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International Algebraic Conference "At the End of the Year" 2022

December 27 – 28, 2022

Kyiv, Ukraine

ABSTRACTS

Kyiv — 2022

Київський національний університет імені Тараса Шевченка

Інститут математики НАН України

Національний університет "Києво-Могилянська академія"

Міжнародна алгебраїчна конференція "Під кінець року" 2022

27 – 28 грудня 2022 р.

Київ, Україна

ТЕЗИ ДОПОВІДЕЙ

Київ — 2022

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C O N T E N T S

З М І С Т

Minimal non-BFC-rings

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Let $(R, +, \cdot)$ be an associative ring (not necessary with unity). By analogy with the group theory, a ring R is called *an FC-ring* if, for any $a \in R$, the centralizer

$$C_R(a) := \{c \in R \mid c \cdot a = a \cdot c\}$$

is a subgroup of finite index in the additive group R^+ of R [1]. In [3] such rings are called FIC. Commutative rings and finite rings are FC-rings. The concept of a Lie FC-ring can be introduced in the same way as for the associative case (see [2]).

A ring R is called *a BFC-ring* (or a PE-ring as in [3] if every set of pairwise non-commuting elements is finite. Every BFC-ring is FC. A ring R is BFC if and only if $|R : Z(R)| < \infty$ (see e.g. [3]).

We obtain a characterization of minimal non-BFC unitary rings of finite characteristics. We also study radical rings R in which every proper subgroup of their adjoint groups R° to be BFC.

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A NON-POLYBOUNDED ABSOLUTELY CLOSED 36-SHELAH GROUP

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A semigroup X is called

- *n-Shelah* for $n \in \mathbb{N}$ if $X = \{a_1 \cdots a_n : a_1, \dots, a_n \in A\}$ for any subset $A \subseteq X$ of cardinality $|A| = |X|$;
- *Shelah* if X is n-Shelah for some $n \in \mathbb{N}$;
- *absolutely* $\mathsf{T_1S}$-*closed* if for any homomorphism $h : X \to Y$ to a T_1 topological semigroup Y the image $h[X]$ is closed in Y;
- *projectively* $\mathsf{T_1S}$-*discrete* if for any homomorphism $h : X \to Y$ to a T_1 topological semigroup Y the image $h[X]$ is a discrete subspace of Y;
- *polybounded* if X is the finite union of *algebraic* subsets, i.e., subsets of the form $\{x \in X : c_0xc_1x \cdots xc_n = b\}$ for some $b \in X$ and $c_0, \dots, c_n \in X^1 = X \cup \{1\}$.

By a result of Protasov (2009), every countable Shelah semigroup is finite. The first example of an uncountable Shelah group was constructed by Shelah in 1980 under the Continuum Hypothesis. His group is 6640-Shelah, simple, and projectively $\mathsf{T_1S}$-discrete. This was the first example of an infinite non-topologizable group. Countable non-topologizable groups were constructed in [3] by Ol'shanskii (1980). Using the approach of Shelah, we prove the following

Main Theorem. *Let κ be a cardinal such that $\kappa^+ = 2^\kappa$. Every group H of cardinality $|H| \leqslant \kappa$ is a subgroup of a non-polybounded absolutely $\mathsf{T_1S}$-closed 36-Shelah group G.*

The following theorem of Banakh and Bardyla implies that the 36-Shelah group G in Main Theorem is projectively $\mathsf{T_1S}$-discrete and hence non-topologizable.

Theorem 1. *Every absolutely $\mathsf{T_1S}$-closed semigroup is projectively $\mathsf{T_1S}$-discrete.*

Main Theorem shows that the "only if" part of following characterization of absolutely $\mathsf{T_1S}$-closed countable groups (due to Banakh and Bardyla) does not extend to uncountable groups.

Theorem 2. *A (countable) group is absolutely $\mathsf{T_1S}$-closed if (and only if) it is polybounded.*

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The most general theory of one-sided fractions

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Ore's method of localizations is an example of a theory of one-sided fractions. The aim of the talk is to introduce the most general theory of one-sided fractions based on the papers [1] and [2].

1. Bavula V. V. Localizable sets and the localization of a ring at a localizable set. J. Algebra, 2022, **610**, no. 15, 38–75.
2. Bavula V. V. Localizations of a ring at localizable sets, their groups of units and saturations. Math. Comp. Sci., 2022, **16**, no. 1, Paper No. 10, 15 pp.

Automorphisms and derivations of associative and Lie algebras of infinite matrices

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Let $\mathbb{F}$ be a ground field, let I be an infinite set, and let $M(I, \mathbb{F})$ denote the associative algebra of $I \times I$ matrices over the field $\mathbb{F}$ having finitely many nonzero entries in each column. If V is a vector space over a field $\mathbb{F}$ of the dimension $|I|$, then the algebra $\mathrm{End}_{\mathbb{F}}(V)$ of all linear transformations of V is isomorphic to $M(I, \mathbb{F})$.

We describe automorphisms and derivations of several important associative and Lie subalgebras of algebras $M(I, \mathbb{F})$ and $M(I, \mathbb{F})^{(-)}$, respectively.

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Self-replicating actions of crystallographic groups

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Self-similar group actions are special actions on the spaces of words that reflect the self-similarity of the space. Self-similar group actions naturally arise in many areas of mathematics: dynamical systems, fractal geometry, algebraic topology, automata theory. For the last twenty years, self-similar actions were studied for many classes of groups: abelian, nilpotent, solvable, free and linear groups, arithmetic groups.

Self-replicating actions is the special case of self-similar actions. There is a nice algebraic criterion: a group G admits a self-replicating action if and only if there is a surjective homomorphism $\phi : H \to G$, where $H < G$ is a subgroup of finite index, and the ϕ-core is trivial. A self-similar action associated to ϕ is obtained by a certain iterated construction.

Every finitely generated virtually abelian group admits a self-similar action. However, not all abelian groups admit self-replicating actions; Nekrashevych–Sidki [1] showed that only free abelian groups have such actions among abelian groups. We consider the question: which crystallographic groups admit a self-replicating action?

A crystallographic group of dimension n is a discrete cocompact group of isometries of $\mathbb{R}^n$. Up to an isomorphism, every crystallographic group G can be given by a pair (P, α), where P is a subgroup of the orthogonal group $O_n(\mathbb{Q})$ (linear part of G) and $\alpha : P \to \mathbb{R}^n/\mathbb{Z}^n$ is a 1-cocycle of P. We got the following criterium:

Theorem. Let G be the crystallographic group given by a pair (P, α). Then G admits a self-replicating action if and only if the normalizer of P in the group $GL_n(\mathbb{Q})$ contains an integer matrix A with the following properties:

1. A^{-1} has no eigenvalues that are algebraic integers;
2. $A(\alpha(p)) \subset \alpha(ApA^{-1})$ for all $p \in P$.

By applying this criterion and computer computations, we show that among 17 crystallographic plane groups only 4 do not admit self-replicating actions, and we have constructed such actions for the other 13 groups.

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Representations of Munn matrix algebras over local algebras

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Let $\mathcal{A}$ be an algebra over a field K. Let m and n be natural numbers and let $P = (p_{ji})$ be a fixed $n \times m$ matrix over $\mathcal{A}$ with at least one invertible entry (such a matrix we call regular). The K-vector space of all $m \times n$ matrices over the algebra $\mathcal{A}$ can be made into an algebra with respect to the following operation ($\circ$): $B \circ C = BPC$. This algebra is called the *Munn* $m \times n$ *matrix algebra over* $\mathcal{A}$ *with sandwich matrix* P and is denoted by $\mathcal{M}(\mathcal{A}; m, n; P)$; see [1].

Theorem 1. *Let K be a field and $\mathcal{A}$ a finite dimensional local split basic K-algebra with Jacobson radical $Rad(\mathcal{A})$ of the nilpotency rank s. Let P be a regular $n \times m$ non-invertible matrix over $\mathcal{A}$ and I_k denotes the $k \times k$ identity matrix (over $\mathcal{A}$).*

I. *The Munn algebra $\mathcal{M}(\mathcal{A}; m, n; P)$ is of finite representation type if and only if $Rad(\mathcal{A})$ is cyclic and one of the following condition holds:*

(a) *$s \in \{1, 2, 3\}$ and $m = n + 1$, P is equivalent to* $\begin{pmatrix} I_n & 0 \end{pmatrix}$,
or $m = n - 1$, P is equivalent to $\begin{pmatrix} I_m \\ 0 \end{pmatrix}$;

(b) *$s = 1$ and $m = n$, P is equivalent to* $\begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$;

(c) *$s > 1$ and $m = n$, P is equivalent to* $\begin{pmatrix} I_{n-1} & 0 \\ 0 & a \end{pmatrix}$ *with a generating $Rad(\mathcal{A})$.*

II. *The Munn algebra $\mathcal{M}(\mathcal{A}; m, n; P)$ is of tame infinite type if and only if $Rad(\mathcal{A})$ is cyclic and one of the following condition holds:*

(d) *$s = 1$ and $m = n + 2$, P is equivalent to* $\begin{pmatrix} I_n & 0 \end{pmatrix}$,
or $m = n - 2$, P is equivalent to $\begin{pmatrix} I_m \\ 0 \end{pmatrix}$;

(e) *$s = 2$ and $m = n$, P is equivalent to* $\begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$.

Now state Theorem 1 in an invariant form (i.e. without equivalence of sandwich matrices).

By the rank r of an $n \times m$ matrix M over a commutative ring A we mean the largest order of any non-zero minor in M and by the corank the pair $(n - r, m - r)$. The rank (corank) of M considered as a matrix over a factor ring A/J is said to be the rank (corank) modulo J.

Theorem 2. *Let K, $\mathcal{A}$, P and s be as in Theorem 1. Denote $R = Rad(\mathcal{A})$.*

I_0. *The Munn algebra $\mathcal{M}(\mathcal{A}; m, n; P)$ is of finite representation type if and only if $Rad(\mathcal{A})$ is cyclic and one of the following condition holds:*

(a) *$s \in \{1, 2, 3\}$ and the corank of P is equal $(0, 1)$ or $(1, 0)$;*

(b) *$s = 1$ and the corank of P modulo R is equal $(1, 1)$;*

(c) *$s > 1$, the corank of P modulo R is equal $(1, 1)$ and modulo R^2 is equal $(0, 0)$;*

II_0. *The Munn algebra $\mathcal{M}(\mathcal{A}; m, n; P)$ is of tame infinite type if and only if $Rad(\mathcal{A})$ is cyclic and one of the following condition holds:*

(d) *$s = 1$ and the corank of P is equal $(0, 2)$ or $(2, 0)$;*

(e) *$s = 2$ and the corank of P modulo R is equal $(1, 1)$.*

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Polynomial similarity of pairs of matrices

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Let K be a field and $M^{(2)}(K)$ the set of all pairs of square matrices of the same size over K. Pairs $P_1 = (A_1, B_1)$ and $P_2 = (A_2, B_2)$ from $M^{(2)}(K)$ are called similar if $A_2 = X^{-1}A_1X$ and $B_2 = X^{-1}B_1X$ for some invertible matrix X over K. Denote by $\mathcal{N}(K)$ the subset of $M^{(2)}(K)$, consisting of all pairs of commuting nilpotent matrices. We study the problem of classifying pairs of matrices from $\mathcal{N}(K)$, up to similarity of special form, namely polynomial similarity. We say that a pair $P = (A, B)$ is *polynomially equivalent* to a pair $\overline{P} = (\overline{A}, \overline{B})$ if $\overline{A} = f(A, B), \overline{B} = g(A, B)$ for some polynomials $f, g \in K[x, y]$ satisfying the next conditions: $f(0, 0) = 0, g(0, 0) = 0$ and $\det J(f, g)(0, 0) \neq 0$, where $J(f, g)$ is the Jacobi matrix of polynomials $f(x, y)$ and $g(x, y)$. Further, pairs of matrices $P = (A, B)$ and $\widetilde{P} = (\widetilde{A}, \widetilde{B})$ from $\mathcal{N}(K)$ will be called polynomially similar if there exists a pair $\overline{P} = (\overline{A}, \overline{B})$ from $\mathcal{N}(K)$ such that P, $\overline{P}$ are polynomially equivalent and $\overline{P}$ and $\widetilde{P}$ are similar. We prove that the problem of classifying pairs of matrices up to polynomial similarity is wild, i.e. contains the classical unsolvable problem of classifying pairs of matrices up to similarity (about wildness see [1], [2]).

Theorem 1. *The problem of classifying the pairs of matrices from $\mathcal{N}(K)$ up to polynomial similarity is wild.*

This result can be reformulated in module language. Let V be a finite dimensional module over the ring $K[x, y]$. If we fix a basis in V over K, then V is uniquely defined by a pair (A, B) of matrices of linear operators on V induced by actions on V of elements x and y respectively. The problem of classifying such modules (up to isomorphism) is equivalent to the problem of classifying pairs of commuting matrices up to similarity and therefore is wild. One can consider weaker equivalence relation *polynomial isomorphism* on the class of finite dimensional $K[x, y]$-modules which a combination of isomorphism and "twisting" modules by an automorphism of $\mathrm{Aut}K[x, y]$. The problem of classifying finite dimensional modules over $K[x, y]$ up to polynomial isomorphism can be reduced to the problem of classifying pairs of matrices up to polynomial similarity and we get the following:

Theorem 2. *The problem of classifying finite dimensional modules over $K[x, y]$ up to polynomial isomorphism is wild.*

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On families of the categories of injective representations

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The representations of (finite) posets over fields, introduced by L. A. Nazarova and A. V. Roiter [1], play an important role in the modern representation theory and its applications. M. M. Kleiner [2] obtained a description of posets of finite representation type in terms of critical posets (the minimal ones of infinite representation type) and Yu. A. Drozd [3] proved that a poset S is of finite representation type if and only if its Tits quadratic form is weakly positive, i.e. positive on the set of non-negative vectors. Posets with positive Tits quadratic form were first studied in [4]. In this note we consider a situation which deals with infinite posets, when the main role is played not by weakly positivity but by positivity of the Tits quadratic form. The situation relates to the study of the categories of representations of a special form, and in this case we use established by the first author a connection between the Tits quadratic forms for posets and commutative quivers (for finite posets, injective representations are studied in [5, 6]).

Let S be an infinite poset (not containing an element designated as 0) and $\mathbb{Z}$ denotes the integer numbers. Denote by $\mathbb{Z}_0^{S\cup 0}$ the subset of the cartesian product $\mathbb{Z}^{S\cup 0} = \{z = (z_i) \,|\, i \in S \cup 0\}$ consisting of all vectors $z = (z_i)$ with finite number of nonzero coordinates. We call *the quadratic Tits form of* S (by analogy with the case of a finite poset) the form $q_S : \mathbb{Z}_0^{S\cup 0} \to \mathbb{Z}$ defined by the equality $q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$. This form is called *positive* if it take positive values for all nonzero $z \in \mathbb{Z}_0^{S\cup 0}$.

A finite poset is said to be of *inj-finite representation type over a field k* if its category of injective representations is of finite type, i.e. has, up to isomorphism, a finite number of indecomposable objects.

Theorem. *Let S be an unlimited poset, i.e. it has no both the minimal and maximal elements, and k be a field. Then the following conditions are equivalent:*

(I) *every finite subposet of S is of inj-finite representation type over k;*

(II) *the Tits quadratic form of S is positive.*

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Filtration and centralizer of the basic Weitzenboeck derivations

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Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $A = \mathbb{K}[x_1, \dots, x_n]$ the polynomial ring and let $W_n(\mathbb{K}) = \mathrm{Der}_{\mathbb{K}} A$ be the Lie algebra of all derivations on A. Recall that a derivation $D \in W_n(\mathbb{K})$ is called linear if it is of the form $D = \sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_i}, a_{ij} \in \mathbb{K}$. The basic Weitzenboeck derivation is a linear derivation whose matrix is a nilpotent Jordan block. It is particularly important among all linear derivations. The kernel of the basic Weitzenboeck derivation (as a subalgebra of A) is finitely generated (see, for example [1]). Using a generating set of this kernel we explicitly provide generating sets for the whole filtration induced by the basic Weitznboeck derivation.

Theorem 1. *Let D be the basic Weitzenboeck derivation on $A = \mathbb{K}[x_1, \dots, x_n]$. Let us choose an arbitrary set of generators $a_1, \dots, a_k$ for the kernel $A_1 = \mathrm{Ker} D$ (as a subalgebra in A) and denote $A_i = \mathrm{Ker} D^i$, $i \geqslant 2$. Then A_i, $i \geqslant 2$ is an A_1-module with the generating sets (as a module) $S_i = \{\hat{D}^{k_1}(a_{i_1}) \dots \hat{D}^{k_t}(a_{i_t}) | a_{i_j} \in \{a_1, \dots, a_k\}, \sum_{j=1}^{t} k_j \leqslant i\}$ and, obviously, A_1 has the set of generators $S_1 = \{1\}$ over A_1.*

We use this result to obtain a generating set of the centralizer of the basic Weitzenboeck derivation in the Lie algebra $W_n(\mathbb{K})$.

Theorem 2. *Let D be the Weitzenboeck derivation on $A = \mathbb{K}[x_1, \dots, x_n]$, let $a_1, \dots, a_k$ be a system of generators of the kernel $A_1 = \mathrm{Ker} D$ (as a subalgebra in A) and*

$$S_n = \{\widehat{D}^{k_1}(a_{i_1}) \dots \widehat{D}^{k_t}(a_{i_t}) | a_{i_j} \in \{a_1, \dots, a_k\}, \sum_{j=1}^{t} k_j \leqslant n\} \cup \{1\}.$$

Then $C_{W_n(\mathbb{K})}(D)$ (as a submodule over $\mathrm{Ker} D$) has the system of generators $\{D_s | s \in S_n\}$, where $D_s = \sum_{i=1}^{n} D^{n-i}(s) \frac{\partial}{\partial x_i}$.

In the more general case when A is a finitely generated domain over $\mathbb{K}$ and D is an arbitrary locally nilpotent derivation we show that the centralizer $C_{\mathrm{Der} A}(D)$ is a "large" subalgebra in $\mathrm{Der} A$. If L is a subalgebra of the Lie algebra $\mathrm{Der}_{\mathbb{K}}(A)$ and R is the field of fractions of A then the dimension $\dim_R RL$ will be called the rank of L over A. Note that some properties of centralizers of locally nilpotent derivations on polynomial rings where studied in [2].

