\tau(\omega, t, y^0)\|_{X_\tau} \leq \|U^\omega_{x^0}(t, 0)x_\tau - U^\omega_{y^0}(t, 0)x_\tau\|_{X_\tau} + \tag{3.16}$$

$$+T \sup_{s,t\in[0,T]} \|\{U^\omega_{x^0}(t,s) - U^\omega_{y^0}(t,s)\}\varphi_\tau(\omega, s, x^0)\|_{X_\tau} \tag{3.17}$$

$$+T \sup_{t\in[0,T]} \|U^\omega_{y^0}(t,s)\{\varphi_\tau(\omega, s, x^0) - \varphi_\tau(\omega, s, y^0)\}\|_{X_\tau} + \tag{3.18}$$

Terms (3.16) and (3.17) are estimated due to (3.15), (3.11) and (3.1). Finally from representation (2.10) and estimate (3.8) with $Q(\cdot) = F^{(s)}(\cdot)$

$$\begin{aligned} &(3.18) \leq Te^{\lambda T} \sup_{t\in[0,T]} \|\varphi_\tau(\omega, t, x^0) - \varphi_\tau(\omega, t, y^0)\|_{X_\tau} \leq \\ &\leq C' \sum_{\gamma_1\cup\ldots\cup\gamma_s=\tau} \sup_{t\in[0,T]} \{1 + \|\xi^0(t, x^0)\|_{\ell_2(a)} + \|\xi^0(t, y^0)\|_{\ell_2(a)}\}^{\mathbf{k}+1} \cdot \\ &\cdot \prod_{i=1}^s (1 + \|\xi_{\gamma_i}(t, x^0)\|_{X_{\gamma_i}} + \|\xi_{\gamma_i}(t, y^0)\|_{X_{\gamma_i}}) \cdot \\ &\cdot \{\|\xi^0(t, x^0) - \xi^0(t, y^0)\|_{\ell_2(a)} + \sum_{i=1}^s \|\xi_{\gamma_i}(t, x^0) - \xi_{\gamma_i}(t, y^0)\|_{X_{\gamma_i}}\} \leq \\ &\leq K'(\omega, \tau, R)\|x^0 - y^0\|_{\ell_2(a)} \end{aligned} \tag{3.19}$$

where $K'(\cdot, \tau, R) \in L_p(\Omega, \mathbf{P}),\ p \geq 1$ appear due to (3.61), (3.62), (3.1) and inductive assumption. ■

The following corollary gives important for further consideration estimates on variations ξ_τ at zero-one initial data (2.11) with uniform on τ, $|\tau| \leq n$ constants. We use this result in Sections 3.2 and 3.3 to obtain integrable majorants and properties of convolutions. Recall that *only at zero-one initial data* variations ξ_τ have sense of derivatives for ξ^0.

Corollary 3.5. *Let* $a, \psi \in I\!\!P$, $tr\, a = 1$ *and* ξ_τ *be a strong solutions to (2.9) with zero-one initial data* $\widetilde{x}_\tau$ *(2.11).*

Then $\forall\, n \geq 1\ \exists\, K_n(\cdot, R, \psi) \in L_p(\Omega, \mathbf{P}),\ p \geq 1$ *such that* $\forall\, \tau\ |\tau| \leq n$

$$\sup_{t\in[0,T]} |\xi_{k,\tau}(t, x^0)| \leq \frac{K_n(\omega, R, \psi)}{a_k^{\frac{\mathbf{k}+1}{2}(|\tau|-1)} \prod_{j\in\tau} \psi_{k-j}} \tag{3.20}$$

$$\sup_{t\in[0,T]} |\xi_{k,\tau}(t, x^0) - \xi_{k,\tau}(t, y^0)| \leq \frac{K_n(\omega, R, \psi)}{a_k^{\frac{\mathbf{k}+1}{2}(2|\tau|-1)} \prod_{j\in\tau} \psi_{k-j}} \|x^0 - y^0\|_{\ell_2(a)} \tag{3.21}$$

with $R = \max(\|x^0\|_{\ell_2(a)}, \|y^0\|_{\ell_2(a)})$.

Proof. Estimates (3.20) and (3.21) are obtained as a consequence of (3.1) and (3.14) at special choice of spaces $\widetilde{X}_\tau = \ell_{m_\tau}(d^{-1}\widetilde{c}_\tau)$ and $\widetilde{Y}_\tau = \ell_{m_\tau}(\widetilde{c}_\tau)$ with vectors $d_k = a_k^{-\frac{\mathbf{k}+1}{2}m_1}$ and $\widetilde{c}_{k,\tau} = a_k^{\frac{\mathbf{k}+1}{2}m_1\frac{|\tau|-1}{|\tau|}} \prod_{j\in\tau} \psi_{k-j}^{m_1/|\tau|}$ for some $m_1 \geq n$. First remark that $\{\widetilde{c}_\tau\}$ fulfills property (2.17) with constant $R_{\tau;\gamma_1\ldots\gamma_s} = 1$. Indeed

$$a_k^{-\frac{\mathbf{k}+1}{2}m_1}[\widetilde{c}_{k,\tau}]^{|\tau|} = a_k^{\frac{\mathbf{k}+1}{2}m_1(|\tau|-2)} \prod_{j\in\tau} \psi_{k-j}^{m_1} \leq a_k^{\frac{\mathbf{k}+1}{2}m_1(|\tau|-s)} \prod_{j\in\tau} \psi_{k-j}^{m_1} =$$

$$= \prod_{i=1}^{s} [a_k^{\frac{\mathbf{k}+1}{2}m_1\frac{|\gamma_i|-1}{|\gamma_i|}} \prod_{b\in\gamma_i} \psi_{k-b}^{m_1/|\gamma_i|}]^{|\gamma_i|} = [\widetilde{c}_{k,\gamma_1}]^{|\gamma_1|} \ldots [\widetilde{c}_{k,\gamma_s}]^{|\gamma_s|} \tag{3.22}$$

where $\tau = \gamma_1 \cup \ldots \cup \gamma_s$, $|\gamma_1| + \ldots + |\gamma_s| = |\tau|$, $s \geq 2$.

(1) We obtain estimate (3.20). For $|\tau| = 1$ $\varphi_\tau \equiv 0$ and $\|\widetilde{x}_\tau\|_{\widetilde{Y}_\tau} = \psi_0$, therefore by representation (3.12)

$$\sup_{|\tau|=1} \sup_{s,t\in[0,T]} \|\xi_\tau(t, x^0, \widetilde{x}_\tau)\|_{\widetilde{Y}_\tau} \leq e^{\lambda T} \|\widetilde{x}_\tau\|_{\widetilde{Y}_\tau} = e^{\lambda T}\psi_0 \tag{3.23}$$

with uniform on τ: $|\tau| \leq n$ constant $\lambda = \lambda(n) \geq \max_{|\tau|\leq n} \|B\|_{\mathcal{L}(\widetilde{Y}_\tau)}$: $\forall\, x \in \widetilde{Y}_\tau$

$$\|Bx\|_{\widetilde{Y}_\tau} = (\sum_{k\in\mathbb{Z}^d} \widetilde{c}_{k,\tau} |\sum_{j:\,|k-j|\leq r_0} b(k-j)x_j|^{m_\tau})^{1/m_\tau} =$$

$$= (\sum_{k\in\mathbb{Z}^d} \widetilde{c}_{k,\tau} |\sum_{|i|\leq r_0} b(i)x_{k-i}|^{m_\tau})^{1/m_\tau} \leq \sum_{|i|\leq r_0} |b(i)| (\sum_{k\in\mathbb{Z}^d} \widetilde{c}_{k,\tau}|x_{k-i}|^{m_\tau})^{1/m_\tau}$$

$$= \sum_{|i|\leq r_0} |b(i)| (\sum_{k\in\mathbb{Z}^d} \frac{\widetilde{c}_{k,\tau}}{\widetilde{c}_{k-i,\tau}} \widetilde{c}_{k-i,\tau}|x_{k-i}|^{m_\tau})^{1/m_\tau} \leq \sum_{|i|\leq r_0} |b(i)|\, \delta_{\widetilde{c}_\tau}^{|i|/m_\tau}\, \|x\|_{\widetilde{Y}_\tau}$$

$$\leq \sum_{|i|\leq r_0} |b(i)|\, \delta_a^{\frac{\mathbf{k}+1}{2}|i|(n-1)}\, \delta_\psi^{|i|}\, \|x\|_{\widetilde{Y}_\tau} = \lambda \|x\|_{\widetilde{Y}_\tau} < \infty \tag{3.24}$$

where we used notation $\delta_\psi = \sup_{|k-j|=1} |\psi_k/\psi_j| < \infty$ for $\psi \in \mathbb{P}$.

We proceed by induction on $|\tau|$. By $\|\widetilde{x}_\tau\|_{\widetilde{Y}_\tau} = 0$, $|\tau| \geq 2$, representation (3.12), base (3.23), inductive assumption and estimate (3.11) we obtain with usage of (3.61) that $\exists\, K_n(\cdot, R, \psi)$ such that for all $|\alpha| \leq n$

$$\sup_{t\in[0,T]} \|\xi_\alpha(\omega, t, x^0)\|_{\widetilde{Y}_\gamma} \leq T e^{\lambda T} \sup_{t\in[0,T]} \|\varphi_\alpha(\omega, t, x^0)\|_{\widetilde{Y}_\alpha} \leq K_n(\omega, R, \psi) \tag{3.25}$$

and its coordinate form leads to (3.20).

(2) The proof of statement (3.21) completely follows the scheme of Theorem 3.4. We only mark the points which give uniform on τ : $|\tau| \leq n$ constant $K_n(\omega, R, \psi)$. By (3.24) and its analog in space $\widetilde{X}_\tau$ we have

$$\sup_{s,t\in[0,T]} \|U_{x^0}^\omega(t,s)\|_{\mathcal{L}(\widetilde{Y}_\tau)},\ \sup_{s,t\in[0,T]} \|U_{x^0}^\omega(t,s)\|_{\mathcal{L}(\widetilde{X}_\tau)} \leq e^{\lambda T} \tag{3.26}$$

and therefore the constant C' in (3.15) is uniform. Estimation of (3.16)–(3.18) is made inductively like in Theorem 3.4 with application of $\|\widetilde{x}_\tau\|_{\widetilde{Y}_\tau} = \psi_0$ at $|\tau| = 1$, zero otherwise, (3.26), (3.19), property $R_{\tau;\gamma_1\ldots\gamma_s} = 1$ and (3.25). We obtain $\exists\, K_n(\cdot, R, \psi)\ \forall\ |\alpha| \le n$

$$\sup_{t\in[0,T]} \|\xi_\alpha(\omega,t,x^0) - \xi_\alpha(\omega,t,y^0)\|_{\widetilde{X}_\alpha} \le K_n(\omega,R,\psi)\|x^0 - y^0\|_{\ell_2(a)}$$

and its coordinate form (3.21). ■

3.2 C^∞ – differentiability of stochastic flow.

In this Section we provide the method to justify property (2.8), i.e. in nonlinear quasi-contractive way we show that at zero-one initial data $\widetilde{x}_\tau$ (2.11) strong solutions ξ_τ, constructed in Theorem 3.1, have a sense of partial derivatives of process ξ^0.

