Taras Shevchenko National University of Kyiv

Institute of Mathematics of NAS of Ukraine

National University of Kyiv-Mohyla Academy

11th International Algebraic Conference in Ukraine dedicated to the 75th anniversary of V. V. Kirichenko

July 3 – 7, 2017

Kyiv, Ukraine

ABSTRACTS

Kyiv — 2017

Київський національний університет імені Тараса Шевченка

Інститут математики НАН України

Національний університет "Києво-Могилянська академія"

11-а Міжнародна алгебраїчна конференція в Україні, присвячена 75-річчю В. В. Кириченка

3 – 7 липня 2017 р.

Київ, Україна

ТЕЗИ ДОПОВІДЕЙ

Київ — 2017

11th International Algebraic Conference in Ukraine dedicated to the 75th anniversary of V. V. Kirichenko. July 3 – 7, 2017, Kyiv, Ukraine. Abstracts. — Kyiv: Institute of Mathematics of NAS of Ukraine, 2017. — 159 p.

11-а Міжнародна алгебраїчна конференція в Україні, присвячена 75-річчю В. В. Кириченка. 3 – 7 липня 2017 р., Київ, Україна. Тези доповідей. — Київ: Інститут математики НАН України, 2017. — 159 с.

Program Committee

Гимн Алгебраистов

С. Д. Берман и Л. А. Калужнин

Поля и кольца, группы и структуры
Владеть всей математикой должны,
Так не нужны пределы, квадратуры,
И интегралы тоже не нужны!

Припев: Отбросив с презреньем дифуры
Вперед устремляем свой путь.
И общие наши структуры
Дают несомненную муть.

Не знаем мы про интеграл двукратный,
Но категорьей нас не испугать.
И применяя аппарат абстрактный
Любой абсурд иы можем доказать.

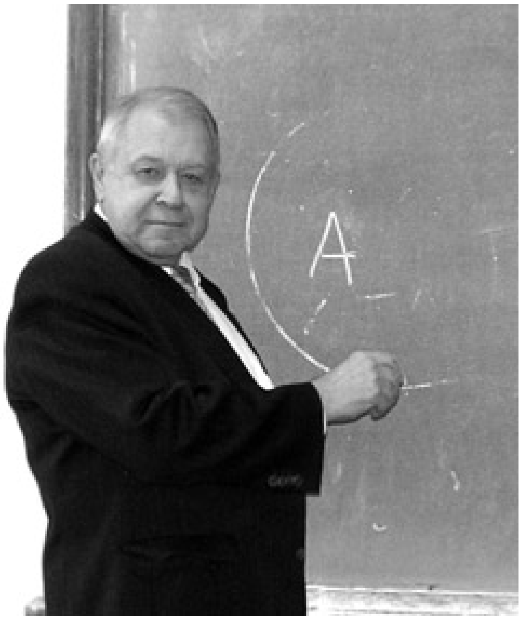
Припев.

чиста сторінка

C O N T E N T S

З М І С Т

Volodymyr Kirichenko



The famous Ukrainian mathematician Volodymyr Kirichenko was born on June 17, 1942. In 1959 he entered the Department of Mechanics and Mathematics of the Kyiv Taras Shevchenko University. In 1961 he joined the group of young mathematicians led by Andrei Roiter and participated the seminars on representation theory and homological algebra. These field of algebra became the area of his own research. In 1964 he graduated with honour from the University and entered the Ph. D. Program at the Institute of Mathematics of the Academy of Sciences of Ukraine. His advisor there was the well-known mathematician Dmitriy Faddeev. In 1967 Volodymyr Kirichenko defended the Ph. D. Thesis "Representations of Hereditary, Fully Decomposable and Bassian Orders" and began his job at the Chair of Algebra and Mathematical Logic of the Department of Mathematics of the Kyiv Taras Shevchenko University. Then he became docent and professor, in 1986 he defended the Doctoral Thesis "Modules and the Structure Theory of Rings." In 1988 he became the Head of the Chair of Geometry.

Volodymyr Kirichenko is known for his deep and original results in the theory of rings and modules. He started from the theory of integral representations of structure of orders. Then his interests were in the theory of semi-chain rings, where he obtained important results about the structure of such rings and modules over them. In particular, he proved the Skornyakov conjecture about semi-chain rings and modules for right noetherian rings. He also developed a new trend in the structure theory of rings concerning their relations to quivers (oriented graphs). In particular, he actively elaborated the notion of the prime quiver of a ring. His results were highly appreciated both in Ukraine and abroad. In 2007 he was awarded with a group of scientists by the State Prize of Ukraine for the series of woks "Representations of Algebraic Structures and Matrix Problems in Linear and Hilbert Spaces". He is the author of several books on structure of algebras and rings, namely:

Finite Dimensional Algebras (with Yu. Drozd), Vyscha Shkola, Kyiv, 1980 (English revised translation: Springer, 1994). (This book is also translated to Chinese and Spanish.)

Rings and Modules (with N. Gubareni), Politechnika Czestochowska, 2001.

Algebras, Rings and Modules (with N. Gubareni and M. Hasewinkel), Springer, vol. 1 — 2004, vol. 2 — 2007.

Algebras, Rings and Modules. Lie Algebras and Hopf Algebras. (with N. Gubareni and M. Hasewinkel), AMS, 2010.

Volodymyr Kirichenko was also a brilliant teacher who opened the way to mathematics for a lot of students. 30 of them defended Ph. D. Theses with Volodymyr Kirichenko as advisor and 5 of them defended Doctoral Theses.

On representation varieties of some HNN extensions of free groups

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Let $G = \langle x_1, x_2, \ldots, x_m \rangle$ be a finitely generated group and K be an algebraically closed field with char $K = 0$. The set $Hom(G, GLn(K))$ has a natural structure of an affine K-variety and is denoted by $R_n(G)$. It is called the representation variety of the group G. The group $GL_n(K)$ acts on $R_n(G)$ by conjugation and the corresponding category factor $R_n(G)//G$ is denoted by $X_n(G)$ and called the character variety of the group G [1].

We consider representation and character varieties of one-relator groups with presentation

$$G = \langle x_1, x_2, \ldots, x_g, t | t(x_1^2 x_2^2 \ldots x_g^2)^p t^{-1} = (x_1^2 x_2^2 \ldots x_g^2)^q \rangle, \tag{1}$$

where $g \geqslant 3$ and $p > |q| \geqslant 1$. Let us denote $d = (p, q)$ and let $\Omega(p, q)$ be the set of matrices A such that A^p and A^q are conjugate. For $A \in \Omega(p, q)$ let us consider varieties

$$L(A) = \langle (x_1, x_2, \ldots, x_g) \in GL_n(K)^g | (x_1^2 x_2^2 \ldots x_g^2) = A \rangle. \tag{2}$$

From results of [2] it is not difficult to obtain the description of $L(A)$. Let n_A be the number of irreducible components of $L(A)$. If $n = 2, g = 3$ and A is scalar, then $n_A = 3$, otherwise $n_A = 2$. Moreover, each irreducible component of $L(A)$ is a rational variety.

For a matrix A we denote by $Z(A)$ its centralizer. Let t_0 be some fixed matrix with $t_0 A^p t_0^{-1} = A^q$. Consider the following morphisms:

$$f_{A,t_0,i} : L_i(A) \times Z(A^d) \times GL_n(K) \to R_n(G), (x_1, x_2, \ldots, x_g, z, T) \mapsto T(x_1, x_2, \ldots, x_g, t_0 z)T^{-1}.$$

It is easy to see that $Imf_{A,t_0,i}$ does not depend on t_0 and $Imf_{A,t_0,i} \subset R_n(G)$. By $W_i(A)$ we denote the Zariski closure of $Imf_{A,t_0,i}$. The following theorems hold.

Theorem 1. *Each variety $W_i(A)$ is an irreducible component of $R_n(G)$ and all irreducible components of $R_n(G)$ are exhausted by $W_i(A)$, where $A \in \Omega(p, q), i = \overline{1, n_A}$. Moreover, each irreducible component of $R_n(G)$ is a rational variety.*

1) If $(n, g) \neq (2, 3)$, then $dimW_i(A) = gn^2 + dimZ(A^d) - dimZ(A)$. The number of irreducible components of $R_n(G)$ is exactly twice more than the number of conjugacy classes in $\Omega(p, q)$.

2) If $n = 2, g = 3$, then $R_2(G)$ consists of $(p - q)(2p - 2d + 5)$ irreducible components of dimension 12 and $(p - q)(d - 1)$ irreducible components of dimension 14.

Let $\pi : R_n(G) \to X_n(G)$ be the factorization morphism.

Theorem 2. *All irreducible components of $X_n(G)$ are exhausted by $\pi(W_i(A))$, where $A \in \Omega(p, q)$, $i = \overline{1, n_A}$.*

1) If $(n, g) \neq (2, 3)$, then each irreducible component $\pi(W_i(A))$ of $X_n(G)$ has dimension $(g - 1)n^2 + 1 + dimZ(A^d) - dimZ(A)$. The number of irreducible components in $X_n(G)$ is exactly twice more than the number of conjugacy classes in $\Omega(p, q)$.

2) If $n = 2, g = 3$, then $X_2(G)$ consists of $(p-q)(2p-2d+5)$ 9-dimensional and $(p-q)(d-1)$ 11-dimensional irreducible components.

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Cube complementary graphs

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All graphs considered are finite simple Graphs. A graph G is called a cube complementary graph if its complement is isomorphic to it cube (cc-graph).

Examples of cc-graphs will be provided. Ways of constructing new graphs out of given ones will be given. Some important properties of cc-graphs will be proved. Some open problems on cc-graphs will be given.

Finally, further properties of regular cc-graphs will be given.

Classification problem for bimodule problems with quasi multiplicative basis

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We have introduced the class $\mathcal{C}$ of a faithful connected finite dimensional one-sided bimodule problems $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ endowed with a quasi multiplicative basis Σ ([1]). By class definition, we exclude from consideration bimodule problems that contain a subproblem from a given list $\mathcal{A}^i = (\mathsf{K}^i, \mathsf{V}^i)$, $i = 1, 2, 3, 4$, with $|\operatorname{Ob} \mathsf{K}| \leqslant 7$ having strictly unbounded type.

The construction of quasi multiplicative basis allows to solve classification problem for a class $\mathcal{C}$ by means of universal covering method [2, 3]. For a bimodule problem $\mathcal{A} \in \mathcal{C}$ the constructed universal covering $\tilde{\mathcal{A}}$ is simply connected, i. e. it is connected and its fundamental group is trivial.

A bimodule problem is called an *infinite line* if its basic bigraph contains infinite solid line without dotted arrows (i. e. corresponding subproblem has trivial category).

Theorem. *There exists the list of critical bimodule problems from the class $\mathcal{C}$ with at most 9 vertices such that bimodule problem $\mathcal{A} \in \mathcal{C}$ is of finite representation type if and only if $\mathcal{A}$ does not include any problem from the given list as a subproblem, and universal covering of $\mathcal{A}$ does not contain infinite line.*

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Homological classification of graded rings

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The properties of modules over a ring affect the properties of the ring, and in some cases they characterize this ring [1]. For the structure theory of graded rings the characterization of graded rings using homological properties of the category of graded modules over them is very important.

In recent years, a number of results that establish a connection between the properties of an associative ring graded by a group and the properties of graded modules over this ring are known. Throughout, following standard practice, the graded analogue of standard definition will be denoted by the prefix "gr-".

The ring R is a graded division ring if and only if all right (left) graded R-modules are gr-free [2].

There is a homological classification of graded semisimple rings [3], regular [4] and semi-perfect rings [5].

Theorem 1. *For graded ring R the following statements are equivalent: (1) R is right gr-Noetherian; (2) each finite generated right graded R-module is gr-Noetherian; (3) direct sum of rigth gr-injective R-modules is gr-injective.*

A ring R is called gr-quasi-Frobenius if it is left and right gr-Artinian and each its one-sided graded ideal is annihilator.

Theorem 2. *For graded ring R the following statements are equivalent: (1) R is gr-quasi-Frobenius; (2) each rigth gr-injective R-module is gr-projective; (3) each rigth gr-flat R-module is gr-injective; (4) each rigth gr-projective R-module is gr-injective.*

This research was supported by Russian Foundation for Basic Research, grant 15-01-01540a.

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On finite groups factorizable by permutable subgroups

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We consider finite groups only. A group G is said to be the product of its pairwise permutable subgroups $A_1, A_2, \dots, A_n$, if $G = A_1 A_2 \cdots A_n$ and $A_i A_j = A_j A_i$ for all integer i and j with $1 \leqslant i, j \leqslant n$. In this case, for every choice of indices $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n$ the product $A_{i_1} A_{i_2} \cdots A_{i_k}$ is a subgroup of the group G.

The groups introduced into the product of their pairwise permutable subgroups have been studied by many authors. By a well-known theorem of P. Hall a finite group is soluble if and only if it is the product of pairwise permutable Sylow subgroups. In 1958–1962 Wielandt and Kegel proved that every finite group which factorized as a product of pairwise permutable nilpotent subgroups is soluble.

Let the group $G = A_1 A_2 \cdots A_n$ be a product of its pairwise permutable subgroups $A_1, A_2, \dots, A_n$. Huppert [1, Theorem VI, 10.2] showed that G is supersolvable if every product $A_i A_j A_k$ supersoluble. In [2] L. S. Kazarin established that if each product $A_i A_j$ soluble, then G is soluble. In this work we continue our research in this direction.

Let $\mathbb{P}$ be the set of all prime numbers. A subgroup H of a group G is called $\mathbb{P}$-subnormal in G whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for every $i = 1, \dots, n$. A group G is called widely supersoluble (briefly, w-supersolvable) if every Sylow subgroup of G is $\mathbb{P}$-subnormal in G. A generalized commutant of a group G is called the smallest normal subgroup N of G such that G/N is a group with abelian Sylow subgroups [3].

Theorem 1. *Let the group $G = A_1 A_2 \cdots A_n$ be the product of pairwise permutable subgroups $A_1, A_2, \dots, A_n$. If $A_i A_j$ is* w*-supersoluble for any $1 \leqslant i, j \leqslant n$ and the generalized commutator subgroup of G is nilpotent, then G is* w*-supersoluble.*

Corollary. *Let the group $G = A_1 A_2 \cdots A_n$ be the product of pairwise permutable subgroups $A_1, A_2, \dots, A_n$. If $A_i A_j$ is supersoluble for any $1 \leqslant i, j \leqslant n$ and the generalized commutator subgroup of G is nilpotent, then G is* w*-supersoluble.*

Theorem 2. *Let the group $G = A_1 A_2 \cdots A_n$ be the product of pairwise permutable subgroups $A_1, A_2, \dots, A_n$. If $A_i A_j$ is supersoluble for all $1 \leqslant i, j \leqslant n$ and derived subgroup G' is nilpotent, then G is supersoluble.*

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On finite dispersive groups with complemented nonmetacyclic subgroups

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A metacyclic group is a group being an extension of a cyclic (in particular, a unit) group by means of a cyclic group. Influence of the properties of metacyclic (nonmetacyclic) subgroups on the structure of the whole group was investigated by V. S. Monakhov [1], V. I. Kovalenko [2], author [3] and others. For example, in [1] it was proved that a finite group, which is the product of two of its subgroups containing cyclic subgroups of indices 1 or 2 is solvable. As a property that all proper nonmetacyclic subgroups of the studied group have, for example, normality [2] and complementarity [3] were chosen.

The problem whether nonmetacyclic subgroups are complementary arose because in the case of a finite group for all its subgroups to be complementary (that is, the group is completely factorizable) it suffices that only elementary Abelian or even cyclic elementary Abelian subgroups are complementary (see [4]). As for finite groups with complemented nonmetacyclic subgroups, they may not be completely factorizable and even nondispersive (see [3]). The following theorem deals with to finite dispersive groups with complemented nonmetacyclic subgroups.

Theorem. *Let G be a finite dispersive nonmetacyclic group with complemented nonmetacyclic subgroups. Let P be a non-Abelian normal Sylow p-subgroup of G. If the group G is not p-decomposable, then $p = 2$, P is the quaternion group of order 8, and the Sylow subgroups of G with respect to the numbers $q \neq p$ are abelian.*

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Generalized Weyl algebras and diskew polynomial rings

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The aim of the talk is to extend the class of generalized Weyl algebras to a larger class of rings (they are also called generalized Weyl algebras) that are determined by two ring endomorphisms rather than one as in the case of 'old' GWAs. A new class of rings, the diskew polynomial rings, is introduced that is closely related to GWAs (they are GWAs under a mild condition). Semisimplicity criteria are given for generalized Weyl algebras and diskew polynomial rings.

THE STAR AND FREQUENTLY SEQUENCES OF SIMPLE GRAPHS

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Let S_k be the star graph defined as the complete bipartite graph $K_{1,k}$. Denote by $S_k(G)$ the number of subgraphs of G that are isomorphic to the star S_k. The sequence

$$2S_1(G), S_2(G), \ldots, S_{n-1}(G)$$

is called the *star sequence* of a graph G.

Let f_i denote the number of vertices of degree $i, i = 1, \ldots, n-1$. The integer sequence

$$f_1, f_2, \ldots, f_{n-1},$$

is called *the frequency sequence* of a graph. There exists a close connection between the star sequence and the frequency sequence of a graph.

Let us recall that two integer sequences $\{a_n\}, \{b_n\}$ is called an *inverse pair* if the following relations hold

$$a_i = \sum_{k=i}^{n} \binom{k}{i} b_k, b_i = \sum_{k=i}^{n} (-1)^{k-i} \binom{k}{i} a_k.$$

The following theorems hold

Theorem 1. *Let G be a simple graph. Then its star and frequency sequences are an inverse pair:*

$$f_i = \sum_{k=i}^{n-1} (-1)^{k-i} \binom{k}{i} S_k(G), 1 < i \leqslant n-1,$$

$$f_1 = S_1(G) + \sum_{k=1}^{n-1} (-1)^{k-i} k S_k(G),$$

and

$$2S_1(G) = \sum_{i=1}^{n-1} i f_i, S_k(G) = \sum_{i=k}^{n-1} \binom{i}{k} f_i, 1 < k \leqslant n-1.$$

Theorem 2.

$$\sum_{i=1}^{n-1} i^m f_i = 2S_1 + \sum_{i=2}^{m} i! \left\{ {m \atop i} \right\} S_i(G),$$

here $\left\{ {m \atop i} \right\}$ *are the Stirling numbers of the second kind.*

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Diagonal limits of linear groups

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It will be considered full linear groups and they subroups: special linear groups, symplectic groups, orthogonal groups and unitary groups. For each type of these groups diagonal inductive limits are well-defined. For the first time such limits were considered by A. E. Zalesskii in [1]. Classification of diagonal limits of full linear groups, special linear groups, symplectic groups, orthogonal groups or unitary groups are determined by using the lattice of Steinitz numbers [2]. Some properties of diagonal limits of these groups will be discussed.

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On finite homomorphic images of commutative Bezout domains

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In the following all rings are assumes to be commutative with $1 \neq 0$. All necessary definitions and facts concerning the topic can be found in [1].

Definition. Let R be a commutative Bezout ring. An element $a \in R$ is said to be semipotent if for any $b \in R$ such that $b \notin J(aR)$ there are noninvertible $r, s \in R$ such that

$$a = rs, \ rR + bR = R, \ rR + sR = R.$$

Theorem 1. *Let R be a commutative Bezout domain. Then a is a semipotent element if and only if R/aR is a semipotent ring.*

Theorem 2. *Let R be a commutative Bezout ring and a is a semipotent element of R. Then zero element is a semipotent element of R/aR.*

Theorem 3. *Let R be a commutative Bezout domain. If zero is a semipotent element of R/aR then a is a semipotent element of R.*

Theorem 4. *A commutative Bezout ring is a semipotent ring if and only if zero element is a semipotent element of R.*

Theorem 5. *Let R be a commutative Bezout domain and $a \in R \backslash \{0\}$. Then R/aR is an indecomposable ring if and only if whenever $a = rs$ for some $r, s \notin U(R)$ then $rR + sR \neq R$.*

There is an open problem: describe necessary and sufficient conditions for an element a such that R/aR is a nonlocal indecomposable ring, where a is a nonzero element of commutative Bezout domain R.

Example. An example of such indecomposable ring is a quotient ring R/xR, where

$$R = \{z_0 + a_1 x + a_2 x^2 + \ldots \mid z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}\}.$$

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Effective duo ring

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All rings considered will be duo-ring and have nonzero identity.

A ring R is called a duo ring if every one-sided ideal of R is a two-sided. A nonzero element a in R is said to be adequate to the element $b \in R$, if we can find such two elements $r, s \in R$ that the decomposition $a = rs$ satisfies the following properties :
1) $rR + bR = R$,
2)$s^{'}R + bR \neq R$, for any noninvertible divisor $s^{'}$ of element s.

A duo Bezout domain R is said to be effective if for any elements $a, b, c \in R$ that $aR + bR + cR = R$ and $aR + bR \neq R$ there exists such element $p \in R$ that element c in R is adequate to the element ap and $pR + bR + cR = R$. A duo ring R is called an exchange ring if for any element $a \in R$ one can find such idempotent $e \in R$ that $e \in aR$ and $(1 - e) \in (1 - a)R$.

Theorem 1. *Effective Hermite duo ring is an elementary divisor ring.*

Theorem 2. *Let R be a duo Bezout domain whose finite homomorphic image R/cR is a exchange ring for any $c \in R$. Then R is an effective ring.*

Theorem 3. *Let R be a duo Bezout domain in which for any elements a, $b, c \in R$ that $aR+bR+cR = R$ there exist such element $p \in R$ that element c in R is adequate to the element ap and $pR + bR + cR = R$. Then R/cR is exchange ring for every $c \in R$.*

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Matrix reduction over Bezout stable range one domains with Dubrovin and Z conditions

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In the following all rings are assumes to be associative with $1 \neq 0$. We study the noncommutative elementary divisor rings whose principal two-sided ideals satisfy some conditions on their generators. All necessary definitions and facts concerning the topic can be found in [3].

Definition 1. It is said that a ring R satisfies Dubrovin condition if for any element $a \in R$ there exists an element $b \in R$ such that $RaR = Rb = bR$.

Definition 2. A domain R is said to be a ring with L condition if whenever $a \in R$ is such that $RaR = R$ then $aR = R$.

Theorem 1. *An elementary divisor ring satisfying L condition also satisfies Dubrovin condition.*

We introduce the generalization of L condition namely Z condition [1, 2].

Definition 3. A domain R is said to be a ring with Z condition if whenever $a \in R$ is such that $RaR = R$ then a is a finite element, i.e. the lattice of right ideals containing element a is finite.

Definition 4. A ring R is said to be a stable range one ring if for any elements $a, b \in R$ such that $aR + bR = R$ there is $x \in R$ such that $(a + bx)R = R$.

Theorem 2. *Let R be a Bezout domain of stable range one satisfying Dubrovin and Z conditions. Then R is an elementary divisor ring.*

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A germinal approach to power sums

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An original criterion for approaching the Fermat equation was devised during a 2005 summer coursework at the Saint-Petersburg State University and stored among the unpublished files by the Italian Society of Authors and Editors for a long time [1]. It consisted of counting the possible pairs $(a; b)$ in the hypothetical equation $a^p + b^p = c^p$ at integer variables a, b, c, p, with $a \leqslant b$ and p prime, in order to find decreasing values with the growth of p. The subsequent concept of progressive restriction for the number of addends in a p-power sum is now proposed with the aim of further analysis and improvement.

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Non-residually finite CAT(0) groups from bireversible automata

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The first example of a non-residually finite CAT(0) group (i.e. group acting properly discontinuously and cocompactly on a CAT(0) space) was constructed by Wise [4]. Shortly after that Burger and Mozes [2] constructed even finitely presented torsion-free simple CAT(0) groups. All these examples are the fundamental groups of complete square complexes.

Glasner and Mozes [3] discovered an interesting connection between square complexes and automata. An automaton-transducer A with the same input-output alphabet X give rise to a square complex Δ_A: one can take a unit square with labeled and oriented edges for each arrow in A and glue these squares to get a complex. All complexes Δ_A have one vertex and belong to the family of VH square complexes introduced in [4]. The fundamental group of Δ_A has presentation

$$\pi_1(\Delta_A) = \langle S, X \mid sx = yt \text{ for each arrow } s \xrightarrow{x|y} t \text{ in } A \rangle.$$

The Δ_A is a complete square complex (the link of a unique vertex is a complete graph) if and only if A is bireversible; in this case $\pi_1(\Delta_A)$ is CAT(0).

In [1] we prove the following statements which relate the residual properties of $\pi_1(\Delta_A)$ with the properties of the automaton group G_A generated by A.

Theorem 1. *Let A be a bireversible automaton over an alphabet X and with the set of states S. If G_A is finite, then $\pi_1(\Delta_A)$ is virtually a direct product of two free groups and therefore residually finite. If G_A is infinite, then the amalgamated free products $\pi_1(\Delta_A) *_S \pi_1(\Delta_A)$ and $\pi_1(\Delta_A) *_X \pi_1(\Delta_A)$ are non-residually finite CAT(0) groups.*

Theorem 2. *Let A be a bireversible automaton with two states or over the binary alphabet. If G_A is infinite, then $\pi_1(\Delta_A)$ is non-residually finite.*

We apply these theorems to prove that certain complete VH square complex with four 2-cells and two complete directed VH square complexes with six 2-cells have non-residually finite CAT(0) fundamental groups, and no smaller examples exist with these properties. This answers to a question of Wise [5, Problem 10.19].

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On isomorphic objects of the category of monomial matrices over a commutative local ring

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Let K be a commutative local ring of principal ideals. Denote by $\mathrm{Mat}(K)$ the category of quadratic matrices over K (i.e. those with objects to be the quadratic matrices over K and the morphisms $A \to B$ to be the matrices X such that $AX = XB$). The full subcategory of $\mathrm{Mat}(K)$ with the objects to be the monomial matrices is denoted by $\mathrm{Mmat}(K)$ and is called the category of monomial matrices over K (by a monomial matrix we mean a matrix, in each row and each column of which there is at most one non-zero element). Finally, denote by $\mathrm{Mmat}_0(K)$ the subcategory of $\mathrm{Mat}(K)$ with monomial objects and monomial morphisms.

To each monomial $n \times n$ matrix $M = (m_{ij})$ over K there corresponds the directed graph with n vertices numbered from 1 to n and arrows $i \to j$ for all $m_{ij} \neq 0$. We call a quadratic monomial matrix g-indecomposable if its graph is connected.

Monomial matrices over commutative local rings were studied in a number of papers (see, e.g., [1]–[4]).

Theorem. *Two g-indecomposable matrices over K are isomorphic in* $\mathrm{Mmat}(K)$ *if and only if they are isomorphic in* $\mathrm{Mmat}_0(K)$.

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New matrix models of finite networks

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In the study of networks and models, there are widely used various algebraic structures and objects: graphs, matrices, quadratic forms, groups, etc. Continuing this trend, we introduce and study the notion of an idempotent-matrix model for finite networks using our results on the theory of representations, summarized in the thesis [1] (the supervisor was the first author). By a network we mean a connected digraph without loops and multiple arrows.

Let R be the field of real numbers and M_m denotes the algebra of all $m \times m$ matrices over R. Given a finite ordered family $P = (P_1, P_2, \ldots, P_n)$ of idempotent matrices from M_m, we associate the generated digraph $G = G(P)$ which has the vertices set $V = \{1, 2, \ldots, n\}$ and the arrow set $A = \{i \to j, i \neq j \mid P_i P_j \neq 0\}$. The graph G, obviously, contains no loops and multiple arrows. The subalgebra $M_m(P) = M_m(P_1, P_2, \ldots, P_n)$ of M_m generated by the matrices $P_1, P_2, \ldots, P_n$ will be called the idempotent-matrix model for G of dimension n, generated by P, or simply an idempotent-matrix model for G.

Theorem. *Let $G = (V, A)$ be a network without oriented cycles, and let $|V| = n > 1$. Then the graph G has an idempotent-matrix model $M_m(P_1, P_2, \ldots, P_n)$ of some dimension m satisfying the following conditions:*

1) *P_i is diagonal for all input (or all output) vertices i;*

2) *$P_i \neq P_j$ if $i \neq j$;*

3) *for any oriented path $i_1 \to i_2 \to \ldots \to i_s$ $(s > 1)$, $P_{i_1} P_{i_2} \ldots P_{i_s}$ is not an idempotent matrix;*

4) *if pathes $i_1 \to i_2 \to \ldots \to i_s$ and $j_1 \to j_2 \to \ldots \to j_k$ are different, then $P_{i_1} P_{i_2} \ldots P_{i_s} \neq P_{j_1} P_{j_2} \ldots P_{j_k}$.*

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The school of Kiev in Colombia and the theory of algorithms of differentiation for posets and its applications

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Investigations carried out in the 1970's in the Kiev's famous seminar were introduced in Colombia by A. G. Zavadskij (a student of Professor V. V. Kirichenko). In his classes Zavadskij told us about the results obtained by Gabriel, Roiter, Nazarova, Bondarenko, Drozd, Kirichenko, Kleiner, Ovsienko, Schkabara and many other remarkable mathematicians. In particular, we recall that Zavadskij introduced seventeen algorithms of differentiation which allow to classify posets with some additional structures, e.g., posets with involution, equipped posets, etc. In this talk, we will describe some of these algorithms and its applications in combinatorics, number theory and information security.