Theorem 3. *Let A be a finitely generated domain over the field $\mathbb{K}$ of characteristic zero and $D \neq 0$ a locally nilpotent derivation on A. Then the centralizer $C_{\mathrm{Der}_{\mathbb{K}} A}(D)$ has rank n over A, where $n = tr.\deg_{\mathbb{K}} A$.*

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New results on algebraic constructions of Extremal Graph Theory and implementations of new algorithms of Postquantum Cryptography

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NIST 2017 tender starts the standardisation process of possible Post-Quantum Public keys aimed for purposes to be

- (i) encryption tools
- (ii) tools for digital signatures.

In July 2020 the Third round of the competition was started. In the category of Multivariate Cryptographyremaining candidates are easy to observe. For the task (i) multivariate algorithm were not selected at all, single multivariate candidate "Rainbow Like Unbalanced Oil and Vinegar"(RUOV) remains in category (ii) with a good chance for the final selection.

Noteworthy that all multivariate NIST candidates were presented by multivariate rule of degree bounded by small constant (2 or 3). In particular, RUOV is given by system of quadratic polynomial equations. We think that NIST outcomes motivate investigations of alternating options in MC oriented on encryption tools:

- (a) to work with encryption transformations of plaintext space $(F_q)^n$ of linear degree cn, where $c > 0$ is a constant as instruments of stream ciphers or public keys,
- (b) to use protocols of Noncommutative Cryptography with platforms of multivariate transformations.

Both approaches as well as combination of (b) and (a) will be used in our talk.

We will use special extremal graphs to generate highly nonlinear automorphisms of $F_q[x_1, x_2, \ldots, x_n]$. They are connected with the problem of approximation of k-regular tree T_k, $k > 2$ by elements of the family of k-regular graphs of increasing order and increasing girth (minimal length of cycle in the graph).

Applications of order series in combinatorics and number theory

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An order series [1] associates to every poset P the following generating function:

$$\sum_{k=r_0(P)}^{\infty} (-1)^{\|P\|-k}\Omega^{\circ}(P,k)x^k$$

where $\Omega^{\circ}(P,n)$ is the Stanley (strict) order polynomial. For example, $\Omega^{\circ}(\{1<2\},k)=\binom{k}{2}$ counts the number of order-preserving labeling maps of the poset $\{1<2\}$, using numbers from 1 to k. Order series are the poset version of Ehrhart series [2].

Let $\zeta(k)=\sum_{n=1}^{\infty}\frac{1}{n^k}$, be the Riemann Zeta function. It is conjectured that the odd zeta values and π are algebraically independent over $\mathbb{Q}$. We study an analogue of order series in which the variable x^n is replaced by $(\zeta(n+1)-1)$:

$$\mathfrak{Z}_{\mathbb{N}}^{+}(P)=\sum_{k=r_0(P)}^{\infty}(-1)^{\|P\|-k}\Omega^{\circ}(P,k)(\zeta(k+1)-1). \tag{1}$$

Consider ψ_n defined by

$$\sum_{n=0}^{\infty}\frac{\psi_n(t)}{(1+t)^{n+1}n!}=\frac{1}{e^{-1}+t}. \tag{2}$$

Using ideas from operad theory, we give a new proof of the following theorem [3]:

Theorem. [Ramanujan 1920, EDC 2022] *Fix n a natural number. Then there exist integers A_k such that*

$$\sum_{k=1}^{\infty}(-1)^{k+1}k^n(\zeta(k+1)-1)=(-1)^n+(-1)^n2^{-n-1}\psi_n(1)+\sum_{k=1}^{n}(-1)^{k+1}A_k\zeta(k+1).$$

More over, we show that series of the form (1), parameterized by a series parallel poset P, are finite sums with integer coefficients on the terms $\{\zeta(2)-1-\frac{1}{2^2},\cdots,\zeta(n+1)-1-\frac{1}{2^{n+1}}\}$ where n is the number of points in the poset P.

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Backström rings

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1. A *Backström pair* is a pair of semiperfect rings $H \supset A$ such that $\operatorname{rad} A = \operatorname{rad} H$. We denote by $C = C(H, A)$ the *conductor* of H in A:

$$C(H, A) = \{\alpha \in A \mid H\alpha \subset A\} = \operatorname{ann}(H/A)$$

(this definition of $C(H, A)$ is left-right equivalent).

2. We call a ring A a (left) *Backström ring* if there is a Backström pair $H \supset A$, where the ring H is left hereditary. If, moreover, both A and H are finite dimensional algebras over a field $\Bbbk$, we call A a *Backström algebra*.

Examples of Backström rings are *Backström orders* considered in [4], in particular, *nodal orders* [1], *nodal algebras* [3], in particular, *gentle* and *skewed gentle* algebras.

The *Auslander envelope* of a Backström pair is the algebra of 2×2 matrices of the form

$$\tilde{A} = \begin{pmatrix} A & H \\ C & H \end{pmatrix}$$

We define the global dimension of the algebra $\tilde{A}$; in particular, it is 2 in the case of non-hereditary Backström rings. We also construct a recollement relating the derived categories of $\tilde{A}$- and A-modules. It shows that $\mathcal{D}(\tilde{A})$ can be considered as a *categorical resolution* of $\mathcal{D}(A)$. We also construct a semi-orthogonal decomposition of $\mathcal{D}(\tilde{A})$ and use it to show that the derived dimension (in the sense of Rouquier) of a Backström algebra is at most 2, and if, moreover, the related hereditary algebra H is of Dynkin type, it is at most 1. We also use this decomposition to establish a representation equivalence between the derived category of finitely generated modules over a Backström algebra and an explicitly described bimodule category.

These results are published in [2].

Acknowledgements. This work was accomplished during the visits of the author to the Max-Plank-Institut of Mathematics and the University of Paderborn, and I am grateful to these institutions for their hospitality and financial support.

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Maximal solvable subalgebras of the Lie algebra $W_n(\mathbb{K})$

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Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and $P_n = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring over $\mathbb{K}$ in n variables. A $\mathbb{K}$-derivation D of P_n is a $\mathbb{K}$-linear mapping $D\colon P_n \to P_n$ such that $D(fg) = D(f)g + fD(g)$ for all $f,\ g \in P_n$. The Lie algebra $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations of P_n is a free module over the polynomial ring P_n. This Lie algebra is an interesting object to study because of connections with the theory of partial differential equations and with geometry. Every derivation of P_n can be considered as a vector field on $\mathbb{K}^n$ with polynomial coefficients (see, for example, [1-3]).

We study solvable (not necessarily finite dimensional) subalgebras of the Lie algebra $W_n(\mathbb{K})$. The known subalgebra of such a type is

$$s_n = \{\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \in W_n(\mathbb{K}) | a_i \in \mathbb{K}[x_1, \ldots, x_{i-1}] + x_i \mathbb{K}[x_1, \ldots, x_{i-1}]\}.$$

The subalgebra s_n is solvable of length $2n$ (see, for example, [3]) and this is the maximal possible length of solvable subalgebras of $W_n(\mathbb{K})$ (see [3]). The Lie algebra s_n obviously contains the triangular Lie algebra $u_n = \mathbb{K}\frac{\partial}{\partial x_1} + P_1 \frac{\partial}{\partial x_2} + \cdots + P_{n-1}\frac{\partial}{\partial x_n}$. The last Lie algebra is locally nilpotent but not nilpotent and consists of locally nilpotent derivations (see [1]).

We got the following result.

Theorem. *The Lie algebra*

$$s_n = (\mathbb{K} + x_1\mathbb{K})\frac{\partial}{\partial x_1} + (P_1 + x_2 P_1)\frac{\partial}{\partial x_2} + \cdots + (P_{n-1} + x_n P_{n-1})\frac{\partial}{\partial x_n}$$

is a maximal solvable subalgebra of $W_n(\mathbb{K})$.

We found also the derivative series of the Lie algebra $s_n(\mathbb{K})$.

Note that in many cases solvable subalgebras of $W_2(\mathbb{K})$ and $W_3(\mathbb{K})$ are isomorphic to subalgebras of $s_2(\mathbb{K})$ or $s_3(\mathbb{K})$ respectively.

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Stable range conditions and diagonalization of matrices

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All rings considered will be commutative and have identity.

We introduce the necessary definitions and facts.

By a Bezout ring we mean a ring in which all finitely generated ideals are principal. An n by m matrix $A = (a_{ij})$ is said to be diagonal if $a_{ij} = 0$ for all $i \neq j$. We say that a matrix A of the dimension n by m admits a diagonal reduction if there exist invertible matrices $P \in GL_n(R), Q \in GL_m(R)$ such that PAQ is a diagonal matrix. We say that two matrices A and B over a ring R are equivalent if there exist invertible matrices P, Q such that $B = PAQ$. Following Kaplansky [1], we say that if every matrix over R is equivalent to a diagonal matrix (d_{ii}) with the property that every (d_{ii}) is a divisor of $d_{i+1,i+1}$, then R is an elementary divisor ring. An element $a \in R$ is called clean if a can be written as the sum of a unit and an idempotent. If each element of R is clean, then we say R is a clean ring [2]. A ring R is said to have stable range 2 if for any $a, b, c \in R$ such that $aR + bR + cR = R$, there exist elements $x, y \in R$ such that $(a + cx)R + (b + cy)R = R$. A ring R is said to have stable range 1 if for any $a, b \in R$ such that $aR + bR = R$, there exists $t \in R$ such that $(a + bt)R = R$.

Definition. *A ring R is said to be a ring of neat range 1 if for any elements $a, b \in R$ such that $aR + bR = R$ and for any nonzero element $c \in R$ there exist such elements $u, v, t \in R$ that $a + bt = uv$, where $uR + cR = R$, $vR + (1 - c)R = R$, and $uR + vR = R$.*

Theorem 1. *Let R be a commutative Bezout ring and let a be an element of R such that for any $c \in R$ there exist elements $u, v, t \in R$ such that $a = uv$ where $uR + cR = R$, $vR + (1 - c)R = R$, and $uR + vR = R$. Then R/aR is a clean ring.*

Theorem 2. *A commutative Bezout ring is an elementary divisor ring if and only if it is a ring of neat range 1.*

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Torsion in linearized Legendrian contact cohomology

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The Legendrian contact homology of a closed Legendrian submanifold Λ of the standard contact vector space $(\mathbb{R}^{2n+1}, \ker(dz - ydx))$ is a modern Legendrian invariant, which can be seen as a version of the symplectic field theory of Eliashberg–Givental–Hofer [2]. It is a homology of the Legendrian contact homology (LCH) differential graded algebra (often called the Chekanov–Eliashberg differential graded algebra). Chekanov–Eliashberg DGA is a unital noncommutative differential graded algebra freely generated by the generically finite set of integral curves of the Reeb vector field ∂_z that start and end on Λ and called Reeb chords. Legendrian contact homology is often defined over $\mathbb{Z}_2$, but if Λ is spin it can be also defined over other fields, over $\mathbb{Z}$ and even more general coefficient rings such as $\mathbb{Z}_2[H_1(\Lambda;\mathbb{Z})]$ or $\mathbb{Z}[H_1(\Lambda;\mathbb{Z})]$.

The Legendrian contact homology DGA is not finite rank, even in fixed degree; the same holds in homology: the graded pieces of the Legendrian contact homology are often infinite dimensional and difficult to compute. In order to deal with this issue Chekanov [1] proposed to use an augmentation of the DGA to produce a generically finite-dimensional linear complex, whose homology is called linearized Legendrian contact homology.

Most of the computations of linearized Legendrian contact homology groups have been done for the Chekanov–Eliashberg algebras with $\mathbb{Z}_2$-coefficients. One can ask whether an arbitrary finitely generated abelian group can be realized as a linearized Legendrian contact (co)homology of some Legendrian.

We provide the following answer to this question in high dimensions:

Theorem. *Given a finitely generated abelian group G and $i \in \mathbb{N}$. There is a Legendrian submanifold Λ in $\mathbb{R}^{2i+7}$ of Maslov number 0 such that the Chekanov-Eliashberg algebra of Λ admits an augmentation $\varepsilon : \mathcal{A}(\Lambda) \to (\mathbb{Z}, 0)$ with $LCH^i_\varepsilon(\Lambda;\mathbb{Z}) \simeq G$.*

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Multivariat Growth and Cogrowth

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The exponent α_H of cogrowth (or relative growth) of a subgroup H of a free group $F_m = \langle a_1, \ldots, a_m \rangle$ (where $A = \langle a_1, \ldots, a_m \rangle$ is a basis of F_m), cogrowth series $H(z)$ associated with H, and cogrowth criterion of amenability for the quotient group F_m/H (when H is normal) or a Schreier graph $\Gamma = \Gamma(F_m, H, A)$ (in general case) were introduced by the first author in [1], [2], [3], [4] and got a big popularity. The formulas relating the spectral radius r of a simple random walk on a quotient group F_m/H (or graph $\Gamma(F_m, H, A)$) and α_H presented in [1], [2], [3], [4] and a formula relating a generating series of probabilities of returns to the original vertex (a Green function) and a cogrowth series presented in [3] were used to prove a criterion of amenability, a criterion for infinite Schreier graph to be Ramanujan (the Ramanujan terminology appeared later) and to prove that in the case when H is normal the cogrowth series $H(z)$ is rational if and only if H is of finite index.

In a recent joint work [6] we consider a multivariate version of α_H and $H(z)$ when instead of the length of elements in H viewed as reduced words over alphabet $A \cup A^{-1}$ we use a vector whose coordinates represent number of occurrences of each symbol a_i (or a^{-1}). We generalize this approach by inventing the same notions for arbitrary formal language $L \subset \Sigma^*$ where Σ^* is a set of all words over a finite alphabet Σ. For important case when L is a regular language (i.e. language accepted by finite automaton) we develop a mechanism for computing the rate of growth $\alpha_L(r)$ of L in the direction $r \in \mathbb{R}^d_{>0}, d = |\Sigma|$. Using the concave condition (Q) of J-F.Quint from [5] and the results of Convex Analysis we represent $\alpha_L(r)$ as a support function of a convex set that is one of the complements to the amoeba determined by the denominator $R(z)$ of the rational function representing a multivariate growth series of L. This allows us to compute $\alpha_L(r)$ in some important cases, like a Fibonacci language or a language of freely reduced words representing elements of a free group. Also we show that the methods of the Large Deviation Theory can be use as an alternative approach, in particular in the case when language L is associated with a subshift of finite type over Σ.

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Maximal subgroups of ample groups

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During the last two decades there was a growing interest in dynamically defined groups. A rich source of such groups are *ample groups* (also known as *topological full groups*). The idea of ampleness in theory of dynamical systems and group theory is quite simple. Given a topological space X and a subgroup G of the group $\mathrm{Homeo}(X)$ of homeomorphisms of X, one can enlarge it by adding those homeomorphisms that locally act as elements of G, thus producing an ample group $\mathcal{G}$. This idea works best in the situation when X is a Cantor set or, more generally, a metrizable compact totally disconnected space. This is because such a space has many *clopen* (i.e., both closed and open) sets, which allows to construct many homeomorphisms that are piecewise elements of G. Still, if G is countable then the ample group $\mathcal{G}$ is also countable.

Maximal subgroups play an extremely important role in group theory. The most remarkable result here is a complete classification of maximal subgroups of finite symmetric groups. Much less is known about maximal subgroups in infinite groups.

Notable subgroups of any transformation group G acting on a set X are *stabilizers* of subsets and partitions. The stabilizer $\mathrm{St}_G(Y)$ of a subset $Y \subset X$ consists of all $g \in G$ such that $g(Y) = Y$. The stabilizer $\mathrm{St}_G(Y_1, Y_2, \dots, Y_k)$ of a partition $X = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_k$ consists of those elements of G that map elements of the partition onto one another.

Recall that all subgroups of the symmetric group S_n are divided into three classes: *intransitive* subgroups (those that leave invariant a nontrivial subset), *imprimitive* subgroups (transitive subgroups that leave invariant a nontrivial partition), and *primitive* subgroups (the remaining ones). It turns out that the maximal intransitive subgroups are stabilizers of certain subsets while the maximal imprimitive subgroups are stabilizers of certain partitions.

We present a number of results on maximal subgroups of ample groups $\mathcal{G} \subset \mathrm{Homeo}(X)$, where X is a Cantor set. The results are mostly parallel to the above classification. Instead of arbitrary subsets and partitions, one needs to consider closed subsets and partitions into closed subsets. Transitivity is replaced by *minimality*, which means absence of nontrivial closed invariant subsets.

Theorem 1. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose H is a maximal subgroup of $\mathcal{G}$ that does not act minimally on X. Then $H = \mathrm{St}_{\mathcal{G}}(Y)$ for some closed set $Y \subset X$ different from the empty set and X. Moreover, the induced action of $\mathrm{St}_{\mathcal{G}}(Y)$ on Y is minimal.*

The condition that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ of a closed set Y acts minimally when restricted to Y implies that Y belongs to one of three classes: (1) finite sets contained in a single orbit of $\mathcal{G}$, (2) infinite sets nowhere dense in X, and (3) clopen sets. For a finite set Y, the converse of Theorem 1 holds for any ample group without finite orbits. In the case of an infinite set Y, we need stronger assumptions. Namely, $\mathcal{G}$ has to act minimally on X and to possess another property that we call *Property NC* (no contraction): if $g(U) \subset U$ for some $g \in \mathcal{G}$ and open set $U \subset X$ then, in fact, $g(U) = U$.

Theorem 2. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X and has Property NC. Suppose U is a clopen set different from the empty set and X. Then $\mathrm{St}_{\mathcal{G}}(U, X \backslash U)$ is a maximal subgroup of $\mathcal{G}$. If U cannot be mapped onto $X \backslash U$ by an element of $\mathcal{G}$ then $\mathrm{St}_{\mathcal{G}}(U) = \mathrm{St}_{\mathcal{G}}(U, X \backslash U)$; otherwise $\mathrm{St}_{\mathcal{G}}(U)$ is a subgroup of index 2 in $\mathrm{St}_{\mathcal{G}}(U, X \backslash U)$.*

ON A SEMITOPOLOGICAL SEMIGROUP $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ WHEN A FAMILY $\mathscr{F}$ CONSISTS OF INDUCTIVE NON-EMPTY SUBSETS OF ω

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Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and $n, m \in \omega$ we put $n-m+F = \{n-m+k \colon k \in F\}$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called *ω-closed* if $F_1 \cap (-n+F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. A subset F of ω is called *inductive* in ω if $n \in F$ implies $n+1 \in F$.