In both Theorems below we proceed in two steps: first show strong differentiability of ξ^0, ξ_τ in specially chosen spaces, and then as application of Corollary 3.5 achieve integral representations.

Theorem 3.6. *Let F fulfill condition (2.2). Then $\forall\, x^0 \in \ell_2(a)$, zero-one data $\widetilde{x}_\tau$ (2.11) and $\forall\, h \in \mathbf{X}_\infty([a,b])$ we have for $\mathbf{P}$ a.e. $\omega \in \Omega$, $t \in [0,T]$*

$$\chi^0(\cdot) = \xi^0(t, x^0 + h(\cdot)) - \xi^0(t, x^0 + h(a)) \in \mathbf{X}_\infty([a,b]) \tag{3.27}$$

In particular, in any space $\ell_p(c)$, $c \in I\!P$, $p \ge 1$,

$$\xi^0(t, x^0 + h(\cdot))\Big|_a^b = \ell_p(c) \int_a^b \sum_{j\in Z\!\!Z^d} \xi_{\{j\}}(t, x^0 + h(s))h_j'(s)ds \tag{3.28}$$

and

$$\frac{d}{ds}\chi^0(\cdot) = \sum_{j\in Z\!\!Z^d} \xi_{\{j\}}(t, x^0 + h(\cdot))h_j'(\cdot) \in L^\infty([a,b], \ell_p(c)) \tag{3.29}$$

Space $\mathbf{X}_\infty([a,b])$ was introduced in (2.36).

Proof. First we prove representation (3.28) for $x^0 \in \ell_{m_1(\mathbf{k}+1)^2}(a)$ and $h \in \mathbf{X}_\infty([a,b])$ in space $X_1 = \ell_{m_1}(c_1)$ with $c_1 \in I\!P$ such that $d_k c_{k,1} \le a_k$, $d_k \ge a_k^{-\frac{\mathbf{k}+1}{2}m_1}$. Inequalities (3.61), (3.62) give that $\forall\, t \in [0,T]$, $\mathbf{P}$ a.e. $\omega \in \Omega$ the map $[a,b] \ni s \to \xi^0(\omega, t, x^0 + h(s)) \in \ell_{m_1}(a)$ is a Lipshitz continuous in $\ell_{m_1}(a)$ and therefore in X_1, thus by theory of absolute continuous functions in Banach space [27]-[29] we have

$$\xi^0(t, x^0 + h(\cdot))\Big|_a^b = \int_a^b X_1 \frac{d}{ds}\xi^0(t, x^0 + h(s))\, ds$$

To reconstruct the strong derivative we prove

$$\sup_{\alpha\in(0,\alpha_0]}\ \sup_{t\in[0,T]} \|\Delta_\emptyset(t,\alpha)\|_{X_1} \le \alpha_0 K(\omega, R) \tag{3.30}$$

where

$$\Delta_{k,\emptyset}(t) = \frac{\xi_k^0(t,y^\alpha) - \xi_k^0(t,y^0)}{\alpha} - \sum_{j\in Z\!\!Z^d} \xi_{k,\{j\}}(t,y^0)h_j'$$

for $y^\alpha = x^0 + h(s+\alpha)$, $h' = h'(s)$ and $K(\cdot, R) \in L_p(\Omega, \mathbf{P})$, $\forall\, p \ge 1$ for

$$R = \max(\|x^0\|_{\ell_{m_1}(a)}, \max_{s\in[a,b]}\{\|h(s)\|_{\ell_{m_1}(a)}, \|h'(s)\|_{\ell_{m_1}(a)}\}).$$

By Theorem 3.1 for strong solutions $\{\xi_{\{j\}}\}_{j\in Z\!\!Z^d}$ and $v \in \bigcap_{p,c} \ell_p(c)$ there is a representation

$$\sum_{j\in Z\!\!Z^d} \xi_{\{j\}}(t)v_j = \sum_{j\in Z\!\!Z^d} \widetilde{x}_j v_j - \int_0^t \sum_{j\in Z\!\!Z^d} v_j(F'(\xi^0) + B)U_{x^0}^\omega(s,0)\widetilde{x}_j ds$$

The strong differentiability of $\sum_{j\in\mathbb{Z}^d}\xi_{\{j\}}(t)v_j$ in X_1 on $t\in[0,T]$ follows from

$$\|\sum_{j\in\mathbb{Z}^d} v_j(F'(\xi^0)+B)U^\omega_{x^0}(s,0)\widetilde{x}_j\|_{X_1}\le$$
$$\le K(1+\|\xi^0\|_{\ell_2(a)})^{\mathbf{k}+1}\sum_{j\in\mathbb{Z}^d}|v_j|\cdot\|U^\omega_{x^0}(s,0)\widetilde{x}_j\|_{\ell_{m_1}(a)}\le$$
$$\le Ke^{\lambda T}(1+\sup_{t\in[0,T]}\|\xi^0\|_{\ell_2(a)})^{\mathbf{k}+1}\sum_{j\in\mathbb{Z}^d}|v_j|a_j$$

Above we applied Proposition 3.3 (3.6) with $d_k c_{k,1}\le a_k$ and properties of $U^\omega_{x^0}$. At last representation $\xi^0(t,y^\alpha)-\xi^0(t,y^0)=\eta^0(t,y^\alpha)-\eta^0(t,y^0)$ with process η^0 defined in (3.63) shows existence of strong derivative $X_1\,\dfrac{d}{dt}\Delta_\emptyset(t)$.

Therefore [27]-[29] for $\mathbf{P}$ a.e. $\omega\in\Omega$, $t\in[0,T]$

$$\frac{d}{dt}\|\Delta_\emptyset(t)\|^{m_1}_{X_1}=m_1\sum_{k\in\mathbb{Z}^d}c_{k,1}\Delta^{m_1-1}_{k,\emptyset}\cdot\{-\frac{F(\xi^0_k(y^\alpha))-F(\xi^0_k(y^0))}{\alpha}+$$
$$+\sum_{j\in\mathbb{Z}^d}F'(\xi^0_k)\xi_{k,\{j\}}h'_j-(B\,\Delta_\emptyset)_k\}=m_1\sum_{k\in\mathbb{Z}^d}c_{k,1}\Delta^{m_1-1}_{k,\emptyset}\{-F'(\xi^0_k)\Delta_{k,\emptyset}-$$
$$-(B\Delta_\emptyset)_k-\int_0^1[\,F'(\zeta_{k,\varepsilon,\alpha})-F'(\xi^0_k)\,]\,\frac{\xi^0_k(y^\alpha)-\xi^0_k(y^0)}{\alpha}\,d\varepsilon\}\le$$
$$\le K'\|\Delta_\emptyset(t)\|^{m_1}_{X_1}+\int_0^1\|[F'(\zeta_{\varepsilon,\alpha})-F'(\xi^0)]\,\frac{\xi^0(y^\alpha)-\xi^0(y^0)}{\alpha}\|^{m_1}_{X_1}d\varepsilon \tag{3.31}$$

with $K'=m_1\|B\|_{\mathcal{L}(X_1)}+m_1-1$, where we used notation $\zeta_{\varepsilon,\alpha}=\xi^0(y^0)+\varepsilon(\xi^0(y^\alpha)-\xi^0(y^0))$, inequalities $F'\ge 0$, $|xy^{m-1}|\le\frac{1}{m}|x|^m+\frac{m-1}{m}|y|^m$ and for $x,y\in\mathbb{R}^1$ formula $f(y)-f(x)=f'(x)(y-x)+\int_0^1\{f'(x+\varepsilon(y-x))-f'(x)\}(y-x)d\varepsilon$.

Inequality (3.31) leads to

$$\frac{d}{dt}\|\Delta_\emptyset(t)\|^{m_1}_{X_1}\le K'\|\Delta_\emptyset(t)\|^{m_1}_{X_1}+\alpha^{m_1}K(\omega,R) \tag{3.32}$$

where $K(\omega,R)$ appears from estimates

$$\|\frac{\xi^0(y^\alpha)-\xi^0(y^0)}{\alpha}\|^{m_1}_{Y_1}\le\frac{1}{\alpha}\|\xi^0(y^\alpha)-\xi^0(y^0)\|^{m_1}_{\ell_{m_1}(a)}\le e^{\lambda Tm_1}(\max_{s\in[a,b]}\|h'\|_{\ell_{m_1}(a)})^{m_1},$$
$$\int_0^1\|F'(\zeta_{\varepsilon,\alpha})-F'(\xi^0)\|^{m_1}_{\mathcal{L}(Y_1,X_1)}d\varepsilon\le$$
$$\le C\int_0^1\|\varepsilon(\zeta_{\varepsilon,\alpha}-\xi^0(y^0))\|^{m_1}_{\ell_2(a)}(1+\|\xi^0\|_{\ell_2(a)}+\|\zeta_{\varepsilon,\alpha}\|_{\ell_2(a)})^{\mathbf{k}m_1}d\varepsilon\le$$
$$\le C'(\alpha\max_{s\in[a,b]}\|h'\|_{\ell_2(a)})^{m_1}(1+2\|\xi^0(y^0)\|_{\ell_2(a)}+\alpha\max_{s\in[a,b]}\|h'\|_{\ell_2(a)})^{\mathbf{k}m_1}$$

Here we used (3.62), differentiability of $h\in\mathbf{X}_\infty([a,b])$, (3.7) with $Y_1=\ell_{m_1}(dc_1)$, assumption $dc_1\le a$ and inequality (3.61).