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On divergence-free and jacobian derivations

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Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{A}$ an associative algebra over the field $\mathbb{K}$. Recall that a linear map $X : A \to A$ is called a $\mathbb{K}$-derivation of the algebra $\mathbb{A}$ if it satisfies the Leibniz rule, i.e. it holds

$$D(ab) = D(a)b + aD(b) \quad \text{for all} \quad a, b \in \mathbb{A}.$$

Let us denote the set of all $\mathbb{K}$-derivations of the algebra $\mathbb{A}$ by $\mathrm{Der}_{\mathbb{K}}(\mathbb{A})$. It is well known that $\mathrm{Der}_{\mathbb{K}}(\mathbb{A})$ is a Lie algebra with the multiplication $[X, Y] = XY - YX$.

Consider the Lie algebra $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x_1, \ldots, x_n])$ of all $\mathbb{K}$-derivations of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. Then the partial derivatives $\frac{\partial}{\partial x_i}$ are $\mathbb{K}$-derivations of the algebra $\mathbb{K}[x_1, \ldots, x_n]$. Divergence of a $\mathbb{K}$-derivation is defined as follows:

$$\mathrm{div} X = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(X(x_i))$$

Locally nilpotent derivations are especially interesting because of their exponents are automorphisms of the polynomial ring. There are some results about those, one of which states that every locally nilpotent derivation is divergence-free (see [1] for example). Divergence-free derivations form a subalgebra of the Lie algebra $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x_1, \ldots, x_n])$.

Let us consider polynomials $f_2, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$. It is easy to see, that the map $J : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[x_1, \ldots, x_n]$ defined by the rule

$$J(f_1) = \det\left(\frac{\partial}{\partial x_j}(f_i)\right), \ f_1 \in \mathbb{K}[x_1, \ldots, x_n]$$

is $\mathbb{K}$-derivation. Such derivations are called jacobian derivations.

The next statement is a generalization of a result given in [2]. It provides representation of divergence-free derivation as a sum of jacobian derivations, these are (in some sense) the simplest among all divergence-free derivations.

Theorem. *Let* $X \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x_1, \ldots, x_n]), \mathrm{div} X = 0.$ *Then there exist jacobian derivations* $J_1, \ldots, J_{n-1}$ *such that* $X = \sum_{i=1}^{n-1} J_i$

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Tilting bundles on weighted projective lines

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Tilting theory, originally introduced in the context of module categories over finite dimensional algebras, plays an important role in the study of many areas of mathematics, including representation theory of finite groups, Lie theory, commutative and non-commutative algebraic geometry. Tilting modules and tilting complexes, as two fundamental concepts in tilting theory, are used widely for constructing equivalences between categories. Besides the classical module categories, there is another standard example of hereditary categories with a tilting object|the category of coherent sheaves on a weighted projective line. In this talk, we introduce some new progress.

This is a joint work with Jianmin Chen, Ping Liu, and Shiquan Ruan.

Locally graded groups with the minimal condition for non-abelian non-complemented subgroups

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Remind that a group, in which every finitely generated subgroup $\neq 1$ possesses a subgroup of finite index $\neq 1$, is called locally graded (S. N. Chernikov, 1970). The class of locally graded groups is very wide. It contains, for instance, all locally finite, solvable and locally solvable, RN-, linear and locally linear, quasi-linear groups. It includes all Kurosh–Chernikov classes of groups.

The following new propositions of the author hold.

Theorem. *Let G be a locally graded non-abelian group, satisfying the minimal condition for non-abelian non-complemented subgroups. Then G is locally finite and also it is Chernikov or metabelian-by-abelian.*

The known Olshanskiy's Examples of infinite simple groups with exclusively abelian proper subgroups show that above the demand: "G must be locally graded" is essential.

Corollary 1. Let G be a non-abelian locally or residually solvable group, satisfying the minimal condition for non-abelian non-complemented subgroups. Then G is solvable locally finite and, moreover, it is Chernikov or metabelian-by-finite.

Corollary 2. Let G be a non-abelian locally or residually finite group, satisfying the minimal condition for non-abelian non-complemented subgroups. Then G is locally finite and, moreover, it is Chernikov or metabelian-by-finite.

Corollary 3. Let G be a non-abelian RN-group, satisfying the minimal condition for non-abelian non-complemented subgroups. Then G is solvable locally finite and, moreover, it is Chernikov or metabelian-by-finite.

Corollary 4. Let G be a linear or quasi-linear group, satisfying the minimal condition for non-abelian non-complemented subgroups. Then G is locally finite and, moreover, it is Chernikov or metabelian-by-finite.

Classification of finite structurally uniform groups

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Let S be a finite semigroup. By $Sub(S)$ we denote the lattice of all its subsemigroups. If $A \in Sub(S)$, then by $h(A)$ we denote the height of the subsemigroup A in the lattice $Sub(S)$. A semigroup S is called **structurally uniform** if, for any $A, B \in Sub(S)$ the condition $h(A) = h(B)$ implies that $A \cong B$.

For a prime number p, by $\mathbb{Z}_p$ denote the corresponding field. The set of all upper triangular matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where a, b, and c are arbitrary elements of the field $\mathbb{Z}_p$, forms a group with respect to an ordinary operation of multiplication, which is called a Heisenberg group over the field $\mathbb{Z}_p$ and denoted by $Heis(\mathbb{Z}_p)$.

Theorem. *Assume that G is a finite group. The group G is structurally uniform if and only if G is:*

1. *either an elementary Abelian p-group, where p is any prime number;*
2. *or a Heisenberg group over the finite field $\mathbb{Z}_p$, where p is an arbitrary odd prime number;*
3. *or quaternion group Q_8;*
4. *or cyclic group C_{p^k} , where p is any prime number.*

Variants of the Rees matrix semigroup over the trivial group with zero

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Let S be a semigroup and $a \in S$. We consider the new multiplication $*_a$ defined by the next equality $x *_a y = xay$ and we will call it sandwich multiplication. Then $*_a$ is an associative operation on S. Semigroup $(S, *_a)$ is called variant of the semigroup S, or sandwich semigroup.

Let G be a trivial group and $G^0 = G \cup 0$ be the group with zero arising from G by the adjunction of a zero element 0. Let I and Λ be arbitrary finite sets. This sets have the next powers $|I| = n$ and $|J| = m$.

By the Rees $n \times m$ matrix over G^0 we mean an $n \times m$ matrix over G^0 having at most one non-zero element. Let $P = \{p_{ji} | p_{ji} \in G^0,\ j \in J,\ i \in I\}$ be an arbitrary but fixed $m \times n$ matrix. Let A and B be an arbitrary Rees $n \times m$ matrix over G^0. We use P to define a binary operation $\circ$ as a sandwich multiplication $A \circ B = A \cdot P \cdot B$. The set of all Rees $n \times m$ matrices over G^0 with respect to the binary operation $\circ$, we call it Rees $n \times m$ matrix semigroup over the group with zero G^0 with sandwich matrix P, and denote by $\mathcal{M}^0(G^0; n, m; P)$.

Proposition 1. *The matrix $P \cdot A_{ij} \cdot P$ by deletion of zero rows and columns could be reduced to the rectangular matrix and all elements of this matris are identities.*

Theorem 1. *Let matrix P' be obtained from P by permutation of rows and columns. Then Rees matrix semigroups $\mathcal{M}^0(G^0; n, m; P')$ and $\mathcal{M}^0(G^0; n, m; P)$ are isomorphic.*

Let A_{ij} be an arbitrary but fixed Rees $n \times m$ matrix over G^0. We consider the variant of the semigroup $\mathcal{M}^0(G^0; n, m; P)$ with the sandwich element A_{ij}.

Proposition 2. *The variant $(\mathcal{M}^0(G^0; n, m; P), *_{A_{ij}})$ is Rees matrix semigroup, with sandwich matrix $P \cdot A_{ij} \cdot P$.*

Theorem 2. *Variants $(\mathcal{M}^0(G^0; n, m; P), *_{A_{ij}})$ and $(\mathcal{M}^0(G^0; n, m; P), *_{A_{lk}})$ are isomorphic if and only if matrix $P \cdot A_{ij} \cdot P$ and $P \cdot A_{lk} \cdot P$ by deletion of zero rows and columns can be reduced to the same matrix.*

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A seven terms exact sequence related to a partial Galois extension of commutative rings

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In [2] S. U. Chase and A. Rosenberg gave an Amitsur cohomology seven terms exact sequence, which was specified by S. U. Chase, D. K. Harrison and A. Rosenberg in [1] to the case of a Galois extension of commutative rings. The latter generalizes the two most fundamental facts from Galois cohomology of fields, the Hilbert's Theorem 90 and the isomorphism of the Brauer Group with the second cohomology group of the Galois group. The proof in [2] used spectral sequences and was not constructive. The first constructive proof was given by T. Kanzaki [7], introducing and applying generalized crossed products. Since then much attention have been payed to the sequence and its parts establishing generalizations and analogues in various contexts.

Partial actions and partial representations were introduced in the theory of operator algebras as crucial ingredients of a new approach in the study of C^*-algebras generated by partial isometries. This influenced numerous algebraic developments, in particular, in [3] a Galois Theory of partial actions was developed and in [4] a group cohomology theory based on partial actions was elaborated.

Using the concept of a partial group cohomology and introducing the notion of the Picard inverse semigroup we constructed in [5], [6] a version of the seven terms exact sequence for a partial Galois extenstion of commutative rings, which generalizes the sequence from [1]. Some details will be presented in our talk.

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Kazimirsky condition for rings of stable range 1

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Let R be an associative ring with identity ($1 \neq 0$). Recall some necessary well-known definitions [1]. A ring R is a ring with right (left) Kazimirsky condition if for any $a \in R$ and any invertible element $u \in R$ the following inclusion holds: $aR \subseteq uaR$ ($Rau \subseteq Ra$). A ring R is a ring of stable range 1 if for any elements $a, b \in R$ the condition $aR + bR = R$ implies that there exists an element $t \in R$ such that $a + bt$ is an invertible element of R. A ring R is right (left) distributive ring if the lattice of right (left) ideals of R is distributive.

Theorem 1. *A ring of stable range 1 with right (left) Kazimirsky condition is a right (left) distributive ring.*

Theorem 2. *Let for any $a \in R$ and any invertible element $u \in R$ there exists $x \in R$ ($y \in R$) such that $1 + ax = u$ ($1 + ya = u$). Then R is a ring with right (left) Kazimirsky condition.*

Also recall that a ring R is an elementary divisor ring if for any $n \times m$ matrix A over R there exist the invertible matrices P and Q of appropriate dimensions such that

$$PAQ = \begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \varepsilon_r & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $R\varepsilon_{i+1}R \subseteq \varepsilon_i R \bigcap R\varepsilon_i$ for any $i \in (1, 2, ..., r-1)$.

Theorem 3. *A ring R of stable range 1 with right (left) Kazimirsky condition is an elementary divisor ring if and only if it is a duo-ring.*

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Nodal curves and quasi-hereditary algebras

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It is a joint work with I. I. Burban.

We consider *nodal curves*, i.e. such non-commutative curves [1] $(X, \mathcal{A})$ over an algebraically closed field $\Bbbk$ that there is a hereditary curve $(X, \mathcal{H})$ such that $\mathcal{I} \subset \mathcal{A} \subset \mathcal{H}$, where

$$\mathcal{I}_x = \begin{cases} \operatorname{rad} \mathcal{H}_x = \operatorname{rad} \mathcal{A}_x & \text{if } \mathcal{A}_x \neq \mathcal{H}_x, \\ \mathcal{H}_x & \text{otherwise} \end{cases}$$

and $\operatorname{length}_{\mathcal{A}}(\mathcal{H} \otimes_{\mathcal{A}} U) \leqslant 2$ for any simple $\mathcal{A}$-module U. We denote by $\mathcal{R}$ the *resolution* of $\mathcal{A}$ which can be considered as the matrix ring

$$\mathcal{R} = \begin{pmatrix} \mathcal{A} & \mathcal{H} \\ \mathcal{I} & \mathcal{H} \end{pmatrix}.$$

Theorem 1. *There is a diagram of functors of derived categories*

$$\mathscr{D}(\mathcal{A}) \begin{array}{c} \xrightarrow{\;\;\mathsf{F}\;\;} \\ \xleftarrow{\;\;\mathsf{G}\;\;} \\ \xrightarrow[\;\;\mathsf{H}\;\;]{} \end{array} \mathscr{D}(\mathcal{R})$$

such that both (F, G) *and* (G, H) *are disjoint pairs,* G *is exact and both natural morphisms* $\mathsf{FG} \to \mathbf{1}_{\mathscr{D}(\mathcal{A})}$ *and* $\mathbf{1}_{\mathscr{D}(\mathcal{A})} \to \mathsf{HG}$ *are isomorphisms. Moreover, there is a semi-orthogonal decomposition* $\mathscr{D}(\mathcal{R}) = \langle \mathscr{D}_1, \mathscr{D}_2 \rangle$*, where* $\mathscr{D}_1 \simeq \mathscr{D}(\mathcal{A}/\mathcal{I})$*,* $\mathscr{D}_2 \simeq \mathscr{D}(\mathcal{H})$*, so* $\operatorname{gl.dim} \mathcal{R} \leqslant 2$ *and* $\mathscr{D}(\mathcal{R})$ *can be considered as a categorical resolution of* $\mathscr{D}(\mathcal{A})$.

If the nodal curve $(X, \mathcal{A})$ is *rational* [1], it is known (ibid.) that $\mathcal{H}$ is derived equivalent to a *Ringel canonical algebra* [4]. Using this fact, we obtain the following result.

Theorem 2. *There is a Ringel canonical algebra* $\boldsymbol{C}$ *and a quasi-hereditary algebra* $\boldsymbol{Q}$ *obtained from* $\boldsymbol{C}$ *by glueing some pairs of vertices and blowing up some other vertices* [3] *such that* $\mathscr{D}(\boldsymbol{Q}) \simeq \mathscr{D}(\mathcal{R})$*. In this case* $\dim \mathscr{D}^b(\mathcal{R}) \leqslant 2$ *and* $\dim \mathscr{D}^{\mathrm{perf}}(\mathcal{A}) \leqslant 2$*, where* 'dim' *denotes the* Rouquier dimension [5]. *If* $\mathcal{H}_x$ *is maximal at all but at most* 2 *points* $x \in X$*, then* $\dim \mathscr{D}^b(\mathcal{R}) \leqslant 1$ *and* $\dim \mathscr{D}^{\mathrm{perf}}(\mathcal{A}) \leqslant 1$.

The structure of $\boldsymbol{Q}$ is defined explicitly from that of $\mathcal{A}$.

Using the results of [2], we also obtain for the algebras appearing in Theorem 2 a criterion of tameness (which coincides in this case with the derived tameness).

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Matrix factorizations of Cohen–Macaulay modules over the plane curve singularity of type T_{44}

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The plane curve singularity of type T_{44} is one of the critical singularities of tame Cohen-Macaulay representation type [1]. It is given by equation $f(x,y)=0$, where $f(x,y)=xy(x-y)(x-\lambda y)$ is polynomial over some algebraically closed field $\mathbf{k}$ and $\lambda \neq 0,1$.

For all Cohen–Macaulay modules over the local ring of the plane curve singularity of type T_{44} we explicitly describe the corresponding matrix factorizations. The calculations are based on the technique of matrix problems, in particular, representations of bunches of chains [2].

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Unicyclic graphs with two main vertices and metric dimension 2

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Let $G = (V, E)$ be a simple unicyclic graph (a simple graph with exactly one cycle) and $G_1 = (V_1, E_1)$ be a subgraph of G, which is a simple cycle. According to the graph G is uniquely determined metric space (V, d_G), defined on the set of vertices V. Metric d_G between two vertices v_1 and v_2 equals 0 if $v_1 = v_2$ and the length of the shortest path between v_1 and v_2 if $v_1 \neq v_2$.

The vertex t is called *distinguished* for pair of vertices x and y if

$$d_G(t, x) \neq d_G(t, y).$$

Definition 1 [1]**.** The subset $M \subset V$ is called *metric generator* of G if for any pair of vertices from V exists $t \in M$ which distinguish them. *The metric basis* is metric generator of G with minimum cardinality. The number of vertices in metric basis is called *metric dimension* of G and denoted $dim(G)$.

Metric generators is used in graph theory, particularly in problems of checking isomorphism of graphs, in problems of searching isometric subspaces, for search the isometry of metric space, which is an extension of isometry of the fixed subspace, in chemistry, biology, robotics and many other disciplines [2].

Definition 2. A vertex $u \in V \backslash V_1$ of graph G is said to be *projected* in the vertex $w \in V_1$ if for any vertex $q \in V_1$ the inequality

$$d_G(u, w) < d_G(u, q)$$

holds.

The vertex with degree 3 from cycle, in which the vertices that have degree 3 and are located outside the cycle are projected, is called *main vertex*.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are simple graphs. Let fix the vertices $v_1 \in V_1$ and $v_2 \in V_2$. A graph G is designed from G_1 and G_2 by *gluing* along the vertices v_1 and v_2 if $G = (V, E)$ has set of vertices $V = V_1 \cup V_2 \backslash v_2$ and set of edges $E = E_1 \cup E_2$ (a vertex v_2 is replaced by v_1 for all edges of G_2). So, we identify vertices v_1 and v_2 of graphs G_1 and G_2.

Definition 3. A unicyclic graph G is called *braided–built* from unicyclic graph G_1 by chains $L_1, ..., L_k$ if G is obtained from the graph G_1 by gluing vertices with degree 2 of cycle and beginings of the chains $L_1, ..., L_k$ and each vertex with degree 2 of cycle glued to the end of exactly one chain.

Theorem. *If a unicyclic graph G is well–braided–built from unicyclic graph G_1 and G_1 has metric dimension 2 then also has a metric dimension 2.*

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On Gao conjecture connected with high order elements in arbitrary finite field

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In [2] Gao gives an algorithm for constructing high order elements [4] for arbitrary extensions F_{q^n} of finite field [3, 4] F_q with lower bound on the order $n^{\frac{\log_q n}{4\log_q(2\log_q n)}-\frac{1}{2}}$. The Gao approach is based on the following proposed by him conjecture.

Conjecture. *Given an integer n, let $m = \lceil \log_q n \rceil$. There exist a polynomial $g(x) \in F_q[x]$, $\deg g(x) \leqslant 2m$ such that $x^{q^m} - g(x)$ has irreducible factor $f(x)$ of degree n.*

If the conjecture holds, then clearly for the coset θ of x the following equality is true: $\theta^{q^m} \equiv g(\theta)$. This fact is used to obtain the lower bound. Conflitti [1] and then Popovych [5] improved the bound to $\binom{n+t-1}{t} \prod_{i=0}^{t-1} \frac{1}{(2m)^i}$, where $t = \lfloor \log_{2m} n \rfloor$.

The conjecture was verified in [2] for $q = 2$ and $n \leqslant 300$. It was also noticed that for these cases $\deg g(x) \leqslant \lceil \log_2 n \rceil + 3 \leqslant 2 \lceil \log_2 n \rceil$.

We have done calculations in Maple for $q = 2$ and $300 < n \leqslant 400$, for $q = 3$ and $n \leqslant 300$, for $q = 5$ and $n \leqslant 200$. The Gao conjecture is confirmed for these cases. Additionally, it was found that $\deg g(x) \leqslant m + 3$. Hence, very likely the bound on the polynomial degree in the conjecture can be strengthened.

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Representation type of Rees semigroup over a cyclic group of prime order in modular case

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Let G be a group. Let B be $n \times m$ matrix over $G \cup \{0\}$ such that in any row and column there is at least one nonzero element. Let $g \in G$ and $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. Denote by $(g)_{ij}$ a $m \times n$ matrix which have g on the position (i, j) and zeros on the other positions. Denote by $\mathcal{R}(G, B)$ the set of all such matrices together with zero matrix and define a multiplication:

$$(g)_{ij} * (g')_{i'j'} = (g)_{ij} \cdot B \cdot (g')_{i'j'}$$

where “$\cdot$” denote the usual multiplication of matrices. The set $\mathcal{R}(G, B)$ together with multiplication “$*$” is called *Rees semigroup over the group G with sandwich matrix B.*

Ponizovskii in article [1] described all Rees semigroups of finite representation type over field F in the situation when $char F$ and order of group G is coprime.

Let $G = C_p$ be a cyclic group of prime order p. We are interested in the representation type of semigroup $\mathcal{R}(C_p, B)$ in modular case i.e. the base field F has characteristic p. The problem of classifying representations of a semigroup $\mathcal{R}(C_p, B)$ is equivalent to the problem of classifying representations of its semigroup algebra $\mathcal{M}(B) = F[\mathcal{R}(C_p, B)]$. It is easy to see that $\mathcal{M}(B)$ is algebra of all $m \times n$ matrices over the group algebra $F[C_p]$ with multiplication $M_1 * M_2 = M_1 B M_2$.

Proposition 1. *If there exist invertible matrices $S \in M_n(F[C_p])$ and $T \in M_m(F[C_p])$ such that $B' = SBT$ then algebras $\mathcal{M}(B)$ and $\mathcal{M}(B')$ are isomorphic.*

It is easy to see that any algebra $\mathcal{M}(B)$ is isomorphic to algebra $\mathcal{M}(D)$ where matrix D has following form:

$$D = \begin{pmatrix} E & 0 & \dots & 0 & 0 \\ 0 & A_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_{p-1} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where $E = diag(e, \dots, e)$, $A_1 = diag(a-e, \dots, a-e)$, ..., $A_{p-1} = diag((a-e)^{p-1}, \dots, (a-e)^{p-1})$ are diagonal matrices; $e \in C_p$ is identity, $a \in C_p$ is element of order p.

Last matrix D we will call simplification of matrix B and will denote $D = \mathfrak{s}(B)$.

Let us formulate the main result in the following theorem.

Theorem. *$\mathcal{M}(B)$ has finite representation type if and only if in case $p = 2$ or $p = 3$: $\mathfrak{s}(B) = (e), (e\ 0), \begin{pmatrix} e \\ 0 \end{pmatrix}, \begin{pmatrix} e & 0 \\ 0 & a-e \end{pmatrix}$, in case $p > 3$ $\mathfrak{s}(B) = (e)$.*

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Classification of low-dimensional Lie algebras

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We plan to present some results related to classification of the low-dimensional ($\dim L \leqslant 5$) Lie algebras as well as low-dimensional ($\dim L \leqslant 5$) nonconjugate subalgebras of Lie algebra of the generalized Poincaré group $P(1,4)$.

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Finite automaton actions of free groups of rank n

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Let X be a finite *alphabet*, $|X| > 1$. Denote by X^n the set of all words over X of length n, $n \geqslant 0$. Notation X^* is used for the set of all finite words over X. For arbitrary words $u, v \in X^*$ the product of u and v is the concatenation uv. A *finite initial automaton* over X is a tuple $A = \langle X, Q, \varphi, \psi, q_0 \rangle$, where Q is a nonempty finite set of inner states, φ and ψ are transition and output functions, which map $Q \times X$ into Q and X, respectively, $q_0 \in Q$ is the initial state. Such an automaton can be defined in terms of labeled oriented graph. The vertex set of this graph is Q and the initial state is somehow highlighted. An arrow from a vertex q_1 to a vertex q_2 with a label $x_1|x_2$ is drawn if and only if $\varphi(q_1, x_1) = q_2, \psi(q_1, x_1) = x_2$. An automaton is called *permutational* if for each its state the restriction of the output function in this state determines some permutation on the alphabet. Each finite permutational initial automaton A over X defines a permutation on X^* by restricting its output function to the initial state. This permutation is called finite automaton permutations over X and denoted by A. All finite automaton permutations over X form a group under superposition which we denote by $FAP(X)$.

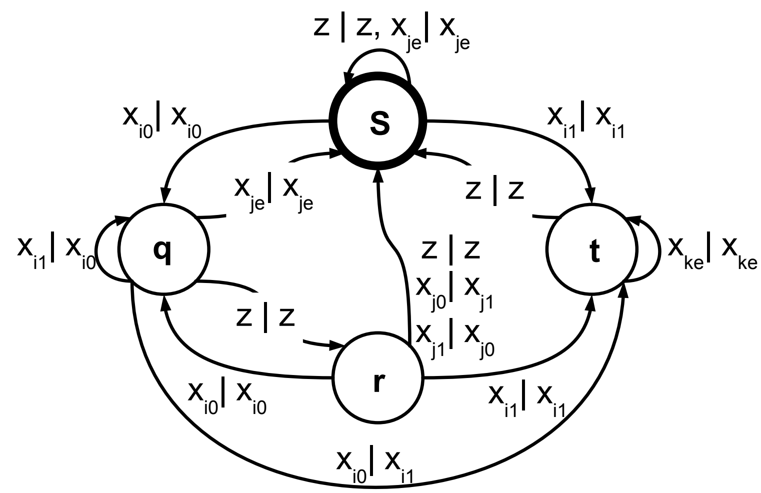

Fig. 1. Automata C_{X_i}, $j \neq i$, $1 \leqslant i, j, k \leqslant n$, $e \in \{0, 1\}$.

Let $\mathsf{X_i} = \{x_{i0}, x_{i1}\}$, $1 \leqslant i \leqslant n$ be $n \geqslant 2$ disjoint alphabets of cardinality 2. Denote by ψ_{X_i} their index functions, i.e. $\psi_{X_i}(x_{ie}) = e$, $e \in \{0, 1\}$. Consider *adding machines* A_{X_i} over the alphabets $\mathsf{X_i}$, $1 \leqslant i \leqslant n$ respectively (see [1, p. 16]). We construct new automata B_{X_i} over the alphabets $\mathsf{X_i}$, $1 \leqslant i \leqslant n$ respectively by adding one new state s and making this state initial. The action of the initial automaton B_{X_i} on a non-empty word xu, $x \in \mathsf{X_i}$, $u \in \mathsf{X}^*$, depends on x. Specifically, if $x = x_{i0}$ then $(xu)^{B_{X_i}} = xu^{A_{X_i}}$ and $(xu)^{B_{X_i}} = xu$ otherwise. Let us construct n initial automata $C_{X_1}, \ldots, C_{X_n}$ over a new alphabet $\mathsf{Z} = \bigcup_{k=1}^n \mathsf{X_i} \cup \{z\}$, where $z \notin \bigcup_{k=1}^n \mathsf{X_i}$. One can obtain these automata by adding one new state to B_X and B_Y correspondingly and extending transition and output functions. (Fig. [1]). Denote by g_i finite automaton transformations defined by initial automata C_{X_i}, $1 \leqslant i \leqslant n$ correspondingly. Let G be a subgroup of $FAP(\mathsf{Z})$ generated by $g_1, \ldots g_n$. The main result is

Theorem. *The group G is a free group of rank n with basis $\{g_1, \ldots, g_n\}$.*

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Relations between orthogonality and retract orthogonality

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In quasigroup theory, the term "orthogonality" refers to several different notions, which are generalizations of orthogonality of binary operations. Here, we will follow [1]. All n-ary and k-ary operations given below are defined on an arbitrary set Q and $m := |Q|$, $n \geqslant 2$, $k \leqslant n$.

A tuple of n-ary operations $f_1, \ldots, f_k$ is called *orthogonal*, if for arbitrary $b_1, \ldots, b_k \in Q$ the system

$$\begin{cases} f_1(x_1, \ldots, x_n) = b_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ f_k(x_1, \ldots, x_n) = b_k \end{cases}$$

has exactly m^{n-k} solutions. Remark that the system should have a unique solution if $n = k$.

Let f be an n-ary operation and let

$$\delta := \{i_1, \ldots, i_k\} \subseteq \overline{1,n}, \quad \{j_1, \ldots, j_{n-k}\} := \overline{1,n} \setminus \delta, \quad \bar{a} := (a_{j_1}, \ldots, a_{j_{n-k}}).$$

An operation $f_{(\bar{a},\delta)}$ which is defined by

$$f_{(\bar{a},\delta)}(x_{i_1}, \ldots, x_{i_k}) := f(y_1, \ldots, y_n),$$

where $y_i := \begin{cases} x_i, & \text{if } i \in \delta, \\ a_i, & \text{if } i \notin \delta, \end{cases}$ is called an $(\bar{a}, \delta)$*-retract* or a δ*-retract* of f.

Operations $f_{1;(\bar{a}_1,\delta)}, f_{2;(\bar{a}_2,\delta)}, \ldots, f_{k;(\bar{a}_k,\delta)}$ are called *similar* δ-retracts of n-ary operations f_1, $f_2, \ldots, f_k$, if $\bar{a}_1 = \bar{a}_2 = \cdots = \bar{a}_k$. Let $\delta \subseteq \overline{1,n}$ and $|\delta| = k$. A k-tuple of n-ary operations is called δ*-retractly orthogonal*, if each tuple of similar δ-retracts of these operations is orthogonal.