The set $\boldsymbol{B}_{\omega} = \omega \times \omega$ with the semigroup operation

$$(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geqslant i_2 \end{cases}$$

is isomorphic to the bicyclic monoid. Let $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_{\omega} \times \mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geqslant i_2. \end{cases}$$

If the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot)$ is a semigroup [1]. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i, j, \varnothing) \colon i, j \in \omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the semigroup

$$\boldsymbol{B}_{\omega}^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [1]. The structure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with an ω-closed family $\mathscr{F}$ of non-empty inductive subsets of ω is described in [1, 2].

Later we assume that $\mathscr{F}$ is an ω-closed family of non-empty inductive subsets of ω.

Theorem 1. *Every Hausdorff shift-continuous topology τ on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is discrete.*

Proposition 2. *Let $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ be a proper dense subsemigroup of a Hausdorff semitopological semigroup S. Then $I = S \setminus \boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a closed ideal of S.*

Theorem 3. *Let S be the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with adjoined zero. Then every Hausdorff locally compact shift-continuous topology on S is either compact or discrete.*

Theorem 4. *Let (S_I, τ) be a Hausdorff locally compact semitopological semigroup, where $S_I = \boldsymbol{B}_{\omega}^{\mathscr{F}} \sqcup I$ and I is a compact ideal of S_I. Then either (S_I, τ) is a compact semitopological semigroup or the ideal I is open.*

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Triameter of trees and block graphs

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Let $G = (V(G), E(G))$ be a finite connected simple graph. Define a metric d_G on the set of vertices $V(G)$ in the next way: for any $u, v \in V(G)$ the distance $d_G(u, v)$ equals the length of the shortest path between u and v.

The *diameter* of a connected graph G is the value $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\}$. A pair of vertices $u, v \in V(G)$ is called *diametral* if $d_G(u, v) = diam(G)$. For every vertices $u, v, w \in V(G)$, define

$$d_G(u, v, w) = d_G(u, v) + d_G(u, w) + d_G(v, w).$$

The *triameter* of a connected graph G is defined as the value

$$tr(G) = \max\{d_G(u, v, w) : u, v, w, \in V(G)\}.$$

The triplet of vertices $u, v, w \in V(G)$ is *triametral* if $d_G(u, v, w) = tr(G)$. The main motivation for studying $tr(G)$ comes from its appearance in lower bounds on radio k-chromatic number of a graph [1, 2] and total domination number of a connected graph [3].

We describe a tight lower bound for the triameter of trees

Theorem. *Let T be a tree with $n \geqslant 4$ vertices and $l \geqslant 3$ leaves. Then*

$$tr(T) \geqslant 6\left[\frac{n-1}{l}\right] + 2\min\{(n-1) \bmod l,\ 3\}.$$

Moreover, this bound is tight for any given pair n, l.

We also show that any triametral triple of vertices contains a diametral pair and that any diametral pair of vertices can be extended to a triametral triple for a connected block graph [4]. Thus, we answer three questions from the paper [5].

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Characteristic subgroups in the group of infinite unitriangular matrices over a field

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A. Bier (2015) described the closed, characteristic subgroups of the group $UT(\infty, K)$ of infinite upper unitriangular matrices over a field K, where $|K| > 2$. We classify all characteristic, strictly characteristic, and fully characteristic subgroups in $UT(\infty, K)$, and provide the first proof that $UT(\infty, K)$ is verbally poor.

Transposed Poisson structures on Block Lie algebras and superalgebras

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We describe transposed Poisson structures [1] on Block Lie algebras $\mathcal{B}(q)$ and Block Lie superalgebras $\mathcal{S}(q)$, where q is an arbitrary complex number (see [2, 4, 5]). More precisely, we show that the transposed Poisson algebra structures on $\mathcal{B}(q)$ are trivial whenever $q \notin \mathbb{Z}$, and for each $q \in \mathbb{Z}$ there is only one (up to an isomorphism) non-trivial transposed Poisson algebra structure on $\mathcal{B}(q)$. The superalgebra $\mathcal{S}(q)$ admits only trivial transposed Poisson superalgebra structures for $q \neq 0$ and two non-isomorphic non-trivial transposed Poisson superalgebra structures for $q = 0$.

This is a joint work [3] with Ivan Kaygorodov (Universidade da Beira Interior, Portugal).

Acknowledgements. Mykola Khrypchenko was partially supported by CMUP, member of LASI, which is financed by national funds through FCT — Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UIDB/00144/2020.

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A categorical viewpoint of partial group actions

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Following the idea in the article by Hu and Vercruysse [1], we introduce partial morphisms in an arbitrary category $\mathbf{C}$, so that partial actions of a group G on a set X correspond to certain functions from G to the set of isomorphism classes of partial morphisms from X to X in the category of sets. Based on that, we generalized the concept of partial group actions to arbitrary categories with pullbacks, and studied the question of the globalization of such partial actions, aiming to find necessary and sufficient conditions in terms of coproducts, coequalizers and pullbacks for a partial action in this sense to be globalizable.

Acknowledgements. The results of this work are a part of the Master's Thesis under the supervision of Mykola Khrypchenko (Federal University of Santa Catarina). I thank the institution FAPESC for their financial support on the composition of this work.

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On nilpotency in the left braces

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A *left brace* is a set A with two binary operations $+$ and $\cdot$ satisfying the following conditions: A is an abelian group by addition, A is a group by multiplication, and $a(b+c) = ab + ac - a$ for every $a, b, c \in A$.

Let A be a left brace. Put $a * b = ab - a - b$.

A left brace A is called *trivial* or *abelian* if $a * b = 0$ or $a + b = ab$ for all elements $a, b \in A$.

The set

$$\zeta(*, A) = \{a|\ a \in A \text{ and } a * x = x * a = 0 \text{ for every element } x \in A\} = \{a|\ a \in A \text{ and } ax = a + x = xa \text{ for every element } x \in A\}$$

is called the $*$-*center* of A. It is possible to prove that the $*$-center of A is an ideal of A.

Starting from the $*$-center we can construct the *upper $*$-central series*

$$\langle 0 \rangle = \zeta_0(*, A) \leqslant \zeta_1(*, A) \leqslant \dots \zeta_\alpha(*, A) \leqslant \zeta_{\alpha+1}(*, A) \leqslant \dots \zeta_\gamma(*, A)$$

of a brace A by the following rule: $\zeta_1(*, A) = \zeta(*, A)$, and recursively $\zeta_{\alpha+1}(*, A)/\zeta_\alpha(*, A) = \zeta(*, A/\zeta_\alpha(*, A))$ for all ordinals α and $\zeta_\lambda(*, A) = \bigcup_{\mu<\lambda} \zeta_\mu(*, A)$ for the limit ordinals λ.

By the definition, each term of this series is an ideal of A. The last term $\zeta_\infty(*, A) = \zeta_\gamma(*, A)$ of this series is called the *upper $*$-hypercenter* of A.

Denote by $\mathrm{zl}(A)$ the length of the upper $*$-central series of A.

If $A = \zeta_\infty(*, A)$, then A is said to be a *$*$-hypercentral brace.*

Let A be a left brace. Put $A^{(1)} = A$, and recursively $A^{(\alpha+1)} = A^{(\alpha)} * A$ for all ordinals α and $A^{(\lambda)} = \bigcap_{\mu<\lambda} A^{(\mu)}$ for limit ordinals λ. And similarly, put $A^1 = A$, and recursively $A^{\alpha+1} = A * A^{(\alpha)}$ for all ordinals α and $A^\lambda = \bigcap_{\mu<\lambda} A^\mu$ for limit ordinals λ.

We say that a left brace A is called *nilpotent in the sense of Smoktunowicz* if there are positive integers n, k such that $A^{(n)} = \langle 0 \rangle = A^k$. These braces have been introduced in the paper of A. Smoktunowicz [1].

Theorem. *Let A be a left brace. Then A has a finite $*$-central series if and only if A is nilpotent in the sense of Smoktunowicz.*

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ON THE STRUCTURE OF THE AUTOMORPHISM GROUPS OF SOME LEIBNIZ ALGEBRAS

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Let L be a Leibniz algebra. As usual, a linear transformation f of L is called an *endomorphism* of L if $f([a,b]) = [f(a), f(b)]$ for all $a, b \in L$. Clearly, a product of two endomorphism of L is also an endomorphism of L, so that the set of all endomorphisms of L is a semigroup by a multiplication.

As usual, a bijective endomorphism of L is called an *automorphism* of L.

Let f be an automorphism of L. Then the mapping f^{-1} is also an automorphism of L. Thus, the set $\mathrm{Aut}_{[,]}(L)$ of all automorphisms of L is a group by a multiplication.

As for other algebraic structures, the study of the structure of the automorphism groups of Leibniz algebras is one of the important problems of this theory.

The automorphism groups of cyclic Leibniz algebras have been studied in [1, 2].

It is natural to study the automorphism groups of Leibniz algebras having low dimension. Here we show a description of the automorphism groups of the following Leibniz algebras that have dimension 3.

Let $L = Fa_1 \oplus Fa_2 \oplus Fa_3$ where

$$[a_1, a_1] = a_3, [a_2, a_2] = \lambda a_3, 0 \neq \lambda \in F,$$
$$[a_1, a_2] = [a_1, a_3] = [a_2, a_1] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

In other words, L is a sum of two nilpotent cyclic ideals $A_1 = Fa_1 \oplus Fa_3$ and $A_2 = Fa_2 \oplus Fa_3$, $[A_1, A_2] = [A_2, A_1] = \langle 0 \rangle$, $\mathrm{Leib}(L) = [L, L] = \zeta^{\mathrm{left}}(L) = \zeta^{\mathrm{right}}(L) = \zeta(L) = Fa_3$.

We say that a field F is 2-*closed*, if an equation $x^2 = a$ has a solution in F for every element $a \neq 0$. We note that if a field F has characteristic 2 and is 2-closed, then a Leibniz algebra of this type cannot exist.

If $\mathrm{char}(F) = 2$, then the automorphism group of L is isomorphic to a subgroup of $\mathrm{GL}_3(F)$, consisting of the matrices of the form

$$\begin{pmatrix} \alpha_1 & \lambda\alpha_2 & 0 \\ \alpha_2 & \alpha_1 & 0 \\ \alpha_3 & \beta_3 & \alpha_1^2 + \lambda\alpha_2^2 \end{pmatrix},$$

$\alpha_1, \alpha_2, \alpha_3, \beta_3 \in F$.

If $\mathrm{char}(F) \neq 2$, then the automorphism group of L is isomorphic to a subgroup of $\mathrm{GL}_3(F)$, consisting of the matrices of the form

$$\begin{pmatrix} \alpha_1 & \lambda\alpha_2 & 0 \\ \alpha_2 & -\alpha_1 & 0 \\ \alpha_3 & \beta_3 & \alpha_1^2 + \lambda\alpha_2^2 \end{pmatrix}$$

where $\alpha_1^2 + \lambda\alpha_2^2 = \lambda^{-1}\beta_1^2 + \beta_2^2$ and $\alpha_1\beta_1 + \lambda\alpha_2\beta_2 = 0$.

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On nilpotency of some modules over group rings

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Let G be a group, R be a ring, and A be an RG-module. If B is an RG-submodule of A, then put $[A/B, G] = ([A, G] + B)/B$. If B, C are the RG-submodules of A such that $B \leqslant C$, then, as usual, C/B is called the *G-factor* of module A. Factor C/B is called *G-perfect* if $[C/B, G] = C/B$. Otherwise, we will say that the factor is not perfect.

A factor C/B is called *G-central*, if $[C, G] \leqslant B$.

An RG-module A is called *G-nilpotent* if A has a finite series of RG-submodules whose factors are G-central. We note that if A is a G-nilpotent module, then, clearly, every factor of A is not G-perfect. And conversely, it is not hard to prove that if A is an RG-module having finite RG-composition series, and A has no non-zero G-perfect factors, then A is G-nilpotent. Therefore, a natural question about modules having no non-zero G-perfect factors arises.

As a first step we will consider the case when a group G is finite. It is possible to prove that for such modules factor-group $G/C_G(A)$ is nilpotent.

The basic case which appear here is the case when $\mathrm{char}(R) = 0$.

Let R be a Dedekind domain of characteristic 0. We say that R is *periodically unlimited* if for every maximal ideal S a field R/S has prime characteristic and orders of elements of the additive group of R/S^n are not bounded whenever $n \in \mathbb{N}$.

Theorem 1. *Let G be a finite group, R be a Dedekind domain, having infinite set of prime ideals, A be an RG-module which is torsion-free as an R-module. If A has no non-zero G-perfect factors, then A is G-nilpotent and $G/C_G(A)$ is nilpotent.*

Theorem 2. *Let G be a finite group, R be a periodically unlimited Dedekind domain, having infinite set of prime ideals, A be an RG-module. If A has no non-zero G-perfect factors, then A is G-nilpotent and $G/C_G(A)$ is nilpotent.*

On the automorphism groups of some nilpotent 3-dimensional Leibniz algebras

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Let L be a Leibniz algebra over a field F. A linear transformation f of L is called an *endomorphism* of L if $f([a,b]) = [f(a), f(b)]$ for all elements $a, b \in L$. A bijective endomorphism of L is called an *automorphism* of L.

The study of the automorphism groups of Leibniz algebras is one of the natural problems of Leibniz algebra theory. One of the first steps is to study the automorphism groups of Leibniz algebras of low dimension. The first type of Leibniz algebras we will consider are nilpotent 3-dimensional Leibniz algebras of nilpotency class 3. There is only one type of such algebras:

$$L_1 = Fa_1 \oplus Fa_2 \oplus Fa_3, \text{ where } [a_1,a_1] = a_2, [a_1,a_2] = a_3,$$
$$[a_1,a_3] = [a_2,a_1] = [a_2,a_2] = [a_2,a_3] = [a_3,a_1] = [a_3,a_2] = [a_3,a_3] = 0.$$

Note that L_1 is cyclic, $\mathrm{Leib}(L_1) = \zeta^{\mathrm{left}}(L_1) = [L_1, L_1] = Fa_2 \oplus Fa_3$, $\zeta^{\mathrm{right}}(L_1) = \zeta(L_1) = Fa_3$.

Theorem 1. *Let G be an automorphism group of Leibniz algebra L_1. Then G is isomorphic to a subgroup of $\mathrm{GL}_3(F)$, consisting of the matrices of the following form:*

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1^2 & 0 \\ \alpha_3 & \alpha_1\alpha_2 & \alpha_1^3 \end{pmatrix}$$

where $\alpha_1 \neq 0$. This subgroup is a semidirect product of normal subgroup T, consisting of the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & 0 \\ \alpha_3 & \alpha_2 & 1 \end{pmatrix}$$

and a subgroup D, consisting of the matrices of the form

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1^2 & 0 \\ 0 & 0 & \alpha_1^3 \end{pmatrix}.$$

Let now L_2 be a nilpotent Leibniz algebra whose nilpotency class is 2 and $\dim_F(\zeta(L)) = 2$. Thus, we obtain the following type of nilpotent Leibniz algebras:

$$L_2 = Fa_1 \oplus Fa_2 \oplus Fa_3, \text{ where } [a_1,a_1] = a_3,$$
$$[a_1,a_2] = [a_1,a_3] = [a_2,a_1] = [a_2,a_2] = [a_2,a_3] = [a_3,a_1] = [a_3,a_2] = [a_3,a_3] = 0.$$

In other words, L_2 is a direct sum of two ideals $A = Fa_1 \oplus Fa_3$ and $B = Fa_2$, $\mathrm{Leib}(L_2) = [L_2, L_2] = Fa_3$, $\zeta^{\mathrm{left}}(L_2) = \zeta^{\mathrm{right}}(L_2) = \zeta(L_2) = Fa_2 \oplus Fa_3$.

Theorem 2. *Let G be an automorphism group of Leibniz algebra L_2. Then G is isomorphic to a subgroup of $\mathrm{GL}_3(F)$, consisting of the matrices of the following form:*

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1^2 \end{pmatrix}$$

where $\alpha_1 \neq 0$, $\beta_2 \neq 0$. In other words, $G = S \leftthreetimes D$, $D \cong F^{\times}$, $S = TC$, T is normal in G, $T \cong F_+ \times F_+$, $C = AB$, A is normal in C, $A \cong F_+ \times F_+$ and $B \cong F^{\times}$.

On Algebra of derivations of cyclic Leibniz algebras of type (II)

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Let L be an algebra over finite field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity

$$[[a,b],c]=[a,[b,c]]-[b,[a,c]] \text{ for all } a,b,c\in L.$$

We show here some basic elementary properties of derivations, which have been proved in [1].

Property 1. Let L be a Leibniz algebra over a field F, and let f be a derivation of L. Then $f(\zeta^{left}(L)\leqslant\zeta^{left}(L)$, $f(\zeta^{right}(L))\leqslant\zeta^{right}(L)$ and $f(\zeta(L))\leqslant\zeta(L)$.

Corollary. Let L be a Leibniz algebra over a field F and f be a derivaion of L. Then $f(\zeta_\alpha(L))\leqslant\zeta_\alpha(L)$ for every ordinal α.

Property 2. Let L be a Leibniz algebra over a field F, and let f be a derivaion of L. Then $f(\gamma_\alpha(L))\leqslant\gamma_\alpha(L)$ for all ordinals α, in particular, $f(\gamma_\infty(L))\leqslant\gamma_\infty(L)$.

Corollary. Let L be a cyclic Leibniz algebra of type (II) over a field F, $L=A\oplus S$, where $A=[L,L]=\mathbf{Leib}(L)$, $S=Fc=\zeta^{right}(L)$. If f is an derivaion of L, then $f(A)\leqslant A$, $f(S)\leqslant S$, in particular, $f(c)=\sigma c$ for some $\sigma\in F$.