Due to $\Delta_\emptyset(0)=0$ from (3.32) we have (3.30) and therefore (3.28) in $\ell_{m_1}(c_1)$ for initial data $x^0\in\ell_{m_1(\mathbf{k}+1)^2}(a)$. The closure to $x^0\in\ell_2(a)$ is made due to (3.62) and estimate

$$\|\int_a^b ds\sum_{j\in\mathbb{Z}^d}\{\xi_{\{j\}}(\omega,t,x^0_n+h(s))-\xi_{\{j\}}(\omega,t,x^0+h(s))h'_j(s)\|_{X_1}\le$$
$$\le(b-a)K_1(\omega,R,\psi)\|x^0_n-x^0\|_{\ell_2(a)}\|\sum_{j\in\mathbb{Z}^d}\frac{1}{\psi_{\cdot-j}}h'_j(s)\|_{\ell_{m_1}(a^{-\frac{\mathbf{k}+1}{2}m_1}c_1)}\le \tag{3.33}$$
$$\le(b-a)K_1(\omega,R,\psi)\|x^0_n-x^0\|_{\ell_2(a)}\sum_{i\in\mathbb{Z}^d}\frac{(\delta_a^{\frac{\mathbf{k}+1}{2}m_1}\delta_{c_1})^{i/m_1}}{\psi(i)}\|h'\|_{\ell_{m_1}(a)}$$

where we applied Corollary 3.5 (3.21) at $|\tau| = 1$ and analogue of (3.24) with $b(i) = 1/\psi_i$. Choice of $\psi \in I\!\!P$ makes the sum convergent and therefore we obtain (3.28) in $X_1 = \ell_{m_1}(c_1)$ for $x^0 \in \ell_2(a)$.

Finally, by theory of absolutely continuous functions in Banach space [27]-[29] representations (3.28) and (3.29) in any space $\ell_p(c)$ would follow from

$$\forall\, p \geq 1,\ c \in I\!\!P \qquad \sum_{j\in \mathbb{Z}^d} \xi_{\{j\}}(\omega, t, x^0 + h(\cdot))h_j'(\cdot) \in L^\infty([a,b], \ell_p(c))$$

To obtain this property one should proceed like in (3.33) with application of (3.20) at $|\tau| = 1$ and $h \in \mathbf{X}_\infty([a,b])$. ■

Now we obtain the high order differentiability of $\xi^0(t,x^0)$ with respect to the initial data.

Theorem 3.7. *Let F fulfill condition (2.2). Then $\forall\, x^0 \in \ell_2(a)$, zero-one data $\widetilde{x}_\tau$ (2.11) and $\forall\, h \in \mathbf{X}_\infty([a,b])$ we have for $\mathbf{P}$ a.e. $\omega \in \Omega$, $t \in [0,T]$ $\forall\, k \in \mathbb{Z}^d$ $\forall\, \tau$*

$$\frac{d}{ds}\xi_{k,\tau}(t, x^0 + h(\cdot)) = \sum_{j\in \mathbb{Z}^d} \xi_{k,\tau\cup\{j\}}(t, x^0 + h(\cdot))h_j'(\cdot) \in L_\infty([a,b], I\!\!R^1) \tag{3.34}$$

and

$$\xi_{k,\tau}(t, x^0 + h(\cdot))\Big|_a^b = \int_a^b \sum_{j\in \mathbb{Z}^d} \xi_{k,\tau\cup\{j\}}(t, x^0 + h(s))h_j'(s)ds$$

Proof. Denote by $X_n = \ell_{m_n}(c_n)$ for $m_n = m_1/n$ for sufficiently large number m_1 and vectors $c_n \in I\!\!P$, such that $\forall\, k \in \mathbb{Z}^d$ $c_{k,1}d_k \leq a_k$ and $\forall\, n \in I\!\!N$ $c_{k,n+1}d_k \leq c_{k,n}$ with $d_k \geq a_k^{-\frac{\mathbf{k}+1}{2}m_1}$, $m_1 \geq 2$. Vectors $c_\tau = c_{|\tau|}$ satisfy condition (2.17) with constant $R_{\tau;\gamma_1,\ldots,\gamma_s} = 1$, therefore by Theorem 3.4 for zero-one initial data $\widetilde{x}_\tau$ (2.11) for $\mathbf{P}$ a.e. $\omega \in \Omega$, $h \in \mathbf{X}_\infty([a,b])$ and $t \in [0,T]$ the map $[a,b] \ni s \to \xi_\tau(\omega, t, x^0 + h(s)) \in X_{|\tau|}$ is Lipschitz continuous in $X_{|\tau|}$. Due to inequality $\|\cdot\|_{X_{|\tau|+1}} \leq const\|\cdot\|_{X_{|\tau|}}$ and theory of absolutely continuous functions in Banach space [27]-[29] we have representation in space $X_{|\tau|+1}$

$$\xi_\tau(t, x^0 + h(\cdot)\Big|_a^b = \int_a^b X_{|\tau|+1}\frac{d}{ds}\xi_\tau(t, x^0 + h(s))\, ds \tag{3.35}$$

I. To reconstruct strong derivative we prove in an inductive on τ $|\tau| \geq 1$ way the following inequality

$$\sup_{\alpha\in(0,\alpha_0]}\sup_{t\in[0,T]} \|\Delta_\tau(t,\alpha)\|_{X_{|\tau|+1}} \leq \alpha_0 K(\omega,\tau,R) \tag{3.36}$$

where
$$\Delta_{k,\tau}(t,\alpha) = \frac{\xi_{k,\tau}(t,y^\alpha) - \xi_{k,\tau}(t,y^0)}{\alpha} - \sum_{j\in \mathbb{Z}^d} \xi_{k,\tau\cup\{j\}}h_j'$$

for $y^\alpha = x^0 + h(s+\alpha)$, $h' = h'(s)$ and $K(\cdot,\tau,R) \in L_p(\Omega,\mathbf{P})$, $p \geq 1$ with

$$R = \max(\|x^0\|_{\ell_{m_1}(a)}, \max_{s\in[a,b]}\{\|h(s)\|_{\ell_{m_1}(a)}, \|h'(s)\|_{\ell_{m_1}(a)}\}).$$

Definition of strong solution with zero-one initial data (2.11) and formula (3.12) imply for $v \in \cap_{p,c}\,\ell_p(c)$, $|\tau| \geq 1$

$$\sum_{j\in \mathbb{Z}^d} \xi_{\tau\cup j}(t)v_j = -\int_0^t \sum_{j\in \mathbb{Z}^d} v_j(F'(\xi^0) + B)\int_0^s U_{x^0}^\omega(s,\sigma)\varphi_{\tau\cup j}(\sigma)\, d\sigma\, ds -$$
$$-\int_0^t \sum_{j\in \mathbb{Z}^d} v_j\varphi_{\tau\cup j}\, ds \tag{3.37}$$

Combining (3.12), (3.11) with $Y_n = \ell_{m_n}(dc_n)$, (3.61) and inequality $\|\widetilde{x}_j\|_{Y_1} = (d_jc_{j,1})^{1/m_1} \leq (tr\, a)^{1/m_1} = 1$ we have $\sup_{j\in \mathbb{Z}^d}\sup_{\sigma\in[0,T]}\|\varphi_{\tau\cup j}(\sigma)\|_{Y_{|\tau|+1}} < \infty$. Due to inequality

$$\|\sum_{j\in \mathbb{Z}^d} v_j(F'(\xi^0) + B)\int_0^s U_{x^0}^\omega(s,\sigma)\varphi_{\tau\cup j}(\sigma)\, d\sigma\|_{X_{|\tau|+1}} \leq$$

$$\leq Te^{\lambda T}(1+\|\xi^0\|_{\ell_2(a)})^{\mathbf{k}+1}\sum_{j\in\mathbb{Z}^d}|v_j|\sup_{\sigma\in[0,T]}\|\varphi_{\tau\cup j}(\sigma)\|_{Y_{|\tau|+1}}$$

we have boundedness of integrands in (3.37) and strong differentiability on $t\in[0,T]$ of $\sum_{j\in\mathbb{Z}^d}v_j\xi_{\tau\cup j}(t)$ and therefore of $\Delta_\tau(t)$ in space $X_{|\tau|+1}$.

Let inequality (3.36) be valid for τ $|\tau|\leq n_0-1$. Because the proof of inductive base at $|\tau|=1$ coincides with the proof of inductive step with $\varphi_\tau=0,\ |\tau|=1$, we conduct both proofs simultaneously.

$$\frac{d}{dt}\|\Delta_\tau(t)\|^{m_{|\tau|+1}}_{X_{|\tau|+1}}=m_{|\tau|+1}\sum_{k\in\mathbb{Z}^d}c_{k,|\tau|+1}\Delta^{m_{|\tau|+1}-1}_{k,\tau}\{-(B\Delta_\tau)_k-$$
$$-\sum_{\gamma_1\cup...\cup\gamma_s=\tau,s\geq1}\frac{1}{\alpha}F^{(s)}(\xi^0_k(\cdot))\xi_{k,\gamma_1}(\cdot)...\xi_{k,\gamma_s}(\cdot)\Big|^{y^\alpha}_{y^0}+$$
$$+\sum_{j\in\mathbb{Z}^d}\sum_{\alpha_1\cup...\cup\alpha_s=\tau\cup\{j\},s\geq1}F^{(s)}(\xi^0_k(y^0))\xi_{k,\alpha_1}(y^0)...\xi_{k,\alpha_s}(y^0)h'_j\}\tag{3.38}$$

First we transform the last term redenoting the indexes of summation and using notation $\xi_\emptyset$ for ξ^0

$$\sum_{j\in\mathbb{Z}^d}\sum_{\alpha_1\cup...\cup\alpha_s=\tau\cup\{j\},\ s\geq1}F^{(s)}(\xi^0_k(y^0))\xi_{k,\alpha_1}(y^0)...\xi_{k,\alpha_s}(y^0)h'_j=$$
$$=\sum_{\gamma_1\cup...\cup\gamma_s=\tau,s\geq1}\Big\{\sum_{q=1}^{s}F^{(s)}(\xi_{k,\emptyset}(y^0))\xi_{k,\gamma_1}(y^0)...(\sum_{j\in\mathbb{Z}^d}\xi_{k,\gamma_q\cup\{j\}}h'_j)...\xi_{k,\gamma_s}(y^0)+$$
$$+F^{(s+1)}(\xi_{k,\emptyset}(y^0))\xi_{k,\gamma_1}(y^0)...\xi_{k,\gamma_s}(y^0)\,(\sum_{j\in\mathbb{Z}^d}\xi_{k,\{j\}}h'_j)\ \Big\}$$