In [2] the retract orthogonality concept was given as a tool of block-wise recursive algorithm for constructing orthogonal n-ary operations. That is why here we describe relations between orthogonality and retract orthogonality.

Theorem 1. *If for some $\delta \subset \overline{1,n}$ a tuple of n-ary operations is δ-retractly orthogonal, then the tuple is orthogonal.*

The inverse statement of Theorem 1 is not true.

Theorem 2. *Let $k < n$. Then there exist k-tuples of orthogonal n-ary operations which are not δ-retractly orthogonal for some δ, where $\delta \subset \overline{1,n}$ and $|\delta| = k$.*

An operation $f(x_1, \ldots, x_n) := \alpha_1 x_1 + \cdots + \alpha_n x_n + a$ is called *central*, if $(Q; +)$ is an Abelian group, $\alpha_1, \ldots, \alpha_n$ are automorphisms of $(Q; +)$ and $a \in Q$.

Theorem 3. *Let $k \leqslant n$ and p be a prime number. n-ary central quasigroups $f_1, \ldots, f_k$ over field $(\mathbb{Z}_p; +, \cdot)$ are orthogonal if and only if there exists δ, such that $|\delta| = k$ and $f_1, \ldots, f_k$ are δ-retractly orthogonal.*

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Construction of Gelfand-Tsetlin modules for $\mathfrak{gl}_n$

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A classical paper of Gelfand and Tsetlin [3] describes a basis of irreducible finite dimensional modules over the Lie algebra $\mathfrak{gl}_n$. This is one of the most remarkable results of the representation theory of Lie algebras which triggered a strong interest and initiated a development of the theory of Gelfand–Tsetlin modules. These modules are related to Gelfand–Tsetlin integrable systems studied by Guillemin and Sternberg [4], Kostant and Wallach [5], [6] and many others. Gelfand–Tsetlin theory had a successful development for infinite dimensional representations [8]. The significance of the class of Gelfand–Tsetlin modules is in the fact that they form the largest subcategory of $\mathfrak{gl}_n$-modules (in particular weight modules with respect to a fixed Cartan subalgebra) where there is some understanding of irreducible modules. The main remaining problem is how to construct explicitly these modules.

We propose a new effective method of constructing explicitly Gelfand–Tsetlin modules for $\mathfrak{gl}_n$ and obtain a large family of irreducible modules (conjecturally all) that have a basis consisting of Gelfand–Tsetlin tableaux, the action of the Lie algebra is given by the Gelfand–Tsetlin formulas and with all Gelfand–Tsetlin multiplicities equal 1. As an application of our construction we prove necessary and sufficient condition for the Gelfand and Graev's continuation construction [2] to define a module which was conjectured by Lemire and Patera [7].

The talk is based on joint results with Luis Enrique Ramirez and Jian Zhang [1].

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Range conditions for abelian rings

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Throughout, all rings are assumed to be associative with identity and $1 \neq 0$. An element a of a ring R is called (von Neumann) regular element, if $axa = a$ for some element $x \in R$. An element a of a ring R is called a left (right) semihereditary element if $Ra(aR)$ is projective. A ring R is a ring of stable range 1, if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $(a + bt)R = R$ [1]. About the different modifications the concept of stable range can be found in [2–4].

The ring R is called abelian ring if every idempotent is central, that is, $ae = ea$ for any $e^2 = e, a \in R$. An abelian ring is directly finite.

Definition 1. *A ring R is said to have a (von Neumann) regular range 1, if for any $a, b \in R$ such that $aR + bR = R$ there exists $y \in R$ such that $a + by$ is a (von Neumann) regular element of R* [5].

Obviously, an example of ring (von Neumann) regular range 1 is a ring of stable range 1.

Theorem 1. *For an abelian ring R the following conditions are equivalent:*

1. R is a ring of stable range 1;

2. R is a ring of (von Neumann) regular range 1.

Definition 2. *A ring R is said to have a semihereditary range 1, if for any $a, b \in R$ such that $aR + bR = R$ there exists $y \in R$ such that $a + by$ is a right semihereditary element of R* [5].

Definition 3. *A ring R is said to have a regular range 1, if for any $a, b \in R$ such that $aR + bR = R$ there exists $y \in R$ such that $a + by$ is a regular element (nonzero divisor) of R* [5].

Theorem 2. *For an abelian ring R the following conditions are equivalent:*

1. R is a ring of regular range 1;

2. R is a ring of semihereditary regular range 1.

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Rings and elements of almost stable range one

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Throughout, all rings are assumed to be commutative with identity and $1 \neq 0$. A ring is a Bezout ring, if every its finitely generated ideal is principal. A ring R is a ring of stable range 1, if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $(a + bt)R = R$ [1]. About the different modifications the concept of stable range can be found in [2–4]. By $J(R)$ we will denote Jacobson radical of a ring R.

Definition 1. *A nonzero element a in a ring R is called a square-free element if having any its decomposition $a = xy$, where$x, y \in R$, one can conclude that $xR + yR = R$.*

It is useful to notice that there are rings without square-free elements, for example such is the ring of all algebraic integers.

Definition 2. *An element a in a ring R is called an element of stable range 1, if for any $b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $(a + bt)R = R$.*

Definition 3. *An element a in a ring R is called an almost stable range 1 element if the stable range of R/aR is equal to 1.*

We say that R is a ring of almost stable range 1 if an arbitrary nonzero noninvertible element of R is an element of almost stable range 1 [3].

Theorem 1. *The square-free elements of commutative Bezout domain are the elements of almost stable range 1.*

Theorem 2. *Let R be a commutative Bezout domain of Krull dimension 2. Then R is a ring of almost stable range 1.*

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Central polynomials codimension growth

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Let A be an associative algebra over a field F of characteristics zero and $Z(A)$ its center. Let $F\langle X\rangle$ be the free associative algebra on a countable set X over F. Recall that a polynomial $f \in F\langle X\rangle$ is a central polynomial of A if for any $a_1, \ldots, a_n \in A$, $f(a_1, \ldots, a_n) \in Z(A)$, the center of A. In case f takes only the zero value, f is a polynomial identity (PI) of A whereas if f takes a non-zero value in $Z(A)$, we say that f is a proper central polynomial.

We compare the growth of the spaces of central polynomials, proper central polynomials and polynomial identities of an algebra in the following sense. Let $Id(A)$ be the T-ideal of polynomial identities of A and, following [1], we let $Id^z(A)$ be the space of central polynomials of A.

Regev in [1] introduced the notion of central codimensions as follows. Let P_n be the space of multilinear polynomials in $x_1, \ldots, x_n$ and set

$$P_n(A) = \frac{P_n}{P_n \cap Id(A)}, \quad P_n^z(A) = \frac{P_n}{P_n \cap Id^z(A)}, \quad \Delta_n(A) = \frac{P_n \cap Id^z(A)}{P_n \cap Id(A)}.$$

We write $c_n(A) = \dim P_n(A)$, $c_n^z(A) = \dim P_n^z(A)$ and $\delta_n(A) = \dim \Delta_n(A)$, respectively.

We also write $exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ in case of existence of this limit. Exponents $exp^z(A), exp^\delta(A)$ are defined similarly. We prove the following results.

Theorem 1. *If A is a finite dimensional algebra, then the proper central PI-exponent $exp^\delta(A)$ exists and is a nonnegative integer.*

Theorem 2. *Let A be a finite dimensional algebra. Then the sequence $\delta_n(A)$, $n = 1, 2, \ldots$, is either polynomially bounded or grows as an exponential function a^n with $a \geqslant 2$.*

Theorem 3. *For any finite dimensional algebra A with $exp(A) \geqslant 2$, the central PI-exponent $exp^z(A)$ exists and is a non-negative integer. Moreover, $exp^z(A) = exp(A)$.*

When $exp(A) = 0$, then A is nilpotent and $exp^z(A) = 0$. In case $exp(A) = 1$, then either $exp^z(A) = 1$ or $exp^z(A) = 0$. If $exp(A) = 1$, then A is not nilpotent and the sequence of codimensions is polynomially bounded. Clearly the same holds for the sequence of central codimensions. Thus $exp^z(A) = 1$ provided $c_n^z(A) \neq 0$ for all n.

The case when $c_n^z(A) = 0$ can be characterized as follows.

Proposition. *Let A be a finite dimensional algebra such that $exp^z(A) = 0$. Then $A = A_1 \oplus A_2$ where A_1 is a nilpotent algebra and A_2 is a commutative algebra.*

Main notions of the numerical PI-theory one can find in [2].

The first author was partially supported by the GNSAGA of INDAM. The second author was supported by the RFBR, grant 16-01-00113.

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On new Catalan identities using Toeplitz–Hessenberg matrices

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The Catalan numbers are a sequence defined directly in terms of binomial coefficients:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}, \quad n \geqslant 0,$$

or recursively as follows:

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}, \quad C_0 = 1.$$

The Catalan numbers have a rich history and many unique properties. They count certain types of lattice paths, permutations, binary trees, and many other combinatorial objects (see [1, 3] and the references given there).

Using Trudi's formula [2] for determinants and permanents of Toeplitz–Hessenberg matrices with Catalan entries, we obtain some new identities for the Catalan numbers.

Proposition. *Let $n \geqslant 1$, except when noted otherwise. The following formulas hold*:

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) C_0^{t_1} C_1^{t_2} \cdots C_{n-1}^{t_n} = \frac{1}{n+1}\binom{2n}{n} = C_n,$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+\cdots+t_n+1} p_n(t) C_1^{t_1} C_2^{t_2} \cdots C_n^{t_n} = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) C_1^{t_1} C_2^{t_2} \cdots C_n^{t_n} = \binom{2n-1}{n} = (2n-1)C_{n-1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+\cdots+t_n+1} p_n(t) C_2^{t_1} C_4^{t_2} \cdots C_{2n}^{t_n} = \frac{1}{4n-1}\binom{4n}{2n} = \frac{2n+1}{4n-1} C_{2n},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) C_1^{t_1} C_3^{t_2} \cdots C_{2n-1}^{t_n} = \frac{1}{n}\binom{3n-2}{n-1} \cdot {}_2F_1(1-n, -4n; 2-3n; -1),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+\cdots+t_n} p_n(t) C_3^{t_1} C_5^{t_2} \cdots C_{2n+1}^{t_n} = \frac{1}{n}\sum_{i=0}^{n+1} 2^i \binom{2n-1+i}{i}\binom{2n-1}{n+1-i}, \quad n \geqslant 2,$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + nt_n = n$,

$$p_n(t) = \frac{(t_1 + t_2 + \cdots + t_n)!}{t_1! t_2! \cdots t_n!}$$

is the multinomial coefficient, and ${}_2F_1(a, b; c; -1)$ is the generalized hypergeometric function.

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q-Virasoro algebra and affine Lie algebra

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In this talk, we study a certain deformation of the Virasoro algebra that was introduced and called q-Virasoro algebra by Belov and Chaltikian, in the context of vertex algebra. In the process, the relation between q-Virasoro algebra and affine Kac-Moody algebra of type $B_l^{(1)}$ was obtained.

This is a joint work with Hongyan Guo, Haisheng Li and Shaobin Tan.

Topological property of Taimanov semigroups

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We shall follow the terminology of [1–4]. A *(semi)topological semigroup* is a topological space endowed with a (separately) continuous semigroup operation. A topology τ on a semigroup S is defined to be *shift-continuous* if for every $a \in S$ the left and right shifts $l_a \colon S \to S$, $l_a \colon x \mapsto ax$, and $r_a \colon S \to S$, $r_a : x \mapsto xa$, are continuous.

A semigroup T is called *Taimanov* if it contains two distinct elements $0_T, \infty_T$ such that for any $x, y \in T$

$$x \cdot y = \begin{cases} \infty_T & \text{if } x \neq y \text{ and } x, y \in T \setminus \{0_T, \infty_T\}; \\ 0_T & \text{if } x = y \text{ or } \{x, y\} \cap \{0_T, \infty_T\} \neq \varnothing. \end{cases}$$

The elements $0_T, \infty_T$ are uniquely determined by the algebraic structure of T: 0_T is a (unique) zero-element of T, and ∞_T is the unique element of the set $TT \setminus \{0_T\}$.

Proposition 1. *Two Taimanov semigroups are isomorphic if and only if they have the same cardinality.*

The following statement generalizes the original result of Taimanov [5].

Proposition 2. *Every shift-continuous T_1-topology τ on any Taimanov semigroup T is discrete.*

A semitopological semigroup S will be called *square-topological* if the map $S \to S$, $x \mapsto x^2$, is continuous. It is clear that each topological semigroup is square-topological.

Theorem 3. *A Taimanov semigroup T is closed in any square-topological semigroup S containing T as a subsemigroup and satisfying the separation axiom T_1.*

Proposition 4. *Any non-isomorphic homomorphic image S of a Taimanov semigroup T is a zero-semigroup.*

Corollary 5. *Every non-isomorphic homomorphic image S of a Taimanov semigroup is a topological semigroup with respect to any topology on S.*

Also we discuss on embeddings of the Taimanov semigroup into compact-like (semi)topological semigroups.

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Hilbert polynomials of the algebras of SL_2-invariants

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Let V_d be the vector space over $\mathbb{C}$ consisting of all binary forms homogeneous of degree d and let $V_{\mathbf{d}} = V_{d_1} \oplus V_{d_2} \oplus \ldots \oplus V_{d_n}, \mathbf{d} = (d_1, \ldots, d_n)$. Denote by $\mathbb{C}[V_{\mathbf{d}}]^{SL_2}$ the algebra of polynomial SL_2-invariant functions on $V_{\mathbf{d}}$. Its well-known that the algebra $\mathcal{I}_{\mathbf{d}} := \mathbb{C}[V_{\mathbf{d}}]^{SL_2}$ is finitely generated and graded: $I_{\mathbf{d}} = (I_{\mathbf{d}})_0 \oplus (\mathcal{I}_{\mathbf{d}})_1 \oplus \ldots \oplus (\mathcal{I}_{\mathbf{d}})_i \oplus \ldots$, where $(\mathcal{I}_{\mathbf{d}})_i$ is a vector $\mathbb{C}$-space of invariants of degree n. The Hilbert function of the algebra $\mathcal{I}_{\mathbf{d}}$ is defined as dimension of the vector space $(\mathcal{I}_{\mathbf{d}})_i : \mathcal{H}(\mathcal{I}_{\mathbf{d}}, i) = \dim(I_{\mathbf{d}})_i$. It is well-known [1]-[3] that the Hilbert function of finitely generated graded K-algebra is equal (starting from some n) to a polynomial of n:

$$\mathcal{H}(\mathcal{I}_{\mathbf{d}}, i) = h_0(i)i^r + h_1(i)i^{r-1} + \ldots,$$

where $h_j(i)$ is some periodic function with values in $\mathbb{Q}$. Then such a polynomial is called the Hilbert polynomial of graded algebra. From combinatorial point of view the Hilbert polynomials are so-called quasi-polynomials, see [4], Chapter 4.

For the case of one binary form ($n = 1$) there exists ([5], [6]) classical Cayley-Sylvester formula for calculation of values of Hilbert function of $\mathcal{I}_{\mathbf{d}}$ We calculated both Hilbert functions and Hilbert polynomials for the following cases:

- algebra $\mathcal{I}_1^{(n)} = \mathbb{C}[\underbrace{V_1 \oplus V_1 \oplus \cdots \oplus V_1}_{n \text{ times}}]$ of joint invariants for the n linear binary forms ($d_1 = \ldots = d_n = 1$);
- algebra $\mathcal{C}_1^{(n)} = \mathbb{C}[\underbrace{V_1 \oplus V_1 \oplus \cdots \oplus V_1}_{n \text{ times}} \oplus \mathbb{C}^2]$ of joint covariants for the n linear binary forms;
- algebra $\mathcal{I}_2^{(n)} = \mathbb{C}[\underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}}]$ of joint invariants for the n quadratic binary forms ($d_1 = \ldots = d_n = 2$);
- algebra $\mathcal{C}_2^{(n)} = \mathbb{C}[\underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}} \oplus \mathbb{C}^2]$ of joint covariants for the n quadratic binary forms.

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Zero-divisors graphs of finite semigroups

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The concept of zero-divisor graph firstly was introduced for the commutative rings (semigroups) and after that Redmond in [4], Akbari and others in [1, 2] had extended this concept to any arbitrary ring (non-commutative semigroups [3, 5]).

In a manner analogous to the commutative case, the zero-divisor graph of a non-commutative semigroup S can be defined as the directed graph $\Gamma(S)$ whose vertices are all non-zero zero-divisors of S in which for any two distinct vertices x and y, $x \to y$ is an edge if and only if $xy = 0$.

We shall discuss the interplay between the properties of a matrix semigroup S over a finite ring R and the graph-theoretic properties of $\Gamma(S)$, $\Gamma(R)$: the connectedness, diameter, existence of sources, sinks, etc.

The author acknowledges professor M. Ya. Komarnytskiy for his suggestions.

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Compactness in abelian categories

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An object c of an abelian category closed under coproducts is called $\mathcal{C}$*-compact* if the covariant functor $\mathrm{Hom}(c, -)$ commutes with all direct sums of objects from a class $\mathcal{C}$ i.e. there is a canonical isomorphism between $\mathrm{Hom}(c, \bigoplus \mathcal{D})$ and $\bigoplus \mathrm{Hom}(c, \mathcal{D})$ in the category of abelian groups for every subsystem $\mathcal{D} \subseteq \mathcal{C}$. The main objective of the talk is to translate several results on compactness from the context of module categories to the case of general abelian categories.

Note that each finitely generated module presents an elementary example of a compact object in a category of modules over a ring, nevertheless the class of all compact modules, which are called *small*, is much larger in general. An important class of compact objects in module categories are so called *self-small* modules, i.e. modules M which are $\mathrm{Add}(M)$-compact in the category of direct sums of copies of M.

Small as well as self-small modules can be characterized by the condition on submodules which cane be formulated in general case of abelian categories $\mathcal{A}$ with class of objects $\mathcal{C}$.

Theorem 1. *The following conditions are equivalent for an object M:*

(1) M is not $\mathcal{C}$-compact,

(2) there exists a countably infinite system $\mathcal{N}_\omega$ of objects from $\mathcal{C}$ and $\varphi \in \mathcal{A}(M, \bigoplus \mathcal{N}_\omega)$ such that $\rho_N \circ \varphi \neq 0$ for every $N \in \mathcal{N}_\omega$,

(2) for every system $\mathcal{G}$ of $\mathcal{C}$-compact objects and every epimorphism $e \in \mathcal{A}(\bigoplus \mathcal{G}, M)$ there exists a countable subsystem $\mathcal{G}_\omega \subseteq \mathcal{G}$ such that $f^c \circ e \circ \nu_{\mathcal{G}_\omega} \neq 0$ for the cokernel f^c of every morphism $f \in \mathcal{A}(F, M)$ where F is a $\mathcal{C}$-compact object.

Our main result describes classes of compact objects closed under products, which generalizes results from the paper [2] and dualizes those presented in [1]. For that purpose we need a notion of a $\prod \mathcal{C}$*-compactly generated* complete Abelian category $\mathcal{A}$ for which there is a set $\mathcal{G}$ of objects of $\mathcal{A}$ that generates $\mathcal{A}$ and every product of a system of objects in $\mathcal{G}$ is $\mathcal{C}$-compact.

Theorem 2. *Let $\mathcal{A}$ be a $\prod \mathcal{C}$-compactly generated category, $\mathcal{M}$ a family of $\mathcal{C}$-compact objects of $\mathcal{A}$. If we assume that there is no strongly inaccessible cardinal, then every product of $\mathcal{C}$-compact objects is $\mathcal{C}$-compact.*

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Indecomposable modules over Kantor superalgebras

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In this talk we plan to give a survey of both classical and recent results obtained in the representation theory of Jordan algebras and superalgebras. Further we will construct indecomposable representations of Kantor superalgebra $Kan(n)$, $n \leqslant 1$. Our main tool is the famous Tits-Kantor-Koecher construction. The representations of superalgebra $Kan(n)$ are given in terms of Ext quiver algebras of the category of representations with the short grading for Poisson superalgebra $\mathfrak{po}_{n+3}$.

This is joint result with Vera Serganova.

On identity in finite rings

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In the [1], authors was shown an algorism to determine whether or not a given finite ring has identity elements. However, in the work [2] the following theorem will give a more practical algorism to determine existence of identity elements.

Theorem.

Let $\{\alpha_{ijk}\}_{i,j,k=1}^{n}$ be a set of structure constants for the Abelian group

$$A = \langle a_1\rangle \oplus \langle a_2\rangle \oplus \cdots \oplus \langle a_n\rangle, \quad \text{where} \quad \langle a_i\rangle \cong C(p^{e_i}), \quad 1 \leqslant e_1 \leqslant e_2 \leqslant \cdots \leqslant e_n. \tag{1}$$

Let R be the ring whose additive group is (1) and whose multiplication is defined by $a_i a_k = \sum_{j=1}^{n} \alpha_{ijk} a_j,\ 1 \leqslant i, k \leqslant n.$

Then:

(I) R has a left (right) identity if and only if there exist integers $c_1, c_2, \cdots, c_n$ such that $0 \leqslant c_i \leqslant p^{e_i} - 1,\ 1 \leqslant i \leqslant n$ and $\sum_{i=1}^{n} c_i \alpha_{ijk} \equiv \delta_{jk} \quad (\sum_{i=1}^{n} c_i \alpha_{kji} \equiv \delta_{jk})\ (mod\ p^{e_j}),\ 1 \leqslant i, k \leqslant n.$

(II) R has an identity if and only if there exist integers $c_1, c_2, \cdots, c_n$ such that $0 \leqslant c_i \leqslant p^{e_i} - 1,\ 1 \leqslant i \leqslant n$ and $\sum_{i=1}^{n} c_i \alpha_{ijk} \equiv \sum_{i=1}^{n} c_i \alpha_{kji} \equiv \delta_{jk}\ (mod\ p^{e_j}),\ 1 \leqslant i, k \leqslant n.$

The is δ_{ij} denotes the Kronecker's delta.

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Self-returning and centroids of polygons of n-ary groups

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The applications of the theory of n-ary groups in affine geometry were first found by D. Vakarelov in [1]. Rusakov S. A. in [2] generalized many of the results of D. Vakarelov and gave a new impetus to the development of this direction. In particular, S. A. Rusakov [2] constructed an affine space $W(G)$ by the method of fundamental sequences of vectors of a semibell n-ary rs-group G. Further development of applications of the theory of n-ary groups in affine geometry was obtained by Yu. I. Kulazhenko (see, for example, [3]). In the same paper [3], a new direction of research, the self-returning of elements of n-ary groups, was reflected.

In the work presented, this line of research is continued, namely, new results are obtained by the analytical methods of determining the centroid of an arbitrary $2k$-gon. It is established that for a partition of an arbitrary $2k$-gon by arbitrary triangles, their centroids form a sequence of points with respect to which an arbitrary element of the n-ary group self-returning.

Theorem 1 *Let G be semiabelien n-ary group, $a_1, a_2, ..., a_{2k}, b$ be arbitrary points from G $(k \in N)$, x_1 be centroid $< a_1, a_2, ..., a_{2k-1}, b >$. If x is centroid $< a_1, a_2, ..., a_{2k} >$, then equality $\overrightarrow{xx_1} = \frac{1}{2k}\overrightarrow{a_{2k}b}$ is satisfied.*

Theorem 2 *Let G be semiabelien n-ary group, $a_1, a_2, ..., a_{2k}, d$ be arbitrary points from G $(k \in N)$. If x_1 is centroid $< a_1, a_2, d >$, x_2 is centroid $< a_2, a_3, d >$,..., x_i is centroid $< a_i, a_{i+1}, d >$, ..., x_{2k} is centroid $< a_{2k}, a_1, d >$, then arbitrary points from G self-returning with the sequence of vertices $2k$-gons $< x_1, x_2, ..., x_{2k} >$.*

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On Lie algebras of derivations with large abelian ideals

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Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $A = \mathbb{K}[x_1, \dots, x_n]$ the polynomial ring and $R = \mathbb{K}(x_1, \dots, x_n)$ the field of rational functions in n variables. The Lie algebra $\widetilde{W}_n(\mathbb{K})$ of all $\mathbb{K}$-derivations of the field R is a vector space of dimension n over R with the standard basis $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$. The structure of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ and its subalgebras is of great interest because in case $\mathbb{K} = \mathbb{R}$ every element of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ is of the form

$$D = \varphi_1(x_1, \dots, x_n)\frac{\partial}{\partial x_1} + \dots + \varphi_n(x_1, \dots, x_n)\frac{\partial}{\partial x_n}$$

for some rational functions $\varphi_1, \dots, \varphi_n$ in R and D can be considered as a vector field on $\mathbb{R}^n$ with rational coefficients. Such Lie algebras were studied by many authors (see, for example, [1-4]). If L is a subalgebra of the Lie algebra $\widetilde{W}_n(\mathbb{K})$, then one can define the rank of L: $\mathrm{rk}_R L = \dim_R RL$ (note that RL being a vector space over the field R is not in general a Lie algebra over R). Finite dimensional (over $\mathbb{K}$) subalgebras of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ of maximal rank n over R are especially interesting from many points of view (see [2]). We study such subalgebras L provided that L contains an abelian ideal of rank n. We prove that such a subalgebra of $\widetilde{W}_n(\mathbb{K})$ is isomorphic (under some restrictions) to a subalgebra of the general affine Lie algebra $ga_n(\mathbb{K})$. Recall that the general affine Lie algebra $ga_n(\mathbb{K})$ is the semidirect product $ga_n(\mathbb{K}) = gl_n(V) \leftthreetimes V$, where V is the n-dimensional vector space over the field $\mathbb{K}$ (with the natural action of $gl_n(V)$ on V).

Theorem. *Let L be a finite dimensional subalgebra of $\widetilde{W}_n(\mathbb{K})$ with $\mathrm{rk}_R L = n$. If L contains an abelian ideal I of rank n over R and there exists an element $D \in L$ such that the linear operator* $\mathrm{ad}D$ *acts nonsingularly on I, then there exist elements $D_1, \dots, D_n \in I$ and $a_1, \dots, a_n \in R$ such that $D_i(a_j) = \delta_{ij} \cdot 1$, $i, j = 1, \dots, n$. Every element of L can be written uniquely in the form $S = f_1(a_, \dots, a_n)D_1 + \dots + f_n(a_, \dots, a_n)D_n$ for some linear polynomials $f_i(x_1, \dots, x_n) \in A$. In particular, the Lie algebra L is isomorphic to a subalgebra of the general affine Lie algebra $ga_n(\mathbb{K})$.*

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On the class of finite groups whose maximal B-groups are all Hall

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A non-nilpotent finite group whose proper subgroups are all nilpotent is called a Schmidt group. Let G be a finite group. Following Berkovich ([1], Definition 1; [2], P. 461), we said that G is a B-group if the factor-group $G/\Phi(G)$ is a Schmidt subgroup. Here $\Phi(G)$ is the Frattini subgroup of G. It is clear that every Schmidt group is a B-group. The dihedral group of order 18 is not a Schmidt group but it is a B-group.

Denote by $\mathfrak{B}$ the class of all finite groups whose maximal B-groups are all Hall. Groups with Hall Schmidt subgroups are studied in [3]. Clearly that all nilpotent groups, all B-groups, and all groups of square free orders belong to $\mathfrak{B}$. Maximal biprimary non-nilpotent groups from $\mathfrak{B}$ are also B-groups. The expansion of an extraspecial group of order 409^3 by a cyclic group of order $5 \cdot 41$ is a triprimary group from the class $\mathfrak{B}$.

Theorem. *The class* $\mathfrak{B}$ *is a normally hereditary homomorph, and every group of* $\mathfrak{B}$ *has a Sylow tower.*

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Effective generation of free groups of permutations

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For arbitrary integer B let us denote by B_3 the unbalanced ternary representation of B. If $B < 0$ we will use signed notation, i.e. $-B_3$. Denote by $S_{10}(B_3)$ the (decimal) sum of all digits in B_3, by $K_{10}(B_3)$ the (decimal) amount of all digits in B_3 and by $I(B)$ the remainder of $S_{10}(B_3) - K_{10}(B_3)$ divided by 4. Let UN_i denotes the string consisting of i consecutive digits 1, $i \geqslant 0$. Denote by $C(B_3)$ the operation of removing the last digit in B_3.

For each $r \in \mathbb{N}$ we define a mapping $\Phi_r : \mathbb{Z} \to \mathbb{Z}$. Let $B \in \mathbb{Z}$, $B_3 = \dots b_3 b_2 UN_{j'} b_1 UN_j$. The definition of $\Phi_r(B)$ splits into the following three types of cases.

Type I.