Put $c=\alpha_2^{\imath 1}(\alpha_2a_1+\ldots+\alpha_na_{n-1}-a_n)$, then $[c,c]=0$, moreover, Fc is a right center of $L, L=[L,L]\oplus Fc$ and $[c,b]=[a_1,b]$ for every element $b\in A$ [2]. In particular, $a_3=[c,a_2]$, $\ldots$, $a_n=[c,a_{n\imath 1}]$, $[c,a_n]=\alpha_2a_2+\ldots+\alpha_na_n$. In this case, we say that L is a ***cyclic algebra of type (II)***.

On the other hand, **Property 1** shows that $f(c)\in Fc$. It is possible, only if $\gamma=0$. In this case, $f(a_1)=\alpha a_2$ and $f(a_2)=\alpha a_2$. In this case, we can see that $\mathbf{Der}(L)\cong F$, in particular, $\mathbf{Der}(L)$ is Abelian and has a dimension 1.

Now, we suppose that $\mathbf{dim}_F(L)>2$.

Proposition 1. Let L be a cyclic Leibniz algebra of type (II) over a field F, and let D be the annihilator of a subspace Fc in algebra $\mathbf{Der}(L)$. Then D is an ideal of $\mathbf{Der}(L)$ and a factor-algebra $\mathbf{Der}(L)/D$ has dimension at most 1.

Proposition 2. Let L be a cyclic Leibniz algebra of type (II) over a field F. If L has a derivaion f such that $f(c)\neq 0$, then $\mathbf{char}(F)$ divides $\mathbf{dim}_F(L)-1$.

Proposition 3. Let L be a cyclic Leibniz algebra of type (II) over a field F, and let D be the annihilator of a subspace Fc in algebra $\mathbf{Der}(L)$. Then D is generated as a vector space by the derivations $\mathfrak{i},\mathfrak{l}_c,\mathfrak{l}_c^2,\ldots,\mathfrak{l}_c^{n-2}$. Moreover, the set $\{\mathfrak{i},\mathfrak{l}_c,\mathfrak{l}_c^2,\ldots,\mathfrak{l}_c^{n-2}\}$ is a basis of D, so that D is Abelian and has a dimension $n-1$.

The proof of these propositions could be found in [3].

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Torsion groups with non-Dedekind locally nilpotent norms of decomposable subgroups

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In group theory the findings related to the study of groups, subgroups (or the system of subgroups) of which have some theoretical group property, have given restrictions, are in the focus. In some cases the presence of one characteristic subgroup with a certain property can be the determining factor for the structure of the group. Different Σ-norms of a group are the subgroups of such a type.

Author continues the study of different classes of groups with non-Dedekind norm of decomposable subgroups, started in [1–2]. *Decomposable subgroup* is a subgroup of a group G representable in the form of the direct product of two nontrivial factors [3]. The intersection N_G^d of normalizers of all decomposable subgroups of the group G is called *the norm of decomposable subgroups*. If $N_G^d = G$, then either all decomposable subgroups are normal in G or the set of such subgroups are empty. Non-Abelian groups with such a property were studied in [3] and called *di-groups*. The characterization of infinite locally finite groups with non-Dedekind locally nilpotent norm N_G^d of decomposable subgroups are given in this paper.

Theorem 1. *The norm N_G^d of infinite periodic locally nilpotent group G is non-Dedekind and does not contain decomposable subgroups if and only if $G = N_G^d$ and G is an infinite quaternion 2-group.*

Theorem 2. *The norm N_G^d of infinite periodic locally nilpotent group G is non-Dedekind and contains decomposable subgroups if and only if G is a p-group of one of the following types*:

1) *G is infinite locally finite di-group, $N_G^d = G$;*
2) *$G = (A \times \langle b \rangle) \leftthreetimes \langle c \rangle \leftthreetimes \langle d \rangle$, where A is a quasicyclic 2-group, $|b| = |c| = |d| = 2$, $[A, \langle c \rangle] = E$, $[b, c] = [b, d] = [c, d] = a_1 \in A$, $|a_1| = 2$, $d^{-1}ad = a^{-1}$ for all $a \in A$; $N_G^d = (\langle a_2 \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle$, $a_2 \in A$, $|a_2| = 4$;*
3) *$G = (A\langle y \rangle)Q$, where A is a quasicyclic 2-group, $[A, Q] = E$, $Q = \langle q_1, q_2 \rangle$, $|q_1| = 4$, $q_1^2 = q_2^2 = [q_1, q_2]$, $|y| = 4$, $y^2 = a_1 \in A$, $y^{-1}ay = a^{-1}$ for all $a \in A$, $[\langle y \rangle, Q] \subseteq \langle a_1, q_1^2 \rangle$; $N_G^d = \langle a_2 \rangle \times Q$, $a_2 \in A$, $|a_2| = 4$.*

Theorem 3. *An infinite locally finite non-locally nilpotent group G has the non-Dedekind locally nilpotent norm N_G^d of decomposable subgroups if and only if $G = (A \times \langle b \rangle) \leftthreetimes \langle c \rangle \leftthreetimes \langle h \rangle$, where A is a quasicyclic p-group (p is odd prime, $p \neq 2^k \cdot 3^l + 1$ for any non-negative integers k and l), $|b| = |c| = p$, $[A, \langle c \rangle] = 1$, $[b, c] = a \in A$, $|a| = p$, $|h| = q^n$ for a prime $q > 3$ and $n \geqslant 1$, q^n divides $(p-1)$, $h^{-1}bh = b^r, h^{-1}ch = c^s$ for integers r and s with $1 < r < p$, $1 < s < p$ such that $r \neq s$ and $rs \not\equiv 1 \,(mod\ p)$, $C_G(y) = \langle h \rangle$ for each non-indentity element $y \in \langle h \rangle$. Moreover, $N_G^d = (A \times \langle b \rangle) \leftthreetimes \langle c \rangle$.*

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About matrix IP quasigroups

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Let K be a commutative ring with a unit and $K^n := K \times \ldots \times K$. The groupoid $(K^n; f)$ being defined by

$$f(\bar{x}, \bar{y}) = \bar{x}A + \bar{y}B + \bar{a}, \tag{1}$$

where A, $B \in M_n(K)$ and $\bar{a} \in K^n$, is called *matrix quasigroup over the ring K* if the matrix A, B are invertible.

A quasigroup $(Q; \circ)$ is called *central*, if there is an abelian group $(Q; +)$, its automorphisms φ, ψ and an element a such that $x \circ y = \varphi(x) + \psi(y) + a$. Each matrix quasigroup is central. Each central quasigroup being isotopic to an elementary abelian group is isomorphic to a matrix quasigroup.

A quasigroup $(Q; \cdot)$ is called: a *left IP quasigroup*, a *right IP quasigroup*, a middle *IP quasigroup*, if there exists a transformation λ, ρ, μ (*invertibility functions*) such that for all x and y the respective equality holds:

$$\lambda(x) \cdot xy = y; \qquad yx \cdot \rho(x) = y; \qquad xy = \mu(yx).$$

Theorem [2]. *Let $(K^n; f, \bar{0})$ be a unitary matrix quasigroup and (1) hold, then:*

1) *$(K^n; f, \bar{0})$ is a middle IP quasigroup if and only if there exists a matrix C such that $C^2 = E$, $B = AC$. Its invertibility function μ is $\mu(\bar{x}) = \bar{x}C$;*

2) *$(K^n; f, \bar{0})$ is a left IP quasigroup if and only if $B^2 = E$. Its invertibility function λ is $\lambda(\bar{x}) = -\bar{x}ABA^{-1}$;*

3) *$(K^n; f, \bar{0})$ is a right IP quasigroup if and only if $A^2 = E$. Its invertibility function ρ is $\rho(\bar{x}) = -\bar{x}BAB^{-1}$.*

For example, consider all central quasigroups of the order 9. All central quasigroup being isotopic to cyclic groups are described in [4]. Another commutative group of the order 9 is $\mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, we have to solve the matrix equation $X^2 = E$ over the field $\mathbb{Z}_3$. All solutions of the equation are

$$\mathcal{M} := \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \right\}.$$

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Poisson superbialgebras

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The purpose of this talk is to introduce and discuss the notion of Poisson superbialgebra as an analogue of Drinfeld's Lie superbialgebras. We extend various constructions dealing with representations on Lie superbialgebras to Poisson superbialgebras. We show an equivalence between Manin triples of Poisson superalgebras and Poisson superbialgebras in terms of matched pairs of Poisson superalgebras.

Moreover, we consider coboundary Poisson superbialgebras based on a combination of the classical Yang–Baxter equation and the associative Yang–Baxter equation.

This talk is based on a joint work with Basdouri, Fadous and Mabrouk.

Topological actions of wreath products

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Let G and H be two groups acting on path connected topological spaces X and Y respectively. Assume that H is finite of order m and the quotient maps $p : X \to X/G$ and $q : Y \to Y/H$ are regular coverings. Then it is well-known that the wreath product $G \wr H$ naturally acts on $W = X^m \times Y$, so that the quotient map $r : W \to W/(G \wr H)$ is also a regular covering. We give an explicit description of $\pi_1(W/(G \wr H))$ as a certain wreath product $\pi_1(X/G) \wr_{\partial_Y} \pi_1(Y/H)$ corresponding to a *non-effective* action of $\pi_1(Y/H)$ on the set of maps $H \to \pi_1(X/G)$ via the boundary homomorphism $\partial_Y : \pi_1(Y/H) \to H$ of the covering map q.

Such a statement is known and usually exploited only when X and Y are contractible, in which case W is also contractible, and thus $W/(G \wr H)$ is the classifying space of $G \wr H$.

The applications are given to the computation of the homotopy types of orbits of typical smooth functions f on orientable compact surfaces M with respect to the natural right action of the groups $\mathcal{D}(M)$ of diffeomorphisms of M on $\mathcal{C}^\infty(M, \mathbb{R})$.

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Algorithmic constructions for groups of automata

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For all definitions about groups of automata see e.g. [1].

Let $\mathcal{A}$ be a finite initial permutational automaton over a finite alphabet X. We present an algorithm that takes as input the automaton $\mathcal{A}$ and a positive integer $n \geqslant 2$. This algorithm outputs n initial automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$ over some finite alphabet Y. Denote by g and $g_1, \ldots, g_n$ finite automaton permutations defined in initaial states of automata $\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n$ correspondingly.

The following statements hold.

Theorem 1. *If the group $\langle g \rangle$ is finite and the discrete logarithm problem is hard in this group then all groups $\langle g_1 \rangle, \ldots, \langle g_n \rangle$ are finite and the discrete logarithm problem is hard in each of them.*

Theorem 2. *The group $\langle g_1, \ldots, g_n \rangle$ splits into the free product of n groups isomorphic to $\langle g \rangle$.*

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Permutation codes over Sylow 2-subgroups $Syl_2(S_{2^n})$ of symmetric groups S_{2^n} with Hamming distance

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The permutation code of length n and with minimum distance d over metric $\mathbf{d}$ is the set of permutations $C \in S_n$ such that for every pair of different permutations $\pi, \sigma \in C$ the distance between π and σ is greater or equal to d. Permutation codes are used as error-correction codes in channels with low power-line communication (see [1], [2]). Bailey in [3] gave efficient decoding algorithms in the case when the permutation codes are subgroup of permutation groups. We study permutation codes over Sylow 2-subgroups $Syl_2(S_{2^n})$ of symmetric groups S_{2^n} with Hamming distance.

Let $C_H(2^n, d)$ be a *code*, which is defined on permutations from Sylow 2-subgroup $Syl_2(S_{2^n})$ of symmetric group S_{2^n} with Hamming distance d such that for every permutations $\pi, \sigma \in Syl_2(S_{2^n})$ we have:

$$\pi, \sigma \in C_H(2^n, d) \text{ if and only if } d_H(\pi, \sigma) \geqslant d.$$

Theorem. *The number of permutation codes $C_H(2^n, 2^n)$ with the maximum Hamming distance can be defined recursively by the formula:*

$$f(n) = \begin{cases} 4, & \text{if } n = 2; \\ f^4(n-1) \cdot (2^{n-1}!)^2, & \text{if } n > 2. \end{cases}$$

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Representations of Munn algebras and related semigroups

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It is a joint work with Yu. Drozd. The results are published in [2].

Let F be a finite dimensional skewfield over a field $\Bbbk$, $m, n, r \in \mathbb{N}$. The *Munn algebra* $\mathbb{M}(F_k, m, n, r)$ is defined as the ring of $(n+r)\times(m+r)$ matrices over F with the multiplication $A \cdot B = A\mu B$, where μ is an $(m+r)\times(n+r)$ matrix of rank r [1, 2]. Let $\mathbb{M} = \prod_{i=1}^{s} \mathbb{M}(F_k, m_k, n_k, r_k)$, $d_k = \dim_{\Bbbk} F_k$ and $\mathfrak{T} = \{(d_k, m_k, n_k) \mid (m_k, n_k) \neq (0, 0\}$. Let $\mathfrak{T} = \mathfrak{T}^- \cup \mathfrak{T}^+ \cup \mathfrak{T}'$, where

$$\mathfrak{T}^- = \{(d_i, 1, 0) \mid 1 \leqslant i \leqslant q\},$$
$$\mathfrak{T}^+ = \{(d_j, 0, 1) \mid q+1 \leqslant j \leqslant s\},$$

$S^- = \sum_{i=1}^{q} d_i,\ S^+ = \sum_{j=q+1}^{s} d_j$ and $S = S^- + S^+$.

Theorem.

1. [1] *$\mathbb{M}$ is representation finite if and only if*

   (a) *either* $\mathfrak{T}' = \varnothing$ *and* $\max\{S^-, S^+\} \leqslant 3$

   (b) *or* $\mathfrak{T}_1 = \{(1, 1, 1)\}$*,* $S \leqslant 3$ *and* $\max\{S^-, S^+\} \leqslant 2$*.*

2. *$\mathbb{M}$ is representation tame if and only if*

   (a) *either* $\mathfrak{T}^+ = \mathfrak{T}^- = \varnothing$ *and* $\mathfrak{T}'$ *is one of the sets*

   $$\{(1, 1, 1), (1, 1, 1)\},\ \{(2, 1, 1)\},\ \{(1, 2, 0)\},\ \{(1, 0, 2)\},$$

   (b) *or* $\mathfrak{T}' = \varnothing$ *and* $\max\{S^- S^+\} = 4$*,*

   (c) *or* $\mathfrak{T}_1 = \{(1, 1, 1)\}$ *and* $S^- = S^+ = 2$*.*

3. *In all other cases $\mathbb{M}$ is representation wild.*

Using this result we establish the representation type of finite Rees matrix semigroups [1], in particular, 0-simple semigroups, and their mutually annihilating unions in the case when the characteristic of the field $\Bbbk$ does not divide the orders of the involved groups.

We devote this work to the memory of I. S. Ponizovskiĭ.

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[1] If the field $\Bbbk$ is algebraically closed, hence all $d_k = 1$, this result coincides with that of Ponizovskiĭ [3, n° 5].

Keyed hash function from large girth expander graphs

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In the paper [1] message authentication codes (MACs) based on graph structures were presented. ThE approach uses a family of expander graphs of large girth, denoted as $D(n,q)$, $n \in \mathbb{N}_{\geqslant 2}$ and q is a prime power. Graphs $D(n,q)$, $n \geqslant 2$ for arbitrary q form a family of q-regular almost Ramanujan graphs ($|\lambda_1(G_i)| \leqslant 2\sqrt{q}$). Expander graphs are known to have excellent mixing properties because they are very dense. The girth of this family of graphs is given by the formula $g_n \geqslant \log_q(q-1)\log_{q-1}(v_n)$, where v_n is the size of the graph $D(n,q)$ [2]. All requirements for a good MAC are satisfied in our method and a discussion about collisions and preimage resistance is also included.

Based on the tests, our graph-based keyed hash functions shows good efficiency in comparison to other techniques - 4 operations per bit of input can be achieved. The number of operations per bit of input for DMAC-1 is given by the formula $\frac{2n+2}{N}\left(1+\frac{r}{l(M)}\right)$, where r is the length of secret key S, N is block size and $l(M)$ is the number of blocks in a message. The outputs closely approximate the uniform distribution and the results we obtained are computationally indistinguishable from random sequences of bits. The algorithm is very flexible and it works with messages of any length. Many existing algorithms output a fixed length tag, while our constructions allow generation of an arbitrary length output.

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On the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ which is generated by the family $\mathscr{F}_n$ of finite bounded intervals of ω

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For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_{\omega}^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [1]. For any $n \in \omega$ we put $\mathscr{F}_n = \{[0; k] \colon k = 0, \ldots, n\}$. It is obvious that $\mathscr{F}_n$ is an ω-closed family of ω.

We study the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, which is introduced in the paper [1], in the case when the family $\mathscr{F}_n$ generated by the set $\{0, 1, \ldots, n\}$. We show that the Green relations $\mathscr{D}$ and $\mathscr{J}$ coincide in $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ of partial convex order isomorphisms of $(\omega, \leqslant)$ of the rank $\leqslant n$, and $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ admits only Rees congruences.

We describe injective endomorphisms of the inverse semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. In particular we show that the semigroup of injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to $(\omega, +)$. Also we describe the structure of the semigroup $\mathfrak{End}(\mathscr{B}_{\lambda})$ of all endomorphisms of the semigroup of $\lambda{\times}\lambda$-matrix units $\mathscr{B}_{\lambda}$.

Theorem 1. *For an arbitrary $n \in \omega$ the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is isomorphic to an inverse subsemigroup of $\mathscr{I}_{\omega}^{n+1}$, namely $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is isomorphic to the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$.*

Proposition. *For any positive integer n every congruence on the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is Rees.*

Theorem 2. *For an arbitrary $n \in \omega$ the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ admits only Rees congruences.*

Theorem 3. *Let n be a non-negative integer and S be a semigroup. For any homomorphism $\mathfrak{h} \colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \to S$ the image $\mathfrak{h}(\boldsymbol{B}_{\omega}^{\mathscr{F}_n})$ is either isomorphic to $\boldsymbol{B}_{\omega}^{\mathscr{F}_k}$ for some $k = 0, 1, \ldots, n$, or is a singleton.*

Theorem 4. *For any positive integer $n \geqslant 2$ the semigroup of injective endomorphisms of the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is isomorphic to the semigroup $(\omega, +)$. In particular the group of automorphisms of $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is trivial.*

For a non-zero cardinal λ we denote by $\mathscr{S}_{\lambda}$ the group of bijective transformations of λ and by $\mathscr{I}\mathscr{T}_{\lambda}$ the semigroup of injective transformation of λ.