Using notation $\zeta_{\varepsilon,\alpha}=\xi(y^0)+\varepsilon(\xi(y^\alpha)-\xi(y^0))$ and formula

$$f(y_0,...,y_s)-f(x_0,...,x_s)=\sum_{i=0}^{s}\frac{\partial f}{\partial i}(\vec{x})(y_i-x_i)+$$
$$+\sum_{i=0}^{s}\int_0^1\{\frac{\partial f}{\partial i}(\vec{x}+\varepsilon(\vec{y}-\vec{x}))-\frac{\partial f}{\partial i}(\vec{x})\}(y_i-x_i)d\varepsilon$$

we rewrite last two terms in (3.38) in the form

$$\sum_{\gamma_1\cup...\cup\gamma_s=\tau,s\geq1}\frac{1}{\alpha}F^{(s)}(\xi^0_k(\cdot))\xi_{k,\gamma_1}(\cdot)...\xi_{k,\gamma_s}(\cdot)\Big|^{y^\alpha}_{y^0}-$$
$$-\sum_{j\in\mathbb{Z}^d}\sum_{\alpha_1\cup...\cup\alpha_s=\tau\cup\{j\},s\geq1}F^{(s)}(\xi^0_k(y^0))\xi_{k,\alpha_1}(y^0)...\xi_{k,\alpha_s}(y^0)h'_j=$$
$$=F'(\xi_{k,\emptyset}(y^0))\Delta_{k,\tau}(t)+\tag{3.39}$$
$$+\sum_{\gamma_1\cup...\cup\gamma_s=\tau,s\geq2}\sum_{q=1}^{s}F^{(s)}(\xi_{k,\emptyset}(y^0))\xi_{k,\gamma_1}(y^0)...\Delta_{k,\gamma_q}...\xi_{k,\gamma_s}(y^0)+\tag{3.40}$$
$$+\sum_{\gamma_1\cup...\cup\gamma_s=\tau,s\geq1}F^{(s+1)}(\xi_{k,\emptyset}(y^0))\xi_{k,\gamma_1}(y^0)...\xi_{k,\gamma_s}(y^0)\Delta_{k,\emptyset}+\tag{3.41}$$
$$+\int_0^1d\varepsilon\sum_{\gamma_1\cup...\cup\gamma_s=\tau,s\geq1}\sum_{q=1}^{s}F^{(s)}(z_{k,\emptyset})z_{k,\gamma_1}...\frac{\xi_{k,\gamma_q}(y^\alpha)-\xi_{k,\gamma_q}(y^0)}{\alpha}...z_{k,\gamma_s}\Big|^{z=\zeta_{\varepsilon,\alpha}}_{z=\xi(y^0)}+\tag{3.42}$$
$$+\int_0^1d\varepsilon\sum_{\gamma_1\cup...\cup\gamma_s=\tau,s\geq1}F^{(s+1)}(z_{k,\emptyset})z_{k,\gamma_1}...z_{k,\gamma_s}\Big|^{z=\zeta_{\varepsilon,\alpha}}_{z=\xi(y^0)}\frac{\xi_{k,\emptyset}(y^\alpha)-\xi_{k,\emptyset}(y^0)}{\alpha}\tag{3.43}$$

To obtain the expression (3.39) we separated in summation $\sum_{\gamma_1\cup...\cup\gamma_s=\tau,s\geq1}$ the first term at $s=1$. Using $F'\geq0$ we continue (3.38)

$$\frac{d}{dt}\|\Delta_\tau(t)\|^{m_{|\tau|+1}}_{X_{|\tau|+1}}=-m_{|\tau|+1}<\{F'(\xi^0)+B\}\Delta_\tau(t),\Delta^\#_\tau>_{X_{|\tau|+1}}-$$
$$-m_{|\tau|+1}<(3.40)+(3.41)+(3.42)+(3.43),\Delta^\#_\tau>\leq$$

$$\leq (m_{|\tau|+1}\|B\|_{\mathcal{L}(X_{|\tau|+1})} + 4(m_{|\tau|+1}-1))\,\|\Delta_\tau(t)\|^{m_{|\tau|+1}}_{X_{|\tau|+1}} +$$

$$+\|3.40\|^{m_{|\tau|+1}}_{X_{|\tau|+1}} + \|3.41\|^{m_{|\tau|+1}}_{X_{|\tau|+1}} + \|3.42\|^{m_{|\tau|+1}}_{X_{|\tau|+1}} + \|3.43\|^{m_{|\tau|+1}}_{X_{|\tau|+1}} \quad (3.44)$$

To finish the proof of (3.36), due to $\Delta_\tau(0)=0$, it remains to show that all terms in (3.44) are estimated by $\alpha_0 K(\omega,\tau,R)$ with $K(\cdot,\tau,R)\in L_p(\Omega,\mathbf{P})$, $p\geq 1$.

To estimate $\|3.40\|$ we use (3.61), (3.1), inductive assumption (3.36) and inequality (3.8) with $n=\ell+1=s, X_{\gamma_n}=X_{|\gamma_q|+1},\ u_{\gamma_n}=\Delta_{\gamma_q},\ \xi^0=\xi_\emptyset(y^0)$ and $X_{\gamma_i}=X_{|\gamma_i|},\ \xi_{\gamma_i}=\xi_{\gamma_i}(y^0)$ for $i\neq q, i=1,...,s,\ \zeta_{\gamma_1}=...=\zeta_{\gamma_\ell}=0$ and achieve

$$\|3.40\|_{X_{|\tau|+1}} \leq \sum_{\gamma_1\cup...\cup\gamma_s=\tau, s\geq 2}\sum_{q=1}^{s} K(1+\|\xi_\emptyset(y^0)\|)^{\mathbf{k}+1}\prod_{i=1, i\neq q}^{s}(1+\|\xi_{\gamma_i}(y^0)\|_{X_{|\gamma_i|}})$$

$$\cdot\{\|\xi_\emptyset(y^0)\|_{\ell_2(a)} + \sum_{i=1,\, i\neq q}^{s}\|\xi_{\gamma_i}(y^0)\|_{X_{|\gamma_i|}}\}\cdot\|\Delta_{\gamma_q}\|_{X_{|\gamma_q|+1}} \leq \alpha_0 K_1(\omega,\tau,R)$$

Estimation of $\|3.41\|_{X_{|\tau|+1}}$ is done in a similar way with application of (3.30) and inequality (3.8) with $n=\ell+1=s+1,\ \xi^0=\xi_\emptyset(y^0),\ X_{\gamma_i}=X_{|\gamma_i|},\ \xi_{\gamma_i}=\xi_{\gamma_i}(y^0)$ for $i=1,...,s,\ X_{\gamma_n}=X_1,\ u_{\gamma_n}=\Delta_\emptyset,\ \zeta_{\gamma_1}=...=\zeta_{\gamma_\ell}=0$.

To estimate $\|3.42\|_{X_{|\tau|+1}}$ we use (3.61), (3.62), (3.1), Theorem 3.4, inequality $\sup_{\alpha\in(0,\alpha_0]}\|h(s+\alpha)-h(s)\|_{\ell_2(a)}\leq\alpha_0 R$ and (3.8) with $n=\ell+1=s,\ \xi^0=\xi_\emptyset(y^0),\ \zeta^0=\zeta_{\varepsilon,\alpha,\emptyset},\ X_{\gamma_n}=X_{|\gamma_q|+1},\ u_{\gamma_n}=(\xi_{\gamma_q}(y^\alpha)-\xi_{\gamma_q}(y^0))/\alpha$ and

for $i=1,q-1\quad X_{\gamma_i}=X_{|\gamma_i|},\ \xi_{\gamma_i}=\xi_{\gamma_i}(y^0),\ \zeta_{\gamma_i}=\zeta_{\varepsilon,\alpha,\gamma_i}$;

for $j=q,s-1\quad X_{\gamma_j}=X_{|\gamma_{j+1}|},\ \xi_{\gamma_i}=\xi_{\gamma_{j+1}}(y^0),\ \zeta_{\gamma_j}=\zeta_{\varepsilon,\alpha,\gamma_{j+1}}$

$$\|3.42\|_{X_{|\tau|+1}} \leq \int_0^1 d\varepsilon\sum_{q=1}^{s}\sum_{\gamma_1\cup...\cup\gamma_s=\tau, s\geq 1} K(1+\|\xi^0(y^0)\|_{\ell_2(a)}+\|\zeta_{\varepsilon,\alpha,\emptyset}\|_{\ell_2(a)})^{\mathbf{k}+1}$$

$$\cdot\prod_{i=1, i\neq q}^{s}(1+\|\xi_{\gamma_i}(y^0)\|_{X_{|\gamma_i|}}+\|\zeta_{\varepsilon,\alpha,\gamma_i}\|_{X_{|\gamma_i|}})\cdot\|\frac{\xi_{\gamma_q}(y^\alpha)-\xi_{\gamma_k}(y^0)}{\alpha}\|_{X_{|\gamma_q|+1}}\cdot$$

$$\cdot(\varepsilon\|\xi^0(y^0)-\xi^0(y^\alpha)\|_{\ell_2(a)}+\sum_{i=1, i\neq q}^{s}\varepsilon\|\xi_{\gamma_i}(y^0)-\xi_{\gamma_i}(y^\alpha)\|_{X_{|\gamma_i|}})\leq\alpha_0 K_2(\omega,\tau,R)$$

Finally, estimation of $\|3.43\|_{X_{|\tau|+1}}$ is done in a similar way, with application of $\frac{1}{\alpha}\|\xi_\emptyset(y^\alpha)-\xi_\emptyset(y^0)\|_{X_1}\leq\frac{1}{\alpha}\|\xi^0(y^\alpha)-\xi^0(y^0)\|_{\ell_2(a)}\leq e^{\lambda T}R$ and (3.8) with $n=\ell+1=s+1,\ \xi^0=\xi_\emptyset(y^0),\ \zeta^0=\zeta_{\varepsilon,\alpha,\emptyset},\ X_{\gamma_n}=X_1,\ u_{\gamma_n}=\frac{1}{\alpha}(\xi_\emptyset(y^\alpha)-\xi_\emptyset(y^0))$ and for $i=1,...,s\quad X_{\gamma_i}=X_{|\gamma_i|},\ \xi_{\gamma_i}=\xi_{\gamma_i}(y^0),\ \zeta_{\gamma_i}=\zeta_{\varepsilon,\alpha,\gamma_i}$.

Therefore we obtain (3.36) and its consequence representation (3.35) for $x^0\in\ell_{m_1(\mathbf{k}+1)^2}(a)$ in form $\forall\, k\in\mathbb{Z}^d$

$$\xi_{k,\tau}(t,x^0+h(\cdot))\Big|_a^b = \int_a^b\sum_{j\in\mathbb{Z}^d}\xi_{k,\tau\cup j}(t,x^0+h(s))h_j'(s)\,ds$$

Closure up to $x^0\in\ell_2(a)$ is simply obtained from $h\in\mathbf{X}_\infty([a,b])$ and Corollary 3.5 (3.21). ■

3.3 Integral representation for partial derivatives of semigroup.

The next theorem shows that for function $f\in C_{\Theta,r}$ function $P_t f$ possesses the partial derivatives (2.7) along the absolutely continuous paths in sense of part 1 in Def.2.3.