1. If $B = 0$ then $\Phi_r(B_3) = -2UN_{r-1}22UN_{r-1}$.
2. If $B_3 = 2UN_j$ and $j \leqslant r-2$ then $\Phi_r(B_3) = -2UN_{r-j-2}22UN_{r-1}$.
3. If $B_3 = -10UN_j$ and $j \leqslant r-2$ then $\Phi_r(B_3) = 2UN_{r-1}$.
4. If $B_3 = -2UN_{j'}00UN_j$ and $j' = j = r-1$, then $\Phi_r(B_3) = 0$.
5. If $B_3 = -2UN_{j'}00UN_j$ and $j' < j \leqslant r-1$, then $\Phi_r(B_3) = 2UN_{j-j'-1}$.

Type II. Assume that $B > 0$.

1. If $I(B) = 0$ then $\Phi_r(B_3) = B_3 0UN_{r-1}22UN_{r-1}$.
2. If $I(B) = 2$ then $\Phi_r(B_3) = B_3 2UN_{r-1}22UN_{r-1}$.
3. If $I(B) = 3$ then $\Phi_r(B_3) = B_3 UN_r 22UN_{r-1}$.
4. If $I(B) = 1$ and $r < j+1$ then $\Phi_r(B_3) = C^r(B_3)02UN_{r-1}$.
5. If $I(B) = 1$, $r > j+1$, and $b_1 = 0$ then $\Phi_r(B_3) = C^{j+1}(B_3)2UN_{r-j-2}22UN_{r-1}$.
6. If $I(B) = 1$, $r > j+1$, and $b_1 = 2$ then $\Phi_r(B_3) = C^{j+1}(B_3)0UN_{r-j-2}22UN_{r-1}$.
7. If $I(B) = 1$, $r = j+1$ and $b_1 = 2$ then $\Phi_r(B_3) = C^r(B_3)12UN_{r-1}$.
8. If $I(B) = 1$, $r = j+1$, $b_1 = 0$, and $b_2 = 1$ then $\Phi_r(B_3) = C^{r+1}(B_3)0UN_{r-1}$.
9. If $I(B) = 1$, $r = j+1$, $b_1 = 0$, and $b_2 = 2$ then $\Phi_r(B_3) = C^{r+1}(B_3)UN_r$.
10. If $I(B) = 1$, $r = j+1$, $b_1 = 0$, $b_2 = 0$, and $r \leqslant j'+1$ then $\Phi_r(B_3) = C^{r+1+j'}(B_3)$.
11. If $I(B) = 1$, $r = j+1$, $b_1 = 0$, $b_2 = 0$, $r > j'+1$ and $b_3 = 0$ then $\Phi_r(B_3) = C^{r+2+j'}(B_3)2UN_{r-j'-2}$.
12. If $I(B) = 1$, $r = j+1$, $b_1 = 0$, $b_2 = 0$, $r > j'+1$ and $b_3 = 2$ then $\Phi_r(B_3) = C^{r+2+j'}(B_3)0UN_{r-j'-2}$.

Type III. Assume that $B < 0$. In this case compute the remainder I of $I(|B|) + 2)$ divided by 4 and then $\Phi_r(B_3) = -\overline{\Phi}_r(|B|)$. Here $\overline{\Phi}_r(|B|)$ is computed as $\Phi_r(|B|)$ in cases of Type II with replacement I instead of $I(|B|)$.

Then the definition is correct and one can check that each Φ_r, $r \geqslant 1$, is a permutation on the set $\mathbb{Z}$. Denote by G the subgroup of the symmetric group on $\mathbb{Z}$, generated by these permutations.

Theorem. *The group G is a free group with basis $\{\Phi_r(B) : r \geqslant 1\}$.*

On Auslander algebra of the symmetric semigroup of degree 2

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Let T_2 denote the symmetric semigroup of degree 2 (i.e. the semigroup of all transformations of the set with 2 elements). There are generators e, a, b with the following defining relations: 1) $e^2 = e, ea = ae = a, eb = be = b$; 2) $a^2 = e$, $b^2 = b$; 3) $ab = b$.

The indecomposable representations of the semigroup T_2 over a field k are classified in [1] if the characteristic of k is not equal to 2, and in [2] if the characteristic is equal to 2.

By definition the Auslander algebra of a semigroup of finite representation type is the algebra of endomorphisms of the direct sum of all indecomposable representations (from each equivalence class of indecomposable representations it need to choose only one representative).

We continue study the representations of the semigroup T_2 and describe the Auslander algebra in the both cases.

These studies were carried out together with Prof. V. M. Bondarenko.

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A criterion for a finite group to be σ-soluble

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All considered groups are finite and G always denotes a finite group. The subgroups A and B of G are called *isoordic* if $|A| = |B|$.

Let σ be some partition of the set of all primes $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$, and following [1, 2], we put $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \varnothing\}$. G is said to be [3]: *σ-primary* if G is a σ_i-group for some i; *σ-soluble* if every chief factor of G is σ-primary.

A subgroup A of G is called *σ-subnormal* [3] in G if there is a subgroup chain

$$A = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$.

Denote by $i_\sigma(G)$ *the number of classes of isoordic non-σ-subnormal subgroups* of G. We prove the following criterion of σ-solubility of groups.

Theorem. *If $i_\sigma(G) \leqslant 2|\sigma(G)|$, then G is σ-soluble.*

In the classical case when $\sigma = \sigma^0 = \{\{2\}, \{3\}, \ldots\}$, we get from Theorem the following result.

Corollary 1 [4, Theorem 1.1(1)]. *If the number of conjugacy classes of non-subnormal subgroups of G is at most $2|\pi(G)|$, then G is soluble.*

In the other classical case when $\sigma = \sigma^\pi = \{\pi, \pi'\}$, we get from Theorem the following

Corollary 2. *If $i_{\sigma^\pi}(G) \leqslant 4$, then G is π-separable.*

Finally, in the case when $\sigma = \sigma^{0\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$, we get from Theorem the following

Corollary 3. *If $i_{\sigma^{0\pi}}(G) \leqslant 2|\sigma^{0\pi}(G)|$, then G is π-soluble.*

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Constructing finite trees for given maps

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Let V be a finite set. Denote by $\mathcal{T}(V)$ the class of all self-maps on V. Also, let $Tree(V)$ denotes the collection of all trees having the vertex set V.

We study the following problem: given a function $\mathcal{F} : Tree(V) \to 2^{\mathcal{T}(V)}$ which assigns to each tree $X \in Tree(V)$ a class of maps $\mathcal{F}(X)$, characterize the union $\bigcup_{X \in Tree(V)} \mathcal{F}(X)$. We consider $\mathcal{F}(X)$ being the class of expansive, anti-expansive, metric, linear, neighborhood maps, automorphisms of X and maps having weakly or strongly connected Markov graphs. In particular, the following results hold.

Proposition 1. *For a map $\sigma \in \mathcal{T}(V)$ there exists a tree $X \in Tree(V)$ such that σ is a neighborhood map on X if and only if each σ-periodic point has a period at most two.*

Proposition 2. *For a map $\sigma \in \mathcal{T}(V)$ there exists a tree $X \in Tree(V)$ such that σ is anti-expansive on X if and only if σ has a unique fixed point.*

Theorem 1. *For a permutation $\sigma \in \mathcal{P}(V)$ there exists a tree $X \in Tree(V)$ such that σ is expansive on X if and only if $|V - fix\,\sigma|$ is even.*

Theorem 2. *For a map $\sigma \in \mathcal{T}(V)$ the following statements are equivalent:*

1. *$fix\,\sigma \neq \varnothing$, or there exists σ-periodic point with period two and all σ-periodic points have even periods;*
2. *there exists a tree $X \in Tree(V)$ such that σ is metric on X;*
3. *there exists a tree $X \in Tree(V)$ such that σ is linear on X;*
4. $\min\limits_{X \in Tree(V)} |A(\Gamma(X,\sigma))| = |Im\,\sigma| - 1.$

Theorem 3. *Let $|V| \geqslant 3$. For a map $\sigma \in \mathcal{T}(V)$ the following statements are equivalent:*

1. *σ is constant or $\sigma = id_V$;*
2. *for every tree $X \in Tree(V)$ the Markov graph $\Gamma(X,\sigma)$ is disconnected;*
3. *for every tree $X \in Tree(V)$ the map σ is metric on X;*
4. *for every tree $X \in Tree(V)$ the map σ is linear on X.*

Proposition 3. *Let $|V| \geqslant 2$. For a map $\sigma \in \mathcal{T}(V)$ the Markov graph $\Gamma(X,\sigma)$ is strongly connected for every tree $X \in Tree(V)$ if and only if n is a prime number and σ is a cyclic permutation.*

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About classification of small length generalized binary functional equations on quasigroups

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A *functional equation* [1] is a universal quantified equality of two terms consisting of functional and individual variables. A *length* of an equation is the number of all functional variable occurrences. A functional equation is called:
— *generalized*, if all functional variables are pairwise different;
— *binary*, if all functional variables are assumed to be binary;
— *quasigroup*, if all considered solutions are supposed to be invertible functions.

Binary quasigroup functional equations of the length not grater than 6 are under consideration. The study of the equations is provided by their classification according to which functional equations belong to the same class, if their solutions are mutually expressible [2]. A number sequence $(m_1, \dots, m_k)$ is called a *type* of a functional equation, if k is the number of different individual variables and m_i is the number of occurrences of the i-th variable.

There are one class, three classes and four classes of equations of the length 1, 2 and 3 respectively [3].

There exist no more than 19 classes of the equations of length 4: two classes of the type (6;0;0), six classes of both types (4;2;0) [4] and (3;3;0) [3] and five equations of the type (2;2;2) [5, 6]. Besides, (2;2;2)-class contains the well-known functional equation of the generalized associativity. Using parastrophic symmetry [7], a connection between parastrophic functional equations and the corresponding varieties is established.

One of functional equations of the length 5 is the well-known functional equation of generalized distributivity, its type is (3;2;2). There exist five classes of equations of the type (3;2;2) [8]. Functional equations of the types (7;0;0), (3;4;0) and (5;2;0) are also studied.

The most investigated functional equations of the length 6 are the functional equation of generalized mediality (its type is (2;2;2;2)) and Bol-Moufang functional equation (its type is (4;2;2;0)). There are eight classes of equations of the type (4;2;2;0).

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On classification of non-trivial binary-ternary quasigroup equations in two functional variables

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A *functional equation* [1] is a universal quantified equality of two terms consisting of functional and individual variables. We assume that the equations are: *generalized* (all functional variables are pairwise different), *quasigroup ones* [2] (all considered solutions are sequence of invertible functions) and *non-trivial*. Non-triviality implies that each individual variable appears at least twice.

Let Q be a set. A mapping $f:\ Q^2 \to Q$ is called a *binary invertible function*, if there exist functions f_1 and f_2 such that for any $x, y \in Q$

$$f(f_1(x,y),y) = x, \quad f_1(f(x,y),y) = x, \qquad f(x,f_2(x,y)) = y, \quad f_2(x,f(x,y)) = y. \qquad (1)$$

A mapping $g:\ Q^3 \to Q$ is called a *ternary invertible function*, if there exist functions g_1, g_2, g_3 such that for any $x, y, z \in Q$ the following identities

$$\begin{aligned} &g(g_1(x,y,z),y,z) = x, \qquad g(x,g_2(x,y,z),z) = y, \qquad g(x,y,g_3(x,y,z)) = z,\\ &g_1(g(x,y,z),y,z) = x, \qquad g_2(x,g(x,y,z),z) = y, \qquad g_3(x,y,g(x,y,z)) = z \end{aligned} \qquad (2)$$

hold.

A functional equation is called *binary-ternary*, if it has both binary and ternary functional variables. Two functional equations are called *parastrophically-primarily equivalent* [3], if one can be obtained from the other in a finite number of applications (1), (2).

Theorem 1. *Each non-trivial binary-ternary functional quasigroup equation in two functional variables is parastrophically-primarily equivalent to exactly one of the following function equations:*

$$F(x,y) = G(x,y,y), \quad (3) \qquad F(y,y) = G(x,x,x), \quad (4)$$

$$F(x,x) = G(x,y,y), \quad (5) \qquad F(x,x) = G(x,x,x). \quad (6)$$

Let $(Q;+,0)$ be a group. A binary function f and a ternary function g are called *linear over* $(Q;+,0)$, if there exist automorphisms α, β, γ_1, γ_2, γ_3 of $(Q;+,0)$ and elements a, b such that:

$$f(x,y) = \alpha x + \beta y + a, \qquad g(x,y,z) = \gamma_1 x + \gamma_2 y + \gamma_3 z + b. \qquad (7)$$

Theorem 2. *A pair (f,g) of linear functions (see (7)) is a solution*

1. *of (3) if and only if $a = b$, $\alpha = \gamma_1$, $\beta = \gamma_2 + \gamma_3$;*
2. *of (4) if and only if $a = b$, $\alpha = -\beta$, $\gamma_1 + \gamma_2 + \gamma_3 = 0$;*
3. *of (5) if and only if $a = b$, $\gamma_2 = -\gamma_3$, $\gamma_1 = \alpha + \beta$;*
4. *of (6) if and only if $a = b$, $\alpha + \beta = \gamma_1 + \gamma_2 + \gamma_3$.*

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Unitary similarity criterion for infinite upper triangular matrices

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The necessary and sufficient conditions that classify unicellular matrices up to unitary similarity are given in [1, 2]. We extend Theorem 3.2 from [1] to indefinite matrices as follows.

Theorem. *Let $A = [a_{ij}]_{i,j=1}^{\infty}$ be an infinite upper triangular matrix such that $a_{11} = a_{22} = a_{33} = \cdots$ and $a_{i,i+1} \neq 0$ for all i (that is, the first superdiagonal of A has only nonzero entries). Then the following statements are equivalent for any infinite upper triangular matrix $A' = [a'_{ij}]_{i,j=1}^{\infty}$:*

- $\|f(A_k)\| = \|f(A'_k)\|$ *for all* $k \in \mathbb{C}[t]$ *and* $k = 1, 2, \ldots$, *where* $A_k := [a_{ij}]_{i,j=1}^{k}$, $A'_k := [a'_{ij}]_{i,j=1}^{k}$, *and* $\|\cdot\|$ *is the spectral matrix norm,*
- $A' = W^*AW$ *for some infinite diagonal unitary matrix* W.

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Serial group rings of simple finite groups

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Let R be a ring and let G be a finite group. The following question is due to Tuganbaev [1, Probl. 16.9]. To find all pairs (R, G) such that the group ring RG is serial. The seriality of RG means that each indecomposable projective right (equivalently left) RG-module has a unique composition series.

We will answer this question in the case when $R = F$ is a field of characteristic p and G is a finite simple group. Most classes of finite simple groups were considered in our previous papers (for instance, see [2]). The only remaining case is given by classical groups defined over a field with even number of elements.

Here is the final list.

Theorem 1. *Let G be a finite simple group and let F be a field of characteristic p dividing the order of G. Then the group ring FG is serial if and only if one of the following holds.*

1) $G = C_p$;

2) $G = \mathrm{PSL}_2(q)$ and $p > 2$ divides $q - 1$;

3) $G = \mathrm{PSL}_2(q)$, $q \neq 2$ or $G = \mathrm{PSL}_3(q)$, where $p = 3$ and $q \equiv 2, 5 \pmod 9$;

4) $G = \mathrm{PSU}_3(q^2)$ and $p > 2$ divides $q - 1$;

5) $G = \mathrm{Sz}(q)$, $q = 2^{2n+1}$, $n \geqslant 1$, where either $p > 2$ divides $q - 1$, or $p = 5$ divides $q + r + 1$, $r = 2^{n+1}$, but 25 does not divide this number;

6) $G = {}^2G_2(q^2)$, $q^2 = 3^{2n+1}$, $n \geqslant 1$, where either $p > 2$ divides $q^2 - 1$, or $p = 7$ divides $q^2 + \sqrt{3}q + 1$, but 49 does not divide this number;

7) $G = M_{11}$, $p = 5$ or $G = J_1$, $p = 3$.

The research was partially supported by BRFFI grant F17RM-063.

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Mathematical modeling container network via a connected graph

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The task was to route messages in a container ship system based on the effective implementation of all processes occurring in this system. The state of the system is determined by the huge number of messages processed and waiting in the queue for processing messages, and you need to determine the optimal time for processing messages in the root node, while spending the minimum time (allocating duplicates).

We represent the network in the form of a connected graph $G = (V, U)$, where V is the set of vertices or nodes, U is the set of edges, and the variable set W of messages (packets) that are generated during network operation. Consider a network with a fixed number of nodes n. Other components of the network are considered random. The graph set $G = (V, U)$ is formed as follows. Random number generated $|U|$ — the power of the set of vertices, starting from the discrete probability distribution of P_U. From the set $V \times V \backslash diag(V \times V)$, we equally choose $|U|$. Different vertices that form a random set U. The generated graph $G = (V, U')$ is connected (from a tree with one vertex to a tree with n vertices), then $U' \in U$.

On the network, each node $v_i, i = 1..n$ is the source of the message flow w_i to some root node $v_j, j = 1..n, j \neq i$, in which the message of this stream should be processed. The flow $w_i \in W$ is a random discrete process with a finite number of message transfer events. The number of events in the stream is a random variable with a discrete probability distribution P_S. Its value does not exceed P. The time intervals τ_i between successive events in the flow are described by the probability distribution functions $A_i(x)$ (how many nodes will be involved in the message transmission).

Primary processes w_i generate threads of duplicate messages, each transmitted message in the network goes through the target processing node. When a message arrives at the target node, it is processed for a time i with a probability distribution function $R_i(x)$ (due to overloading the central node and creating a FIFO queue).

Totally, the primary processes form a random process W with a finite number of events. This process is not uniquely determined by the set of primary processes $\{w_i\}$. It depends on the routing algorithm R. The optimal routing is used, based on the forecast of waiting times and the processing of the message from the source node to the target node. The main tasks of the analysis are to study such characteristics of a distributed random process W:

1) time α responses to the message from the moment of generation by the source node to the moment it was received by the target node;

2) time β processing the stream of duplicate messages (from the generation of the first message to the receipt of the last message from one Bays;

3) the ratio of the number of messages in the process W to the total possibility of arrival of events in the primary processes $\{w_i\}$.

The probability distribution functions were calculated under the given conditions, $R_M(x) = P\{\alpha \leqslant x\}$ and $R_S(x) = P\{I\beta \leqslant x\}$, for the number of nodes in the network $n = 1...20$.

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Thomsen's figure, centroids and self-returning elements of n-ary groups

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As the n-ary analogue of parallelism, the notion of a parallelogram of the n-ary group G introduced by S. A. Rusakov in [1] is taken. Theorem 1 establishes the existence of a sequence of parallelograms that are characteristic of the above-mentioned Thomsen's figure. Theorem 2 establishes that the vertices of the constructed Thomsen's figure taken in a certain sequence form such triangles that their centroids coincide.

Of interest, in our opinion, are the corollaries obtained from Theorem 2. Corollary 2.1 confirms that an arbitrary point is self-returning with elements of the sequence of vertices of a specially constructed hexagon: a Thomsen figure. In Corollary 2.2, the property of a specially constructed hexagon (Thomsen's figure) is established, connected with its centroid.

Theorem 1. *If G is semiabelien n-ary group, a, b, c are arbitrary points from G, points x, y such, that tetragons $< c, x, S_{S_b(a)}(a), b >$, $< x, S_b(a), a, c >$, $< S_b(a), S_c(a), y, S_{S_b(a)}(a) >$ are parallelograms G, then tetragon $< y, b, c, S_{S_c(a)}(c) >$ is parallelogram G.*

Theorem 2. *Let a, b, c be arbitrary points from n-ary group G. n-Ary group G is semiabelien if and only if centroids of at least two of the following triangles $< c, S_b(a), [S_c(a)a^{[-2]}\overset{2n-4}{a}b] >$, $< a, S_{S_b(a)}(b), S_{S_c(a)}(c) >$, $< b, [ca^{[-2]}\overset{2n-4}{a}S_b(a)], S_c(a) >$ coincide.*

Consequence 2.1. *Let a, b, c be arbitrary points from semiabelien n-ary group G. An arbitrary point $p \in G$ is self-returning with elements of the sequence of vertices of a hexagon $< c, b, S_b(a), [ca^{[-2]}\overset{2n-4}{a}S_b(a)], [S_c(a)a^{[-2]}\overset{2n-4}{a}b], S_c(a) >$.*

Consequence 2.2. *Let a, b, c be arbitrary points from semiabelien n-ary group G. Centroids of triangles $< c, S_b(a), [S_c(a)a^{[-2]}\overset{2n-4}{a}b] >$, $< b, [ca^{[-2]}\overset{2n-4}{a}S_b(a)], S_c(a) >$ and hexagon $< c, b, S_b(a), [ca^{[-2]}\overset{2n-4}{a}S_b(a)], [S_c(a)a^{[-2]}\overset{2n-4}{a}b], S_c(a) >$ coincide.*

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Leibniz algebras whose subideals are ideals

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An algebra L over a field F is said to be a ***Leibniz algebra*** (more precisely a ***left Leibniz algebra***) if it satisfies the Leibniz identity

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]] \text{ for all } a, b, c \in L.$$

Leibniz algebras are generalizations of Lie algebras.

Leibniz algebra appeared first in the papers of A. M. Bloh [1], in which he called them D-algebras. Real interest in Leibniz algebras arose only after two decades thanks to the work of J. L. Loday [2].

A Leibniz algebra L is called a ***T-algebra***, if a relation "to be an ideal" is transitive. In other words, if A is an ideal of L and B is an ideal of A, then B is an ideal of L. It follows that in a Leibniz T-algebra every subideal is an ideal.

Lie algebras T-algebras have been studied by I. Stewart [3].

Let L be a Leibniz algebra. The subalgebra $\mathbf{Nil}(L)$ generated by all nilpotent ideals of L is called the ***nil radical*** of L. Clearly $\mathbf{Nil}(L)$ is an ideal of L. If $L = \mathbf{Nil}(L)$, then L is called a Leibniz ***nil-algebra***. Every nilpotent Leibniz algebra is a nil-algebra, but converse is not true even for a Lie algebra. Note also that if L is a finite dimensional nil-algebra, then L is nilpotent.

A Leibniz algebra L is called ***hyperabelian*** if it has an ascending series

$$< 0 >= L_0 \leqslant L_1 \leqslant \ldots \leqslant L_\alpha \leqslant L_{\alpha+1} \leqslant \ldots \leqslant L_\gamma = L$$

of ideals whose factors $L_{\alpha+1}/L_\alpha$ are abelian for all $\alpha < \gamma$. If this series is finite, then L is called a ***soluble Leibniz algebra***.

The structure of Leibniz T-algebras essentially depends of the structure of its nil-radical.

Theorem 1. *Let L be a hyperabelian Leibniz T-algebra over a field F. If $\mathbf{char}(F) \neq 2$, then $\mathbf{Nil}(L)$ is abelian.*

We say that a field F is **2-closed**, if the equation $x^2 = a$ has a solution in F for every element $a \neq 0$. We note that every locally finite (in particular, finite) field of characteristic 2 is 2-closed.

Theorem 2. *Let L be a hyperabelian Leibniz T-algebra over a field F. Suppose that L is non-nilpotent and $\mathbf{Nil}(L)$ is non-abelian. If a field F is 2-closed and $char(F) = 2$, then $L = (Fe \oplus Fc) \oplus Fv$ where*

$$[e, e] = c, [c, e] = [e, c] = [c, v] = [v, c] = 0, [v, v] = 0, [v, e] = e + \gamma c = [e, v], \gamma \oplus F.$$

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Lattice of subvarieties of the variety generated by the Grassmann algebra

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Let G be the Grassmann algebra on a countable set of anticommuting generators $\{e_1, e_2, \dots \mid e_i e_j = -e_j e_i\}$ over a field F of characteristic 0. It is well known [1, 2] that the basis for identities of G is formed by the identities $(xy)z = x(yz)$ of associativity and $\big[[x, y], z\big] = 0$ of Lie nilpotency of step 2. Let $\mathcal{V}$ be a variety of algebras over F defined by these two identities and $\mathfrak{A} = F_{\mathcal{V}}[X]$ be a free algebra of $\mathcal{V}$ on a countable set $X = \{x_1, x_2, \dots\}$ of free generators. The skew-symmetry property $[x_i, x_j][x_k, x_\ell] = -[x_i, x_k][x_j, x_\ell]$ yields that an additive base of $\mathfrak{A}$ can be formed by the polynomials

$$[x_{i_1}, x_{i_2}] \dots [x_{i_{2t-1}}, x_{i_{2t}}] x_{i_{2t+1}} \dots x_{i_n},$$
$$i_1 < \dots < i_{2t}, \quad i_{2t+1} < \dots < i_n, \quad t = 0, 1, \dots, \left[\tfrac{n}{2}\right].$$

In the present paper, we describe the lattice $\mathcal{L}(\mathcal{V})$ of subvarieties of $\mathcal{V}$ and compute topological ranks [3, 4] of the elements of $\mathcal{L}(\mathcal{V})$.

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On Minimal Composition Hartley Functions of Partially Composition Fitting Classes

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All groups considered are finite. All unexplained notations and terminologies are standard (see [1–3]). Recall that a group class $\mathfrak{F}$ closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups is called a Fitting class. For any group G we denote by $\mathrm{Com}(G)$ the class of all simple abelian groups A such that $A \cong H/K$, where H/K is a composition factor of G.

For every group class $\mathfrak{F} \supseteq (1)$, by $G^{\mathfrak{F}}$ we denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. In particular, we write $O^{\omega}(G) = G^{\mathfrak{G}_{\omega}}$ and $C_p(G) = G^{\mathfrak{G}_{cp}}$. The symbols (1), $\mathfrak{G}_{\omega}$ and $\mathfrak{G}_{cp}$ denote, respectively, the class of all identity groups, the class of all ω-groups and the class of all groups in which every p-chief factor, that is, a chief factor of p power order (where p is prime), is central.

Let f be a function of the form

$$f : \omega \cup \{\omega'\} \to \{\text{Fitting classes}\}. \tag{1}$$

According to [2] (see [3]) we consider the groups class

$$CR_{\omega}(f) = (G \mid O^{\omega}(G) \in f(\omega') \text{ and } C_p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(\mathrm{Com}(G))).$$

If a Fitting class such that $\mathfrak{F} = CR_{\omega}(f)$ for a function f of the form (1), then $\mathfrak{F}$ is said to be ω-composition and f is said to be an ω-composition Hartley function (shortly, an ω-composition H-function) of $\mathfrak{F}$ (see [2, 3]).

Let $\{f_i \mid i \in I\}$ be a collection of ω-composition H-functions. By $\bigcap_{i\in I} f_i$ we denote the ω-composition H-function f such that $f(a) = \bigcap_{i\in I} f_i(a)$ for all $a \in \omega \cup \{\omega'\}$. Let $\{f_i \mid i \in I\}$ be the collection of all ω-composition H-function of a Fitting class $\mathfrak{F}$. Since the lattice of all ω-composition Fitting class c_{ω} is complete, we conclude that $f = \bigcap_{i\in I} f_i$ is an ω-composition H-function of $\mathfrak{F}$. The H-function f is called minimal (see [2, 3]). We write $c_{\omega}\mathrm{fit}(\mathfrak{X})$ to denote the intersection of all ω-composition Fitting classes containing a collection of groups $\mathfrak{X}$. Thus, $\mathrm{fit}(\mathfrak{X})$ is the intersection of all Fitting classes containing a collection of groups $\mathfrak{X}$.

Theorem. *Let $\mathfrak{X}$ be a non-empty collection of groups, $\mathfrak{F} = c_{\omega}\mathrm{fit}(\mathfrak{X})$, let $\pi = \omega \cap \pi(\mathrm{Com}(\mathfrak{X}))$, and f the minimal ω-composition H-function of $\mathfrak{F}$. Then*:

1) $f(\omega') = \mathrm{fit}(G^{\mathfrak{G}_{\omega}} \mid G \in \mathfrak{X})$;
2) $f(p) = \mathrm{fit}(C_p(G) \mid G \in \mathfrak{X})$ *for all* $p \in \pi$;
3) $f(p) = \varnothing$ *for all* $p \in \omega\backslash\pi$;
4) $\mathfrak{F} = CR_{\omega}(h)$, *where* $h(\omega') = \mathfrak{F}$ *and* $h(p) = f(p)$ *for all* $p \in \pi$.

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Matrix reduction via invertible Hankel matrices and square stable range one

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In the following all rings are assumes to be commutative with $1 \neq 0$. All necessary definitions and facts concerning the topic can be found in [2].

Definition 1. A matrix $A \in M_2(R)$ over a ring R is called a Hankel matrix if there are elements a, b, $c \in R$ such that

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Below we use the notion of square stable range one [1] and also we introduce its modification: unit square stable range one.

Definition 2. A commutative ring R is called a (unit) square stable range one ring if for any pair of coprime elements $a, b \in R$ there is an (unit) element $x \in R$ such that $a^2 + bx \in U(R)$.

Definition 3. A commutative Bezout ring R is called a Hankel ring if for any elements a, $b \in R$ there is an invertible Hankel matrix H and element $d \in R$ such that $(a, b)H = (d, 0)$.