Theorem 5. *The semigroup $\mathfrak{End}^{\mathrm{inj}}(\mathscr{B}_{\lambda})$ of injective endomorphisms of $\mathscr{B}_{\lambda}$ is isomorphic to $\mathscr{I}\mathscr{T}_{\lambda}$, and moreover the group $\mathfrak{Aut}(\mathscr{B}_{\lambda})$ of automorphisms of $\mathscr{B}_{\lambda}$ is isomorphic to $\mathscr{S}_{\lambda}$.*

By $\mathfrak{End}^{\mathrm{ann}}(\mathscr{B}_{\lambda})$ we denote the semigroup of all annihilating endomorphisms of $\mathscr{B}_{\lambda}$.

Theorem 6. *The semigroup $\mathfrak{End}(\mathscr{B}_{\lambda})$ of all endomorphisms of the semigroup of $\lambda{\times}\lambda$-matrix units $\mathscr{B}_{\lambda}$ is the union of the semigroups $\mathfrak{End}^{\mathrm{inj}}(\mathscr{B}_{\lambda})$ and $\mathfrak{End}^{\mathrm{ann}}(\mathscr{B}_{\lambda})$. Moreover, $\mathfrak{End}^{\mathrm{inj}}(\mathscr{B}_{\lambda})$ a left cancellative semigroup and $\mathfrak{End}^{\mathrm{ann}}(\mathscr{B}_{\lambda})$ is the minimal ideal of $\mathfrak{End}(\mathscr{B}_{\lambda})$ which is a right zero semigroup.*

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On the group of automorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$

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Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and $n, m \in \omega$ we put $n - m + F = \{n - m + k \colon k \in F\}$ if $F \neq \varnothing$ and $n - m + \varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called *ω-closed* if $F_1 \cap (-n + F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$.

On the set $\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}$, where $\boldsymbol{B}_{\mathbb{Z}}$ is the extended bicyclic semigroup and $\mathscr{F}$ is an ω-closed subfamily of $\mathscr{P}(\omega)$, we define the semigroup operation "$\cdot$" by formula

$$(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geqslant i_2. \end{cases}$$

In [1] it is proved that $(\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i, j, \varnothing) \colon i, j \in \mathbb{Z}\}$ is an ideal of the semigroup $(\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ we define the following semigroup

$$\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_{\mathbb{Z}} \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F}. \end{cases}$$

Theorem. *Let $\mathscr{F}$ be an ω-closed family of inductive nonempty subsets of ω. Then the group of automorphisms $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is isomorphic to the additive group of integers $(\mathbb{Z}, +)$.*

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A NOTE ON A MINIMAL SOLUTION OF THE MATRIX POLYNOMIAL EQUATION $A(\lambda)X(\lambda) - Y(\lambda)B(\lambda) = C(\lambda)$

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Let $\mathbb{F}$ be a field. Denote by $\mathbb{F}_{m,n}[\lambda]$ the set of $m \times n$ matrices over the polynomial ring $\mathbb{F}[\lambda]$. A matrix $A(\lambda) = \sum_{i=0}^{s} A_i \lambda^{s-i} \in \mathbb{F}_{k,k}[\lambda]$ is said to be regular if $\det A_0 \neq 0$ (in the sense of Gantmacher [1]).

Let $A(\lambda) \in \mathbb{F}_{m,m}[\lambda]$, $B(\lambda) \in \mathbb{F}_{n,n}[\lambda]$ and $C(\lambda) \in \mathbb{F}_{m,n}[\lambda]$. Consider the matrix equation

$$A(\lambda)X(\lambda) - Y(\lambda)B(\lambda) = C(\lambda), \tag{1}$$

where $X(\lambda), Y(\lambda) \in \mathbb{F}_{m,n}[\lambda]$ are unknown matrices. It is said that equation (1) has a minimal solution $\{X_0(\lambda), Y_0(\lambda)\}$ if $\deg X_0(\lambda) < \deg B(\lambda)$ or $\deg Y_0(\lambda) < \deg A(\lambda)$.

Barnett [2] considered the case in which $A(\lambda)$ and $B(\lambda)$ are regular polynomial matrices and proved that equation (1) has a unique minimal solution if and only if $\deg C(\lambda) \leqslant \deg A(\lambda) + \deg B(\lambda) - 1$ and $(\det A(\lambda), \det B(\lambda)) = 1$. Feinstein and Bar-Ness [3] proved that Barnett's conditions for uniqueness are true in the case when only $A(\lambda)$ or $B(\lambda)$ (not necessarily both) is regular.

In [5] the following statement was proved. Let $A(\lambda) \in \mathbb{F}_{m,m}[\lambda]$ and $B(\lambda) \in \mathbb{F}_{n,n}[\lambda]$ be nonsingular matrices and matrix $B(\lambda)$ admits the representation $B(\lambda) = W(\lambda)D(\lambda)$, where $W(\lambda) \in GL(n, \mathbb{F}[\lambda])$ and $D(\lambda) \in \mathbb{F}_{n,n}[\lambda]$ is a monic polynomial matrix ($\deg D(\lambda) < \deg B(\lambda)$) (see [4, 6]). If $(\det A(\lambda), \det B(\lambda)) = 1$, then equation (1) has a unique solution $\{X_0(\lambda), Y_0(\lambda)\}$ such that $\deg X_0(\lambda) < \deg D(\lambda)$. We note that similar problem was investigated in [7].

Purpose of this report is to present the following statement.

Theorem. *Let* $A(\lambda) = \begin{bmatrix} a_1(\lambda) & 0 & \dots & \dots & 0 \\ a_{21}(\lambda) & a_2(\lambda) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1}(\lambda) & a_{m2}(\lambda) & \dots & a_{m,m-1}(\lambda) & a_m(\lambda) \end{bmatrix} \in \mathbb{F}_{m,m}[\lambda]$, $B(\lambda) \in \mathbb{F}_{n,n}[\lambda]$ *and* $C(\lambda) \in \mathbb{F}_{m,n}[\lambda]$.

The matrix equation (1) has a unique solution $\{X_0(\lambda), Y_0(\lambda)\}$ *such that degrees of elements of the k-th row* $\begin{bmatrix} y_{k1}(\lambda) & y_{k2}(\lambda) & \dots & y_{kn}(\lambda) \end{bmatrix}$ *of the matrix* $Y_0(\lambda)$ *are smaller than the degree of the element* $a_k(\lambda)$ *for all* $k = 1, 2, \dots, m$; *if and only if* $(\det A(\lambda), \det B(\lambda)) = 1$.

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Affine Courant algebroid, its coadjoint orbits and related integrable flows

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Poisson structures related with the affine Courant algebroid are analyzed. The coadjoint action orbits are studied, infinite hierarchies of the Casimir functionals are described. A wide class of integrable flows on functional manifolds is constructed.

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Palais–Smale sequences for the prescribed Ricci curvature functional

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On homogeneous spaces, solutions to the prescribed Ricci curvature equation coincide with the critical points of the scalar curvature functional subject to a constraint. We provide a complete description of Palais–Smale sequences for this functional. As an application, we obtain a new existence result for the prescribed Ricci curvature equation, which enables us to observe previously unseen phenomena.

Finite one-sided distributive structures and GAP

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We study algebraic structures called nearrings. Nearrings naturally arise in the study of systems of nonlinear mappings, and have been studied for many decades.

The classification of all nearrings up to certain orders (i.e. producing their complete and irredundant list up to equivalency) is an open problem. It requires extensive computations, and the most suitable platform for their implementation is the computational algebra system GAP [1].

For the researchers in nearrings, the list of all 698 local nearrings of order at most 31 up to isomorphism is provided by the GAP package SONATA [2]; however, classifying nearrings of order 32 and more is a significant challenge.

Presently, the library of local nearrings of the package LocalNR [3] contains local nearrings of orders at most 361 (except several orders described above). All nearrings in the library are local nearrings. The library of local nearrings is arranged in archived files. They can be used to obtain any necessary information concerning such nearrings. New data libraries will be included in the next version of the LocalNR package (possibly as optional downloads for extremely large collections). For example, the library of zero-symmetric local nearrings of order 128 on 2-generated groups can be extracted from [4] using the package LocalNR.

The initial idea for the project was motivated by the need of having a database of examples of moderately sized nearrings with identity to search for examples and counterexamples. Unfortunately, the number of nearrings with identity is so much bigger, and most of them bare so little structure, that new techniques to store and handle such nearrings had to be developed. Of course, the first step was to actually construct some classes of nearrings. However, it is not true that any finite group is the additive group of a nearring with identity. Therefore it is important to determine such groups and to classify some classes of nearrings with identity on these groups, for example, local nearrings.

Acknowledgements. The authors would like to thank IIE-SRF for supporting of our fellowship at the University of Warsaw.

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Weight modules of quantum Weyl algebras

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We classify simple weight modules over quantum Weyl algebras. The quantum Weyl algebra contains a maximal commutative subalgebra. Weight modules are then modules on which this commutative subalgebra acts diagonally.

This is joint work with V. Futorny and A. Solotar.

Elementary divisor rings with Dubrovin-Komarnytskii conditions

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Let R be an associative ring with non-zero unit. A ring R is called an *elementary divisor ring* if for an arbitrary matrix A over R there exists invertible matrices P and Q of suitable sizes such that $PAQ = D$ is a diagonal matrix $D = (d_i)$ where d_{i+1} is a total divisor of d_i, i.e. $Rd_{i+1}R \subseteq d_iR \cap Rd_i$ for each i [1]. A right (left) Bezout ring is a ring in which every finitely generated right (left) ideal is principal. If a ring is both left and right Bezout then it is called a *Bezout ring*. A ring R is a *ring of stable range 1* if for any $a, b \in R$ such that $aR + bR = R$ we have $(a + bt)R = R$ for some $t \in R$ [2]. Condition for which that for any element $a \in R$ there exists the element $a_* \in R$ such that $RaR = a_*R = Ra_*$ is called Dubrovun's condition [3]. From now on me assume that R is domain in which every factor of an invariant element is invariant element; this condition is said Komarnytskii condition.

Theorem 1. *Let R be an elementary divisor domain with Dubrovin and Komarnytskii conditions. Then any matrix A over is equivalent to matrix $diag(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$, where $R\varepsilon_{i+1}R \subseteq R\varepsilon_i \cap \varepsilon_iR$ for all $i = 1, \ldots, k-1$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{k-1}$ are invariant elements.*

Theorem 2. *Let R be a principal ideal domain. Then R is an elementary divisor ring with Dubrovin-Komarnytskii condition if and only if R is a ring with Dubrovin-Komarnytskii condition.*

Theorem 3. *A Bezout domain R is an elementary divisor ring with Dubrovin-Komarnytskii condition if and only if 2×2 matrices are equivalent to the matrix $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & a \end{smallmatrix}\right)$ where $RaR \subseteq \varepsilon R = R\varepsilon$ or $\varepsilon = 0$ and $a \in R$.*

Theorem 4. *Let R be a Bezout domain with Dubrovin and Komarnytskii condition. Then R is an elementary divisor ring if and only if for any $a, b, c \in R$ such that $RaR + RbR + RcR = R$ exists $p, q \in R$ such that $paR + (pb + qc)R = R$.*

Let R be a Bezout domain with Dubrovin and Komarnytskii condition and $a \in R$ such that $RaR = R$. We say that element a is redusible if for any $b, c \in R$ there are such $p, q \in R$ that $paR + (pb + qc)R = R$.

Theorem 5. *Let R be a Bezout domain of stable range 1 with Dubrovin and Komarnytskii conditions. Then R is an elementary divisor ring with Dubrovin-Komarnytskii conditions if and only if every nonzero element is a redusible.*

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2-STATE ZC-AUTOMATA GENERATING CYCLIC GROUPS

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Let Z be the set of integers. A permutational automaton $\mathcal{A} = \langle \mathsf{Z}, Q, \varphi, \psi \rangle$ over alphabet Z is called ZC-automaton (see [1]), if in any inner state $q \in Q$ the output function ψ_q defines a shift by some integer c_q:

$$\psi_q(z) = z + c_q, z \in \mathsf{Z}. \tag{1}$$

In every inner state a ZC-automaton $\mathcal{A}$ determines a permutation on the set of infinite integer sequences. The group generated by all these permutations ia called the group of the automaton $\mathcal{A}$.

Consider 2-state ZC-automata with states q_1 and q_2. Such an automaton $\mathcal{A}$ is determined by two partitions of the set Z, $\mathsf{Z} = A_1 \cup A_2$ and $\mathsf{Z} = B_1 \cup B_2$, and by two integers a and b (see Fig. 1). Hence each 2-state ZC-automaton can be uniquely determined as the quadruple $\langle A_1, B_1, a, b \rangle$, where $A_1, B_1 \subset \mathsf{Z}$, $a, b \in \mathsf{Z}$.

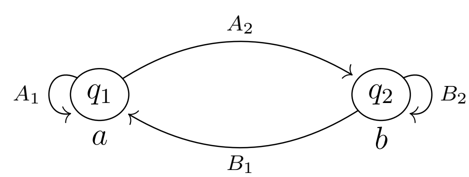

Fig 1. 2-state ZC-automaton

Theorem 1. *If 2-state ZC-automaton $\mathcal{A} = \langle A_1, B_1, a, b \rangle$ generates a cyclic group, and $f_2 = {f_1}^m$, with $m \in N$, then $m = 1$.*

Theorem 2. *If the group of 2-state ZC-automaton $\mathcal{A} = \langle A_1, B_1, a, b \rangle$ is cyclic as a permutation group on the words of length 2, then it is cyclic as a permutation group on Z^*.*

Theorem 2 allows for to construct the following ZC-automata. Fix a natural $a \neq 0$. Define a 2-state ZC-automaton $\mathcal{A}$, specified by a quadruple $\langle A_1, B_1, 1, -a \rangle$. Sort the elements of A_2 in the ascending order. Then if z_1, z_2 are nearby elements of A_2, then $|z_1 - z_2| \geqslant a$. For each $z \in Z$, $z \in B_2$ if $z - i \in A_1$, $i = \overline{1, a}$, and $z \in B_1$ in other way.

Theorem 3. *The group of 2-state ZC-automaton $\mathcal{A}$ is cyclic.*

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Almost ω-Euclidian domain

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Let R denote a commutative domain with a nonzero unit element.

Let $\varphi : R \to \mathbb{N} \cup \{0\}$ be a function satisfying the following condition: $\varphi(a) = 0$ if and only if $a = 0$; $\varphi(a) > 0$ for any nonzero and $\varphi(ab) \geqslant \varphi(a)$ for any arbitrary elements $a, b \in R$. This function is called the *norm* over domain R.

A *k-stage division chain* [4] for any arbitrary elements $a, b \in R$ with $b \neq 0$ is understood as the sequence of equalities $a = bq_1 + r_1,\ b = r_1q_2 + r_2,\ \ldots,\ r_{k-2} = r_{k-1}q_k + r_k,$ (1) with $k \in \mathbb{N}$. Domain R is called *ω-Euclidean domain* [4] with respect to the norm φ, if for any arbitrary elements $a, b \in R$, $b \neq 0$, there exists a k-stage division chain (1) for some k, such as $\varphi(r_k) < \varphi(b)$. Clearly, the 2-Euclidean domain is ω-Euclidean domain.

A ring R is called *a ring with elementary reduction of matrices* [4] if an arbitrary matrix over R possesses elementary reduction, i.e. for an arbitrary matrix A over the ring R there exist such elementary matrices over R , $P_1, \ldots, P_k, Q_1, \ldots, Q_s$ of respectful sizes such that

$$P_1 \cdots P_k \cdot A \cdot Q_1 \cdots Q_s = diag(\varepsilon_1, \ldots, \varepsilon_r, 0, \ldots, 0),$$

where $R\varepsilon_{i+1}R \subseteq R\varepsilon_i \cap \varepsilon_i R$ for any $i = 1, \ldots, r-1$.

A ring R is called *a ring of stable range 1* [3] if for any elements $a, b \in R$ the equality $aR + bR = R$ implies that there is some $x \in R$ such that $(a + bx)R = R$.

An element $a \neq 0$ of a commutative ring R is called *an element of almost stable range 1* [1] if the stable range of R/aR is equal to 1. If all nonzero elements of a ring R are elements of almost stable range 1, then we say that R is *a ring of almost stable range 1.*

Definition. *An element $a \neq 0$ of a commutative domain R is called an element of almost ω-Euclidian if R/aR is ω-Euclidean domain. If all nonzero elements of a domain R are elements of almost ω-Euclidian, then we say that R is an almost ω-Euclidian domain.*

Theorem 1. *Let R be a commutative Bezout domain. If R is an ω-Euclidian domain, then R is an almost ω-Euclidian domain.*

Theorem 2. *Let R be a commutative Bezout domain. If R is a ring of almost stable range 1, then R is an almost 2-Euclidian domain.*

Theorem 3. *Let R be a commutative almost 2-Euclidian domain. Then R is an ω-Euclidian domain if and only if R is a ring with elementary reduction of matrices.*

You can see more results about rings with elementary reduction of matrices in [2, 4].

We denote by R_n the ring of all $n \times n$ matrices over R. Then we have next theorem.

Theorem 4. *Let R be a commutative almost 2-Euclidian domain. Then R_2 is an almost right 2-Euclidian domain and an almost left 2-Euclidian domain.*

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On connections between pre-Lie rings and braces

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In 2014, Wolfgang Rump presented a connection pathway from pre-Lie algebras to braces. This pathway can also be described using the group of flows of a pre-Lie algebra. An advantage of this construction is that the additive group of the pre-Lie algebra and the obtained brace are the same. It is not yet clear if every brace of cardinality p^n for $p > n$ can be obtained from a pre-Lie ring in this way. An affirmative answer to this question would yield an extension of the classical Lazard correspondence between p-adic Lie groups and p-adic Lie rings to the correspondence between braces and pre-Lie rings.