Theorem 3.8. Let F fulfill (2.2) and $\Theta=\Theta^1\cup...\cup\Theta^n$ be a quasi-contractive array with parameter $\mathbf{k}\geq -1$. Then $\forall\, f\in C_{\Theta,r}\ \forall\,\tau:\ |\tau|\leq n$ functions $\partial_\tau P_t f(\cdot)\in C(\ell_2(a),\mathbb{R}^1)$ and satisfy relations (2.33), (2.34) of Definition 2.3: $\forall\, h\in\mathbf{X}_\infty([a,b])$

$$(P_t f)(x+h(\cdot))\Big|_a^b = \int_a^b\sum_{k\in\mathbb{Z}^d}\partial_k P_t f(x+h(s))h_k'(s)ds \quad (3.45)$$

and $\forall\,|\tau|\le n-1$

$$\partial_\tau P_t f(x+h(\cdot))\Big|_a^b = \int_a^b \sum_{k\in\mathbb{Z}^d} \partial_{\tau\cup\{k\}} P_t f(x+h(s)) h'_k(s) ds \tag{3.46}$$

Proof. I. Corollary 3.5 (3.21), assumption $\partial_\tau f(\cdot)\in C(\ell_2(a),\mathbb{R}^1)$ (Def. of $C_{\Theta,r}$) and (3.62) give $\mathbf{P}$ a.e. $\omega\in\Omega,\ t\in[0,T]$ continuity on $x^0\in\ell_2(a)$ of expression

$$\partial_{\{j_1\dots j_s\}} f(\xi^0(t,x^0))\xi_{j_1,\gamma_1}(t,x^0)\dots\xi_{j_s,\gamma_s}(t,x^0) \tag{3.47}$$

Continuity $\partial_\tau P_t f\in C(\ell_2(a),\mathbb{R}^1)$ follows by representation (2.7) from estimate

$$|\partial_{j_1\dots j_s} f(x)| \le \frac{1}{(G^1_{j_1}\dots G^s_{j_s})^{1/2}}\, p_s(\|x\|^2_{\ell_2(a)})\,\|f\|_{C_{\Theta,r}}$$

with some weight $(p_s,\mathcal{G}^s=G^1\otimes\dots\otimes G^s)\in\Theta^s\subset\Theta$ and inequality (3.20) with $\psi_k=e^{M|k|}$ for sufficiently large M. Together with (3.61) this gives uniform on balls $x^0\in\ell_2(a)$ summable on $j_1,\dots,j_s\in\mathbb{Z}^d$ and $\mathbf{P}$-integrable majorant for (3.47).

II. By Theorem 3.6 path $\chi^0(\cdot)=\xi^0(x^0+h(\cdot))-\xi^0(x^0+h(a))\in\mathbf{X}_\infty([a,b])$ with $\mathbf{P}$ a.e. derivative (3.29). Therefore for $f\in C_{\Theta,r}$ from (2.4) and Definition 2.3 (2.33) we have

$$(P_t f)(x^0+h(\cdot))\Big|_a^b = \mathbf{E}[f(\chi^0(\cdot)+\xi^0(t,x^0+h(a)))]\Big|_a^b =$$

$$= \mathbf{E}\int_a^b \sum_{j\in\mathbb{Z}^d} \partial_j f(\xi^0(t,x^0+h(a))+\chi^0(s))\cdot(\chi^0_j)'(s)\,ds =$$

$$= \mathbf{E}\int_a^b \sum_{j\in\mathbb{Z}^d} \partial_j f(\xi^0(t,x^0+h(s))) \sum_{k\in\mathbb{Z}^d} \xi_{j,\{k\}}(t,x^0+h(s))h'_k(s)\,ds =$$

$$= \int_a^b \sum_{k\in\mathbb{Z}^d} h'_k(s)\mathbf{E} < \partial f(\xi^0),\xi_{\{k\}} >\ ds = \int_a^b \sum_{k\in\mathbb{Z}^d} \partial_k P_t f(x^0+h(s))h'_k(s)\,ds$$

Estimate $|\partial_j f(x)|\le (G^1_j)^{-1/2} p_1(\|x\|^2_{\ell_2(a)})\|f\|_{C_{\Theta,r}}$ with some weight $(p_1,\mathcal{G}^1=G^1)\in\Theta^1\subset\Theta$, estimate (3.20) with $\psi_k=e^{M|k|}$ with sufficiently large M and property $h'\in\cap_{p,c}\ell_p(c)$ give majorant and justify the application of Fubini theorem above.

III. First remark, that due to (2.35) for $f\in C_{\Theta,r}$ the Definition 2.3 (2.34) implies property $\forall\,|\tau|\le n-1$

$$\partial_\tau f(x^0+h(\cdot))\pi(\cdot)\Big|_a^b = \int_a^b \sum_{k\in\mathbb{Z}^d} \partial_{\tau\cup k} f(x^0+h(s))h'_k(s)\pi(s)\,ds +$$

$$+\int_a^b \partial_\tau f(x^0+h(s))\pi'(s)\,ds$$

where $h\in\mathbf{X}_\infty([a,b])$ and $\pi\in AC_\infty([a,b],\mathbb{R}^1)$ (2.36).

Using Theorem 3.7 (3.34) and $\pi(\cdot)=\{\xi_{j_1,\gamma_1}\dots\xi_{j_s,\gamma_s}\}(x^0+h(\cdot))\in AC_\infty([a,b],\mathbb{R}^1)$ we achieve

$$(\partial_\tau P_t f)(x^0+h(\cdot))\Big|_a^b =$$

$$= \sum_{s=1}^{|\tau|} \sum_{\gamma_1\cup\dots\cup\gamma_s=\tau} \mathbf{E} \sum_{j_1\dots j_s\in\mathbb{Z}^d} \{\partial_{j_1}\dots\partial_{j_s} f(\xi^0)\xi_{j_1,\gamma_1}\dots\xi_{j_s,\gamma_s}\}(t,x^0+h(\cdot))\Big|_a^b =$$

$$= \sum_{s=1}^{|\tau|} \sum_{\gamma_1\cup\dots\cup\gamma_s=\tau} \mathbf{E} \sum_{j_1\dots j_s\in\mathbb{Z}^d} \int_a^b ds\cdot\{ \sum_{j\in\mathbb{Z}^d} \partial_{j_1\dots j_s,j} f(\xi^0)(\sum_{k\in\mathbb{Z}^d} \xi_{j,\{k\}}h'_k)\xi_{j_1,\gamma_1}\dots\xi_{j_s,\gamma_s} +$$

$$+ \sum_{q=1}^{s} \partial_{j_1\dots j_s} f(\xi^0)\xi_{j_1,\gamma_1}\dots(\sum_{k\in\mathbb{Z}^d} \xi_{j_q,\gamma_q\cup\{k\}}h'_k)\dots\xi_{j_s,\gamma_s}\}(t,x^0+h(s)) = \tag{3.48}$$

$$= \int_a^b ds \sum_{k\in\mathbb{Z}^d} h'_k(s)\Big\{ \sum_{\gamma_1\cup...\cup\gamma_s=\tau,\ s=1,|\tau|} \mathbf{E}\{ \sum_{j_1...j_s,j\in\mathbb{Z}^d} \partial_{j_1...j_s,j} f(\xi^0)\xi_{j_1,\gamma_1}...\xi_{j_s,\gamma_s}\xi_{j,\{k\}}\}+$$

$$+\sum_{q=1}^{s}\mathbf{E}\{ \sum_{i_1...i_s\in\mathbb{Z}^d} \partial_{i_1}...\partial_{i_s} f(\xi^0)\xi_{i_1,\gamma_1}..\xi_{i_q,\gamma_q\cup\{k\}}...\xi_{i_s,\gamma_s}\}\Big\}(t,x^0+h(s)) =$$

$$= \int_a^b ds \sum_{k\in\mathbb{Z}^d} h'_k(s) \sum_{\gamma_1\cup...\cup\gamma_s=\tau\cup\{k\},\ s=1,|\tau|+1} [\mathbf{E} \sum_{j_1...j_s\in\mathbb{Z}^d} \partial_{j_1...j_s} f(\xi^0)\xi_{j_1,\gamma_1}...\xi_{j_s,\gamma_s}](x^0+h(s))$$

$$= \int_a^b \sum_{k\in\mathbb{Z}^d} h'_k(s)\partial_{\tau\cup k}P_t f(x^0+h(s))\, ds$$

Above we redenoted indexes of summation. The interplacement of summations and integrations in (3.48) was done due to Fubini Theorem with majorant obtained from $h'\in\cap_{p,c}\ell_p(c)$, (3.20) with $\psi_k = e^{M|k|}$ for sufficiently large M and estimates

$$|\partial_{j_1...j_s} f(x)| \le (G^1_{j_1}...G^s_{j_s})^{-1/2} p_s(\|x\|_{\ell_2(a)})\|f\|_{C_{\Theta,r}}$$

for some weights $(p_s, \mathcal{G}^s = G^1\otimes...\otimes G^s)\in\Theta^s\subset\Theta$ and (3.61). ■

3.4 Quasi-contractive properties of semigroup P_t in scales $C_{\Theta,r}$ and ergodicity in variations.

In this section we end the proof of Theorem 2.4 and show, as an application of nonlinear estimate (2.18), the quasi-contractive property of semigroup P_t in the scales of smooth functions $C_{\Theta,r}$. Moreover, we obtain uniform on balls exponential ergodicity in derivatives for semigroup.

The theorem below states one of the central results of the paper about the preservance of spaces $C_{\Theta,r}$ under the action of semigroup P_t.

Theorem 3.9. Let F fulfill (2.2) with parameter $\mathbf{k}\ge -1$ and $\Theta=\Theta^1\cup...\cup\Theta^n$ be a quasi-contractive array of order $n\in\mathbb{N}$ with parameter $\mathbf{k}$:

Then $\forall\, r\ge 0$ the Feller semigroup P_t (2.4) preserves space $C_{\Theta,r}(\ell_2(a))$, i.e. $\forall\, t\ge 0\ \ P_t : C_{\Theta,r}\to C_{\Theta,r}$ and

$$\exists\, K,M>0\ \ \forall\, f\in C_{\Theta,r}\quad \|P_t f\|_{C_{\Theta,r}} \le K e^{Mt}\|f\|_{C_{\Theta,r}} \tag{3.49}$$

Proof. Due to Theorem 3.8 and preservance of spaces $Lip_r(\ell_2(a))$ under the action of semigroup P_t (Theorem 3.11) it remains to prove estimate (3.49).