Theorem 1. *Every unimodular row of length 2 over commutative square stable range one ring R is completable to an invertible Hankel matrix.*

Theorem 2. *A commutative Hermite ring is a Hankel ring if and only if it is a ring of square stable range one.*

Theorem 3. *Over commutative elementary divisor ring of square stable range one every 2×2 matrix is diagonalizable via left and right multiplication by invertible Hankel matrices.*

Corollary 1. *Over commutative elementary divisor ring of square stable range one every invertible 2×2 matrix decomposes as a product of invertible Hankel matrices.*

Theorem 4. *Let R be a commutative Hermite ring of unit square stable range one. Then R is an elementary divisor ring and every 2×2 matrix over R is diagonalizable via invertible Hankel matrices.*

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Cramer's Rule for Two-sided Restricted Quaternionic Matrix Equation

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Denote by $\mathbb{H}_r^{m\times n}$ the set of all $m\times n$ matrices with a rank r over the quaternion skew field $\mathbb{H}$. Let $\mathbf{A}_{M,N}^{\dagger}$ be the weighted Moore-Penrose inverse of $\mathbf{A}\in\mathbb{H}_r^{m\times n}$ with weights $\mathbf{M}$ and $\mathbf{N}$ which are Hermitian positive definite matrices of order m and n, respectively. It means $\mathbf{A}_{M,N}^{\dagger}$ denote a unique solution $\mathbf{X}$ of the following equations,

$$\mathbf{AXA}=\mathbf{A},\ \ \mathbf{XAX}=\mathbf{X},\ \ (\mathbf{MAX})^*=\mathbf{MAX},\ \ (\mathbf{NXA})^*=\mathbf{NXA}.$$

Weighted Singular Value Decomposition (WSVD) of quaternion matrices and representation of the weighted Moore-Penrose inverse over the quaternion skew field by WSVD, and then by using this representation, obtaining its limit and determinantal representations have been obtained recently in [1]. Within the framework of the theory of noncommutative column and row determinants introduced in [2], in this study we give determinantal representations of the weighted Moore-Penrose inverse solution of the restricted matrix equation,

$$\mathbf{AXB}=\mathbf{C}, \tag{1}$$

$$\mathcal{R}_r(\mathbf{X})\subset\mathbf{N}^{-1}\mathcal{R}_r(\mathbf{A}^*),\ \mathcal{N}_r(\mathbf{X})\supset\mathbf{P}^{-1}\mathcal{N}_r(\mathbf{B}^*),\ \mathcal{R}_l(\mathbf{X})\subset\mathcal{R}_l(\mathbf{A}^*)\mathbf{M},\ \mathcal{N}_l(\mathbf{X})\supset\mathcal{N}_l(\mathbf{B}^*)\mathbf{Q}, \tag{2}$$

where $\mathbf{A}\in\mathbb{H}_r^{m\times n}$, $\mathbf{B}\in\mathbb{H}_t^{p\times q}$, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{P}$, and $\mathbf{Q}$ are Hermitian positive definite matrices of order m, n, p, and q, respectively. $\mathcal{R}_r(\mathbf{X})$, $\mathcal{N}_l(\mathbf{X})$ are the right column space and the left null space of $\mathbf{X}$. Denote $\mathbf{A}^{\sharp}=\mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}$ and $\mathbf{B}^{\sharp}=\mathbf{Q}^{-1}\mathbf{B}^*\mathbf{P}$. There are cases, when $\mathbf{A}^{\sharp}\mathbf{A}$ and $\mathbf{B}\mathbf{B}^{\sharp}$ both or one of them are Hermitian, and when they are non-Hermitian. In this abstract, we consider only the following.

Theorem. *Let* $\mathbf{A}^{\sharp}\mathbf{A}$ *and* $\mathbf{B}\mathbf{B}^{\sharp}$ *be Hermitian. If* $\mathbf{C}\subset\mathcal{R}_r\left(\mathbf{A}\mathbf{A}^{\sharp},\mathbf{B}^{\sharp}\mathbf{B}\right)$ *and* $\mathbf{C}\subset\mathcal{R}_l\left(\mathbf{A}^{\sharp}\mathbf{A},\mathbf{B}\mathbf{B}^{\sharp}\right)$, *then the unique solution of (1)-(2) is* $\mathbf{X}=\mathbf{A}_{M,N}^{\dagger}\mathbf{C}\mathbf{B}_{P,Q}^{\dagger}$ *and it possess the following determinantal representation*

$$x_{ij}=\frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\operatorname{cdet}_i\left(\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{.\,i}\left(\mathbf{c}_{.j}^{\mathbf{B}}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,n}}\left|\left(\mathbf{A}^{\sharp}\mathbf{A}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{t,p}}\left|\left(\mathbf{B}\mathbf{B}^{\sharp}\right)_{\alpha}^{\alpha}\right|},$$

where $\mathbf{c}_{.j}^{\mathbf{B}}=\left(\sum\limits_{\alpha\in I_{t,p}\{j\}}\operatorname{rdet}_j\left(\left(\mathbf{B}\mathbf{B}^{\sharp}\right)_{j.}\left(\tilde{\mathbf{c}}_{k.}\right)\right)_{\alpha}^{\alpha}\right)\in\mathbb{H}^{n\times 1}$ is the column vector, $k=1,\ldots,n$. $\tilde{\mathbf{d}}_{i.}$ is the ith row of $\tilde{\mathbf{C}}=\mathbf{A}^{\sharp}\mathbf{C}\mathbf{B}^{\sharp}$ for all $i=1,\ldots,n$, $j=1,\ldots,p$.

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Semiperfect principal ideal rings

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Definition. A ring A is called a **principal ideal ring** if all its right ideals are right principal and all its left ideals are left principal.

Recall that the ring $\mathcal{O}$ (not necessary commutative) is called a **principal ideal domain** if it has no zero divisors and all its right and left ideals are principal.

Theorem 1. *If every two-sided ideal in an Artinian ring A is a right principal ideal and also a left principal ideal then A is isomorphic to a finite direct product of full matrix rings over Artinian uniserial rings.*

Note that, conversely, each right ideal in such a ring is a right principal ideal and every left ideal is a left principal ideal.

Theorem 2. *Let A be a semiperfect ring such that every two-sided ideal in A is both a right principal ideal and a left principal ideal. Then A is a principal ideal ring isomorphic to a direct product of a finite number of full matrix rings over Artinian uniserial rings and local principal ideal domains. Conversely, all such rings are semiperfect principal ideal rings.*

On the normal structure of a transitive Sylow p-subgroup of the finitary symmetric group

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This is a continuation of the joint work with Agnieszka Bier and Vitaliy Sushchansky started in [1], where the group $\mathrm{Aut}_f\, D_k$ of finitary automorphisms of a k-adic restricted parabolic tree D_k was studied.

If p is a prime, every two Sylow p-subgroups of $\mathrm{Aut}_f\, D_p$ are isomorphic (they are conjugate in the group $\mathrm{Aut}\, D_p$ of all automorphisms of D_p). Let us fix one such Sylow p-subgroup $\mathcal{P}$. This group is isomorphic to a so called D-wreath product of infinitely many copies of $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$ (additive group) and it can be considered as the group of all almost zero sequences

$$[a_1(\bar{x}), a_2(\bar{x}), \ldots, a_n(\bar{x}), 0, 0, \ldots],$$

where $\bar{x} = (x_1, x_2, \ldots)$ and $a_i(\bar{x}) = a_i(x_{i+1}, x_{i+2}, \ldots)$ is a function with finite support from $\mathbb{Z}_p^\omega$ into $\mathbb{Z}_p$.

Given a monomial $\bar{x}^{\vec{\alpha}}$ define its weighted degree by $\partial(\bar{x}^{\vec{\alpha}}) = \langle \vec{w}, \vec{\alpha} \rangle$ with weights

$$\vec{w} = (p^{-1}, p^{-2}, \ldots, p^{-i}, \ldots).$$

The weighted degree induces a natural ordering on monomials and one can define the class of so called parallelotopic subgroups in $\mathcal{P}$ (see [2]). Every such subgroup R can be describe by its indicatrix, a sequence

$$\langle {}^{\epsilon_1}k_1, {}^{\epsilon_2}k_2, \ldots, {}^{\epsilon_i}k_i, \ldots \rangle_{i=1}^{\infty},$$

where k_i is a real number and $\epsilon_i \in \{\uparrow, \downarrow\}$. The depth $\mathrm{dp}(R)$ of the subgroup R is the index j such that $k_j \neq 0$ and $k_i = 0$ for all $i > j$ (if such index does not exist, the depth of the subgroup is defined to be ∞).

Theorem. *Let R be a parallelotopic subgroup of $\mathcal{P}$.*

1) *If $\mathrm{dp}(R) = r$ then R is a normal subgroup of $\mathcal{P}$ if and only if*

$$k_i \geqslant p^{-i} - p^{-r}$$

for all $i \in \{1, 2, \ldots, r-1\}$ and $\epsilon_i = \uparrow$ if $k_i = p^{-i} - p^{-r}$.

2) *If $\mathrm{dp}(R) = \infty$ then R is a normal subgroup of $\mathcal{P}$ if and only if $k_i = p^{-i}$ for all $i \in \mathbb{N}$.*

Corollary. *There are continuum many normal parallelotopic subgroups of infinite depth in $\mathcal{P}$.*

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Factorization of noncommutative polynomials

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I will talk on the research direction, pursued together with A. Heinle, M. Giesbrecht and J. Bell (University of Waterloo, Canada), which resulted in a series of recent papers.

As in the classical commutative case, we are interested in factorizing polynomials over noncommutative rings. Let us start with a field K and a finitely presented associative K-algebra A, which is a domain, i. e. A has no nontrivial zero-divisors.

It turns out, that there are at least two distinct notions of a factorization of polynomials over A. One of them originates from the ring theory (N. Jacobson, P. M. Cohn) and uses a weak notion of association relation (called left or right similarity), what is at the same time hard to approach algorithmically. On the contrary, in applications we'd like to use the classical association relation, i.e. when two elements differ by a factor, which is nonzero central unit.

The results from [1] give long-seeked conditions for a given algebra A to be a *finite factorization domain*, i. e. a domain, where every nonunit has at most finite number of factorizations. Over such domains a factorization procedure thus becomes into an algorithm. Examples, bounds and counterexamples will be given. Over the well-known class of ubiquitous G-algebras (or PBW algebras), we provide a factorization algorithm [2] , its' smarter graded-driven version for graded algebras [3, 4] and a factorizing Gröbner algorithm [2]. All of these are implemented in Singular:Plural [5]. We view the factorizing Gröbner algorithm as the only general possibility to obtain a weaker analogon to the primary decomposition from the commutative algebra. Applications of the mentioned algorithms will be presented as well.

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A relationship between determinant and paradeterminant

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The paradeterminant of a triangular matrix (a_{ij}), where $a_{ij} = 0$ if $i < j$, is the number

$$ddet(A) = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s, p_1+\ldots+p_{s-1}+1}\},$$

where $\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}$ and the summation of over the set of natural solutions of the equality $p_1 + \ldots + p_r = n$.

Theorem. For each triangular matrix

$$ddet \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n = \det \begin{pmatrix} b_{11} & 1 & 0 & \ldots & 0 \\ b_{21} & b_{22} & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \ldots & 1 \\ b_{n1} & b_{n2} & b_{n3} & \ldots & b_{nn} \end{pmatrix},$$

where $b_{ij} = \{a_{ij}\}$ for each $1 \leqslant j \leqslant i \leqslant n$.

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Infinite periodic groups with the locally nilpotent non-Dedekind norm of decomposable subgroups

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Let Σ be a system of all subgroups of a group G with a certain theoretical group property. Σ-norm of a group G is an intersection of normalizers of all subgroups of a group G which belong to the system Σ.

The authors consider one of such Σ-norms: *the norm N_G^d of decomposable subgroups of a group G* – and continue the study of infinite periodic groups with non-Dedekind locally nilpotent norm of decomposable subgroups [1]. Recall that a subgroup of the group G representable in the form of the direct product of two nontrivial factors is called *decomposable* [2].

It is clear, that in the case when $N_G^d = G$, all decomposable subgroups are normal in the group G or the system of such subgroups is empty. The non-Dedekind groups with such property were studied in [2] and called *di-groups*.

The following statements describe the structure of infinite locally finite groups with locally nilpotent non-Dedekind norm N_G^d.

Theorem 1. *An infinite periodic locally nilpotent group G has a non-Dedekind norm $N_G^d \neq G$ if and only if it is a 2-group of one of the following types:*

1) $G = (A\times\langle b\rangle)\leftthreetimes\langle c\rangle\leftthreetimes\langle d\rangle$, where A is a quasicyclic 2-group, $|b| = |c| = |d| = 2$, $[A,\langle c\rangle] = 1$, $[b,c] = [b,d] = [c,d] = a_1 \in A$, $|a_1| = 2$, $d^{-1}ad = a^{-1}$ for any element $a \in A$; $N_G^d = (\langle a_2\rangle\times\langle b\rangle)\leftthreetimes\langle c\rangle$, $a_2 \in A$, $|a_2| = 4$;

2) $G = (A\langle y\rangle)Q$, where A is a quasicyclic 2-group, $[A,Q] = E$, $Q = \langle q_1,q_2\rangle$, $|q_1| = 4$, $q_1^2 = q_2^2 = [q_1,q_2]$, $|y| = 4$, $y^2 = a_1 \in A$, $|a_1| = 2$, $y^{-1}ay = a^{-1}$ for any element $a \in A$, $[\langle y\rangle,Q] \subseteq \langle a_1\rangle\times\langle q_1^2\rangle$; $N_G^d = \langle a_2\rangle\times Q$, $a_2 \in A$, $|a_2| = 4$.

Theorem 2. *An infinite locally finite non-locally nilpotent group G has a non-Dedekind locally nilpotent norm N_G^d of decomposable subgroups if and only if it is a Frobenius group $G = G_p\leftthreetimes\langle h\rangle$, where subgroups G_p and $\langle h\rangle$ satisfy the following conditions:*

1) $G_p = (A\times\langle b\rangle)\leftthreetimes\langle c\rangle$, where A is a quasicyclic p-group, $|b| = |c| = p$, $[A,\langle c\rangle] = 1$, $[b,c] = a_1 \in A$, $|a_1| = p$, (p is odd prime, $p \neq 2^n + 1$);

2) $Z(G) = E$ and the centralizer of any element $x \in N_G^d$ is contained in G_p;

3) the subgroup G_p contains decomposable subgroups, which are non-invariant in G;

4) $\langle h\rangle$ is a cyclic q-group, q is prime, $q|(p-1)$, $q \neq 2$.

Moreover, $N_G^d = G_p$.

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On derivations for Chebyshev polynomials of the first kind and related identities

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The derivative of the Chebyshev polynomials of the first kind can be expressed in the terms of the polynomials $T_n(x)$ as follows (see, for example, [1]).

$$\frac{d}{dx}T_n(x) = n\left(\sum_{k=1}^{n-1}\left(1-(-1)^k\right)T_{n-k}(x) + \frac{1-(-1)^n}{2}T_0(x)\right). \tag{1}$$

We are interested in finding polynomial identities of the form $P(T_0(x), T_1(x), \ldots, T_n(x)) = 0$ where $P(x_0, x_1, \ldots, x_n)$ is a polynomial of $n+1$ variables.

Using the approach proposed in [2], we introduce the differential operator $\mathcal{D}_{\mathcal{T}}$ of the ring $\mathbb{C}[x_0, x_1, \ldots, x_n]$ by

$$\mathcal{D}_{\mathcal{T}} := n\left(\sum_{k=1}^{n-1}\left(1-(-1)^k\right)x_{n-k} + \frac{1-(-1)^n}{2}x_0\right). \tag{2}$$

The derivation $\mathcal{D}_{\mathcal{T}}$ is locally nilpotent. We call it the *Chebyshev derivation of the first kind.* Each element of the kernel $\ker \mathcal{D}_{\mathcal{T}}$ defines a polynomial identity.

It is known [3], that for an arbitrary locally nilpotent derivation D the element

$$\sigma_D(x_n) = \sum_{k=0}^{\infty} D^k(x_n)\frac{\lambda^k}{k!}, \text{ where } D(\lambda) = -1, \tag{3}$$

belongs to the kernel of the derivation D and is called the *Cayley element* of $\ker D$.

Put in (3) $\lambda = -\frac{x_1}{x_0}$ and using the expression for the k-th derivatives of the Chebyshev first kind polynomials given in [4], we obtain the following theorem:

Theorem. *The Cayley element $C_n = \sigma_{\mathcal{D}_{\mathcal{T}}}(x_n)$ of the Chebyshev derivation $\mathcal{D}_{\mathcal{T}}$ has the form:*

$$C_n = x_n x_0^{n-1} + \sum_{k=1}^{n-1}\frac{2^k n}{k!}\left(\sum_{i=0}^{\left[\frac{n-k}{2}\right]} A_{k,i}\, x_{n-k-2i} x_1^k x_0^{n-1-k} - (1+(-1)^{n-k})\frac{2^{k-2}n}{(k-1)!}B_{k,n}\, x_0^{n-k}\right), \tag{4}$$

where

$$A_{k,i} = \frac{(n-i-1)!\binom{k+i-1}{k-1}}{(n-k-i)!},\ B_{k,n} = \left(\frac{\left(\frac{n+k}{2}-1\right)!}{\left(\frac{n-k}{2}\right)!}\right)^2.$$

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On one property of modular representations of the dihedral group of order 8

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We consider the matrix representations of the dihedral group

$$G_8 = \langle a, b \,|\, a^2 = 1, b^2 = 1, (ab)^4 = 1\rangle$$

of order 8 over an infinite field k of characteristic 2. A matrix representation T of the group G_8 is called a representation of constant rank if the rank of the matrix $\alpha(E + T_a) + \beta(E + T_b)$, where $\alpha, \beta \in k$, $(\alpha, \beta) \neq (0, 0)$, is independent of α and β (E denotes the identity matrix).

Theorem. *Let k be as above and n be a natural number that is divisible by 4. Then in dimension n the group G_8 has infinitely many indecomposable pairwise non-equivalent matrix k-representations of constant rank.*

These studies were carried out together with Prof. V. M. Bondarenko.

Investigation and analysis of algorithms for solving NP-complete problems

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In the theory of algorithms, a problem from the class NP, to which any other problem from this class can be reduced in polynomial time, is called an NP-complete problem. Thus, NP-complete problems form in a certain sense a subset of typical problems in the class NP: if for some of them a “polynomially quick” solution algorithm is found, then any other problem in the NP class can also be solved "quickly". Having a small amount of input data, that algorithm may work, whose operating time is expressed by an exponential function. Sometimes it is possible to identify important special cases that can be solved in polynomial time.

There is a number of methods for solving these problems, which are still not sufficiently effective or optimal. Practically all methods that exist in our time are heuristic. In most of them there is not the most effective solution, but an approximate one. Most often so-called any-time-algorithms are taken, i.e. algorithms that gradually improve some current approximate solution.

Many problems of practical interest are NP-complete. It is unlikely for them to find an exact algorithm with a polynomial time of operation.

The purpose of our work is to study and analyze algorithms for approximate solution of NP-complete problems.

Complexity classes are the set of computational problems, approximately equal in complexity of computation, i.e. the complexity classes are the set of predicates that use approximately the same number of resources to calculate.

For detailed analysis, we chose methods from classes of pseudopolynomial and heuristic algorithms. The control of the results is carried out in the correlation of the results obtained with the work of the branch-and-bound method, which belongs to the class of the exact and, at the same time, it is faster and more efficient than the full-search method.

In a series of experiments, individual problems were considered that were solved by the branch-and-bound method, the dynamic programming method, and the genetic algorithm. The experiment showed that for finding the optimal solution classical algorithms use more calculations than the genetic one to create a population. However, the average accuracy of the classical algorithm is higher. The obtained results allow us to talk about the need to search for more accurate approximate methods for solving NP-complete problems. One of such directions is the use of the so-called “method of experience” for both the genetic algorithm and the algorithm of the ant colony.

The algorithms considered are different in their idea, in their implementation and reliability. And accordingly, each algorithm is suitable for different tasks.

The developed theory of NP-completeness has developed a number of pragmatic recommendations for researchers engaged in solving applied problems. In those cases when the task of the developer of practical algorithms is NP-complete, it makes sense to try to construct an effective algorithm for any modification or special case that is acceptable from a practical point of view. When one cannot find such a modification, it makes sense to try to construct an approximate effective algorithm for the problem, which guarantees finding a solution that differs from the optimal one no more than a predetermined number of times. Such algorithms are often used in practice.

2-good Bezout duo rings and stable range conditions

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Let R be duo ring with nonzero unit. A ring R is called a right (left) Bezout ring if every finitely generated right (left) ideal in R is principal. Let $J(R)$ denote the Jacobson radical and let $U(R)$ denote the group of units of the ring R.

Definition 1 [1]**.** *We say that R is a ring of unit stable range 1 if the condition $aR+bR = R$ for arbitrary elements $a, b \in R$ yields the existence of an invertible element $u \in R$ such that $a + bu \in U(R)$.*

Definition 2 [3]**.** *We say that R is a 2-good ring if an arbitrary element of R is the sum of two invertible elements.*

Definition 3. *An element $a \neq 0$ of the right Bezout duo domain R is called a right adequate element if for any element $b \in R$ there exists the elements $r, s \in R$ so that:*

1) $a = rs$;

2) $bR + rR = R$;

3) $\forall\ s' \in R$, $sR \subset s'R \neq R \Rightarrow bR + s'R \neq R$.

Definition 4. *We say that an element a of a duo ring R is an element of almost unit stable range 1 if the quotient ring R/aR is a ring of unit stable range 1.*

We say that R is a ring of almost unit stable rank 1 if an arbitrary nonzero noninvertible element of R is an element of almost unit stable rank 1.

Theorem 1. *Let R be right Bezout duo-ring and let a be an right adequate element of the ring R such that $2R + aR = R$. Then the quotient ring R/aR is a 2-good ring.*

Theorem 2. *Let R be a duo ring of unit stable range 1. Then R is a ring of almost unit stable range 1.*

Theorem 3. *Let R be a duo ring of almost unit stable range 1 with nonzero Jacobson radical $J(R)$. Then R is a ring of unit stable range 1.*

Theorem 4. *Let R be a duo ring of almost unit stable range 1 with nonzero Jacobson radical. Then R is a 2-good ring.*

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π-Normal products of Fitting classes with given properties

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All groups considered are finite and soluble. For notation and terminologies we refer to [1].

A class of groups $\mathfrak{F}$ is called a Fitting class if the following conditions are fulfilled:

1) $\mathfrak{F}$ is closed under taking normal subgroups.

2) If $H_1 \trianglelefteq G$, $H_2 \trianglelefteq G$, $H_1 \in \mathfrak{F}$, $H_2 \in \mathfrak{F}$ and $H_1H_2 = G$, then $G \in \mathfrak{F}$.

For every non-empty Fitting class $\mathfrak{F}$, every group G has a largest normal $\mathfrak{F}$-subgroup which is called the $\mathfrak{F}$-radical of G and denoted by $G_{\mathfrak{F}}$.

Recall that a Fitting class $\mathfrak{F} \neq (1)$ is called normal [2] if a subgroup $G_{\mathfrak{F}}$ of G is a maximal $\mathfrak{F}$-subgroup of G for all groups G.

Let π be a non-empty set of primes and $\mathfrak{F}$ be a non-identity Fitting class. The class $\mathfrak{F}$ is called *normal in the class* $\mathfrak{S}_\pi$ of all π-groups or π-*normal* (we denote $\mathfrak{F} \trianglelefteq \mathfrak{S}_\pi$)[3] if $\mathfrak{F} \subseteq \mathfrak{S}_\pi$ and the $\mathfrak{F}$-radical of every π-group G is $\mathfrak{F}$-maximal among the subgroups of G. Note that if $\pi = \mathbb{P}$, then π-normal Fitting class is normal.

Recall that the product $\mathfrak{F}\mathfrak{H}$ of two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G : G/G_{\mathfrak{F}} \in \mathfrak{H})$.

In the theory of classes of groups the problems (see [4, Problems 9.58 and 11.25(a)]) of the existence of local formations and local Fitting classes factorizing by non-local factors, are well-known. This problem were solved in [5, 6]. In connection with this, an analogue of such problems for the products of π-normal Fitting classes arises.

Problem. *Are there π-normal products of Fitting classes, which can be factorized by not π-normal Fitting classes?*

In this paper we give a positive answer to this question. It is proved the following

Theorem. *Let $\mathfrak{X}$ be a Fitting class such that $\mathfrak{X} \subset \mathfrak{S}_\pi$ and $\mathfrak{X} = \mathfrak{X}^2$. Let $\mathfrak{Z}^{\sigma(\mathfrak{X})} = (G : \mathrm{Soc}_{\sigma(\mathfrak{X})}(G) \leqslant Z(G))$, where $\sigma(\mathfrak{X}) = \cup\{\sigma(X) : X \in \mathfrak{X}\}$ and $\sigma(X) = \{p \in \mathbb{P} : p \mid |X|\}$. Then $\mathfrak{X}$ and $\mathfrak{Z}^{\sigma(\mathfrak{X})}$ are not π-normal Fitting classes, but $\mathfrak{X}\mathfrak{Z}^{\sigma(\mathfrak{X})}$ is π-normal Fitting class.*

Research is supported by the Belarusian Republican Foundation for Fundamental Research (F17M-064).

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(Co-)torsion via stable functors

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For a long time, this speaker has been calling for a study of stable categories. Most often this term refers to the category of modules modulo projectives. Its objects are modules, but the morphisms are quotients of the usual homomorphisms by the subgroup of homomorphisms factoring through projectives. This tool has numerous uses in diverse areas of representation theory, group cohomology, algebraic number theory, algebraic geometry, commutative algebra, and even topology (in fact, this concept originated in the work of Eckmann and Hilton on duality in homotopy theory). But stable categories don't seem to have been studied for their own sake. The first attempt at a phenomenological study of such categories was recently undertaken in joint work of the first author and Dali Zangurashvili (J. Pure Appl. Algebra 219 (2015), no. 9, 4061–4089). It became immediately clear that there were surprisingly tight and unexpected connections between the properties of the ring and the properties of its projectively stable category.

In the last few months it has transpired that additive functors defined on stable categories, also known as stable functors, bring significant additional power to the study of rings and modules. This talk will concentrate on two new applications of such functors. The first one is a definition of the torsion submodule of a module, the second is a definition of the cotorsion quotient module of a module. This will be done in utmost generality: for any module over any ring. The new definitions are remarkably simple but, for a person not used to working with functors, may seem counterintuitive. One of the goals of this talk is to demystify these definitions and convince the audience that the language of functors is simple, convenient, and natural. It leads to new insights even in the classical setting of abelian groups. Time permitting, we shall see that the Auslander-Gruson-Jensen functor sends the cotorsion functor to the torsion functor (of opposite chirality). If the injective envelope of the ring is finitely presented, then the right adjoint of the AGJ functor sends the torsion functor back to the cotorsion functor. In particular, this correspondence establishes a duality between torsion and cotorsion on the categories of all modules over an artin algebra.

This is joint work with Jeremy Russell.

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On Keigher semimodules

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Let R be a commutative semiring and M be a subsemimodule over R. An additive map $\delta\colon R \to R$ is called a *derivation on* R if $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$ [3]. An additive map $d\colon M \to M$ is called a *semimodule derivation* of M, associated with the semiring derivation $\delta\colon R \to R$ (δ-derivation) if $d(rm) = \delta(r)m + rd(m)$ for any $m \in M$, $r \in R$. A semiring R equipped with a derivation δ is called *differential* with respect to δ [2], a semimodule M together with the semimodule derivation d is called *differential*. A subsemimodule N of M is called *differential* if $d(m) \in N$ whenever $m \in N$.

Let (M, d) be a differential semimodule over the differential semiring (R, δ). For a subset X of M we define its *differential* $X_\#$ to be the set $X_\# = \{x \in M \,|\, d^n(x) \in X \text{ for all } n \in \mathbb{N} \cup \{0\}\}$. If for any prime subtractive subsemimodule N of M the subsemimodule $N_\#$ is prime, then the semimodule M is called a *Keigher semimodule*.

Theorem 1. *A differential semimodule M over the differential semiring R containing $\mathbb{Q}_+$ is a Keigher semimodule.*

Theorem 2. *For a differential semimodule M the following conditions are equivalent:*

1. M is a Keigher semimodule;

2. If N is a differential subtractive subsemimodule of M, then so is $rad(N)$;

3. Any differential substractive subsemimodule of M, which is maximal among differential subsemimodules not meeting some Sm-system of M, is prime;

4. Any prime subtractive subsemimodule, minimal over some differential subsemimodule, is differential.

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On square roots of integer matrices

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Let $\mathbb{Z}^{n\times n}$ be the set of $(n \times n)$-matrices over the ring of integers $\mathbb{Z}$, I_n be an identity matrix of size $n \times n$, $0_{n\times n}$ be a zero $(n \times n)$-matrix.