In this talk we will show that if A is a brace of cardinality p^n where $p > n+1$ then the brace $A/ann(p^4)$ is obtained as the group of flows of some left nilpotent pre-Lie ring. This answers the above question up to elements whose additive order is at most p^4. Here $ann(p^4)$ denotes the set of elements whose additive order is p^i for $i \leqslant 4$.

Rump introduced braces in 2007. They are a generalisation of Jacobson radical rings with the two-sided braces being exactly the Jacobson radical rings. One of the main motivations for investigating braces is their connections with set theoretic solutions of the Yang–Baxter equation. Another is the relationship of braces to homological group theory since braces are exactly groups with bijective 1-cocycles. The theory of braces is also connected to algebraic number theory and its generalisations through the concept of Hopf–Galois extensions of abelian type (which was demonstrated by David Bachiller).

Some of this talk relates to work done in collaboration with Aner Shalev.

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About orthogonality and strong orthogonality of medial quasigroups

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An n-ary operation f defined on Q of order $m < \infty$ is called *invertible* and the pair $(Q; f)$ is a *quasigroup*, if for all $a_1, \ldots, a_n$ of Q each of the terms $f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$, $i = 1, \ldots, n$, defines a permutation of Q.

Theorem 1. [1] *An n-ary quasigroup $(Q; f)$ is medial if and only if there exists an abelian group $(Q; +)$, its pairwise commuting automorphisms $\varphi_1, \varphi_2, \ldots, \varphi_n$ and $a \in Q$ such that*

$$f(x_1, x_2, \ldots, x_n) = \varphi_1 x_1 + \varphi_2 x_2 + \ldots + \varphi_n x_n + a. \tag{1}$$

n-ary operations $f_1, f_2, \ldots, f_n$ defined on a set Q is called:

- *orthogonal*, if for all $a_1, a_2, \ldots, a_n \in Q$ the following system of equations has a unique solution
$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = a_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(x_1, x_2, \ldots, x_n) = a_n; \end{cases}$$
- *strongly orthogonal* if each n-tuple of the operations $f_1, \ldots, f_n, e_1, \ldots, e_n$ is orthogonal, where
$$e_i(x_1, x_2, \ldots, x_n) := x_i, \quad i = 1, \ldots, n.$$
The operations $e_1, \ldots, e_n$ are called *selectors.*

For every permutation $\sigma \in S_n$ a *σ-parastrophe* ${}^\sigma f$ of an invertible ternary operation f is defined by
$${}^\sigma f(x_{1\sigma}, x_{2\sigma}, \ldots, x_{n\sigma}) = x_{(n+1)\sigma} :\Longleftrightarrow f(x_1, x_2, \ldots, x_n) = x_{n+1}.$$
A σ-parastrophe is called *principal* if $(n+1)\sigma = n+1$. A quasigroup having $(n+1)!$ pairwise different parastrophes is called *asymmetric*. A quasigroup is called *totally-parastrophic orthogonal* (*self-orthogonal*) if each n-tuple of (principal) parastrophes are orthogonal.

We propose algorithms for constructing totally-parastrophic orthogonal and self-orthogonal asymmetric ternary medial quasigroups. For this, we prove that self-orthogonality is reduced to invertibility-valued of three polynomials over the set $\{\varphi_1, \varphi_2, \varphi_3\}$, strongly self-orthogonality is reduced to invertibility-valued of five polynomials over the set $\{\varphi_1, \varphi_2, \varphi_3\}$, totally-parastrophic orthogonality is reduced to invertibility-valued of ten polynomials over the set $\{\varphi_1, \varphi_2, \varphi_3, J\}$.

The considered concepts are different as the following example shows. Let $\mathbb{Z}_m$ be a ring of integers modulo m and the ternary operation f is defined by:

$$f(x, y, z) := x + 2y + 3z.$$

If m is relatively prime to 6, then $(\mathbb{Z}_m; f)$ is a quasigroup. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup, if m is not divisible by 6; $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup, but it is not strongly self-orthogonal if m is not divisible by 6 and m is divisible by 5 or 7; $(\mathbb{Z}_m; f)$ is a strongly self-orthogonal ternary quasigroup, if m is not divisible by 2, 3, 5 and 7.

Theorem 2. *n-ary strongly self-orthogonal linear quasigroups exist if and only if $n = 2, 3$.*

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Naturally ordered endotopism semigroups preserving an equivalence relation

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For an arbitrary semigroup S the binary relation $\leqslant$ defined by $a \leqslant b$ iff $a = bx = yb$, $a = ax$ for some $x, y \in S$, is a partial order which called the *natural partial order* of S [1].

An ordered pair (φ, ψ) of transformations φ and ψ of a nonempty set X is called an *endotopism* [2] of $\rho \subseteq X \times X$ if for all $a, b \in X$ the condition $(a, b) \in \rho$ implies $(a\varphi, b\psi) \in \rho$. The set of all endotopisms of ρ is a semigroup with respect to the componentwise multiplication operation. This semigroup is called the *endotopism semigroup* of ρ and it is denoted by $Et(X, \rho)$.

Let α be an arbitrary equivalence relation on the set X and X/α denote the quotient set of X. It is known that $Et(X, \alpha)$ is a correspondence of the endomorphism semigroup $End(X, \alpha)$ [3]. For every $f \in End(X, \alpha)$, let $T(f) = \{f^{-1}(A) | A \in X/\alpha \text{ and } f^{-1}(A) \neq \varnothing\}$. Then $T(f)$ is a partition of X. Obviously, x, y are contained in the same $U \in T(f)$ if and only if $(xf, yf) \in \alpha$. Besides, for $(\phi, \psi) \in Et(X, \alpha)$ we have $B(\phi) = B(\psi)$.

Theorem. *Let $(\phi_1, \psi_1), (\phi_2, \psi_2) \in Et(X, \alpha)$. Then $(\phi_1, \psi_1) \leqslant (\phi_2, \psi_2)$ if and only if the following statements hold:*
(i) for any $A \in X/\alpha$ there exists $B \in X/\alpha$ such that $A\phi_1 \subseteq B\phi_2, A\psi_1 \subseteq B\psi_2$;
(ii) for any $V \in T(\phi_2)$ there exists $U \in T(\phi_1)$ such that $V \subseteq U$, and analogously, for any $V \in T(\psi_2)$ there exists $U \in T(\psi_1)$ such that $V \subseteq U$;
(iii) for any $x, y \in X$ the condition $x\phi_2 = y\phi_2$ implies $x\phi_1 = y\phi_1$, and analogously, $x\psi_2 = y\psi_2$ implies $x\psi_1 = y\psi_1$;
(iv) for $x \in X$ the condition $x\phi_2 \in X\phi_1$ implies $x\phi_1 = x\phi_2$, and analogously, $x\psi_2 \in X\psi_1$ implies $x\psi_1 = x\psi_2$.

In addition, we study the maximal and minimal elements of the endotopism semigroups of an equivalence. The similar problems for endomorphism semigroups of an equivalence relation were considered in [4].

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On matrices with all minors of some fixed order being equal

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Matrices with all principal minors of some fixed order being equal were investigated by R. C. Thompson in [1] and [2]. In [1], a classification was found for symmetric matrices A over an arbitrary field, for which all $t \times t$ principal minors of A are equal, for three consecutive values of t less than the rank of A. A similar theorem classifying the real symmetric matrices in which the condition on the principal minors is weakened to requiring that all $t \times t$ principal minors of A be equal, for two consecutive values of t less than the rank of A, and in which a sign condition is imposed on the nonprincipal $t \times t$ minors for these two consecutive values of t, was given in [2]. The paper [2] also classifies all square matrices A (over an arbitrary field and not necessarily symmetric) in which the condition on the principal minors of A is weakened to requiring that all $t \times t$ principal minors of A be equal for one value of t less than the rank of A, and for this value of t the condition on the nonprincipal $t \times t$ minors of A is strengthened to requiring that they all be equal.

Discussed in this report is a class $\mathfrak{M}$ of matrices over an arbitrary field in which all minors of some fixed order k are equal and nonzero.

Theorem. *Let P be an arbitrary field and A be a $m \times n$-matrix over P in which all minors of order k are equal and nonzero. Then: (i)* $\operatorname{rank} A = k$*; (ii) $k \leqslant m, n \leqslant k+1$.*

Corollary 1. Let A be a $k \times (k+1)$-matrix over the field P. All minors of order k of the matrix A are equal and nonzero iff the following conditions 1)-2) hold:

1) $\operatorname{rank} A = k$;
2) $(k+1)$-th column A^{k+1} of the matrix A is expressed as the linear combination:
   $A^{k+1} = \sum\limits_{j=1}^{k} (-1)^{k+2-j} A^j$ where A^j is a j-th column of the matrix A, $1 \leqslant j \leqslant k$.

Corollary 2. Let A be a $(k+1) \times (k+1)$-matrix over the field P. All minors of order k of the matrix A are equal and nonzero iff the following conditions 1)-3) hold:

1) $\operatorname{rank} A = k$;
2) $(k+1)$-th column A^{k+1} of the matrix A is expressed as the linear combination:
   $A^{k+1} = \sum\limits_{j=1}^{k} (-1)^{k+2-j} A^j$ where A^j is a j-th column of the matrix A, $1 \leqslant j \leqslant k$.
3) $(k+1)$-th row A_{k+1} of the matrix A is expressed as the linear combination:
   $A_{k+1} = \sum\limits_{i=1}^{k} (-1)^{k+2-i} A^i$ where A_i is a i-th row of the matrix A, $1 \leqslant i \leqslant k$.

Using the necessary and sufficient condition for a matrix to have all minors of order k equal and nonzero, one can easily classify all matrices for fixed values of k.

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Simple torsion free modules for the algebras A_2, C_2, G_2 with infinite dimensional weight spaces

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We describe the centralizers of Cartan subalgebras of simple finite-dimensional Lie algebras. Then we apply this result to rank 2 Lie algebras and construct all torsion free tame Gelfand-Tsetlin modules with infinite-dimensional weight spaces. This is a joint project with Vyacheslav Futorny, Carlos Martins da Fonseca and Milica Andelic.

On irreducible induced representations of certain minimax nilpotent groups

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If a group G has a finite series each of whose factor is either cyclic or quasi-cyclic then G is said to be minimax. Let G be a group, let k be a field and let M be a kG-module. Let H be a subgroup of the group G and let U be a kH-submodule of M. The module M is said to be induce from the submodule U if $M = U \otimes_{kH} kG = \oplus_{t \in T} Ut$, where T is a right transversal to the subgroup H in G.

Let φ be a representation of G over k and let M be an kG-module of the representation φ. The representation φ is said to be faithful if $Ker\varphi = 1$. If M is induced from some FH-module U, where H is a subgroup of the group G, then we say that the representation φ is induced from a representation ϕ of the subgroup H, where U is the module of the representation ϕ. The module M and the representation φ are said to be primitive if there are no subgroups $H < G$ such that M is induced from an FH-submodule.

In [1] we proved that among minimax nilpotent groups of nilpotency class 2 only finitely generated groups may have faithful irreducible primitive representations over a finitely generated field of characteristic zero. In [2] we proved that any irreducible representation of a finitely generated nilpotent G over a finitely generated field of characteristic zero is induced from a primitive representation of some subgroup of G. Now, we prove the following theorem.

Theorem 1. *Let G be a minimax nilpotent group of nilpotency class 2. Let k be a finitely generated field of characteristic zero and let M be an irreducible kR-module. Then there are a subgroup H and a primitive kH-module U such that $M = U \otimes_{kH} kG$ and the quotient group $H/C_{kH}(U)$ is finitely generated.*

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On new results of Algebraic Geometry and their impact on Extremal Graph Theory

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Classical Extremal Graph Theory developed by P. Erdos' and his school had been started with the following problem formulated by Turan. What is the maximal value $ex(v, C_{2n})$ for the size (number of edges) of graph on v vertices without cycles C_n of length $2n$?

Other important question is about maximal size $ex(v, C_3, C_4, \dots, C_{2n}, C_{2n+1})$ of a graph of order v without cycles of length $3, 4, \dots, 2n+1$, i.e. graphs of girth $\geqslant 2n+2$. Recall that girth of the graph is minimal length of its cycle. According to Erdos Even Circuit Theorem $ex(v, C_{2n}) = O(v^{1+1/n})$. Studies of lower bounds for $ex(v, C_{2n})$ and $ex(v, C_3, C_4, \dots, C_{2n}, C_{2n+1})$ form important direction of Extremal Graph Theory.

Classical objects of Algebraic Geometry are algebraic graphs, i. e. simple graphs of binary relations defined over algebraic varieties over field F such that their edge sets are also algebraic varieties over F. Studies of algebraic graphs with prescribed girth and diameter form classical direction of Geometry.

For example classical projective plane is a graph of girth 6 and diameter 3. Its vertex set is a disjoint union of one dimensional and two dimensional vector spaces of F^3. J. Tits defined generalised m-gons as a bipartite graph of girth $2m$ and diameter m. Noteworthy that geometries of Chevalley groups $A_2(F)$, $B_2(F)$ and $G_2(F)$ are generalised m-gons for $m = 3, 4$ and 6.

Algebraic bipartite graphs $A(n, F)$ with partition sets isomorphic to F^n are given by the following relation. Point $(x_1, x_2, \dots, x_n)$ is incident to line $[y_1, y_2, \dots, y_n]$ if and only if $x_2 - y_2 = y_1x_1$, $x_3 - y_3 = x_1y_2$, $x_4 - y_4 = y_1x_3$, $x_5 - y_5 = x_1y_4$, ... (see [1] and further references).

We prove (see [2]) the following statement.

Theorem. *The girth of $A(n, F)$, $F \neq F_2$ is $2n$ or $2n+2$.*

Counting the size of q-regular graphs $A(n, F_q)$, $n = 2, 3, \dots$ gives the following proposition.

Corollary. $ex(v, C_{2n}) \geqslant ex(v, C_3, C_4, \dots, C_{2n+1}) \geqslant (1/2)^{1+1/(n+1)}v^{1+1/(n+1)}$.

This is strong improvement of previously known lover bounds for $n \geqslant 6$.

We see that $C'v^{1+1/(n+1)} \leqslant ex(v, C_{2n}) \leqslant Cv^{1+1/n}$ for some positive constants C and C' if $n = 4$ or $n \geqslant 6$.

Conjecture. If $n = 4$ or $n \geqslant 6$ then $ex(v, C_{2n}) = O(v^{1+1/(n+1)})$.

Remark 1. If the conjecture is true then new bound is sharp, i.e. $ex(v, C_{2n}) <=> Cv^{1+1/(n+1)}$ for some positive C in the case of $n = 4$ or $n \geqslant 6$.

Remark 2. Generalised m-gons, $m = 3, 4, 6$ with automorphism groups $A_2(F_q)$, $B_2(F_q)$, $G_2(F_q)$ support the sharpness of Erdos' bound, i.e $ex(v, C_{2n}) <=> Cv^{1+1/n}$ for $n = 2, 3, 5$.

Acknowledgements: This research is supported by Fellowship of British Academy for Researchers at Risk 2022.

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Divisor function of the Gaussian integers weighted by the Kloosterman sum

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Let G denote ring of the Gaussian integers. For $\gamma \in G$, let G_γ denote the residue class ring modulo γ, and through G_γ^* let denote the multiplicative group of this ring.

For $\alpha, \beta, \gamma \in G$ the Kloosterman sum $K(\alpha, \beta; \gamma)$ is determined by equality

$$K(\alpha, \beta; \gamma) = \sum_{x \in G_\gamma^*} exp\left(2\pi i Re\left(\frac{\alpha x + \beta x^{-1}}{\gamma}\right)\right),$$

where x^{-1} is the multiplicative inverse modulo γ for x.

In this work we obtain the asymptotic formula for mean of the divisor function $\tau(\omega)$, $\omega \in G$, weighted by the Kloosterman sum.

Theorem 1. *Let $f(\omega)$ be a multiplicative function over G for which the series $\sum\limits_{\omega \in G} f(\omega)N(\omega)^{-s}$ converges absolutely. Then in semiplane $Res > 1$ the equality*

$$\sum_{\omega \in G} \frac{f(\omega)K(1, \omega; \gamma)}{N(\omega)^s} = \sum_{\delta | \gamma_1} \mu(\delta) \sum_{\substack{\omega \in G \\ (\omega, \gamma) = \delta}} \frac{f(\omega)}{N(\omega)^s} K\left(1, \omega\delta^{-1}; \frac{\gamma}{\delta}\right)$$

holds.

Theorem 2. *Let $g(\omega)$ be completely multiplicative function over G and let the Dirichlet series $\sum\limits_{\omega \in G} g(\omega)N(\omega)^{-1}$ converge absolutely in semiplane $Res > 1$. Then for every $\alpha, \gamma \in G$, $N(\gamma) > 1$, $(\alpha, \gamma) = 1$, we have*

$$\sum_{\omega \in G} \tfrac{f(\omega)K(1, \alpha\omega; \gamma)}{N(\omega)^s} =$$

$$= \sum_{\delta | \gamma_1} \mu(\delta) \sum_{\substack{t_1, t_2 \in G \\ t_1 t_2 | \frac{\gamma}{\delta}}} \tfrac{\mu(t_1)\mu(t_2)}{N(t_1)^s N(t_2)^s} \times$$

$$\times \sum_{S(C)} g(\delta) Z_g\left(s; 0; \tfrac{\alpha_1 \delta_1 \delta^{-1}}{\frac{\gamma}{\delta}}\right) Z_g\left(s; 0; \tfrac{\alpha_2 \delta_2 \delta^{-1}}{\frac{\gamma}{\delta}}\right),$$

where $f(\omega) = \sum\limits_{\delta | \omega} g(\omega)$, γ_1 denotes a square-free part of γ,

$$C := \left\{\alpha_1, \alpha_2 \in G^*_{\frac{\gamma}{\delta}},\ \alpha_1\alpha_2 \equiv 1 \pmod{\frac{\gamma}{\delta}}\right\};\ \delta^{-1} \pmod{\frac{\gamma}{\delta}}$$

(from now on listing $S(C)$ under a sign of sum implicate that the summing up under condition of C which describe separate).

These two assertions allow us to construct the asymptotic formulas for the sum of values of the divisor function over the ring of Gaussian prime numbers under some regions of complex plane.