First remark that weights

$$\begin{aligned} \widetilde{p}_i &= q(z)(1+z)^{\frac{\mathbf{k}+1}{2}(m_1/i - m_1/|\tau|)}, \\ \widetilde{c}_{k,\gamma} &= a_k^{\frac{\mathbf{k}+1}{2}m_1\frac{|\gamma|-1}{|\gamma|}} \prod_{j\in\tau}\psi_{k-j}^{m_1/|\gamma|},\quad \gamma=\{j_1,...,j_s\} \end{aligned} \tag{3.50}$$

fulfull (2.16) and (2.17) with constants $K_p = R_{\gamma;\alpha_1,...\alpha_s} = 1$ (3.22). By Theorem 3.4 and (3.62) the nonlinear estimate (2.18) is closable up to $x^0\in\ell_2(a)$ with generalized solution ξ^0. Its application with zero-one initial data $\widetilde{x}_\tau$ (2.11) due to $\|\widetilde{x}_\gamma\|_{\ell_{m_\gamma}(\widetilde{c}_\gamma)} = 0,\ |\gamma|\ge 2$ gives at $\tau=\{i_1,...,i_n\},\ |\tau|=n$

$$\mathbf{E}\sum_{s=1}^{n}\widetilde{p}_s(\|\xi^0_t\|^2_{\ell_2(a)}) \sum_{\gamma\subset\tau,\,|\gamma|=s}\|\xi_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} \le e^{Mt}\widetilde{p}_1(\|x^0\|^2_{\ell_2(a)})\sum_{s=1}^{n}\|\widetilde{x}_{\{i_s\}}\|^{m_1}_{\ell_{m_1}(\widetilde{c}_{\{i_s\}})}$$

Substituting expressions of $\widetilde{p}_i$ and $\widetilde{c}_\gamma$ and omiting in l.h.s. terms for $s<n$ we achieve coordinate form

$$\mathbf{E}\{q(\|\xi^0(t,x^0)\|^2_{\ell_2(a)})\,|\xi_{k,\tau}(t)|^{m_\tau}\} \le$$

$$\leq \frac{K_{|\tau|}e^{tM_{|\tau|,\psi}}\, q(\|x^0\|^2_{\ell_2(a)})(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}m_1\frac{|\tau|-1}{|\tau|}}}{a_k^{\frac{\mathbf{k}+1}{2}m_1\frac{|\tau|-1}{|\tau|}}\prod\limits_{j\in\tau}\psi_{k-j}^{m_1/|\tau|}} \tag{3.51}$$

with $K_{|\tau|} = |\tau|\psi$. Constant $M_{|\tau|,\psi}$ above is uniform on $\tau,\ |\tau| \leq n$ because of (3.24) (uniform finiteness of norm $\|B\|$) and $K_p = R_{\gamma;\alpha_1,...,\alpha_s} = 1$ in representaions (2.29) and (2.30).

To obtain (3.49) it is sufficient to show that $\forall\, \Theta^m \subset \Theta$

$$\max_{m=1,...,n} \|\partial^{(m)}P_t f\|_{\Theta^m} \leq K e^{Mt}\|f\|_{C_{\Theta,r}} \tag{3.52}$$

The quasi-contractivity of array Θ implies $\forall\, s \leq m$ and subdivision $\beta_1 \cup ... \cup \beta_s = \{1,...,m\}$ there is a pair $(p_s, \widetilde{\mathcal{G}}^{(s)} = \widetilde{G}^1 \otimes ... \otimes \widetilde{G}^s) \in \Theta^s$ such that (2.31)

$$\forall\, s < m \quad p_s(z)(1+z)^{(m-s)\frac{\mathbf{k}+1}{2}} \leq L\, p_m(z) \tag{3.53}$$

and

$$\forall\, k_1,...,k_s \in \mathbb{Z}^d \qquad \prod_{i=1}^{s} a_{k_i}^{-(\mathbf{k}+1)(|\gamma_i|-1)} G_{k_i}^{(\beta_i)} \leq L \prod_{i=1}^{s} \widetilde{G}^i_{k_i}, \tag{3.54}$$

with $G_k^{(\beta)} = \prod\limits_{i\in\beta} G^i_k,\ k \in \mathbb{Z}^d$.

The multifunction $\partial^{(m)}P_t f(x^0) = \{\partial_{k_1...k_m}P_t f(x^0)\}_{k_1...k_m\in\mathbb{Z}^d}$ with coordinates (2.7) can be represented as a finite sum

$$\partial^{(m)}P_t f(x^0) = \sum_{s=1}^{m}\ \sum_{\beta_1\cup...\cup\beta_s=\{1,...,m\}} \mathbf{E} < \partial^{(s)}f(\xi^0), \vec{\xi}_{\beta_1}...\vec{\xi}_{\beta_s} >$$

where m^{th} order multifunctions $\mathbf{E} < \partial^{(s)}f(\xi^0), \vec{\xi}_{\beta_1}...\vec{\xi}_{\beta_s} >,\ \beta_1 \cup ... \cup \beta_s = \{1,...,m\}$ have coordinates

$$\{\mathbf{E} < \partial^{(s)}f(\xi^0), \vec{\xi}_{\beta_1}...\vec{\xi}_{\beta_s} >\}_{j_1...j_m\in\mathbb{Z}^d} = \mathbf{E} < \partial^{(s)}f(\xi^0), \xi_{\gamma_1}...\xi_{\gamma_s} > \tag{3.55}$$

with $\gamma_i = \{j_t, t \in \beta_i\},\ i = 1,...,s$, for given points $j_1,...,j_m \in \mathbb{Z}^d$.

Then for $z_t = \|\xi^0(t,x^0)\|^2_{\ell_2(a)}$, $z_0 = \|x^0\|^2_{\ell_2(a)}$

$$\frac{|\partial^{(m)}P_t f(x^0)|_{\mathcal{G}^{(m)}}}{p_m(z)} =$$

$$= \frac{1}{p_m(z)}\Big|\sum_{s=1}^{m}\ \sum_{\beta_1\cup...\cup\beta_s=\{1...m\}} \mathbf{E}[\sum_{k_1,..,k_s\in\mathbb{Z}^d} (\partial_{k_1...k_s}f)(\xi^0)\vec{\xi}_{k_1,\beta_1}...\vec{\xi}_{k_s,\beta_s}]\Big|_{\mathcal{G}^{(m)}} \leq$$

$$\leq \sum_{s=1}^{m}\ \sum_{\beta_1\cup...\cup\beta_s=\{1...m\}} \Big|\sum_{k_1...k_s\in\mathbb{Z}^d} \vec{B}^{\beta_1,..,\beta_s}_{k_1,..,k_s}(\mathbf{E}\frac{|\partial_{k_1}..\partial_{k_s}f(\xi^0)|^2}{p_s^2(z_t)})^{1/2}\Big|_{\mathcal{G}^{(m)}} \tag{3.56}$$

where m^{th} order multifunction $\vec{B}^{\beta_1,..,\beta_s}_{k_1,..,k_s} = \dfrac{1}{p_m(z_0)}(\mathbf{E}p_s^2(z_t)|\vec{\xi}_{k_1,\beta_1}...\vec{\xi}_{k_s,\beta_s}|^2)^{1/2}$ has coordinates: for $\gamma_i = \{j_t,\ t \in \beta_i\}$

$$\{\vec{B}^{\beta_1,..,\beta_s}_{k_1,..,k_s}\}_{j_1...j_m\in\mathbb{Z}^d} = B^{\gamma_1...\gamma_s}_{k_1...k_s} = \frac{1}{p_m(z_0)}(\mathbf{E}p_s^2(z_t)|\xi_{k_1,\gamma_1}...\xi_{k_s,\gamma_s}|^2)^{1/2}$$

By the Holder inequality with $h_i = p_s^{2|\gamma_i|/m}|\xi_{k_i,\gamma_i}|^2,\ q_i = m/|\gamma_i|,\ i = 1,...,s,\ \sum_{i=1}^{s} 1/q_i = 1$ and nonlinear estimate (3.51) with $q = p_s^2,\ m_\gamma = 2m/|\gamma|$ and $m_1 = m_\gamma \cdot |\gamma| = 2m$ we obtain

$$B^{\gamma_1...\gamma_s}_{k_1...k_s} \leq \frac{1}{p_m(z_0)}(\mathbf{E}p_s^2(z_t)|\xi_{k_1,\gamma_1}|^{\frac{2m}{|\gamma_1|}})^{\frac{|\gamma_1|}{2m}}...(\mathbf{E}p_s^2(z_t)|\xi_{k_s,\gamma_s}|^{\frac{2m}{|\gamma_s|}})^{\frac{|\gamma_s|}{2m}} \leq$$

$$\leq \frac{K_m^{1/2}e^{\frac{1}{2}M_{n,\psi}t}}{p_m(z_0)}\prod_{\ell=1}^{s}\Big(\frac{p_s^2(z_0)(1+z_0)^{\frac{\mathbf{k}+1}{2}m_1\frac{|\gamma_\ell|-1}{|\gamma_\ell|}}}{a_{j_\ell}^{\frac{\mathbf{k}+1}{2}m_1\frac{|\gamma_\ell|-1}{|\gamma_\ell|}}\prod\limits_{j\in\gamma_\ell}\psi_{k_\ell-j}^{m_1/|\gamma_\ell|}}\Big)^{\frac{|\gamma_\ell|}{2m}} =$$

$$= K_m^{1/2} e^{\frac{1}{2}M_{n,\psi}t} \frac{p_s(z_0)(1+z_0)^{(m-s)\frac{\mathbf{k}_{+1}}{2}}}{p_m(z_0)} \cdot \prod_{\ell=1}^{s} (a_{k_\ell}^{-\frac{\mathbf{k}_{+1}}{2}(|\gamma_\ell|-1)} \prod_{j\in\gamma_\ell} \psi_{k_\ell-j}^{-1}) \leq$$

$$\leq L\, K_m^{1/2} e^{\frac{1}{2}M_{n,\psi}t} \prod_{\ell=1}^{s} (a_{k_\ell}^{-\frac{\mathbf{k}_{+1}}{2}(|\gamma_\ell|-1)} \prod_{j\in\gamma_\ell} \psi_{k_\ell-j}^{-1}) \tag{3.57}$$