A matrix $B \in \mathbb{Z}^{n\times n}$ is called a square root of the matrix $A \in \mathbb{Z}^{n\times n}$ if $B^2 = A$. Therefore finding conditions under which a square root of the matrix $A \in \mathbb{Z}^{n\times n}$ exists is equivalent to solving the matrix equation $X^2 = A$. Solvability of this equation is equivalent to existence of factorization $A(\lambda) = (I_n\lambda - B)(I_n\lambda + B)$ for a polynomial matrix $A(\lambda) = I_n\lambda^2 - A$. Thus we obtain

$$\det A(\lambda) = a(\lambda) = b(\lambda)\widetilde{b}(\lambda), \tag{1}$$

where $b(\lambda), \widetilde{b}(\lambda) \in \mathbb{Z}[\lambda]$ are unital polynomials of degree n. If $b(\lambda) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} \cdots + b_{n-1}\lambda + b_n$, then it follows from (1) that $\widetilde{b}(\lambda) = \lambda^n - b_1\lambda^{n-1} + b_2\lambda^{n-2} \cdots + (-1)^k b_k\lambda^{n-k} + \cdots + b_2\lambda^{n-2} \cdots + b_{n-1}\lambda + b_n$.

Note that (1) is a necessary condition for existence of square root for the matrix $A \in \mathbb{Z}^{n\times n}$. It is easy to see that not every nonsingular matrix $A \in \mathbb{Z}^{n\times n}$ has a square root. This report aims to give conditions under which a square root for a matrix $A \in \mathbb{Z}^{n\times n}$ does exist. Note that this problem for matrices over the ring of integers is little studied.

In what follows we consider nonsingular matrices from $\mathbb{Z}^{n\times n}$ of even order, i. e. $n = 2k$, for which the condition (1) holds. To the matrix $A \in \mathbb{Z}^{2k\times 2k}$ and the polynomial $b(\lambda)$, defined by the equality (1), we put in correspondence matrices

$$K = A^{k-1}b_1 + A^{k-3}b_3 + \cdots + Ab_{n-1} \quad \text{and} \quad M = -(A^k + A^{k-2}b_2 + \cdots + A^2 b_{n-2} + I_n b_n).$$

Theorem. *Let for a nonsingular matrix $A \in \mathbb{Z}^{n\times n}$ of even order the determinant of $A(\lambda) = I_n\lambda^2 - A$ may be presented as a product (1), where $b(\lambda) \in \mathbb{Z}[\lambda]$ is a unital polynomial of degree n. If the matrix equation $KX = -M$ is solvable, then there exists a square root for the matrix A, i. e. there exists a matrix $B \in \mathbb{Z}^{n\times n}$ such that $B^2 = A$ and $\det(I_n\lambda - B) = b(\lambda)$.*

Remark. Square roots for nonsingular matrices $B \in \mathbb{Z}^{n\times n}$ of odd order can be found using the following method. Choose a number $a \in \mathbb{Z}$ such that $(a, \det B) = 1$ and build a matrix $A = \begin{bmatrix} a^2 & 0_{1,n} \\ 0_{n,1} & B \end{bmatrix}$. For the determinant of the matrix $A(\lambda) = I_n\lambda^2 - A$ we build a presentation as a product (1), provided one of the conditions $b(a) = 0$ or $b(-a) = 0$ holds. Use the above theorem for it and the matrix A.

Finding square roots of nonsingular integer matrix is algorothmical in character, so this problem can be solved in finite number of steps. In order to find a solution of a matrix equation $KX = -M$ from the Theorem the method given in the work [1] can be used.

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Groups with $\mathfrak{A}^2$-subnormal subgroups

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We consider only finite groups. All notations and terminology are standard [1]. By $\mathfrak{N}$, $\mathfrak{A}$ and $\mathfrak{E}$ we denote the class of all abelian, nilpotent and all groups, respectively; $F(G)$ denotes the Fitting subgroup of a group G.

Let $\mathfrak{F}$ be a formation, G be a group. The subgroup $G^{\mathfrak{F}} = \bigcap\{N \lhd G : G/N \in \mathfrak{F}\}$ is the smallest normal subgroup of G with quotient in $\mathfrak{F}$, and it is called the $\mathfrak{F}$-residual of G. A subgroup H of a group G is called $\mathfrak{F}$-subnormal if there is a chain of subgroups

$$H = H_0 <\cdot\, H_1 <\cdot\, \ldots <\cdot\, H_n = G$$

such that $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all i, that is equivalent to $H_i^{\mathfrak{F}} \leqslant (H_{i-1})_{H_i}$. Here $Y_X = \bigcap_{x\in X} Y^x$ denotes the core of Y in X, $H_{i-1} <\cdot\, H_i$ denotes that H_{i-1} is a maximal subgroup of H_i.

If $\mathfrak{X}$ and $\mathfrak{F}$ are s-closed formations, then the product

$$\mathfrak{X}\mathfrak{F} = \{\, G \in \mathfrak{E} \mid G^{\mathfrak{F}} \in \mathfrak{X}\},$$

by [1, p. 337], is an s-closed formation. When $\mathfrak{X} = \mathfrak{F}$, we write $\mathfrak{X}^2$ instead of $\mathfrak{X}\mathfrak{F}$.

Groups with various collections of $\mathfrak{F}$-subnormal subgroups are investigated by many authors, see references of [2–4].

It is easy to prove that every Sylow subgroup of any soluble group is $\mathfrak{A}\mathfrak{N}$-subnormal. Therefore in the universe of all soluble groups the class of groups with $\mathfrak{F}$-subnormal Sylow subgroups should be investigated when $\mathfrak{F}$ does not contain $\mathfrak{A}\mathfrak{N}$.

Theorem. *In a group G every Sylow subgroup is $\mathfrak{A}^2$-subnormal in G if and only if G is soluble and every Sylow subgroup of $G/F(G)$ is abelian.*

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On the quasinilpotent hypercenter of a finite group

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All groups considered here are finite. Let $\mathfrak{X}$ be a class of groups. Recall [1, p. 6–8] that a chief factor H/K of a group G is called $\mathfrak{X}$-central if $(H/K)\rtimes G/C_G(H/K) \in \mathfrak{X}$. A normal subgroup N of G is said to be $\mathfrak{X}$-hypercentral in G if $N = 1$ or $N \neq 1$ and every chief factor of G below N is $\mathfrak{X}$-central. The $\mathfrak{X}$-hypercenter $\mathrm{Z}_{\mathfrak{X}}(G)$ is the product of all normal $\mathfrak{X}$-hypercentral subgroups of G. So if $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups then the hypercenter $\mathrm{Z}_\infty(G) = \mathrm{Z}_{\mathfrak{N}}(G)$ for every group G.

In [2] R. Baer showed that $\mathrm{Z}_\infty(G)$ coincides with the intersection of all maximal nilpotent subgroups of G. L. A. Shemetkov possed the following problem on the Gomel Algebraic Seminar in 1995: "Describe all hereditary saturated formations $\mathfrak{F}$ such that $\mathrm{Z}_{\mathfrak{F}}(G)$ coincides with the intersection of all $\mathfrak{F}$-maximal subgroups of G for every group G". This problem was solved by A. N. Skiba in [3].

Recall that the class of all quasinilpotent groups $\mathfrak{N}^*$ is non-saturated formation.

Theorem. *Let G be a group. Then the intersection of all maximal quasinilpotent subgroups of G is $\mathrm{Z}_{\mathfrak{N}^*}(G)$.*

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Multi-relational algebras as semantics of quasiary logics

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The notion of predicate is a main semantic notion of logic. There are two conventional approaches for predicate representation: function-theoretic and relational. The first one treats a predicate as a mapping from a predicate domain to the set of logical values, while the second specifies a predicate as a relation on the predicate domain. Both approaches have their advantages and disadvantages. Function-theoretic approach looks more natural for operational treatment of predicates; relational approach, being a special case of set-theoretic approach to formalization of mathematical notions, is well-developed and mature. In the case of total single-valued (deterministic) predicates there is a natural bijection between classes of predicates and relations. The situation becomes more difficult if we consider partial non-deterministic predicates which often appear in computer science [1]. In this case we should use three relations to represent a predicate: a relation for the truth domain of the predicate, a relation for the falsity domain, and a relation for the undefinedness domain.

We distinguish two levels of predicates: propositional level (elements of predicate domains are unstructured objects) and quasiary level (elements of predicate domains are sets of named values). In the latter case a relation can be treated as a partial table [2] called quasi-relation. Thus, a partial non-deterministic quasiary predicate can be represented by three quasi-relations.

In our previous work [3] we studied relational quasiary predicates represented by two quasi-relations; we constructed bi-quasirelational algebras and investigated their relationship with logics of relational quasiary predicates. The constructed algebras are generalizion of cylindric algebras [4].

The aim of this paper is to construct triple-quasirelational algebras and investigate their relationship with logics of partial non-deterministic quasiary predicates.

We introduce the following operations (compositions), induced by operation on predicates: negation $\neg$, disjunction $\vee$, renomination $R_{\bar{x}}^{\bar{v}}$, and existential quantification $\exists x$ [3].

Different subclasses of triple-quasirelational algebras are defined and their relationship with algebras of partial single-valued, total many-valued, partial many-valued, monotone and antitone quasiary predicates are investigated.

The isomorphism between the triple-quasirelational algebras and the first-order algebras of partial non-deterministic quasiary predicates is proved. This means that such algebras can be considered as semantics of corresponding logics. Sequent rules for such logics are defined, their validity is proved.

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Self-similar closures of free products of groups

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Let $X = \{0, 1, \ldots, n\}$, $n \geqslant 1$, be an alphabet. Consider an automaton A_n over X, defined in [1]. The set of states of A_n is $\{a_i, b_i : 1 \leqslant i \leqslant n\}$. The transition φ and output ψ functions of A_n are defined by the equalities

$$\varphi(a_i, j) = \varphi(b_i, j) = \begin{cases} a_i, & j = 0 \text{ or } j = i \\ b_i, & j \neq 0 \text{ and } j \neq i \end{cases},$$

$$\psi(a_i, j) = \begin{cases} i, & j = 0 \\ 0, & j = i \\ j, & j \neq 0 \text{ and } j \neq i \end{cases}, \varphi(b_i, j) = j,$$

where $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n$. In [1] it is shown that the subgroups of the group of automaton permutations over X generated by the sets $\{a_i : 1 \leqslant i \leqslant n\}$ and $\{b_i : 1 \leqslant i \leqslant n\}$ split into the free product of n copies of cyclic groups of order 2. The group of the automaton $G(\mathsf{A}_n)$ is the self-similar closure [2] of both these subgroups. The characterization of this self-similar group is obtained.

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Commuting graphs of metabelian groups

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Let G be a group. Denote by $\mathcal{Z}(G)$ its center. The commuting graph $\Gamma(G)$ of G is defined as a simple non-oriented graph such that the vertex set of $\Gamma(G)$ is $G \backslash \mathcal{Z}(G)$ and vertices a, b are connected by an edge if and only if $ab = ba$ [1].

For arbitrary positive integers $n, m > 1$ consider the abstract wreath product $\mathbb{Z}_n \wr \mathbb{Z}_m$. Denote by $\Gamma_{n,m}$ the commuting graph of this metabelian group. In [2] it is established a criterion when the graph $\Gamma_{n,m}$ is connected and in case of connectivity its diameter is computed.

The further combinatorial and graph-theoretical properties of graphs $\Gamma_{n,m}$ are obtained. Generalizations for other metabelian groups are presented.

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Honey encryption applied to Rabin cryptosystem

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The concept of honey encryption as a general cryptographic approach to protect data with low entropy parameters in password-based encryption schemes was introduced in [1]. Since that a few concrete realizations and other applications were proposed.

We apply honey encryption to protect secret keys in Rabin cryptosystem. The hardness of the problem to brake the Rabin cryptosystem is known to be equivalent to the hardness of the problem to factorize integers [2]. The other applications of proposed method are discussed.

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Algorithms for computations in Sylow 2-subgroups of symmetric and alternating groups

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The structure of the Sylow 2-subgroup $Syl_2(Alt(2^n)), n \geqslant 1$, of the alternating group of degree 2^n and its basic properties were described in [1]. For these purposes the tuples of length n of reduced polynomials over the field $GF(2)$ to represent elements of $Syl_2(Alt(2^n))$ were used.

To provide algorithmic computations in $Syl_2(Alt(2^n))$ a representation of its elements in terms of binary strings is proposed. Using this representation it was designed the following algorithms.

- Algorithm of conversion between permutations and binary strings.
- Algorithm of determining the parity of a permutation.
- Algorithm of finding inverse elements.
- Algorithm of multiplication.

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Distances in Cayley graphs of alternating group

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The problem of computing distances in the directed Cayley graphs of alternating groups $Alt(n), n \geqslant 3$, defined by a natural generating set $S = \{(123), (124), \ldots, (12n)\}$ is considered. *Canonical decomposition of a permutation* corresponding to this generating set S is defined. To find such a decomposition sets of words $\Omega_{n,k}$, $k \geqslant n-2$ are defined. Their elements correspond to even permutations via function ϕ. The sets $T_m(\overrightarrow{t}, \overrightarrow{l})$, where $m \geqslant 1$, $\overrightarrow{t}, \overrightarrow{l}$ are vectors over $\mathbb{Z}_+$ are used to classify even permutations over the generating set S.

For each $w \in \Omega_{n,n-1}$ there exist exactly two instances of the same latter. Denote by p_1 and p_2 corresponding positions and let $\rho_d(w) = p_2 - p_1 - 1 \geqslant 0$. In this case we have the following results.

Theorem 1. *Let $w \in \Omega_{n,n-1}, 2|n$.*

1. *Let $\rho_d(w)$ be an odd number and $\rho_d(w) \neq 1$. Then:*
   If p_1 is odd and $\rho_d(w) \in \{3, \ldots, n-6\}$ then:
   $$\phi(w) \in (T_3((1,2,0),(l_1+1, l_2+1, l_3)),\ l_1 = \tfrac{n-1}{2}, l_2 \in \overline{1, n-l_1-4}, l_3 = n-l_1-l_2-2).$$
   If p_1 is odd and $\rho_d(w) = n-4$ then: $\phi(w) \in T_2((1,0),([\frac{n}{2}]+1, [\frac{n}{2}]-1))$.
   If p_1 is even and $\rho_d(w) \in \{3, \ldots, n-6\}$ then:
   $$\phi(w) \in (T_3((1,2,0),(l_1+1, l_2+1, l_3)),\ l_1 \in \overline{1, n-l_1-4}, l_2 = \tfrac{n-1}{2}, l_3 = n-l_1-l_2-2).$$
   If p_1 is even and $\rho_d(w) = n-4$, then: $\phi(w) \in T_2((2,0),([\frac{n}{2}]+1, [\frac{n}{2}]-1))$.

2. *If $\rho_d(w) = 1$ then $\phi(w) \in Alt(n-1)$.*

3. *Let $\rho_d(w)$ be odd number. Then:*
   If p_1 is odd then $\phi(w) \in (T_1(t,n), t \in \overline{[\frac{n}{2}]+3, n+1})$.
   If p_1 is even then: $\phi(w) \in (T_1(\bar{t},n), t \in \overline{[\frac{n}{2}]+4, n+1})$.

Theorem 2. *Let $w \in \Omega_{n,n-1}, 2 \nmid n$.*

1. *Let $\rho_d(w)$ be an odd number. Then:*
   If p_1 is odd and $\rho_d(w) \neq 1$ then:
   $$\phi(w) \in (T_2((n+2,0),(l+2, n-l-2)),\ l \in \{\tfrac{n}{2}-1, \ldots, n-4\}).$$
   If p_1 is even and $\rho_d(w) \neq 1$ then:
   $$\phi(w) \in (T_2((\overline{\tfrac{n}{2}+3}, 0),(l+2, n-l-2)),\ l \in \{\tfrac{n}{2}, \ldots, n-4\}).$$
   If $\rho_d(w) = 1$, then $\phi(w) \in Alt(n-1)$.

2. *Let $\rho_d(w)$ be an even number. Then:*
   If p_1 is odd and $\rho_d(w) \in \{0, \ldots, n-6\}$ then:
   $$\phi(w) \in (T_2((1,2),(n-l-1, l+1)), l < \tfrac{n}{2}-1).$$
   If p_1 is odd and $\rho_d(w) = n-4$, then $\phi(w) \in T_1(1, n-1)$.
   If p_1 is even and $\rho_d(w) \in \{0, \ldots, n-6\}$ then:
   $$\phi(w) \in (T_2((1,2),(l+1, n-l-1)), l < \tfrac{n}{2}-1).$$
   If p_1 is even and $\rho_d(w) = n-4$ then $\phi(w) \in T_1(2, n-1)$.

On threefactorized finite groups

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Only finite groups are considered. According to Kegel [1], a group G is said to be threefactorized if it contains subgroups A, B and C such that $G = AB = AC = BC$. The threefactorization of G occurs in a natural way if G has three subgroups A, B and C whose indices are pairwise relatively prime in G. Concerning such groups it is well known that G is abelian if A, B and C are abelian. Wielandt [2] proved that the solvability of the factors A, B and C implies the solvability of the whole group. Kegel [1] established, the group G preserves the nilpotency property in the case of nilpotency of its subgroups A, B and C. However, in the general case, the properties of subgroup factors are not carried over to the whole group. For example, the supersolubility of G does not follow from the supersolubility of A, B, C and $|G : A| = |G : B| = |G : C| = 1$.

We consider the problem of studying the structure of a group G with three supersoluble subgroups whose indexes are pairwise coprime in G.

A subgroup H of a group G is called $\mathbb{P}$-subnormal in G whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for every $i = 1, \ldots, n$. A group G is called w-supersoluble if every Sylow subgroup og G is $\mathbb{P}$-subnormal in G [3].

Theorem 1. *Let G be a group with supersoluble subgroups G_1, G_2 and G_3, whose indices are pairwise coprime in G.*

(1) *If $G_i \ne G$ for every $i = 1, 2, 3$, then $|\pi(G)| \geqslant 3$.*

(2) *G is* w*-supersoluble.*

(3) *G has the Fitting length at most 3.*

(4) *If p is a smallest prime divisor of order G, then the commutator subgroup G' is p-decomposable.*

Theorem 2. *Let G be a group with supersoluble subgroups G_1, G_2 and G_3, whose indices are pairwise coprime in G. Suppose that one of the following statements holds.*

(1) *The commutator subgroup of G' of G is contained in $G_1 \cap G_2 \cap G_3$.*

(2) *G_i is $\mathfrak{U}$-subnormal in G for every $i = 1, 2, 3$.*

(3) *G_i is $\mathbb{P}$-subnormal in G for every $i = 1, 2, 3$.*

Then G is supersoluble.

Corollary [4]**.** *A group G is a supersoluble group whose order has at last three different prime divisors if and only if there exist three maximal supersoluble subgroups of G whose indices are three different primes.*

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Homomorphisms of matrix groups above the associative rings

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Let R and K be arbitrary associative rings with 1, W be left (not necessary free) K-module, $E(n,R)$ be the subgroup generated by transvections $t_{ij}(r) = 1 + e_{ij}$, $r \in R$, $1 \leqslant i \neq j \leqslant n$, $t_{ij} = t_{ij}(1)\, t_{ji}(-1)\, t_{ij}(1)$, $a_{ij} = t_{ij}t_{ij}(-1)$, $GL(W)$ is the group of module automorphisms of W. If $\sigma \in GL(W)$, then $R(\sigma) = (\sigma - 1)\, W$, $P(\sigma) = \ker(\sigma - 1)$.

Homomorphism $\Lambda : G \to GL(W)$, $E(n,R) \subseteq G \subseteq GL(n,R)$, $n \geqslant 2$ has standard description if there exist left K-modules L and P and isomorphism $g : W \to \underbrace{L \oplus \cdots \oplus L}_{n} \oplus P$ such as that $\Lambda(x) = g^{-1}\left[\overline{\delta}(x)\, e + \overline{\nu}(x)^{-1}(1-e) + e_1\right] g$ where $x \in E(n,R)$, $\overline{\delta}$ be the ring homomorphism and $\overline{\nu}$ be the ring anti-homomorphism of $R_n = End\left(\underbrace{L \oplus \cdots \oplus L}_{n}\right)$ ring, which are induced by ring homomorphism $\delta : R \to EndL$ and a ring anti-homomorphism $\nu : R \to EndL$, e is an idempotent of this ring, e_1 is a unit of a ring $EndP$.

Theorem 1. *Let R and K be associative rings with 1, $2 \in K^*$, $\Lambda : G \to GL(W)$, $E(n,R) \subseteq G \subseteq GL(n,R)$, $n \geqslant 3$, be a group homomorphism is such that $P(\Lambda t_{12}^2) = P(\Lambda t_{12})$. Then Λ has standard description.*

Theorem 2. *Let R and K be associative rings with 1, $3 \in K^*$, $\Lambda : G \to GL(W)$, $E(n,R) \subseteq G \subseteq GL(n,R)$, $n \geqslant 4$, be a group homomorphism is such that $R(\Lambda a_{12}) \bigcap R(\Lambda a_{34}) = 0$. Then Λ has standard description.*

Theorem 1 was first proved in [1]. From it therefore, in particular, comes the results of the [2] and [3]. Theorem 2 is new.

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On locally nilpotent Lie algebras of derivations

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Let $\mathbb{K}$ be a field of characteristic zero and A an integral domain over $\mathbb{K}$. Denote by R the fraction field of the algebra A. The set $\operatorname{Der}_{\mathbb{K}} A$ of all $\mathbb{K}$-derivations of A is a Lie algebra over $\mathbb{K}$ with multiplication $[D_1, D_2] = D_1 D_2 - D_2 D_1$ for all $D_1,\ D_2 \in \operatorname{Der}_{\mathbb{K}} A$. Any derivation D of A can be uniquely extended to a derivation of R. One can define a $\mathbb{K}$-derivation rD by setting $(rD)(x) = r \cdot D(x)$ for all $r \in R$, $D \in \operatorname{Der}_{\mathbb{K}} A$ and $x \in A$. The Lie algebra $\operatorname{Der}_{\mathbb{K}} A$ is naturally embedded into the subalgebra $R \operatorname{Der}_{\mathbb{K}} A = \mathbb{K}\langle rD | r \in R, D \in \operatorname{Der}_{\mathbb{K}} A\rangle$ of the Lie algebra $\operatorname{Der}_{\mathbb{K}} R$ of all derivations on R. We denote the Lie algebra $R \operatorname{Der}_{\mathbb{K}} A$ by $W(A)$.

Let L be a subalgebra of $W(A)$. The rank $\operatorname{rk}_R L$ of L over R is defined as $\dim_R RL$, where RL is an R-linear hull of all derivations of L. The field of constants $F = F(L)$ for L consists of all $r \in R$ such that $D(r) = 0$ for all $D \in L$. The vector space FL is defined analogously to RL, and it is a Lie algebra over F. If L is a locally nilpotent Lie algebra, then FL has the same property as a Lie algebra over F.

Nilpotent subalgebras of $W(A)$ were studied in [2] and [3]. Our goal is to describe locally nilpotent subalgebras of $W(A)$. For instance, the Lie algebras $u_n(\mathbb{K})$ of triangular polynomial derivations are locally nilpotent, but not nilpotent (see [1]).

Theorem 1. *Let L be a locally nilpotent subalgebra of $W(A)$ with $\operatorname{rk}_R L = n$ and F the field of constants for L. Then*

(1) L contains a series of ideals $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ such that $\operatorname{rk}_R L_s = s$ and L_s/L_{s-1} is an abelian factor algebra for all $s = 1, 2, \ldots, n$. Moreover, $\dim_F FL/FL_{n-1} = 1$.

(2) There exists a basis $\{D_1, \ldots, D_n\}$ of L over R such that $L_s = (RD_1 + RD_2 + \cdots + RD_s) \cap L$ for all $s = 1, 2, \ldots, n$.

Using the theorem above, we give a characterization of maximal with respect to inclusion locally nilpotent Lie algebras of rank 3 over R.

Theorem 2. *Let L be a maximal locally nilpotent subalgebra of $W(A)$ with $\operatorname{rk}_R L = 3$ and F the field of constants for L. Then L is a Lie algebra over F, $L = FL$, and L is one of the following Lie algebras:*

(1) L is a nilpotent Lie algebra of finite dimension over F;

(2) $L = F\langle D_1, \{\frac{a^i}{i!} D_2\}_{i=0}^{\infty}, \{\frac{a^i}{i!} D_3\}_{i=0}^{\infty}\rangle$, where $D_1,\ D_2,\ D_3 \in L$ and $a \in R$ such that $D_1(a) = 1$, $D_2(a) = D_3(a) = 0$, and $[D_i, D_j] = 0$ for $i, j = 1, 2, 3$;

(3) $L = F\langle D_1, \{\frac{a^i}{i!} D_2\}_{i=0}^{\infty}, \{\frac{a^i b^j}{i! j!} D_3\}_{i,j=0}^{\infty}\rangle$, where $D_1,\ D_2,\ D_3 \in L$ and $a,\ b \in R$ such that $D_1(a) = D_2(b) = 1$, $D_1(b) = D_2(a) = D_3(a) = D_3(b) = 0$, and $[D_i, D_j] = 0$ for $i, j = 1, 2, 3$.

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A bound on degrees of solutions of the matrix linear bilateral polynomial equation

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We consider the matrix polynomial equation $A(\lambda)X(\lambda)+Y(\lambda)B(\lambda)=C(\lambda)$, where $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are given $n\times m$, $n\leqslant m$ matrices over a polynomial ring $\mathcal{F}[\lambda]$, where $\mathcal{F}$ is a field, $X(\lambda)$ and $Y(\lambda)$ are unknown $m\times m$ and $n\times n$ matrices, respectively. It is obvious, if this matrix polynomial equation is solvable, it has solutions of unbounded above degrees. To describe solutions of this equation and to develop an effective method of their construction, it is important to estimate their possible degrees, in particular, their minimal degrees. Some bounds on the degrees of the solutions of this matrix polynomial equation are known, when both matrices $A(\lambda), B(\lambda)$ are regular or if at least one of the matrices $A(\lambda)$ or $B(\lambda)$ is regular.

We reduce the matrix polynomial equation to the equivalent equation

$$T^A(\lambda)\widetilde{X}(\lambda)+\widetilde{Y}(\lambda)T^B(\lambda)=\widetilde{C}(\lambda), \tag{1}$$

where $T^A(\lambda)=QA(\lambda)R^A(\lambda)$, $T^B(\lambda)=QB(\lambda)R^B(\lambda)$ are the triangular forms with invariant factors $\mu_i^A(\lambda), \mu_i^B(\lambda)$ of the matrices $A(\lambda), B(\lambda)$ on the main diagonals with respect to semiscalar equivalence [1, 2], $\widetilde{X}(\lambda)=(R^A(\lambda))^{-1}X(\lambda)R^B(\lambda)$, $\widetilde{Y}(\lambda)=QY(\lambda)Q^{-1}$, $\widetilde{C}(\lambda)=QC(\lambda)R^B(\lambda)$, $Q\in GL(n,\mathcal{F})$, $R^A(\lambda)$ and $R^B(\lambda)\in GL(m,\mathcal{F}[\lambda])$.

Theorem. *Let*

$$S^A(\lambda)=\operatorname{diag}(\mu_1^A(\lambda),\dots,\mu_p^A(\lambda),\mu_{p+1}^A(\lambda),\dots,\mu_{p+q}^A(\lambda),\mu_{p+q+1}^A(\lambda),\dots,\mu_n^A(\lambda))$$

be the Smith normal form of the matrix $A(\lambda)$, *where* $\deg\mu_i^A(\lambda)=0$, *i.e.,* $\mu_i^A(\lambda)=1$, *if* $i=1,\dots,p$, $\deg\mu_i^A(\lambda)=1$ *if* $i=p+1,\dots,p+q$, *and* $\deg\mu_i^A(\lambda)>1$ *if* $i=p+q+1,\dots,n$.

Let the matrix equation (1) *be solvable. Then this equation has the solution* $\widetilde{X}_1(\lambda)=[\widetilde{x}_{ij}^{(1)}(\lambda)]_1^n$, $\widetilde{Y}_1(\lambda)=[\widetilde{y}_{ij}^{(1)}(\lambda)]_1^n$ *such that*

(i) $\widetilde{y}_{ij}^{(1)}(\lambda)=0$, *for* $i=1,\dots,p$, $j=1,\dots,n$,

(ii) $\widetilde{y}_{ij}^{(1)}(\lambda)=\widetilde{y}_{ij}^{(1)}\in F$, *for* $i=p+1,\dots,p+q$, $j=1,\dots,n$,

(iii) $\deg\widetilde{y}_{ij}^{(1)}(\lambda)<\deg\mu_i^A(\lambda)-\deg(\mu_i^A(\lambda),\mu_1^B(\lambda))$, *for* $i=p+q+1,\dots,n$, $j=1,\dots,n$.

The similar bounds for the degrees of the elements of $\widetilde{X}_2(\lambda)$ of the solution $\widetilde{X}_2(\lambda)=[\widetilde{x}_{ij}^{(2)}(\lambda)]_1^n$, $\widetilde{Y}_2(\lambda)=[\widetilde{y}_{ij}^{(2)}(\lambda)]_1^n$ of the equation (1) are established.