The Kloosterman sums on the ellipse

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The main point of our research is to obtain the estimates for Kloosterman sums $\widetilde{K}(\alpha, \beta; h, q; k)$ considered on the ellipse bound for the case of the integer rational module q and for some natural number k with conditions $(\alpha, q) = (\beta, q) = 1$ on the integer numbers of imaginary quadratic field. These estimates can be used to construct the asymptotic formulas for the sum of divisors function $\tau_\ell(\alpha)$ for $\ell = 2, 3, \ldots$ over the ring of integer elements of imaginary quadratic field in arithmetic progression.

Let $\alpha, \beta \in \mathbb{Z}[\theta]$, $h \in \mathbb{Z}$, $q \in \mathbb{N}$, $q > 1$, $(h, q) = 1$. Let us assume

$$\widetilde{K}(\alpha, \beta; h, q) := \sum_{\substack{x,y (mod\, q)\\ N(xy) \equiv h (mod\, q)}} e_q\left(\frac{1}{2}Sp(\alpha x + \beta y)\right)$$

and call it the Kloosterman sum over the ellipse $u^2 + dv^2 \equiv 1 \pmod{p^m}$.

Theorem 1. *Let $(h, p) = 1$. Then*

$$\widetilde{K}(\alpha, \beta; h, p^n) \ll (p^{m_\alpha}, p^{m_\beta}, p^n)^{\frac{1}{2}} \cdot p^{\frac{3n}{2}}$$

with absolute constant in symbol "$\ll$".

For natural $k > 1$ we define the generalized Kloosterman sum

$$\widetilde{K}(\alpha, \beta; h, q; k) := \sum_{\substack{x,y \in \mathbb{G}_q\\ N(xy) \equiv h (mod\, q)}} e_q\left(\frac{1}{2}Sp(\alpha x^k + \beta y^k)\right).$$

Theorem 2. *Let p be irreducible, $h \in \mathbb{Z}$, $(h, p) = 1$, $k \in \mathbb{N}$, $t = (k, p-1)$. Then for any of integer numbers α, β, $(\alpha, \beta, p) = 1$ over the ring $\mathbb{Z}[\theta]$ the following estimate*

$$\left|\widetilde{K}(\alpha, \beta; h, p; k)\right| \ll \begin{cases} t^2 p^{\frac{3}{2}}, & if \quad t - 1 \leqslant \sqrt[4]{p}, \\ dp^2, & if \quad t \geqslant \sqrt[4]{p} + 1. \end{cases}$$

holds.

Theorem 3. *Let $\alpha, \beta \in \mathbb{Z}[\theta]$ and let $h, q, k, n \in \mathbb{N}$, $k \geqslant 2$, $(k, q) = (h, q) = 1$. Then for $(\alpha, q) = (\beta, q) = 1$ we have*

$$\widetilde{K}(\alpha, \beta; h, q; k) \ll D(k, q) q^{\frac{3}{2}},$$

where

$$D(k, q) = \prod_{\substack{p | q\\ p \equiv 1 (q)}} d^6(k, p) \cdot \prod_{\substack{p^n \| q\\ p \equiv 3 (q)}} d^3(k, p) \log p^n,$$

$$d(k, p) = (k, p-1).$$

On the derivations of some Leibniz algebras

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A linear transformation f of a Leibniz algebra L is called a *derivation*, if

$$f([a,b]) = [f(a),b] + [a,f(b)] \text{ for all } a,b \in L.$$

Let $\mathbf{Der}(L)$ be the subset of all derivations of L. It is possible to prove that $\mathbf{Der}(L)$ is a subalgebra of a Lie algebra $\mathbf{End}_F(L)$. $\mathbf{Der}(L)$ is called the *algebra of derivations* of a Leibniz algebra L.

The influence of algebra of derivations on the structure of Leibniz algebras is very essential. The next result shows it: if A is an ideal of a Leibniz algebra, then the factor-algebra of L by the annihilator of A is isomorphic to some subalgebra of $\mathbf{Der}(L)$ [1, Proposition 3.2]. In the paper [2] has been described the algebra of derivations of infinite dimensional cyclic Leibniz algebra. Here we show the description of algebra of derivations of nilpotent cyclic Leibniz algebra. It is an algebra L, having a basis $a_1,\dots,a_n$ satisfying the following conditions: $[a_1,a_1] = a_2$, $[a_1,a_{j\text{ı}1}] = a_j$, $3 \leqslant j \leqslant n$, $[a_1,a_n] = 0$, $[a_m,a_k] = 0$ for all $m > 1$, $1 \leqslant k \leqslant n$.

The algebra of derivations of L is isomorphic to the Lie algebra of matrices algebra $\mathbf{M}_n(F)$, consisting of the matrices having the following form

$$\begin{pmatrix}
\gamma_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\gamma_2 & 2\gamma_1 & 0 & 0 & \dots & 0 & 0 & 0 \\
\gamma_3 & \gamma_2 & 3\gamma_1 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \cdots & \gamma_2 & (n-1)\gamma_1 & 0 \\
\gamma_n & \gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \cdots & \gamma_3 & \gamma_2 & n\gamma_1
\end{pmatrix}$$

This algebra is a direct sum of abelian ideal, which is isomorphic to the subalgebra of $\mathbf{M}_n(F)$, consisting of the matrices having the following form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\gamma_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\gamma_3 & \gamma_2 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \cdots & \gamma_2 & 0 & 0 \\
\gamma_n & \gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \cdots & \gamma_3 & \gamma_2 & 0
\end{pmatrix}$$

and one-dimensional subalgebra.

I would like to say a special thank you to my supervisor, Kurdachenko L.A. His expertise and knowledge have been invaluable, I greatly appreciate his all-round support in my scientific endeavour.

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On classical prime subacts and classical Kato spectrum of classical duo-act

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Let S be a duo-monoid with zero, A is S-act.

Each right S-act is classical duo-act, so all its subacts are two-sided.

The set of all two-sided subacts of act A that are classical prime is called classical prime spectrum $CKSpec(A)$ of act A over monoid.

We define an almost Zariski topology on act A. Let B be subact of nonzero classical duo-act A and we define a classic variety $V(B)$ over B. $V(B)$ is the set of all classical prime subacts P of act A such that $N \subseteq P$. Then $V(N) = \varnothing$, $V(0) = CKSpec(A)$, $\bigcap_{i \in I} V(N_i) = V(\sum_{i \in I} N_i)$ for all $i \in I$, $V(N) \cup V(L) \subseteq V(N \cap L)$, $N, L, N_i \leqslant M$.

Let $C(A)$ be the family of all subsets $V(N)$ of set $CKSpec(A)$.

S-act A is called top-act if the set $C(A)$ is closed under finite unions, that is for any subacts N and L of act A exists subact K of act A such that $V(N) \cup V(L) = V(K)$.

Then $C(A)$ satisfies the axioms for closed subsets of topological space. All finite intersections of complements of sets in $C(A)$ are the base of open subsets of space $CKSpec(A)$.

Let X be a topological space. A subset $A \subset X$ is called a blob if there exists $a \in X$ such that A is the intersection of all open subsets of X which contain a.

Theorem. *A topological space X is homeomorphic to $CKSpec(A)$ for some top-duo-act A if and only if the following properties hold:*

(i) X is T_0-spase;

(ii) the set of open blobs of spase X is a base of X which contains X and is closed under finite intersections;

(iii) every intersection of irreducible closed subsets of space X is the closure of a unique point and X also satisfies condition: if $\{U_\lambda : \lambda \in \Lambda \subset U\}$ is a collection of open blobs in X and U is an open set with $\bigcap_\lambda U_\lambda \subset U$, then there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $\bigcap_{i=1}^{n} U_{\lambda_i} \subset U$.

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Growth functions of algebras

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This is a joint work with Be'eri Greenfeld. We will discuss growth functions of nil algebras and growth functions that oscillate between two functions. As an application we answer some questions about multiplicativity of Gelfand–Kirillov dimension.

The least n-nilpotent dimonoid congruences on the free trioid

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The notion of a trioid first appeared in the work of J.-L. Loday and M. O. Ronco [1] in the context of algebraic topology. Recall the construction of the free trioid.

As usual, $\mathbb{N}$ denotes the set of all positive integers. Let X be an arbitrary nonempty set, and let $F[X]$ be the free semigroup on X. For every word ω over X the length of ω is denoted by ℓ_ω. For any $n, k \in \mathbb{N}$ and $L \subseteq \{1, 2, ..., n\}, L \neq \varnothing$, we let $L + k = \{m + k \,|\, m \in L\}$. Define operations $\dashv$, $\vdash$, and $\perp$ on the set

$$F = \{(w, L) \,|\, w \in F[X],\ L \subseteq \{1, 2, ..., \ell_w\},\ L \neq \varnothing\}$$

by

$$(w, L) \dashv (u, R) = (wu, L), \quad (w, L) \vdash (u, R) = (wu, R + \ell_w),$$
$$(w, L) \perp (u, R) = (wu, L \cup (R + \ell_w))$$

for all $(w, L), (u, R) \in F$. By Lemma 7.1 and Theorem 7.1 from [2], the algebra $(F, \dashv, \vdash, \perp)$ is the free trioid.

If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that two operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and it is a dimonoid, we say that ρ is a dimonoid congruence [3]. A dimonoid congruence ρ on a trioid $(T, \dashv, \vdash, \perp)$ is called a $d^{\perp}_{\dashv}$-congruence (respectively, $d^{\perp}_{\vdash}$-congruence) [3] if the operations $\dashv$ and $\perp$ (respectively, $\vdash$ and $\perp$) of $(T, \dashv, \vdash, \perp)/\rho$ coincide. A dimonoid congruence ρ on a trioid $(T, \dashv, \vdash, \perp)$ will be called n-nilpotent if $(T, \dashv, \vdash, \perp)/\rho$ is an n-nilpotent dimonoid [4]. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that the operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and $(T, \dashv, \vdash, \perp)/\rho$ is an n-nilpotent semigroup, we say that ρ is an n-nilpotent semigroup congruence.

We characterize the least n-nilpotent $d^{\perp}_{\dashv}$-congruence, the least n-nilpotent $d^{\perp}_{\vdash}$-congruence and the least n-nilpotent semigroup congruence on the free trioid.

The author was supported by a Special Research Fellowship of the Erwin Schrödinger International Institute for Mathematics and Physics at the University of Vienna.

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On endomorphisms of free g-dimonoids of rank 1

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An algebraic system $(D, \dashv, \vdash)$ with two binary operations $\dashv$ and $\vdash$ is called a *g-dimonoid* [1] if for all $x, y, z \in D$ the following conditions hold:

$$(x \dashv y) \dashv z = x \dashv (y \dashv z),$$

$$(x \dashv y) \dashv z = x \dashv (y \vdash z),$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z),$$

$$(x \vdash y) \vdash z = x \vdash (y \vdash z).$$

It is clear that g-dimonoids are a generalization of dimonoids [2]. A construction of the free g-dimonoid was described in [1], in particular, a monogenic case was given separately.

Let e be an arbitrary symbol and $E = \{0, 1\}$. Take a natural number $n > 1$ and put

$$I^1 = \{e\}, \;\; I^n = E^{n-1} = \underbrace{E \times E \times ... \times E}_{n-1}, \;\; I = \bigcup_{m \geqslant 1} I^m.$$

Define operations $\dashv$ and $\vdash$ on the set I as follows:

$$(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n-1}) \dashv (\theta_1, \theta_2, ..., \theta_{m-1}) = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n-1}, \underbrace{1, 1, ..., 1}_{m}),$$

$$(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n-1}) \vdash (\theta_1, \theta_2, ..., \theta_{m-1}) = (\theta_1, \theta_2, ..., \theta_{m-1}, \underbrace{0, 0, ..., 0}_{n}).$$

The algebra $(I, \dashv, \vdash)$ is a g-dimonoid isomorphic to the free monogenic g-dimonoid [1]. We study endomorphisms of free monogenic g-dimonoids and construct a semigroup which is isomorphic to the endomorphism semigroup of the free monogenic g-dimonoid. The similar problem for free dimonoids of rank 1 was considered in [3].

The first author was supported by a Special Research Fellowship of the Erwin Schrödinger International Institute for Mathematics and Physics at the University of Vienna.

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Нові тотожності для симетричних многочленів Шура

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Нехай $\mathcal{P}_n$ – множина всіх розбиттів довжини не більше n. Розбиття $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ є впорядкований за спаданням набір невід'ємних цілих чисел. Розглянемо частковий порядок $\leqslant$ на $\mathcal{P}_n$ поклавши $\lambda \leqslant \mu$ якщо $\lambda_i \leqslant \mu_i$ для всіх i.

Многочлен Шура $\boldsymbol{s}_\lambda(\boldsymbol{x})$, що відповідає розбиттю $\lambda \in \mathcal{P}_n$ є многочленом від змінних $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ який визначається наступним чином (див. [1], [2]):

$$\boldsymbol{s}_\lambda(\boldsymbol{x}) = \frac{\det(x_j^{\lambda_i+n-i})}{\det(x_j^{n-i})} = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \ldots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_1+n-2} & x_2^{\lambda_1+n-2} & \ldots & x_n^{\lambda_1+n-2} \\ \vdots & \vdots & \ldots & \vdots \\ x_1^{\lambda_1} & x_2^{\lambda_1} & \ldots & x_n^{\lambda_1} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & 1 & \ldots & 1 \end{vmatrix}}.$$

Справедлива наступна теорема

Теорема.

1. Нехай $f_i = f_i(\boldsymbol{y})$ довільна сім'я многочленів і $\sum\limits_{i=0}^{\infty} f_i z^i = F(\boldsymbol{y}, z)$. Тоді
$$\sum_{\lambda \in \mathcal{P}_n} \boldsymbol{s}_\lambda(\boldsymbol{x}) \begin{vmatrix} f_{\lambda_1} & f_{\lambda_1+1} & f_{\lambda_1+2} & \ldots & f_{\lambda_1+n-1} \\ f_{\lambda_2-1} & f_{\lambda_2} & f_{\lambda_2+1} & \ldots & f_{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ f_{\lambda_n-(n-1)} & f_{\lambda_n-(n-2)} & f_{\lambda_n-(n-3)} & \ldots & f_{\lambda_n} \end{vmatrix} = \prod_{i=1}^{n} F(\boldsymbol{y}, x_i).$$

2. Нехай $t_1, t_2, \ldots, t_n$ деякий набір змінних. Тоді
$$\sum_{\lambda \in \mathcal{P}_n} \boldsymbol{s}_\lambda \det(t_j^{\lambda_i+j-i}) = \frac{\det\left(\dfrac{x_j^{n-i}}{1 - x_j t_i}\right)}{\det(x_j^{n-i})}.$$

3. Нехай $\sum\limits_{i=0}^{a} f_i z^i = F(\boldsymbol{y}, z, a), a \in \mathbb{N}$. Тоді
$$\sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda \leqslant (a_1, a_2, \ldots, a_n)}} \boldsymbol{s}_\lambda(\boldsymbol{x}) \det(f_{\lambda_i-i+j}) = \frac{\det\left(x_j^{n-i} F(\boldsymbol{y}, x_i, a_i)\right)}{\det(x_j^{n-i})}.$$

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Узагальнення теореми Де Марко-Орсатті для ко-мультиплікаційних та вторинно-мультиплікаційних модулів

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Нехай R – асоціативне кільце з $1 \neq 0$, M – лівий R-модуль. Той факт, що N є підмодулем M позначимо як $N \leqslant M$.

Означення 1. *R-модуль M називається* ***вторинним модулем****, якщо $M \neq 0$ і ${}_RAnn(M) ={}_R Ann(M/N)$ для кожного власного підмодуля $N \leqslant M$.*

Означення 2. *Підмодуль N лівого R-модуля M називається* ***вторинним підмодулем****, якщо він сам по собі є вторинним модулем.*

Множину усіх вторинних підмодулів модуля M позначаємо через $Spec^s(M)$ і називаємо **вторинним спектром** модуля M.

Означення 3. *Модуль M називається* ***мультиплікаційним модулем****, якщо для кожного $N \leqslant M$ існує такий двосторонній ідеал I кільця R, що $N = MI$.*

Означення 4. *Модуль M називається* ***ко-мультиплікаційним модулем****, якщо для кожного підмодуля $N \leqslant M$ існує такий двосторонній ідеал I кільця R, що $N = (0 :_M I)$, де $(0 :_M I) = \{m \in M \mid Im = 0\}$.*

Означення 5. *Модуль M називається* ***вторинно-мультиплікаційним модулем (s-мультиплікаційним модулем)****, якщо або M не має жодних вторинних підмодулів, або для кожного вторинного підмодуля $S \leqslant M$ існує такий двосторонній ідеал I кільця R, що $S = MI$.*

Означення 6. *Модуль M називається sm-модулем, якщо кожен вторинній підмодуль модуля M міститься в єдиному максимальному підмодулі.*

Теорема 1. *Нехай M вторинно-мультиплікаційний модуль. Відображення $\Phi\colon Spec^s(M) \to Max(M)$, котре кожному вторинному підмодулю M ставить у відповідність максимальний підмодуль, котрий його містить, є неперервним та сюр'єктивним.*

Теорема 2. *Нехай M ко-мультиплікаційний модуль і $Max(M)$ є ретрактом простору $Spec^s(M)$. Тоді M є sm-модулем.*

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Кільця ω-евклідового рангу 1

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Нехай R — комутативне кільце з відмінною від нуля одиницею. Під *елементарними матрицями* з елементами кільця R розуміємо квадратні матриці таких трьох типів: матриці, відмінні від одиничної наявністю деякого ненульового елемента поза головною діагоналлю; діагональні матриці з оборотними елементами на головній діагоналі; матриці перестановки, тобто матриці, які отримуються з одиничної перестановкою двох рядків чи стовпчиків. Множину усіх елементарних матриць другого порядку з елементами кільця R позначимо через $GE_2(R)$. Якщо для довільних елементів $a, b \in R$ існують такий елемент $d \in R$ і така матриця $P \in GE_2(R)$, що $(a, b)P = (d, 0)$, то кільце R називають *елементарно головним* [1]. *Норму* над кільцем R визначимо як функцію φ: $R \to \mathbb{N} \cup \{0\}$, яка задовольняє умовам $\varphi(0) = 0$, $\varphi(a) > 0$ для будь-якого $a \in R\backslash\{0\}$, $\varphi(ab) > \varphi(a)$ для довільних $a, b \in R$ таких, що $ab \neq 0$. Елемент a кільця R називається *ω-евклідовим*, якщо для довільного ненульового елемента b цього кільця існують норма φ та послідовність рівностей $a = bq_1 + r_1$, $b = r_1q_2 + r_2$, $r_1 = r_2q_3 + r_3$, $r_{k-2} = r_{k-1}q_k + r_k$, такі, що $\varphi(r_k) < \varphi(b)$ для деякого натурального k. Кільце R називається *кільцем ω-евклідового рангу* 1, якщо для довільних елементів $a, b \in R$, де $aR + bR = R$, існує такий елемент $y \in R$, що $a + by$ — ω-евклідовий елемент.