Above we used the condition of hierarchy (3.53). Substituting (3.57) in (3.56) we have

$$\frac{|\partial^{(m)} P_t f(x^0)|_{\mathcal{G}^{(m)}}}{p_m(z)} \leq \sum_{s=1}^{m} \sum_{\beta_1\cup\ldots\cup\beta_s=\{1,\ldots,m\}} L\, K_m^{1/2} e^{\frac{1}{2}M_{n,\psi}t} [\sum_{j_1,..,j_m\in\mathbb{Z}^d} G_{j_1}^1 \ldots G_{j_m}^m$$

$$(\sum_{k_1,..,k_s\in\mathbb{Z}^d} (\mathbf{E}\frac{|\partial_{k_1}\ldots\partial_{k_s} f(\xi^0)|^2}{p_s^2(z_t)})^{1/2} \prod_{\ell=1}^{s} \{a_{k_\ell}^{-\frac{\mathbf{k}_{+1}}{2}(|\gamma_\ell|-1)} \prod_{j\in\gamma_\ell} \psi_{k_\ell-j}^{-1}\})^2]^{1/2} \tag{3.58}$$

with $\gamma_\ell = \{j_i,\, i\in\beta_\ell\}$ and $\beta_1\cup\ldots\cup\beta_s = \{1,\ldots,m\}$. Applying combinatorial Lemma 3.12 (see Appendix) with $b_a = \psi_a^{-1}$,

$$x_{k_1\ldots k_s} = \prod_{\ell=1}^{s} a_{k_\ell}^{-\frac{\mathbf{k}_{+1}}{2}(|\gamma_\ell|-1)} (\mathbf{E}\frac{|\partial_{k_1\ldots k_s} f(\xi^0(t))|^2}{p_s^2(z_t)})^{1/2}$$

and choosing $\psi\in\mathbb{P}$ so that $K_\psi = \sup_{m=1\ldots n} \sup_{(p_m,\mathcal{G}^{(m)})\in\Theta^m} \sup_{s=1,\ldots,m} \sum_{a\in\mathbb{Z}^d} \psi_a^{-1}[\delta_{G^s}]^{|a|} < \infty$ we have estimate on each term in (3.58)

$$(3.58)_{\beta_1,\ldots,\beta_s} \leq L\, K_m^{1/2} e^{\frac{1}{2}M_{n,\psi}t}(1+K_\psi)^m \cdot$$

$$\cdot \Big\{ \sum_{k_1\ldots k_s\in\mathbb{Z}^d} G_{k_1}^{(\gamma_1)}\ldots G_{k_s}^{(\gamma_s)} | \prod_{\ell=1}^{s} a_{k_\ell}^{-\frac{\mathbf{k}_{+1}}{2}(|\gamma_\ell|-1)} (\mathbf{E}\frac{|\partial_{k_1\ldots k_s} f(\xi^0)|^2}{p_s^2(z_t)})^{1/2}|^2 \Big\}^{1/2} =$$

$$= L\, K_m^{1/2} e^{\frac{1}{2}M_{n,\psi}t}(1+K_\psi)^m \cdot (\mathbf{E} \sum_{k_1\ldots k_s\in\mathbb{Z}^d} \prod_{i=1}^{s} \{a_{k_i}^{-(\mathbf{k}+1)(|\gamma_i|-1)} G_{k_i}^{(\gamma_i)}\} \frac{|\partial_{k_1\ldots k_s} f(\xi^0)|^2}{p_s^2(z_t)})^{1/2}$$

From hierarchy (3.54) we obtain

$$\frac{|\partial^{(m)} P_t f(x^0)|_{\mathcal{G}^{(m)}}}{p_m(z)} \leq K_m^{1/2} L^{3/2} e^{\frac{1}{2}M_{n,\psi}t}(1+K_\psi)^n \sum_{s=1}^{m} \sum_{\beta_1\cup\ldots\cup\beta_s=\{1,\ldots,m\}} \sup_{\xi^0} \frac{|\partial^{(s)} f(\xi^0)|_{\mathcal{G}^{(s)}}}{p_s(\|\xi^0\|^2_{\ell_2(a)})}$$

Therefore (3.49) holds with $M = \frac{1}{2}M_{n,\psi}$. ■

Below we apply the technique of nonlinear quasi-contractive estimates to state the ergodicity in derivatives for semigroup, i.e. we show that the more monotone is function F, the more derivatives $\partial^{(s)} P_t f(x)$ converge to zero uniformly on balls in $\ell_2(a)$ when $t\to\infty$. Such convergence develops the notion of ergodicity, recently studied in [41] and [14], i.e. that there is some Gibbs measure μ such that $P_t f(x) \to <f>_\mu$ uniformly on x in balls in $\ell_2(a)$ when $t\to\infty$.

Theorem 3.10. Let monotone function $F\in C^\infty(\mathbb{R}^1):\ F(0)=0$ be such that

$$\exists\, K,\mathbf{k}\ \forall\, i=1,\ldots,n \quad |F^{(i)}(x) - F^{(i)}(y)| \leq K|x-y|(1+|x|+|y|)^{\mathbf{k}} \tag{3.59}$$

and $\Theta = \Theta_1\cup\ldots\cup\Theta_m$ be a quasi-contractive array with parameter $\mathbf{k}\geq -1$.

There are constants $\{a_i(\Theta,\mathbf{k},K)\}_{i=1}^n$ such that if for some $i_0\in\{1,\ldots,n\}$

$$a_{i_0+1}(\Theta) \geq \inf_{x\in\mathbb{R}^1} F'(x) > a_{i_0}(\Theta)$$

then the semigroup $P_t f$ fulfills

$$\forall\, s=1,\ldots,i_0\ \forall\,(p,\mathcal{G}^{(s)})\in\Theta\ \exists\,\varepsilon>0\ \ \forall\, R\geq 0\ \forall\, f\in\mathcal{E}_{\Theta,r}$$

$$\sup_{\|x\|_{\ell_2(a)}\le R} |\partial^{(s)} P_t f|_{\mathcal{G}^{(s)}} \le K e^{-\varepsilon t} p(R) \tag{3.60}$$

with polynomial function p, which gives the exponential ergodicity in derivatives at $t \to \infty$.

Remark Functions $F(x)$ and $F_a(x) = F(x) + ax$ have the same K and $\mathbf{k}$ in (3.59), because $F'(x) = F_a'(x) + a$ and $F^{(i)}(x) = F_a^{(i)}(x)$, $i \ge 2$. Thus the growth of $a > 0$ in SDE (2.1) with F_a instead of F makes more derivatives converge to zero at $t \to \infty$.

Proof. The structure of $\|\cdot\|_{C_{\Theta,r}}$ (2.35) shows, that the required statement will follow from (3.49), written in the form: $\forall\, m = 1, ..., i_0\ \forall\, (p, \mathcal{G}^{(m)}) \in \Theta$

$$|\partial^{(m)} P_t f(x)|_{\mathcal{G}^{(m)}} \le K e^{M_{i_0} t} p(\|x\|^2_{\ell_2(a)}) \, \|f\|_\Theta$$

We only have to ensure $M_{i_0} < 0$. Recall that by the proof of Theorem 3.9 parameter M_{i_0} appears from estimate (3.51). Thus to end the proof we must find conditions, when the constant M_{i_0} is negative. Note that by the proof of Theorem 2.2 the parameter $\lambda_F = \inf_{x\in I\!R^1} F'(x)$ appears only at the step (2.24) in the form

$$m_\gamma < \frac{d}{dt}\xi_\gamma(t), [\xi_\gamma(t)]^\# > \le$$

$$\le m_\gamma(\|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))} - \lambda_F)\|\xi_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} + m_\gamma| < \varphi_\gamma, [\xi_\gamma]^\# >$$

and it do not influence the estimation of φ_γ, because here work only condition (3.59). By the proof of Theorem 2.2 the constant $K_1(F)$ in (2.29) changes to

$$K_1(F) = 2M_{p_i} + \sup_{\gamma\subset\tau}(m_\gamma\|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))} - m_\gamma\lambda_F + (m_\gamma - 1)2^{|\gamma|^2})$$

and gives additional factor $-m_{|\tau|}\lambda_F$ in (2.29). The growth of λ_F leads constant $M_{|\tau|}$ in (2.18) and thus in (3.51) to decay linearly on λ_F which gives the statement. ■

3.5 Appendix.

Using technique of [13, 14] it is not difficult to prove the following theorem.

Theorem 3.11. *$\forall\, m \ge 2$ and $\forall\, x^0 \in \ell_m(a)$, $tr\, a = 1$ there is a unique generalized solution $\xi^0(t, x^0)$ to (2.1), which is $\mathbf{P}$ – a.e. $\ell_m(a)$ - continuous on $t \in [0, T]$ and $\exists\, M_m\ \exists\, K_m(\cdot) \in L_p(\Omega, \mathbf{P}), p \ge 1\ \forall\, x^0 \in \ell_m(a)\ \forall\, \mathbf{P}$a.e. $\omega \in \Omega$*

$$\sup_{t\in[0,T]} \|\xi^0(t, x^0)\|_{\ell_m(a)} \le e^{M_m T}\|x^0\|_{\ell_m(a)} + K_m(\omega) \tag{3.61}$$

$$\sup_{t\in[0,T]} \|\xi^0(t, x^0) - \xi^0(t, y^0)\|_{\ell_m(a)} \le e^{M_m T}\|x^0 - y^0\|_{\ell_m(a)} \tag{3.62}$$

Moreover, the space $Lip_r(\ell_2(a))$, $r \ge 0$ (2.6) is preserved under the action of semigroup P_t (2.4).

Proof. Following [13, 14], for $x^0 \in \ell_{m(\mathbf{k}+1)^2}(a)$ one has representation $\xi^0(t, x^0) = \eta^0(t, x^0) + W(t)$, where $\eta^0(t, x^0)$ is $\ell_{m(\mathbf{k}+1)}(a)$-valued solution to equation

$$\eta^0(t, x^0) = x^0 - \int_0^t (F + B)(\eta^0(s, x^0) + W(s))ds \tag{3.63}$$

in space $\ell_m(a)$. Due to $\mathbf{P}$ - a.e. continuity of $W(t)$ in any $\ell_p(a)$, $p \ge 1$, from [13], Th4.1 it follows that for $\mathbf{P}$ a.e. $\omega \in \Omega$ function $\eta^0(t, x^0)$ is uniformly on $t \in [0, T]$ bounded in space $\ell_{m(\mathbf{k}+1)}(a)$. Then for $\mathbf{P}$ *a.e.* $\omega \in \Omega$

$$(F + B)(\eta^0(\cdot\,, x^0) + W(\cdot\,)) \in L_\infty([0, T], \ell_m(a)) \tag{3.64}$$

and representation (3.63) implies that for $\mathbf{P}$ a.e. $\omega \in \Omega$ function $\eta^0(t, x^0)$ is absolutely continuous in $\ell_m(a)$ with respect to $t \in [0, T]$. Using the quasi-monotonicity of map $F + B$ we have for $\mathbf{P}$ a.e. $\omega \in \Omega$

$$\frac{d}{dt}\|\eta^0(t, x^0)\|^m_{\ell_m(a)} = -m < (F + B)(\eta^0 + W), [\eta^0]^\# > \le \tag{3.65}$$

$$\leq (m\|B\| + m - 1)\|\eta^0\|^m_{\ell_m(a)} + \|(F+B)(W(t))\|^m_{\ell_m(a)}$$