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Peirce decomposition, Peirce idempotents and rings

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The topic of my talk is the joint results with G. Birkenmeier and L. W. Wyk.

We use the Peirce decomposition for inventing Peirce idempotents and the quite big class of the so-called Peirce rings possessing a satisfactory structure theory.

Commutative Bezout rings in which zero is adequate

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In the following all rings are assumed to be commutative with $1 \neq 0$. All necessary definitions and facts can be found in [1–3].

Definition. A commutative Bezout ring R is a ring in which *zero is adequate* if for any element $b \in R$ there exist elements $r, s \in R$ such that $rs = 0$ and $rR + bR = R$ and for any noninvertible divisor s' of element s it follows that $s'R + bR \neq R$.

Generalizing [2] we obtained the following results.

Theorem 1. *Let R be a ring in which zero is an adequate element. Then for any nonzero and noninvertible element $b \in R$ exists an idempotent $e \in R$ such that $be \in J(R)$ and $eR + bR = R$.*

Theorem 2. *A commutative Jacobson semisimple Bezout ring is a ring in which zero is an adequate element if and only if it is a von Neumann regular ring.*

Theorem 3. *A commutative Bezout ring is a ring in which zero is an adequate element if and only if it is semiregular.*

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On the max-indecomposable exponent matrices

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A square matrix $A = (\alpha_{ps})$ is called *exponent* if its diagonal entries are equal to zero and for all possible indices i, j, k, the so-called *ring inequalities*

$$\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$$

hold (see [1] and [2, pp. 349–350]).

Non-negative $n \times n$-exponent matrixes $\mathcal{E}_n$ form a semiring with respect to addition $\oplus$ (also called tropical addition) defined as entry-wise maximum, and multiplication $\odot$ (also called tropical multiplication) defined as entry-wise addition. By [3] the generators of this semiring are the (0, 1)-matrices $T_{\mathcal{I}} = (t_{pq})$, defined for all possible proper subsets $\mathcal{I} \subset \{1, \ldots, n\}$ by $t_{ij} = 1$ if and only if $i \in \mathcal{I}$ and $j \notin \mathcal{I}$.

Theorem 1. *A non-negative exponent matrix $A = (\alpha_{ij})$ is a generator of the semigroup $(\mathcal{E}_n, \oplus)$ if and only if there exists a permutation $\sigma \in S_n$, $s \in \mathbb{N}$ and $\{k_1, \ldots, k_s\} \in \{1, \ldots, n-1\}$ such that $\sigma \cdot A = \sum\limits_{t=1}^{s} T_{\{1,\ldots,k_t\}}$, where $\sigma \cdot A$ is defined as $\sigma \cdot A = (\alpha_{\sigma(i)\sigma(j)})$.*

The following proposition specifies Theorem 1.

Proposition 1. *The following statements for a non-negative $n \times n$-matrix $A = (\alpha_{pq})$ are equivalent:*

1. *There exist natural numbers $k_1, \ldots, k_s$ such that $A = \sum\limits_{t=1}^{s} T_{\{1,\ldots,k_t\}}$.*
2. *A is an upper triangular $\oplus$-indecomposable exponent matrix.*
3. *A is of the block form (A_{pq}), $1 \leqslant p, q \leqslant m$, where each diagonal block is square and there exist $x_1, \ldots, x_m$, $0 < x_1 < \ldots < x_m$, such that all the entries of each block A_{pq} equal* $\max\{0,\, x_q - x_p\}$.
4. *$\alpha_{ij} = [j > i](\alpha_{1j} - \alpha_{1i})$ for all i, j, where $[\cdot]$ denotes the boolean function.*
5. *$A = \sum\limits_{i=1}^{n-1} \alpha_{i,i+1} T_{\{1,\ldots,i\}}$.*

Notice, that Proposition 1 permits us to find the quiver of any reduced $\oplus$-indecomposable exponent matrix in the sense of definition in Chapter 14.7 of [2], and to characterize all the quivers which come from the reduced $\oplus$-indecomposable exponent matrices. Our proof of Proposition 1 implies an alternative proof of the description from [3] of the generators of the max-plus algebra of exponent matrices.

This is a joint work with M. Dokuchaev (University of São Paulo, São Paulo, Brazil), G. Kudryavtseva (University of Ljubljana, Ljubljana, Slovenia) and V. Kyrychenko (Taras Shevchenko National University of Kyiv, Kyiv, Ukraine).

The author was supported by Fapesp of Brazil, Proc. 2013/11350-2.

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On cohomologies of the Kleinian 4-group

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Let $G = \langle a, b | a^2 = b^2 = e, ab = ba \rangle$ be the Kleinian 4-group. Let $F = \mathbb{Z}G[x, y]$, the polynomial ring over $\mathbb{Z}G$, F_n be the group of homogeneous polynomial of degree n and define the differential $d : F_n \longrightarrow F_{n-1}$ by the rule

$$dx^n = \begin{cases} (a-1)x^{n-1} & \text{if } n \text{ is odd,} \\ (a+1)x^{n-1} & \text{if } n \text{ is even,} \end{cases}$$

$$dy^n = \begin{cases} (b-1)y^{n-1} & \text{if } n \text{ is odd,} \\ (b+1)y^{n-1} & \text{if } n \text{ is even,} \end{cases}$$

$$dx^k y^m = (a + (-1)^k)x^{k-1}y^m + (-1)^k(b + (-1)^m)x^k y^{m-1}.$$

We prove that F is a resolution of the trivial G-module $\mathbb{Z}$ and use it to calculate $H^2(G, M)$, where M is dual to a $\mathbb{Z}G$-lattice with at most 3 irreducible components.

These calculation can be used in the study of Chernikov p-groups, according to [1].

Note that our resolution generalizes the result of Shapochka [2] for the second cohomology.

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Parastroph orthogonality of alinear quasigroups

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We continue the study of orthogonal linear and alinear quasigroups [2–4]. Necessary concepts and definitions can be found in [1, 4].

We recall, in [4] the following theorem is proved. Here $x + Ix = 0$, $J_t x = t + x - t$ for all $x \in Q$ and some fixed $t \in Q$.

Theorem. *An alinear quasigroup $(Q, \cdot)$ of the form $x \cdot y = I\alpha x + I\beta y + c$ and an alinear quasigroup $(Q, \circ)$ of the form $x \circ y = I\gamma y + I\delta x + d$, both defined over a group $(Q, +)$, where $\alpha, \beta, \gamma, \delta \in Aut(Q, +)$, are orthogonal if and only if the mapping $(\beta^{-1}\alpha - J_t\gamma^{-1}\delta)$ is a permutation of the set Q for any element $t \in Q$.*

We research conditions of parastroph orthogonality of alinear quasigroups defined over dihedral groups of small orders.

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Normal bases and elements of high order in finite field extensions based on cyclotomic polynomials

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Let q be a power of a prime number p, and F_q be a finite field with q elements. For any integer m, a normal basis of F_{q^m} over F_q is a basis of the form $\{\alpha, \alpha^q, ..., \alpha^{q^{m-1}}\}$ for some $\alpha \in F_{q^m}$. In this case the element $\alpha \in F_{q^m}$ is called normal over F_q [1, 2].

Let $r = 2n+1$ be a prime number coprime with q. Let q be a primitive root modulo r, that is the multiplicative order of q modulo r equals to $r-1$. Set $F_q(\theta) = F_{q^{r-1}} = F_q[x]/\Phi_r(x)$, where $\Phi_r(x) = x^{r-1} + ... + x + 1$ is the r-th cyclotomic polynomial and $\theta \equiv x \ (\mathrm{mod}\, \Phi_r(x))$. It is clear that the equality $\theta^r = 1$ holds. We have the following tower of finite fields: $F_q \subset F_{q^n} \subset F_{q^{2n}}$.

Theorem. *Let b be such element of the field F_q that $2nb \neq 1(\mathrm{mod} p)$. Then the following statements are true*:
(a) *element $\theta + b \in F_{q^{2n}}$ is normal over F_q*;
(b) *element $\theta + \theta^{-1} + 2b \in F_{q^n}$ is normal over F_q.*

Note that for $b = 0$ the order of θ equals only to r. But for $b \neq 0$ the element $\theta + b \in F_{q^{2n}}$ has high order according to [3, Theorem 1 (a), (d)]. Also if $2b = (a^2 + 1)a^{-1}$ and $b \neq 0$, then the element $\theta + \theta^{-1} + 2b = (\theta^{-f} + a)(\theta^f + a)$ has high order according to [3, Theorem 1 (b)].

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Automata that generate metabelian groups

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For each $k \geqslant 0$ consider an automaton $S_{2k+1,2} = \langle Q_k, \tau_{2k+1}, \rho_{2k+1} \rangle$ over a binary alphabet $X = \{0,1\}$ (Fig. 1), where $Q_k = \{s_0, \ldots, s_{2k}\}$ — its set of inner states, $\tau_{2k+1} : Q_k \times X \to Q_k$ – transition function, $\rho_{2k+1} : Q_k \times X \to X$ – output function.

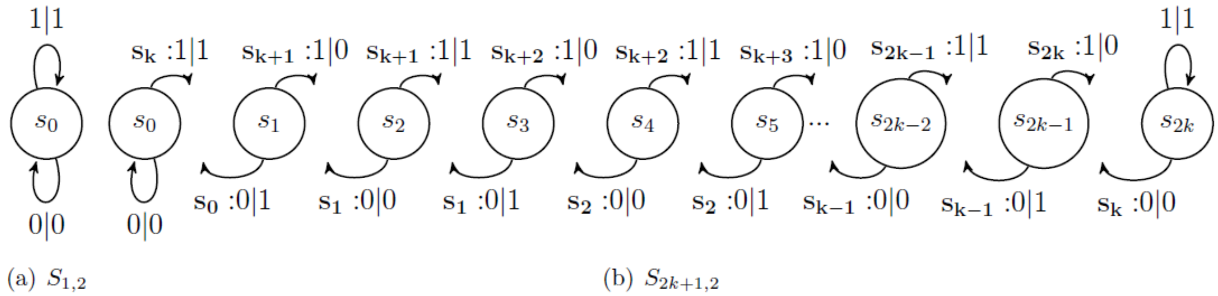

Figure 1: Automata $S_{1,2}$ and $S_{2k+1,2}$, $k > 0$

Denote by G_k the self-similar group [1], generated by the automaton $S_{2k+1,2}$, $k > 0$. Let

$$BS(1,m) = \langle a, t \,|\, tat^{-1} = a^m \rangle, m > 1,$$

be the metabelian Baumslag-Solitar group [2].

Theorem [3]**.** *For $k > 0$ the self-similar group G_k is isomorphic to the metabelian Baumslag-Solitar group $BS(1, 2k+1)$.*

The automata $S_{2k+1,2}$, $k \geqslant 0$, admit a recursive definition. Indeed, assume that for some $m_1 = 2k_1 + 1$ the automaton $S_{m_1,2}$ is defined. Then for $m_2 = m_1 + 2$ the automaton $S_{m_2,2}$ satisfies the following equalities:

$$\begin{array}{ll}
\tau_{m_2}(s,1) = \tau_{m_1}^{+1}(s,1), s \in Q_k, & \tau_{m_2}(s,0) = \tau_{m_1}(s,0), s \in Q_k, \\
\tau_{m_2}(s_{2k_1+1},1) = \tau_{m_2}(s_{2k_1+2},1) = s_{2k_1+2}, & \tau_{m_2}(s_{2k_1+1},0) = \tau_{m_1}(s_{2k_1},0), \\
\tau_{m_2}(s_{2k_1+2},0) = \tau_{m_1}^{+1}(s_{2k_1},0), & \\
\rho_{m_2}(s,1) = \rho_{m_1}(s,1), s \in Q_k, & \rho_{m_2}(s,0) = \rho_{m_1}(s,0), s \in Q_k, \\
\rho_{m_2}(s_{2k_1+1},1) = 0, & \rho_{m_2}(s_{2k_1+1},0) = 1, \\
\rho_{m_2}(s_{2k_1+2},1) = 1, & \rho_{m_2}(s_{2k_1+2},0) = 0,
\end{array}$$

where for each state s such that $s = s_i$ for some $i \geqslant 0$ the state s_{i+1} is denoted by s^{+1}.

The further properties of the automata $S_{2k+1,2}$, $k \geqslant 0$, are established.

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ON SOLVABILITY OF THE MATRIX EQUATION $AX = XB$ OVER INTEGRAL DOMAINS

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Let $\mathbb{D}$ be an integral domain, i.e. $\mathbb{D}$ is a commutative ring without zero divisors which contains an identity element e. Denote by $\mathbb{D}^{m\times n}$ the set of m-by-n matrices with entries from the domain $\mathbb{D}$, I_n is the identity of n-by-n matrix.

Consider the equation

$$AX = XB, \tag{1}$$

where $A \in \mathbb{D}^{m\times m}$, $B \in \mathbb{D}^{n\times n}$ and X is an unknown m-by-n matrix having elements belonging to the domain $\mathbb{D}$. Put $a(\lambda) = \det(I_m\lambda - A)$ and $b(\lambda) = \det(I_n\lambda - B)$.

The general theory of solvability of matrix equation (1) over a field is discussed in [1, 2]. The main purpose of this note is to establish the conditions for the solvability of matrix equation (1) over an integral domain $\mathbb{D}$. We note that in such generality this is a difficult problem. (See Chapter 2 in [2] for information about solvability of matrix equation (1) over a Bezout domain.) The following statements are the main results of this note.

Theorem 1. *The following statements are equivalent*

1) *the equation (1) has a non-zero solution $X_0 \in \mathbb{D}^{m\times n}$,*

2) *the matrix $a(B)$ is singular,*

3) *the matrix $b(A)$ is singular.*

Theorem 2. *Let a non-zero matrix $X_0 \in \mathbb{D}^{m\times n}$ be a solution of equation (1). Then*

1) $\operatorname{rank} X_0 \leqslant m - \operatorname{rank} a(B)$,

2) $\operatorname{rank} X_0 \leqslant n - \operatorname{rank} b(A)$.

Let $\mathbb{D} = \mathbb{B}$ be a Bezout domain (see Chapter 1 in [2]). We describe the structure of non-zero solutions of matrix equation (1) over the domain $\mathbb{B}$. We also propose conditions of similarity of matrices $A, B \in \mathbb{B}^{n\times n}$.

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Free products in groups of infinite unitriangular matrieces

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Let R be an associative ring with a unity of prime characteristic $p \geqslant 2$. Consider the group $UT_\infty(R)$ of infinite unitriangular matrices over R ([1]).

For any square matrix A over R denote by $U(A)$ the infinite unitriangular matrix of the form

$$U(A) = \begin{pmatrix} E & A & O & O & \dots \\ O & E & A & O & \dots \\ O & O & E & A & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

where E and O stand for identity and zero matrices respectively of the same dimension as the matrix A is.

Lemma. *If a matrix A is nilpotent of degree p^k, $k \geqslant 1$, then the matrix $U(A)$ has order p^k in the group $UT_\infty(R)$.*

For fixed natural $n \geqslant 2$ and $t \geqslant 1$ consider n pairwise disjoint subsets of nonzero elements $V_1, V_2, \dots, V_n$ of the direct power R^t and n square matrices $A_1, A_2, \dots, A_n$ of dimension t such that:

- for any $i, 1 \leqslant i \leqslant n$, the matrix A_i is nilpotent of degree p^{k_i} for some $k_i \geqslant 1$;
- for any $i, j (1 \leqslant i, j \leqslant n, i \neq j)$, $v_i \in V_i$ and natural number l $(1 \leqslant l \leqslant 2^{k_i} - 1)$ we have $v_i A_j^l \in V_j$.

Theorem. *The subgroup of $UT_\infty(R)$, generated by matrices $U(A_1), U(A_2), \dots, U(A_n)$, is a free product of n cyclic groups of orders $p^{k_1}, p^{k_2}, \dots, p^{k_n}$ respectively.*

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On locally finite groups whose cyclic subgroups are GNA-subgroups

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Let G be a group. We recall that a subgroup H of G is said to be *abnormal* in G if $g \in \langle H, H^g \rangle$ for each element $g \in G$. Note that normal and abnormal subgroups are antipodes. In fact, a subgroup H of G is normal and abnormal in G if and only if $H = G$. There are many generalizations of normal and abnormal subgroups (for example, pronormal subgroups, self-conjugate permutable subgroups and others). We recall also that an abnormal subgroup H of G is self-normalizing, that is $H = N_G(H)$. Taking this into account, we obtain the following natural generalization of normal and abnormal subgroups.

Definition [1]. A subgroup H of a group G is said to be *GNA-subgroup* (generalized normal and abnormal) of G if for every element $g \in G$ either $H^g = H$ or $N_K(N_K(H)) = N_K(H)$ where $K = \langle H, g \rangle$.

We recall that a group G is said to be a *T-group* if every subnormal subgroup of G is normal in G. A group G is said to be a *$\bar{T}$-group*, if every subgroup of G is a T-group. Recently, in the paper [1] was proved that if G is a locally finite group such that every subgroup of G is GNA-subgroup, then G is a $\bar{T}$-group. The next step is to consider the locally finite groups whose cyclic subgroups are GNA-subgroups.

If G is a group then we let $\Pi(G)$ denote the set of prime divisors of the orders of the elements of G.

Theorem. *Let G be a locally finite group and L be a locally nilpotent residual of G. If every cyclic subgroup of G is GNA-subgroup, then the following conditions hold:*

(i) *L is abelian;*

(ii) $2 \notin \Pi(L)$ *and* $\Pi(L) \cap \Pi(G/L) = \varnothing$;

(iii) *G/L is a Dedekind group;*

(iv) *every subgroup of $C_G(L)$ is G-invariant.*

Conversely, if a group G satisfies conditions (i)–(iv), every subgroup of G is GNA-subgroup.

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On Wiegold's function

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One of the central results in the theory of infinite groups is the so-called Schur's theorem, which demonstrates the connection between the central factor-group $G/Z(G)$ of an arbitrary group G and the derived subgroup of G.

Theorem 1 [1]. *Let G be a group and suppose that $G/Z(G)$ is finite. Then the derived subgroup $[G,G]$ of G is finite.*

Taking this result into account, it is natural to consider the relationship between the orders $|G/Z(G)|$ and $|[G,G]|$. In other words, is there a function f such that $|[G,G]| \leqslant f(|G/Z(G)|)$? J. Wiegold in [2] obtained the best result.

Theorem 2 [2]. *Let G be a group. Suppose that the central factor-group $G/Z(G)$ is finite and has order t.*

(i) Then $|[G,G]| \leqslant w_1(t)$ where $w_1(t) = t^m$ and $m = \frac{1}{2}(log_2 t - 1)$.

(ii) If $t = p^n$ where p is a prime, then $[G,G]$ is a p-group of order at most $w_2(p,n) = p^{\frac{1}{2}n(n-1)}$.

(iii) For each prime p and each integer $n > 1$ there exists a p-group G with $|G/Z(G)| = p^n$ and $|[G,G]| = w_2(p,n)$.

Surprising is the fact that since 1965 Wiegold's function has not been improved. Here the following two situations are possible: either this estimate is very good, or the algebraists simply did not study this question. We conducted a mathematical experiment that shows that the second assertion is true. We illustrate only a few examples that clearly demonstrate that Wiegold's function is far from the real picture.

We denote by (m,n) the n-th group of order m in the SmallGroup library in GAP [3]. We begin with the case of non p-groups.

Group	(96,41)	(132,3)	(272,15)	(336,55)	(406,2)	(644,3)
$\lvert G/\zeta(G)\rvert$	48	66	136	168	203	322
$\lvert[G,G]\rvert$	**12**	**33**	**68**	**28**	**29**	**161**
w_1	**>7148**	**>38837**	**>3118310**	**>12955126**	**>48991203**	**>1557548105**

Below are the results for p-groups.

Group	(2^8,10338)	(3^7,4349)	(5^5,27)	(7^5,32)	(11^4,7)	(13^4,7)
$\lvert G/\zeta(G)\rvert$	64	729	625	2401	1331	2197
$\lvert[G,G]\rvert$	**4**	**27**	**25**	**343**	**121**	**169**
w_2	**32768**	**14348907**	**15625**	**117649**	**1331**	**2197**

These observations indicate that it is necessary to conduct additional and more in-depth studies of the above question. Moreover, it is obvious that it is necessary to use a different technique in comparison with J. Wiegold.

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Equivalence of bundles and topological isomorphisms of function spaces

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Let $\{X_s : s \in S\}$ be a family of subspaces of Tychonoff space X and $\{Y_s : s \in S\}$ be a family of subspaces of Tychonoff space Y. We say that the bundle $(X, \{X_s : s \in S\})$ is M-equivalent to the bundle $(Y, \{Y_s : s \in S\})$, if there exists topological isomorphism $h\colon F(X) \to F(Y)$ of the free topological groups, such that $h(\langle A_s \rangle) = \langle B_s \rangle$ for all $s \in S$. (we will write shortly $(X, \{X_s : s \in S\}) \overset{M}{\sim} (Y, \{Y_s : s \in S\})$).

For a topological space X and topological group G denote by $C(X;G)$ the group of continuous mappings from X to G equipped with the operation $f \cdot g(x) = f(x) \cdot g(x)$. Group $C(X;G)$ equipped with the topology of pointwise convergence is denoted $C_p(X;G)$.

Proposition 1. *Let G be a topological group, $\{G_s : s \in S\}$ be a family of subgroups in G, $\{X_s : s \in S\}$ be a family of subspaces of topological space X. Then the subset $\{f \in C(X,G) : \forall s \in S\ f(X_s) \subseteq G_s\}$ is a subgroup in $C(X;G)$.*

Proposition 2. *Let G be a topological group, $\{\tau_s : s \in S\}$ be a family of infinite cardinals, $\{X_s : s \in S\}$ be a family of subspaces of topological space X. Then the subset $\{f \in C(X,G) : \forall s \in S\ nw(f(X_s)) \leqslant \tau_s\}$ is a subgroup in $C(X;G)$.*

Proposition 3. *Let G be a topological group, $\{\tau_s : s \in S\}$ be a family of infinite cardinals, and $\{X_s : s \in S\}$ be a family of subspaces of topological space X. Then the subset $\{f \in C(X,G) : \forall s \in S\ |f(X_s)| \leqslant \tau_s\}$ is a subgroup in $C(X;G)$.*

Theorem 1. *Let G be a topological group, $\{G_s : s \in S\}$ be a family of subgroups in G, $\{\tau_s : s \in S\}$ be a family of infinite cardinals, $\{X_s : s \in S\}$ be a family of subspaces of topological space X, $\{Y_s : s \in S\}$ be a family of subspaces of topological space Y. If $(X, \{X_s : s \in S\}) \overset{M}{\sim} (Y, \{Y_s : s \in S\})$, then there exists a topological isomorphism $h\colon C_p(X;G) \to C_p(Y;G)$ such that*

$$h(\{f \in C(X,G) :\ \forall s \in S\ f(X_s) \subseteq G_s\}) = \{g \in C(Y,G) :\ \forall s \in S\ g(Y_s) \subseteq G_s\};$$

$$h(\{f \in C(X,G) :\ \forall s \in S\ nw(f(X_s)) \leqslant \tau_s\}) = \{g \in C(Y,G) :\ \forall s \in S\ nw(g(Y_s)) \leqslant \tau_s\};$$

$$h(\{f \in C(X,G) :\ \forall s \in S\ |f(X_s)| \leqslant \tau_s\}) = \{g \in C(Y,G) :\ \forall s \in S\ |g(Y_s)| \leqslant \tau_s\}.$$

Let A be a subspace of Tychonoff space X, G be a topological group. Denote by $CE(A,X,G)$ the set of all continuous mappings from A to G which admits a continuous extension onto X. (CE — continuous extendible)

Proposition 4. *Let A be a subspace of Tychonoff space X, G — topological group. Then subset $CE(A,X,G)$ is a subgroup in $CE(A,G)$.*

Theorem 2. *Let $(X,A) \overset{M}{\sim} (Y,B)$, subspace A is P-embedded in X, subspace B is P-embedded in Y. Then there exists a topological isomorphism $h\colon C_p(A,G) \to C_p(B,G)$ such that $h(CE(A,X,G)) = CE(B,Y,G)$.*

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On exponential sums involving the divisor function over $\mathbb{Z}[i]$

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We apply the van der Corput transform to investigate the sums of view $\sum r(n)g(n)e(f(n)))$, where $r(n)$ is the number of representations of n as the sum of two squares of integer numbers. Such sums have been studied by M. Jutila, O. Gunyavy, M. Huxley and etc. Depending on differential properties of the functions $g(n)$ and $f(n)$ there have been obtained different kinds of error terms in bounds of the considered sums. In the special case, O. Gunyavy improved the result of M. Jutila in the problem on estimate the exponential sum involving the divisor function $\tau(n)$. We obtain the asymptotic formula of the sum $\sum \tau_k(\alpha)e\left(\frac{a}{q}N(\alpha)\right)$, $k = 2, 3$ over the ring of Gaussian integers which is an analogue of the asymptotic formulas obtained by M. Jutila and O. Gunyavy.

The main results of our investigation are represented by the following theorems

Theorem 1. *Let α_0, β be the Gaussian integers, $(\alpha_0, \beta) = 1$, and $\tau(\alpha)$ be the divisor function over the ring of Gaussian numbers. Then for $N(\beta) \ll x^{\frac{1}{4}-\varepsilon}$ the following asymptotic formula*

$$\sum_{N(\alpha)\leqslant x} \tau(\alpha)e^{2\pi i N(\frac{\alpha_0\alpha}{\beta})} = C_1(\beta)\frac{x\log x}{N(\beta)} + C_2(\beta)\frac{x}{N(\beta)} + O\left(x^{\frac{3}{4}+\varepsilon}\right) + O\left(x^{\frac{1}{2}+\varepsilon}N(\beta)\right)$$

where $C_i(\beta)$ are computable constants, $N(\beta)^{-\varepsilon} \ll C_i(\beta) \ll N(\beta)^{\varepsilon}$, $i = 1, 2$, holds.

Theorem 2. *Let a and q be the positive integers, $(a, q) = 1$. Then for $x \to \infty$*

$$\sum_{N(\alpha)\leqslant x} \tau_3(\alpha)e\left(\frac{aN(\alpha)}{q}\right) = \frac{x}{q}P_2(\log x) + O\left(x^{\theta_0}\right),$$

where $P_2(u)$ is a polynomial of two degree with the fixed coefficients $\theta_0 = \frac{2-\theta}{3-2\theta}$, $\theta = \frac{1792}{3615}$.

Local nearrings of order p^3

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A nearring with identity is called local if the set of all its non-invertible elements is a subgroup of its additive group [1].

It was proved in [2] that, up to an isomorphism, there is a unique local nearring whose additive group is cyclic, namely the local ring of quotients $\mathbb{Z}/p^n\mathbb{Z}$ with a prime p and $n \geqslant 1$. Furthermore, it follows from [3, 4] that for every non-cyclic abelian p-group G of order $|G| > 4$ there exists a local nearring on G which is not a ring and there are p non-isomorphic local nearrings with a non-cyclic additive group of order p^2 which are not nearfields. In particular, together with the paper [5] this gives a complete description of all local nearrings of order p^2. However the general problem of determining the number of non-isomorphic local nearrings on a group G of order $|G| > p^2$ remains open.

It is well-known that for each p there is 5 non-isomorphic group of order p^3 two of which are non-abelian. These two groups cannot be the additive group of a local nearring for $p = 2$, as it was shown in [6]. On the other hand, the following assertion follows from our results [7, 8].

Proposition 1. *Each non-abelian group of order p^3 with $p > 2$ is the additive group of a local nearring.*

It follows from [9] that the number of non-isomorphic local nearrings of order p^3 on a non-abelian group of order p^3 with a cyclic subgroup of index p is equal to 4 for $p = 3$ and 2 for $p > 3$. The following result concerns local nearrings on a non-abelian additive group of order p^3 and exponent p.

Theorem 1. *Let G be a non-abelian group of order p^3 and exponent p with $p > 2$. Then the number of all non-isomorphic local nearrings whose additive group is isomorphic to G is equal to $p + 1$.*

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Gelfand–Tsetlin modules with arbitrary characters

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Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. The algebra $U = U(\mathfrak{g})$ contains a maximal commutative subalgebra Γ called the Gelfand-Tsetlin subalgebra. A Gelfand-Tsetlin module is a U-module on which the action of Γ is locally finite. Gelfand-Tsetlin modules form a large subcategory of the category of representations of $\mathfrak{gl}(n, \mathbb{C})$. Explicit examples of Gelfand-Tsetlin modules are *generic* GT-modules, which are have a basis indexed by talbeaux, and over which the action of U is given by rational functions in the entries of these tableaux.

Any Gelfand-Tsetlin module decomposes as a direct sum of Γ-modules, and each of these Γ-modules has an associated character $\chi : \Gamma \to \mathbb{C}$; characters are indexed by points in $\mathbb{C}^{\binom{n}{2}}$ (up to finite multiplicity). The singularities of the rational functions giving the action of U lie in a certain hyperplane arrangement. Points lying outside this arrangement (and the corresponding characters) are called *generic*; points outside this arrangement (and the corresponding characters) are called *singular*. Up to now, there were only explicit constructions of Gelfand-Tsetlin modules with generic characters, or some special cases of singular characters. We extend this constructions to cover all possible characters and study some basic features of the Gelfand-Tsetlin modules thus obtained, such as an explicit basis, formulas for the action of U on this basis, and multiplicities of the characters.