Теорема 1. *Якщо R — кільце ω-евклідового рангу* 1*, то для довільних елементів $a, b \in R$ таких, що $aR + bR = R$, існують такі елемент $d \in R$ і матриця $P \in GE_2(R)$, що $(a, b)P = (d, 0)$.*

Теорема 2. *Якщо R — кільце ω-евклідового рангу* 1*, для будь-яких елементів $a, b \in R$ таких, що $aR + bR = R$, і довільного ненульового елемента $c \in R$ існують такі елементи $y, d \in R$ і матриця $P \in GE_2(R)$, що $(a + by, c)P = (d, 0)$.*

Кільце, в якому довільний скінченнопороджений ідеал є головним, називають *кільцем Безу* [3].

Теорема 3. *Кільце Безу ω-евклідового рангу* 1 *є елементарно головним.*

Теорема 4. *Довільна оборотна матриця над кільцем ω-евклідового рангу* 1 *розкладається у скінченний добуток елементарних матриць.*

Кільце R називають *кільцем Ерміта* [4], якщо для довільних елементів $a, b \in R$ існують такий елемент $d \in R$ і така оборотна матриця Q другого порядку, що $(a, b)Q = (d, 0)$.

Теорема 5. *Кільце Безу ω-евклідового рангу* 1 *є кільцем Ерміта.*

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Групи неперервних перетворень відрізка, пов'язані з різними системами кодування дійсних чисел, і їх фрактальні підгрупи

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Традиційно перетворенням відрізка $I = [0;1]$ називається бієктивне відображення цього відрізка на себе. Відомо, що множина G всіх перетворень відрізка відносно операції «композиція» (суперпозиція) утворює групу (групу перетворень відрізка I). Неперервним перетворенням відрізка є неперервна функція, визначена на цьому відрізку, яка строго зростає або строго спадає. Множина C всіх неперервних перетворень відрізка $I = [0;1]$ є нескінченною підгрупою групи G.

Нехай A — алфавіт (набір цифр), скінченний або нескінченний; $L = A \times A \times \ldots$ — простір послідовностей елементів алфавіту. *Кодуванням (зображенням)* дійсних чисел відрізка I засобами алфавіту A називається сюр'єктивне відображення $\varphi : L \to I$, а саме

$$L \ni (\alpha_n) \xrightarrow{\varphi} x = \Delta^{\varphi}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \in I.$$

При цьому множина $\Delta^{L}_{c_1c_2\ldots c_m} = \{(a_1, a_2, \ldots, a_n, \ldots) : a_i = c_i, \ i = m\}$ називається *циліндром рангу m з основою $c_1c_2\ldots c_m$* у просторі L. *Образ* $\Delta^{\varphi}_{c_1c_2\ldots c_m} = \varphi\left(\Delta^{L}_{c_1c_2\ldots c_m}\right)$ циліндра $\Delta^{L}_{c_1c_2\ldots c_m}$ при відображенні φ називається циліндром рангу m з основою $c_1c_2\ldots c_m$ у множині I. Послідовність $(\alpha_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) \in L$, яка відповідає числу x, називається його *φ-зображенням*, а α_n — *n-ою цифрою* цього зображення і записується $x = \Delta^{\varphi}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$. Кажуть, що зображення має *нульову (екстранульову) надлишковість*, якщо кожне число має не більше двох зображень (має єдине зображення).

Казатимемо, що *функція $y = f(x)$ зберігає хвости φ*-зображення чисел відрізка I, якщо для будь-якого $x = \Delta^{\varphi}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ і його образу $y = f(x) = \Delta^{\varphi}_{\beta_1\beta_2\ldots\beta_n\ldots}$ існують невід'ємні цілі числа k і m такі, що $\alpha_{k+j} = \beta_{m+j}$ для будь–якого $j \in N$.

У доповіді представляються результати дослідження груп перетворень відрізка $[0;1]$, які пов'язані з різними системами кодування чисел зі скінченним та нескінченним алфавітами. Основна увага приділяється двосимвольним системам кодування чисел. Серед головних інваріантів, що визначають підгрупу групи перетворень є:

1) фрактальна розмірність Гаусдорфа–Безиковича, 2) хвости зображення чисел, 3) частоти цифр, 4) нормальні властивості зображення чисел, 5) параметри динамічних систем.

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Підгрупи групи функцій, означених в термінах Q_2-зображення дійсних чисел

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Нехай $A = \{0, 1\}$ — алфавіт, $L = A \times A \times \ldots$ — простір послідовностей елементів алфавіту, $q_0 \in (0; 1)$, $q_0 + q_1 = 1$. Тоді [1] для довільного $x \in [0; 1]$ існує $(\alpha_n) \in L$ така, що

$$x = \alpha_1 q_{1-\alpha_1} + \sum_{n=2}^{\infty} (\alpha_n (1 - q_{\alpha_n}) \prod_{j=1}^{n-1} q_{\alpha_j}) \equiv \Delta^{Q_2}_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots}.$$

Розклад числа x в такий ряд називається Q_2-представлення, а скорочений запис $\Delta^{Q_2}_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots}$ — його Q_2-зображенням. Існують числа, що мають два Q_2-зображення. Це числа виду $\Delta^{Q_2}_{\alpha_1 \ldots \alpha_{n-1} \alpha_n (0)} = \Delta^{Q_2}_{\alpha_1 \ldots \alpha_{n-1} [\alpha_n - 1](1)}$. Їх ми називаємо Q_2-бінарними числами. Решта чисел, що мають одне Q_2-зображення, ми називаємо Q_2-унарними.

Розглядається клас функцій $f_{(n_k)}$, означенних рівністю:

$$f_{(\varphi_n)}(x = \Delta^{Q_2}_{\alpha_1 \alpha_2 \ldots . \alpha_n \alpha_{n+1} \ldots}) = \Delta^{G_2}_{\beta_1 \beta_2 \ldots . \beta_n \ldots}, \tag{1}$$

де $\beta_n = \varphi_n(\alpha_n, \alpha_{n+1})$, $\varphi_n(A^2) \to A$, $\Delta^{Q_2}_{\alpha_1 \alpha_2 \ldots . \alpha_n \alpha_{n+1} \ldots}$ і $\Delta^{G_2}_{\beta_1 \beta_2 \ldots . \beta_n \ldots}$ — два Q_2-зображення (породжених параметрами q_0, g_0) аргумента і значення функції $f_{(\varphi_n)}$ [2].

Означення функції $f_{(\varphi_n)}$, рівністю (1) є не коректним. Домовившись не використовувати одне із зображень Q_2-бінарних чисел дану коректність усуваємо. Клас функцій $f_{(\varphi_n)}$, породжених параметром q_0, g_0 і послідовністю відображень (φ_n) (фіксованих та змінних) є континуальним.

Функцію φ_n зручно асоціювати з матрицею $\begin{pmatrix} a^{(n)}_{00} & a^{(n)}_{01} \\ a^{(n)}_{10} & a^{(n)}_{11} \end{pmatrix}$, елементами якої є числа 0 та 1, а саме: $a^{(n)}_{ij} = \varphi_n(i, j) \in A$. Тоді кожну з функцій $f_{\overline{\varphi}}$ можна ототожнювати з послідовністю матриць $M_k = \begin{pmatrix} a^{(k)}_{00} & a^{(k)}_{01} \\ a^{(k)}_{10} & a^{(k)}_{11} \end{pmatrix}$.

Теорема 1. *Множина функцій φ (f_φ) разом з операцією $\varphi_i * \varphi_j = |\varphi_i(a; b) - \varphi_j(a; b)|$, $(a; b) \in A^2$ ($f_{\varphi_i} \star f_{\varphi_j} = f_{\varphi_i * \varphi_j}$) утворює комутативну групу, нейтральним елементом якої є $\varphi(a; b) = 0$ ($f_\varphi = 0$), а оберненим кожен елемент сам до себе.*

Теорема 2. *Множина функцій f_φ, що зберігають хвости зображення, тобто існує $k, m \in N$ таке, що $\alpha_{k+n}(x) = \beta_{m+n}(y)$ для $n \in N$, разом з операцією $f_{\varphi_i} \star f_{\varphi_j} = f_{\varphi_i * \varphi_j}$ утворює підгрупу групи перетворень.*

У доповіді пропонуються результати дослідження структурних, фрактальних, диференціальних властивостей функцій підгруп групи функцій $f_{(\varphi_n)}$.

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Квазі-мономи відносно підгруп афінної групи площини

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Сім'я многочленів $\{B_{m,n}(x,y)\}$ називається *квазі-мономіальною* відносно підгрупи H афінної групи $\mathrm{Aff}(2)$ площини, якщо $\{B_{m,n}(x,y)\}$ утворює такий базис векторного простору многочленів від двох змінних, що в цьому базисі лінійні оператори, якими діє H, мають таку саму матрицю, яку вони мають в стандартному мономіальному базисі $\{x^m y^m\}$. В статті [1] автори довели, що сім'я многочленів $B_{m,n}(x,y) = H_m(x)H_n(x)$, де $H_n(x)$ – многочлени Ерміта, є квазі-мономіальною відносно групи обертань та групи паралельних перенесень площини. В [2] дано опис всіх сімей многочленів, квазі-мономіальних відносно обертань площини у термінах їхніх породжуючих функцій.

Для деяких інших підгруп афінної групи площини нами отримано схожий опис відповідних квазі-мономіальних сімей многочленів.

Теорема. Сім'я многочленів $\{B_{m,n}(x,y)\}$ визначена експоненціальною породжуючою функцією

$$G = \sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!}$$

є квазі-мономіальною відносно:

- *групи розтягів* тоді і тільки, коли G є функцією двох змінних xu, yv:

$$G = G\left(xu, yv\right);$$

- *групи трансляцій* тоді і тільки тоді, коли функція G має вигляд:

$$G = C(u,v)e^{xu+yv},$$

де C – довільний степеневий ряд від змінних u, v;

- *підгрупи породженої поворотами та рівномірними розтягами* тоді і тільки, коли G є функцією двох змінних $ux + vy$ та $(x^2+y^2)(u^2+v^2)$:

$$G = G(ux+vy, (x^2+y^2)(u^2+v^2));$$

- *підгрупи породженої поворотами та трансляціями* тоді і тільки тоді, коли функція G має вигляд:

$$G = C(u^2+v^2)e^{xu+yv},$$

де $C(u^2+v^2)$ довільний многочлен від u^2+v^2.

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Функція $R(n)$ на асимптотичній прогресії

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Майже 200 років тому К. Ф. Гаусс і Й. Діріхле почали вивчати проблему круга

$$\sum_{\substack{u,v\in\mathbb{Z}\\ u^2+v^2\leqslant x}} 1, \sum_{\substack{u,v\in\mathbb{N}\\ uv\leqslant x}} 1$$

(число точок з цілими координатами в крузі радіуса $x^{1/2}$ і, відповідно, число точок з цілими координатами в першій чверті під гіперболою $uv \leqslant x$).

Український математик Г. Ф. Вороний в 1908–1910 роках розробив аналітичні методи розв'язання цих задач. В 1959 році К. Хулі почав вивчати проблему дільників на арифметичній прогресії, а в 1968 році П. Варбанець вивчав задачу круга в арифметичній прогресії [1]. Отримані ними оцінки залишкових членів $O(x^{1/3}) + O(\frac{x^{1/2}}{q^{1/4}})$ були покращені П. Варбанцем в 2020 році у результаті збільшення області нетривіальності відповідних асиметричних формул.

В теперішній час виникло багато аналогів задач кола і дільників на спеціальних послідовностях. Ми вивчаємо аналог тривимірних задач кола і дільників в арифметичній прогресії $n \equiv l(modq), n \leqslant x$, коли знаменник прогресії q зростає разом з x до нескінченності.

Нехай $R(n)$ — кількість представлень n у вигляді $n = (u^2 + v^2)w$, де $u, v \in \mathbb{Z}, w \in \mathbb{N}$. Нашою метою є побудова асимптотичної формули для сум

$$\sum_{\substack{n\equiv l(modq)\\ n\leqslant x}} R(n), (x \to \infty).$$

Ця сума є аналогом суми значень тривимірної функції дільників $\tau_3(n) = \sum_{\substack{n=n_1n_2n_3\\ n_i\in\mathbb{N}}} 1$. Для середнього значення $\tau_3(n)$ в арифметичній прогресії Р. Хіт-Браун в 1986 році отримав оцінку залишкового члена в асимптотичній формулі суми $\sum_{\substack{n\equiv l(modq)\\ n\leqslant x}} \tau_3(n)$, нетривіальної в області $q \ll x^{1/2+1/81+2}$ [2].

В нашій роботі, використовуючи оцінку спеціальної тригонометричної суми(яке є узагальненням двовимірної суми Клостермана)

$$\widetilde{K}(a, b, c, q) = \sum_{(u^2+v^2)w\equiv l(modq)} \exp^{2\pi i \frac{au+bv+cw}{q}},$$

доведено асимптотичну формулу

$$\sum_{\substack{n\equiv l(modq)\\ n\leqslant x}} R(n) = \frac{\pi x A_0(l, q)}{q} \log x + \frac{A_1(l, q)}{q} x + \frac{A_2(l, q)}{q} x \log q + O(\frac{x^{1+\epsilon}}{q^{4/3}}) + O(\frac{x^{3/5+\epsilon}}{q^{1/5}}),$$

де $A_i(l, q), i = \overline{1, 3}$, — ненульові обчислювані функції від l та q, обмежені по абсолютному значенню числом 2.

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Σ-ФУНКЦІЇ НІЛЬПОТЕНТНИХ НАПІВГРУП

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Нехай $T : a \to T(a)$, $a \in S$ — матричне зображення над полем K напівгрупи S. Позначимо через $d(T)$ максимальне число вільних параметрів однорідної системи лінійних рівнянь $T(a)X = XT(a)$, де a пробігає S, відносно елементів матриці X, яке дорівнює розмірності алгебри ендоморфізмів $End_{Rep_K(S)}(T)$ зображення T в категорії $Rep_K(S)$ матричних зображень напівгрупи S. Якщо S — напівгрупа скінченного зображувального типу над K, тобто, за означенням, має скінченне число класів еквівалентності нерозкладних зображень, а $T = \{T_1, T_2, \ldots, T_m\}$ — множина представників усіх таких класів (яка називається хребтом категорії $Rep_K(S)$), то для $n \in [1, m] := \{1, 2, \ldots, m\}$ покладемо

$$d_n(T) := \sum_{i_1 < i_2 < \ldots < i_n} d(T_{i_1} \oplus T_{i_2} \oplus \ldots \oplus T_{i_n}), \quad \Sigma_{S,K}(n) := d_n(T).$$

Введена функція $\Sigma_{S,K} : [1, m] \to \mathbb{N}$ називається *Σ-функцією категорії $Rep_K(S)$* або *Σ-функцією напівгрупи S над K* [1].

Зауважимо, що однією із форм задання категорії зображень є алгебра Аусландера як алгебра ендоморфізмів зображення $T_0 = T_1 \oplus T_2 \oplus \cdots \oplus T_m$ для хребта $T = \{T_1, T_2, \ldots, T_m\}$. І, отже, $\Sigma_{S,K}(m)$ — розмірність цієї алгебри, а $\Sigma_{S,K}(i)$ для $i < m$ — комбінаторні характеристики її канонічних підалгебр.

Приклад. Нехай $S_1^{(2)}$ — циклічна напівгрупа, породжена елементом a таким, що $a^2 = 0$. За об'єкти хребта категорії $Rep_K S$ візьмемо клітки Жордана розміру 1×1 і 2×2 з власним числом 0. Тоді матрична алгебра Аусландера складається з матриць вигляду

$$X = \left(\begin{array}{c|cc} x_{11} & 0 & x_{13} \\ \hline x_{21} & x_{22} & x_{23} \\ 0 & 0 & x_{22} \end{array}\right),$$

де $x_{11}, x_{13}, x_{21}, x_{22}, x_{23}$ пробігають поле K, і $\Sigma_{S,K}(1) = 3$, $\Sigma_{S,K}(2) = 5$.

Наступна теорема описує Σ-функцію довільної циклічної нільпотентної напівгрупи.

Теорема. *Σ-функція напівгрупи $S = S_1^{(m)} = \{a | a^m = 0\}, m \geqslant 1$, над довільним полем K задається наступною формулою:*

$$\Sigma_{S,K}(n) = \begin{cases} \frac{m(m+1)}{2}, & \textit{якщо} \quad n = 1; \\ C_{m-1}^{n-1} \frac{m(m+1)}{2} + C_{m-2}^{n-2} \frac{(m-1)m(m+1)}{3}, & \textit{якщо} \quad 1 < n < m; \\ \frac{m(m+1)(2m+1)}{6}, & \textit{якщо} \quad n = m. \end{cases}$$

Результати отримано у співавторстві з проф. В. М. Бондаренком.

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International Algebraic Conference “At the End of the Year” 2022

December 27 – 28, 2022

Kyiv, Ukraine

ABSTRACTS

Kyiv — 2022

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Міжнародна алгебраїчна конференція “Під кінець року” 2022

27 – 28 грудня 2022 р.

Київ, Україна

ТЕЗИ ДОПОВІДЕЙ

Київ — 2022

Комп’ютерна верстка
І. Ю. Раєвська, М. Ю. Раєвська

Відповідальні за випуск
А. П. Петравчук, І. Ю. Раєвська, М. Ю. Раєвська