Above for $u \in \ell_m(a)$ we use notation $u^{\#} = \|u\|^{m-2}_{\ell_m(a)}\mathcal{F}u$, $\|u^{\#}\|_{\ell^*_m(a)} = \|u\|^{m-1}_{\ell_m(a)}$, with duality map $\mathcal{F}$ in $\ell_m(a)$. Estimate (3.65) implies

$$\sup_{t\in[0,T]} \|\xi^0(t,x^0)\|_{\ell_m(a)} \leq e^{M_mT}\|x^0\|_{\ell_m(a)} + \sup_{t\in[0,T]} \|W(t)\|_{\ell_m(a)} +$$

$$+e^{M_mT}\Big(\int_0^T \|(F+B)(W(s))\|^m_{\ell_m(a)} ds\Big)^{1/m} \leq e^{M_mT}\|x^0\|_{\ell_m(a)} + K_m(\omega)$$

with

$$K_m(\omega) = (1 + e^{M_mT}T^{1/m}\|B\|) \sup_{t\in[0,T]} \|W(t)\|_{\ell_m(a)} +$$

$$+e^{M_mT}T^{1/m} \sup_{t\in[0,T]} \|F(W(t))\|_{\ell_m(a)}$$

Integrability of $K_m(\cdot,T)$ follows from the Doob inequality, properties of map F and estimates

$$\forall\, q \geq p \quad \mathbf{E} \sup_{t\in[0,T]} \|W_t\|^q_{\ell_p(a)} \leq \mathbf{E} \sup_{t\in[0,T]} \|W_t\|^q_{\ell_q(a)} \leq$$

$$\leq \sum_{k\in\mathbb{Z}^d} a_k \mathbf{E} \sup_{t\in[0,T]} |W_k(t)|^q \leq \sum_{k\in\mathbb{Z}^d} a_k \Big(\frac{q}{q-1}\Big)^q \mathbf{E}|W_k(T)|^q < \infty$$

$$\|F(W)\|_{\ell_m(a)} \leq C(1 + \|W\|^{\mathbf{k}+1}_{\ell_{m(\mathbf{k}+1)}(a)})$$

Inequality (3.62) is proved in a similar way. The closures on $x^0_n \to x^0$ in $\ell_m(a)$ for $x^0_n \in \ell_{m(\mathbf{k}+1)^2}(a)$ and the continuity on $t \in [0,T]$ of $\xi^0(t,x^0_n)$ for $x^0_n \in \ell_{m(\mathbf{k}+1)^2}(a)$ finish the proof of estimates (3.61), (3.62). The preservance of $Lip_r(\ell_2(a))$ follows from

$$|P_tf(x)| = \Big|\mathbf{E}\frac{f(\xi(t,x^0))}{(1+\|\xi^0\|_{\ell_2(a)})^{r+1}}(1+\|\xi^0\|_{\ell_2(a)})^{r+1}\Big| \leq$$

$$\leq C\|f\|_{Lip_r}(1+\|x^0\|_{\ell_2(a)})^{r+1}$$

and

$$|P_tf(x^0) - P_tf(y^0)| = \mathbf{E}[f(\xi(t,x^0)) - f(\xi(t,y^0))] \leq$$

$$\leq \|f\|_{Lip_r}\mathbf{E}\Big\{\|\xi^0(x^0) - \xi^0(y^0)\|_{\ell_2(a)}(1 + \|\xi^0(x^0)\|_{\ell_2(a)} + \|\xi^0(y^0)\|_{\ell_2(a)})^r\Big\} \leq$$

$$\leq C'\|f\|_{Lip_r}\|x^0 - y^0\|_{\ell_2(a)}(1 + \|x^0\|_{\ell_2(a)} + \|y^0\|_{\ell_2(a)})^r \quad \blacksquare$$

The following combinatorial in nature lemma we used in Section 3.4 for convolutional estimation of derivatives for semigroup P_t.

Lemma 3.12. *Let $\delta_d \stackrel{def}{=} \sup_{|k-j|=1} |d_k/d_j|$ and $b \in I\!P$ be such that $|b(k)| \leq 1$, $k \in \mathbb{Z}^d$. Suppose that for $d^{(i)} \in I\!P$, $i = 1,...,n$ constant*

$$C_b(d) = \prod_{\ell=1}^n \{1 + \sum_{a\in\mathbb{Z}^d} b_a|\delta_{d^{(\ell)}}|^{|a|}\} < \infty \tag{3.66}$$

Then $\forall\, \beta_1 \cup ... \cup \beta_s = \{1,...,n\}$, $s \geq 1$ we have inequality

$$\Big(\sum_{j_1...j_n\in\mathbb{Z}^d} d^{(1)}_{j_1}...d^{(n)}_{j_n}\Big|\sum_{k_1...k_s\in\mathbb{Z}^d} x_{k_1...k_s}\prod_{i=1}^s\prod_{\ell\in\beta_i} b_{k_i-j_\ell}\Big|^2\Big)^{1/2} \leq \tag{3.67}$$

$$\leq C_b(d)\cdot\Big(\sum_{k_1...k_s\in\mathbb{Z}^d} d^{(\beta_1)}_{k_1}...d^{(\beta_s)}_{k_s}|x_{k_1...k_s}|^2\Big)^{1/2}$$

where $d^{(\beta)}_k = \prod_{i\in\beta} d^{(i)}_k$.

Proof. Let $\alpha(i) = \min\{m : m \in \beta_i\}$, $i = 1..s$. Introduce new indexes of summation $a_i = k_i - j_{\alpha(i)}$, $i = 1, ..., s$ and rewrite expression (3.67) in form

$$(\sum_{j_1...j_n \in \mathbb{Z}^d} d_{j_1}^{(1)}...d_{j_n}^{(n)} |\sum_{k_1...k_s \in \mathbb{Z}^d} x_{k_1...k_s} \prod_{i=1}^{s} \prod_{\ell\in\beta_i} b_{k_i - j_\ell}|^2)^{1/2} =$$

$$= (\sum_{j_1...j_n \in \mathbb{Z}^d} d_{j_1}^{(1)}...d_{j_n}^{(n)} |\sum_{k_1...k_s \in \mathbb{Z}^d} x_{k_1...k_s} \prod_{i=1}^{s} b_{k_i - j_{\alpha(i)}} \prod_{\ell\in\beta_i\backslash\alpha(i)} b_{k_i - j_\ell}|^2)^{1/2} =$$

$$= (\sum_{j_1...j_n \in \mathbb{Z}^d} d_{j_1}^{(1)}...d_{j_n}^{(n)} |\sum_{a_1...a_s \in \mathbb{Z}^d} b_{a_1}...b_{a_s} x_{a_1+j_{\alpha(1)}\,...\,a_s+j_{\alpha(s)}} \prod_{i=1}^{s} \prod_{\ell\in\beta_i\backslash\alpha(i)} b_{a_i+j_{\alpha(i)}-j_\ell}|^2)^{1/2} \leq$$

$$\leq \sum_{a_1...a_s \in \mathbb{Z}^d} b_{a_1}...b_{a_s} (\sum_{j_1...j_n \in \mathbb{Z}^d} d_{j_1}^{(1)}...d_{j_n}^{(n)} \{\prod_{i=1}^{s} \prod_{\ell\in\beta_i\backslash\alpha(i)} b^2_{a_i+j_{\alpha(i)}-j_\ell})\} |x_{a_1+j_{\alpha(1)}\,...\,a_s+j_{\alpha(s)}}|^2)^{1/2} =$$

$$= \sum_{a_1...a_s \in \mathbb{Z}^d} b_{a_1}...b_{a_s} (\sum_{j_1...j_n \in \mathbb{Z}^d} [\prod_{\substack{t=1,...,n \\ t\neq\alpha(i),\, i=1...s}} d_{j_t}^{(t)}] \prod_{i=1}^{n} d^{(\alpha(i))}_{j_{\alpha(i)}-a_i} \cdot$$

$$\cdot \prod_{i=1}^{s} \prod_{\ell\in\beta_i\backslash\alpha(i)} b^2_{j_{\alpha(i)}-j_\ell} |x_{j_{\alpha(1)}\cdots j_{\alpha(s)}}|^2)^{1/2} \tag{3.68}$$

To proceed further we rewrite the coefficient

$$[\prod_{\substack{t=1,...,n \\ t\neq\alpha(i),\, i=1...s}} d_{j_t}^{(t)}] \prod_{i=1}^{n} d^{(\alpha(i))}_{j_{\alpha(i)}-a_i} = d_{j_1}^{(1)}...d_{j_n}^{(n)} \prod_{i=1}^{n} \frac{d^{(\alpha(i))}_{j_{\alpha(i)}-a_i}}{d^{(\alpha(i))}_{j_{\alpha(i)}}}$$

therefore

$$(3.68) \leq \sum_{a_1...a_s \in \mathbb{Z}^d} b_{a_1}...b_{a_s} \prod_{i=1}^{s} \delta^{|a_i|}_{d^{(\alpha(i))}} \cdot$$

$$\cdot(\sum_{j_1...j_n \in \mathbb{Z}^d} d_{j_1}^{(1)}...d_{j_n}^{(n)} \prod_{i=1}^{s} \prod_{\ell\in\beta_i\backslash\alpha(i)} b^2_{j_{\alpha(i)}-j_\ell} |x_{j_{\alpha(1)}\cdots j_{\alpha(s)}}|^2)^{1/2} = \prod_{i=1}^{s} (\sum_{a\in\mathbb{Z}^d} b_a \delta^{|a|}_{d^{(\alpha(i))}}) \cdot$$

$$\cdot(\sum_{j_1...j_n \in \mathbb{Z}^d} [\prod_{i=1}^{s} \prod_{\ell\in\beta_i} d^{(\ell)}_{j_{\alpha(i)}}] \cdot [\prod_{i=1}^{s} \prod_{\ell\in\beta_i\backslash\alpha(i)} \frac{d^{(\ell)}_{j_\ell}}{d^{(\ell)}_{j_{\alpha(i)}}} b^2_{j_{\alpha(i)}-j_\ell}] \, |x_{j_{\alpha(1)}\cdots j_{\alpha(s)}}|^2)^{1/2} \leq$$

$$\leq \prod_{i=1}^{s} (\sum_{a\in\mathbb{Z}^d} b_a \delta^{|a|}_{d^{(\alpha(i))}}) \cdot \prod_{i=1}^{s} \prod_{\ell\in\beta_i\backslash\alpha(i)} (\sum_{a\in\mathbb{Z}^d} \delta^{|a|}_{d^{(\ell)}} b_a^2)^{1/2} \cdot$$

$$\cdot(\sum_{j_{\alpha(1)}\cdots j_{\alpha(s)} \in \mathbb{Z}^d} \prod_{i=1}^{s} \prod_{\ell\in\beta_i} d^{(\ell)}_{j_{\alpha(i)}} |x_{j_{\alpha(1)}\cdots j_{\alpha(s)}}|^2)^{1/2} \leq \tag{3.69}$$

Using (3.66) we finally have

$$(3.69) \leq C_b(d) (\sum_{k_1...k_s} d_{k_1}^{(\beta_1)}...d_{k_s}^{(\beta_s)} |x_{k_1...k_s}|^2)^{1/2} \qquad \blacksquare$$

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