The Smith normal form of the lest common multiples of one class of matrices

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Let R be a commutative principal ideal domain with $1 \neq 0$ and A, B be matrices over R. If $A = BC$, then the matrix B is a left divisor of the matrix A and the matrix A is a right multiples of the matrix B. The matrix M is a common right multiples of the matrices A and B, if $M = AA_1$ and $M = BB_1$. Moreover, the matrix M is the least common right multiples of the matrices A and B, if the matrix M is a left divisor every other common right multiples of the matrices A and B (by notation $[A, B]_r$).

The method for finding the least common right multiples of matrices A and B was proposed by C. MacDuffee [1] in 1933. B. Stewart [2] showed that the least common right multiples of two given matrices are uniquely determined up to invertible right factors. M. Newman formulated the problem to establish of the relationship between the Smith normal forms of two given matrices and the Smith normal form of their least common multiples over commutative principal ideal domain. R. Thompson [3] showed some divisibility conditions between the invariant factors of two given matrices and the invariant factors of their least common multiples.

Let A, B be an 3×3 non-singular matrices over R. For the matrices A and B there exist invertible matrices P_A, P_B and Q_A Q_B, such that

$$P_A A Q_A = E = diag(\varepsilon_1, \varepsilon_2, \varepsilon_3), \ \ where \ \ \varepsilon_i \,|\varepsilon_{i+1} \ , \ i = 1, 2.$$

$$P_B B Q_B = \Delta = diag(\delta_1, \delta_2, \delta_3), \ \ where \ \ \delta_i \,|\delta_{i+1} \ , \ i = 1, 2.$$

The matrices E and Δ are called the canonical diagonal forms or Smith normal forms, P_A, P_B and Q_A, Q_B are called left and right transforming matrices for matrices A and B, respectively.

Denote by $\mathbf{P}_A$ the set of all left transforming matrices for matrix A.

Theorem. *Let*

$$A \sim E = diag(1, \varepsilon, \varepsilon), \quad B \sim \Delta = diag(1, 1, \delta)$$

and let $P_B P_A^{-1} = \|s_{ij}\|_1^3$, *where* $P_A \in \boldsymbol{P}_A$, $P_B \in \boldsymbol{P}_B$. *Then the Smith normal form of the least common right multiples of the matrices* A *and* B *has the form:*

$$\Omega = diag(\frac{(\varepsilon, \delta)}{(\varepsilon, \delta, s_{31})}, \varepsilon, [\varepsilon, \delta]).$$

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When $R(X)$ and $R\langle X\rangle$ are ω-Euclidean domain

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Let R be a commutative domain with nonzero unit element. For a polynomial f in $R[X]$, we let $c(f)$ be the ideal of R generated by the coefficients of f. Set $S = \{f \in R[X] : c(f) = R\}$, a multiplicatively closed subset of $R[X]$ consists of the regular elements. The *Nagata ring* over R is the ring $R(X) = R[X]_S$. Another interesting localization of $R[X]$ is given by the multiplicatively closed subset $W = \{f \in R[X] : f \text{ is monic}\}$. We denote $R\langle X\rangle = R[X]_W$. *A Bezout ring* is a ring in which every finitely generated is principal.

Let $\varphi : R \to \mathbb{N} \cup \{0\}$ be a function satisfying the following condition: $\varphi(a) = 0$ if and only if $a = 0$; $\varphi(a) > 0$ for any nonzero and $\varphi(ab) \geqslant \varphi(a)$ for any arbitrary elements $a, b \in R$. This function is called the *norm* over domain R.

A *k-stage division chain* for any arbitrary elements $a, b \in R$ with $b \neq 0$ is understood as the sequence of equalities

$$a = bq_1 + r_1, b = r_1q_2 + r_2, \dots, r_{k-2} = r_{k-1}q_k + r_k, \tag{1}$$

with $k \in \mathbb{N}$.

Domain R is called *a ω-Euclidean domain* with respect to the norm φ, if for any arbitrary elements $a, b \in R$, $b \neq 0$, there exists a k-stage division chain (1) for some k, such as $\varphi(r_k) < \varphi(b)$.

An elementary $n \times n$ matrix with entries from R is a square $n \times n$ matrix of one of the types below: 1) diagonal matrix with invertible diagonal entries; 2) identity matrix with one additional non diagonal nonzero entry; 3) permutation matrix, i.e. result of switching some columns or rows in the identity matrix.

A ring R is called *a ring with elementary reduction of matrices* in case of an arbitrary matrix over R possesses elementary reduction, i.e. for an arbitrary matrix A over the ring R there exist such elementary matrices over R , $P_1, \dots, P_k, Q_1, \dots, Q_s$ of respectful size that

$$P_1 \cdots P_k \cdot A \cdot Q_1 \cdots Q_s = diag(\varepsilon_1, \dots, \varepsilon_r, 0, \dots, 0),$$

where $R\varepsilon_{i+1}R \subseteq R\varepsilon_i \cap \varepsilon_i R$ for any $i = 1, \dots, r-1$. All other necessary definitions and facts can be found in [1–4].

Theorem 1. *A domain R is ω-Euclidean if and only if $R(X)$ is ω-Euclidean.*

Theorem 2. *Let R be a integral domains. The following are equivalent:*
1) R is a Bezout domain;
2) $R(X)$ is a ω-Euclidean domain;
3) $R(X)$ is a ring with elementary reduction of matrices.

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Summatory formula for Fourier coefficients of cusp forms over square

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Let $f(z)$ be a holomorphic cusp form of weight $k \geqslant 12$ for the full modular group $SL(2,\mathbb{Z})$, $z \in H$, $H = \{z \in \mathbb{C} | Im(z) > 0\}$ is the upper half plane. We suppose that $f(z)$ is a normalized eigenfunction for the Hecke operators $T(n)(n \geqslant 1)$. In this case, $f(z)$ has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z},$$

where $a_f(1) = 1$, and $T(n)f = a_f(n)f$ for every $n \in \mathbb{N}$.

Hafner J. L., Ivić A. [1] obtained on O-estimate and $\Omega_{\pm}$-results for $\sum_n a_f(n)$. Rankin, Selberg [3] investigated the second moment $\sum_{n \leqslant x} |a_f(n)|^2$. In [2] Lau Y. K., Lü G. S., Wu J. studied the summation $\sum_{n \leqslant x} a_f(n)^j$, where $3 \leqslant j \leqslant 8$. We consider summatory function associated with the arithmetic convolution.

$$T_f(x) = \sum_{nm \leqslant x} a_f(n)^2 \tau_{12}(m),$$

where $\tau_{12}(m)$ is the number of representations of $n \in \mathbb{N}$ as $n = n_1 n_2$, $n_1, n_2 \in \mathbb{N}$.

Theorem. *For $x \to \infty$ we have*

$$T_f(x) = A_1 x \log x + A_0 x + O\left(x^{\frac{2}{3}+\epsilon}\right).$$

Here A_1, A_0 are constants, which may be explicitly evaluated, constant in O-term depends only of ϵ.

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On Some Algebraic Structures Associated with Infinite-Dimensional Dynamical Systems

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Consider the following operator equation

$$N(u) = 0, \qquad u \in D(N), \tag{1}$$

where $N\colon D(N) \subseteq U \to V$ is a Gâteaux differentiable operator, U, V are linear normed spaces over the field of real numbers $\mathbb{R}$, $D(N)$ is the domain of the operator N.

In the paper we will use notations and notions of [1–3].

Let us consider an infinitesimal transformation

$$\overline{u} = u + \varepsilon S(u), \tag{2}$$

where $S\colon D(N) \to D(N'_u)$ is a generator of the transformation.

If S_1, S_2 are generators of transformation (2), then their $(\mathcal{S}, \mathcal{T})$-product can be defined as [1, p. 91]

$$(S_1, S_2)(u) = S'_{1u}\mathcal{S}_u S_2(u) - S'_{2u}\mathcal{T}_u S_1(u), \tag{3}$$

the corresponding $\mathcal{G}$-commutator is given by

$$[S_1, S_2]_{\mathcal{G}}(u) = S'_{1u}\mathcal{G}_u S_2(u) - S'_{2u}\mathcal{G}_u S_1(u) \tag{4}$$

and the commutator is

$$[S_1, S_2](u) = S'_{1u} S_2(u) - S'_{2u} S_1(u). \tag{5}$$

Definition. Transformation (2) is called a symmetry of equation (1), if for any sufficiently small ε and any solution u of this equation function $\overline{u}$ (2) is also a solution of this equation.

In this case the operator S is also called a generator of the symmetry of equation (1).

Theorem 1. *If $\mathcal{S}_u, \mathcal{T}_u$ are recursion operators and $\forall u \in D(N)$, $\forall h, v \in D(N'_u)$*

$$N''_u(h, \mathcal{S}_u v) = N''_u(v, \mathcal{T}_u h), \qquad \mathcal{G}'_u(h; \mathcal{G}_u v) = \mathcal{G}'_u(v; \mathcal{G}_u h),$$

where $\mathcal{G}_u \equiv \mathcal{S}_u + \mathcal{T}_u$, then generators of symmetries of equation (1) form a Lie-admissible algebra under $(\mathcal{S}, \mathcal{T})$-product (3).

Theorem 2. *If $\mathcal{G}_u$ is a recursion operator and $\forall u \in D(N)$, $\forall h, v \in D(N'_u)$*

$$N''_u(h, \mathcal{G}_u v) = N''_u(v, \mathcal{G}_u h), \qquad \mathcal{G}'_u(h; \mathcal{G}_u v) = \mathcal{G}'_u(v; \mathcal{G}_u h),$$

then generators of symmetries of equation (1) form a Lie algebra under $\mathcal{G}$-commutator (4).

Theorem 3. *Generators of symmetries of equation (1) form a Lie algebra under commutator (5).*

Similar results are established for variational symmetries and associated algebraic structures.

The work is supported by the Russian Foundation for Basic Research (grant No. 16-01-00450_a).

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On the semigroup ID_∞

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We shall follow the terminology of [3–6]. A *(semi)topological semigroup* is a topological space endowed with a (separately) continuous semigroup operation.

By ID_∞ we denote the semigroup of all partial cofinite isometries of integers $\mathbb{Z}$. The semigroup ID_∞ defined in [1] and its algebraic properties is studied in [1, 2].

We show that ID_∞ is an F-inverse semigroup and describe the minimum group congruence on ID_∞.

Theorem 1. *The semigroup ID_∞ is isomorphic the the semidirect product $\mathsf{Iso}(\mathbb{Z}) \ltimes_{\mathfrak{h}} \mathscr{P}_\infty(\mathbb{Z})$ of the free semilattice with unit $(\mathscr{P}_\infty(\mathbb{Z}), \cup)$ by the group $\mathsf{Iso}(\mathbb{Z})$ of all isometries of $\mathbb{Z}$.*

Theorem 2. *Let τ ba a Baire T_1-topology on ID_∞ such that left (right) translations in (ID_∞, τ) are continuous maps. Then the group of units $H(1)$ is a discrete subspace in (ID_∞, τ).*

We shall say that a topology τ on an inverse semigroup S is *left (right) E-Baire* if for any idempotent $e \in S$ the subspace eS (Se) of S is Baire.

Theorem 3. *Let τ be a left (right) E-Baire T_1-topology on ID_∞ such that right (left) translations on (ID_∞, τ) are continuous maps. Then τ is discrete.*

On ID_∞ there exists a non-discrete non-Baire Hausdorff topology τ_{NB} such that $(ID_\infty, \tau_{\mathsf{NB}})$ is a topological semigroup.

Theorem 4. *If ID_∞ is a dense discrete subsemigroup of a T_1-semitopological semigroup S such that $I = S \setminus ID_\infty \neq \varnothing$ then I is a two-sided ideal in S.*

Theorem 5. *If a Hausdorff topological semigroup S contains ID_∞ as a dense discrete subsemigroup then the square $S \times S$ is not a feebly compact space.*

Theorem 6. *If a Hausdorff topological semigroup S contains ID_∞ with an isolated point in ID_∞ then the square $S \times S$ is not a countably compact space.*

Theorem 7. *There exists no a feebly compact quasiregular T_1-topological inverse semigroup which contains ID_∞ as a dense subsemigroup.*

Theorem 8. *The semigroup ID_∞ does not embed into a countably compact T_3-topological inverse semigroup.*

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Noncommutative versions of Noether's Problem

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We are going to present an overview of results on reformulations of the classical Noether's Problem for the Weyl Algebra and it's quantization. As application, we will discuss consequences of the results to the Gelfand–Kirillov Conjecture and some of it's analogues.

Finite groups with systems of K-$\mathfrak{F}$-subnormal subgroups

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In what follows, G is a finite group. Let $\mathfrak{F}$ be a class of groups. A subgroup A of G is said to be $\mathfrak{F}$-*subnormal in G in the sense of Kegel* or K-$\mathfrak{F}$-*subnormal* in G if there is a subgroup chain

$$A = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i} \in \mathfrak{F}$ for all $i = 1, \ldots, n$. A formation $\mathfrak{F}$ is said to be K-lattice provided in every group G the set of all its K-$\mathfrak{F}$-subnormal subgroups forms a sublattice of the lattice of all subgroups of G.

We consider some new applications of the theory of K-lattice formations. In particular, we prove the following

Theorem. *Let $\mathfrak{F}$ be a hereditary K-lattice saturated formation containing all nilpotent groups.*

(i) *If every $\mathfrak{F}$-critical subgroup H of G is K-$\mathfrak{F}$-subnormal in G with $H/F(H) \in \mathfrak{F}$, then $G/F(G) \in \mathfrak{F}$.*

(ii) *If every Schmidt subgroup of G is K-$\mathfrak{F}$-subnormal in G, then $G' \in \mathfrak{F}$.*

Corollary 1 (Semenchuk [1]). *If every Schmidt subgroup of G is subnormal in G, then G is metanilpotent.*

Corollary 2 (Monakhov and Knyagina [2]). *If every Schmidt subgroup of G is subnormal in G, then $G/F(G)$ is abelian.*

Corollary 3 (Al-Sharo and Skiba [3]). *If every Schmidt subgroup of G is σ-subnormal in G (in the sense of* [4]*),then $G/F_\sigma(G)$ is abelian.*

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On groups whose subgroups of infinite special rank are transitively normal

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Groups with certain prescribed properties of subgroups form one of the central subjects of research in group theory. Their investigation introduced many important notions such as finiteness conditions, local solubility, group rank, and others. Choosing specific prescribed properties and concrete families of subgroups, which possess these properties, we come to distinct classes of groups. There is an enormous array of papers devoted to these topics. We have considered of a family of subgroups of finite special rank and the one of transitively normal groups on a group structure.

A group G is said to have a finite special rank r if every finitely generated subgroup of G can be generated by at most r elements and there exists a finitely generated subgroup H, which has exactly r generators [1]. The theory of groups of finite special rank is one of the most profoundly developed parts of the group theory [2-4]. In a paper [5] M. R. Dixon, M. J. Evans and H. Smith have considered groups whose subgroups of infinite special rank have some fixed property P. A bunch of authors expanded the research area taking into account distinct natural properties P [4]. We focus on groups whose subgroups of infinite special rank are transitively normal.

A subgroup H of a group G is transitively normal if H is normal in every subgroup $K \geqslant H$, in which H is subnormal [6].

Theorem. *Let G be a periodic soluble group of infinite special rank whose subgroups of infinite special rank are transitively normal. Then every subgroup of G is a transitively normal one.*

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Improved triangular form of the matrix and its invariants

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Let $A(x)$ and $B(x)$ be $n \times n$ matrices with entries in the ring of polynomials $\mathbf{C}[x]$. Then $A(x)$ is said to be semiscalarly equivalent to $B(x)$ if there exist matrices S in $GL(n, \mathbf{C})$ and $R(x)$ in $GL(n, \mathbf{C}[x])$ (so $det(R(x))$ is a nonzero complex number) such that $B(x) = SA(x)R(x)$ [1]. The concept of semiscalarity is of interest as it occurs naturally in diverse applications in applied mathematics, but finding a complete set of (computable) invariants for it is a very difficult problem. In this report, the problem is partly solved for the case of matrices $A(x)$ with only one characteristic root. Then the rank of matrix $A(x)$ is full. Without loss of generality, we can assume that first invariant multiplier of considered matrix is identity and its characteristic root is zero. Notation $a^{(t)}(\alpha)$ is the value at $x = \alpha$ of the t-th derivative of the polynomial $a(x)$. In what follows the abbreviation GCD means greatest common divisor.

Proposition. *In the class $\{SA(x)R(x)\}$ of semiscalarly equivalent matrices exist a matrix of the form*

$$\left\| \begin{array}{ccccc} 1 & & & & 0 \\ a_{21}(x) & x^{k_1} & & & \\ a_{31}(x) & a_{32}(x) & x^{k_2} & & \\ \dots & \dots & \dots & \dots & \\ a_{n1}(x) & a_{n2}(x) & a_{n3}(x) & \dots & x^{k_{n-1}} \end{array} \right\|, \tag{1}$$

where $dega_{il}(x) < k_{i-1}$, $a_{i1}(0) = 0$, $i = 2, 3, \dots, n$, $l < i$, $a^{(k_j)}_{j+2,\, j+1}(0) = \dots = a^{(k_j)}_{n,\, j+1}(0) = 0$, $j = 1, 2, \dots, n-2$.

Theorem. *The following quantities are invariants of the matrix (1) with respect to semiscalar equivalent transformations:*

$$GCD(a_{n1}(x), x^{k_1}),$$
$$GCD(a_{n1}(x), a_{n-1,1}(x), x^{k_1}), \dots,$$
$$GCD(a_{n1}(x), a_{n-1,1}(x), \dots, a_{21}(x), x^{k_1}),$$
$$GCD(a_{n1}(x), a_{n2}(x), x^{k_2}),$$
$$GCD(a_{n1}(x), a_{n2}(x), a_{n-1,1}(x), a_{n-1,2}(x), x^{k_2}), \dots,$$
$$GCD(a_{n1}(x), a_{n2}(x), a_{n-1,1}(x), a_{n-1,2}(x), \dots, a_{31}(x), a_{32}(x), x^{k_2}), \dots,$$
$$GCD(a_{n1}(x), \dots, a_{n,n-2}(x), x^{k_{n-2}}),$$
$$GCD(a_{n1}(x), \dots, a_{n,n-2}(x), a_{n-1,1}(x), \dots, a_{n-1,n-2}(x), x^{k_{n-2}}),$$
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Helmer lemma for Bezout domains of stable range 1.5

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The study of elementary divisor rings has a rich history. I. Kaplansky defined the ring R to be an elementary divisor ring (e.d.r.) if every (not necessary square) matrix A over R admits canonical diagonal reduction, that is, there exist invertible matrices P, Q such that

$$PAQ = \operatorname{diag}(\varphi_1, \ \ldots, \ \varphi_k, \ 0, \ \ldots, \ 0) = \Phi,$$

where φ_i is a full divisor of φ_{i+1} for each i.

Any e.d.r. is a Bezout ring that is a ring in which every finitely generated ideal is a principal one. L. Gilman and M. Henriksen constructed an example of a commutative Bezout ring which is not an e.d.r. This leads to the problem whether arbitrary commutative Bezout domain is an elementary divisor ring. The concept of stable range is effective enough to solve this problem.

Definition. A ring R has **stable range 1.5** if the condition

$$aR + bR + cR = R,$$

$a, b, c \in R$, $c \neq 0$ implies the existence of $r \in R$ such that

$$(a + br)R + cR = R.$$

All factorial rings, principal ideal rings, adequate rings, 2×2 matrices over commutative Bezout domain has stable range 1.5.

Theorem 1. *Let R be a commutative Bezout domain of stable range 1.5 and*

$$(a_1, \ldots, a_n) = 1,$$

$n \geqslant 2$. Let ψ be an arbitrary fixed nonzero element of the ring R. Then there exist $u_1, \ldots, u_n$, which simultaneously satisfy the following equalities:

1) $u_1 a_1 + \ldots + u_n a_n = 1$;

2) $(u_1, \ldots, u_i) = 1$ *for arbitrary fixed i, $2 \leqslant i \leqslant n$;*

3) $(u_i, \psi) = 1$ *for arbitrary fixed i, $2 \leqslant i \leqslant n$.*

O. Helmer proved that an adequate rings are e.d.r.. To prove this fact he used the auxiliary results: for every matrix A with maximal rank over an adequate ring there is a row $u = \| \, 1 \ \ u_2 \ \ \ldots \ \ u_n \, \|$ such that g.c.d. of the elements of uA and A coincide (Helmer lemma). V. Petrychkovych extended this statement for matrices whose range is greater than one.

The purpose of this paper is a generalization of Helmer's lemma to commutative Bezout domains of stable range 1.5.

Theorem 2. *Let R be a commutative Bezout domain. The following are equivalent:*

1) *R has stable range* 1.5;

2) *for each $n \times m$ matrix A over R,* $\operatorname{rank} A > 1$, *there is a row $u = \| \, 1 \ \ u_2 \ \ \ldots \ \ u_n \, \|$ such that*

$$uA = \| \, b_1 \ \ b_2 \ \ \ldots \ \ b_m \, \|,$$

where $(b_1, \, b_2, \ \ldots, \ b_m)$ is g.c.d. of elements A.

On languages accepted by quantum finite 1-way 1-qubit automata

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Achievements in quantum computing have stimulated formation of quantum automata theory, which investigates acceptors defined in terms of quantum Turing machine (QTM). Special class of these acceptors consists of those that have been defined by 1-way QTM [1]. The most known of them are MO-1QFA, MM-1QFA, N-QFA, CL-QFA, L-QFA and kQFA.

For 1-qubit QTM exists geometrical interpretation, namely Bloch's sphere [2], where there are fixed three pair-wise orthogonal axes: the x-axe, the y-axe and the z-axe. It is well known that any special unitary operator acting in $\mathbb{C}^2$ can be presented as superposition of rotations around these axes. It is evident that rotations around the fixed axe correspond to unitary operators satisfying to the commutativity law.

In the given presentation we investigate some characteristics of languages accepted by finite 1-way 1-qubit automata (either with given probability, or with given mistake), under supposition that there are extracted free input subsemigroups with elements that correspond to unitary operators that satisfy to the commutativity law.

Let X^* be any free input subsemigroup, such that unitary operators that correspond to free generators satisfy to the commutativity law. Then the unitary operator defined for a string $w \in X^*$ can be presented in the standard form

$$sfuo(w) = U_1^{r_1} \dots U_m^{r_m},$$

where U_i corresponds to free generator x_i and r_i is the number of occurrences of $x_i \in X$ in the string w.

Let π be the partition of X^* defined as follows:

$$w_1 \equiv w_2 (mod\, \pi) \Leftrightarrow sfuo(w_1) = sfuo(w_1).$$

We get the following

Theorem. *Let L be the language accepted (either with given probability, or with given mistake) by any of models MO-1QFA, MM-1QFA, N-QFA, CL-QFA, L-QFA or kQFA, and X^* be any free input subsemigroup, such that unitary operators that correspond to free generators satisfy to the commutativity law. Then $L \cap X^*$ is the union of some blocks of the partition π.*

The case when X^* is any free input subsemigroup, such that unitary operators that correspond to free generators are rotations around the fixed axe of Bloch's sphere is investigated in detail. Conditions when the partition π is finitary are established. Some classes of commutative semigroups of unitary operators acting in $\mathbb{C}^2$ are characterized.

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On compositions of attributed transition systems

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The notion of attributed transition system (ATS) is one of the basic notions of Insertion modeling [1]. We define an ATS without hidden transitions as some system $\mathcal{S} = (S, A, U, T, \varphi)$, where S is finite set of states, A is the set of actions, $T \subseteq S \times A \times S$ is the transition relation, U is the set of attributes, and $\varphi : S \to U$ is the function of states marking. Any finite sequence of transitions $s_1 \to_{a_1} \ldots \to_{a_{k-1}} s_k$ ($k \in \mathbb{N}$) is a history of functioning of $\mathcal{S}$, and corresponding sequence $\varphi(s_1) \to_{a_1} \ldots \to_{a_{k-1}} \varphi(s_k)$ is a trace. For any subsets $S_1, S_2 \subseteq S$ we denote $\mathrm{His}(S_1, S_2)$ the set of all histories started in the set S_1 and terminated in the set S_2, and $\mathrm{Tr}(S_1, S_2)$ all traces with initial label $\varphi(s_1) \in \varphi(S_1)$ and final label $\varphi(s_k) \in \varphi(S_2)$. Operation of concatenation on the sets of histories, as well as on the sets of traces are defined in usual way. We deal with ATS $\mathcal{S}$ as with some acceptor $\mathfrak{S} = (\mathcal{S}, S_{in}, S_{fin}, S_{frb})$, where S_{in}, S_{fin}, and S_{frb} are the sets, correspondingly, of initial, final and forbidden states. The acceptor $\mathfrak{S}$ is history-safe, if $\mathrm{His}(S_{in}, S_{frb}) = \mathrm{His}(S_{frb}, S_{fin}) = \varnothing$, and trace-safe, if $\mathrm{Tr}(S_{in}, S_{frb}) = \mathrm{Tr}(S_{frb}, S_{fin}) = \varnothing$. Similarly, the acceptor $\mathfrak{S}$ is history-correct, if $\mathrm{His}(S_{in}, S_{frb})\mathrm{His}(S_{frb}, S_{fin}) = \varnothing$, and trace-correct, if $\mathrm{Tr}(S_{in}, S_{frb})\mathrm{Tr}(S_{frb}, S_{fin}) = \varnothing$.

Algorithms for checking safeness and correctness of the acceptor $\mathfrak{S}$ are proposed. These algorithms are based on two-sided design of corresponding trees.

Compositions, intended to present union, intersection, concatenation and iteration of languages presented by initial acceptors are defined. The main result is the following one.

Theorem. *If initial acceptors satisfy to condition*

$$X \in \{history - safe, trace - safe, history - correct, history - safe\},$$

then each of compositions, intended to present union, intersection, concatenation and iteration of their languages satisfy to the condition X.

It is established that similar results hold for acceptors, that present ω-languages (in the sense of [2]).

More general case takes the place for ATS with hidden transitions. They differ in the fact that $T \subseteq S \times A \times S \cup S \times S$, where elements of the set $T \cap S \times S$ are the hidden transitions. Procedure for reducing analysis of these ATS to the previous case is proposed.

The author would like to thank Academician of NASU, Prof. Alexander Adolfovich Letichevsky for the statement of the problem, and discussion on receiving some results of this presentation.

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On growth function of initial invertible automata

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We will discuss some results on growth functions of initial invertible automata. In particular, the class of two-state automata over two-letter alphabet will be considered. For such automata we will give the exact formula of the corresponding growth functions. Also there will be a discussion on application of current approach to other automata classes. Notice that in [1–3] there are some results for non-initial automata. We will compare how the growth function differs between initial automaton and the corresponding non-initial one.

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On linearity of isotopes of Abelian groups

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An algebra $(Q;\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot})$ is called a *quasigroup*, if the operations $(\overset{\ell}{\cdot})$ and $(\overset{r}{\cdot})$ are *left* and *right inverse* to $(\cdot)$, i.e., the identities $(x\cdot y)\overset{\ell}{\cdot}y=x$, $(x\overset{\ell}{\cdot}y)\cdot y=x$, $x\cdot(x\overset{r}{\cdot}y)=y$, $x\overset{r}{\cdot}(x\cdot y)=y$ hold. In this case, the operation $(\cdot)$ is called *invertible*. Including $(\cdot)$, $(\overset{\ell}{\cdot})$, $(\overset{r}{\cdot})$, there are six operations which are inverse to $(\cdot)$ and all of them are defined by

$$x_{1\sigma}\overset{\sigma}{\cdot}x_{2\sigma}=x_{3\sigma}\ :\Leftrightarrow\ x_1\cdot x_2=x_3,$$

where $\sigma\in S_3:=\{\iota,\ell,r,s,s\ell,sr\}$, $\ell:=(13)$, $r:=(23)$, $s:=(12)$. The operation $(\overset{\sigma}{\cdot})$ is called a *σ-parastrophe* of $(\cdot)$ and the algebra $(Q;\overset{\sigma}{\cdot},\overset{\ell\sigma}{\cdot},\overset{r\sigma}{\cdot})$ is *σ-parastrophe* of a quasigroup $(Q;\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot})$. Each parastrophe of a quasigroup is a quasigroup that is why there are six concepts to every introduced concept. These six concepts are called parastrophes of the considered concept. Some of them can coincide. Here, we describe all concepts being parastrophic to a left linearity.

A quasigroup $(Q;\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot})$ is a *group isotope* [1], if $x\cdot y=\gamma_3^{-1}(\gamma_1x\circ\gamma_2y)$ for a group $(A;\circ)$ and for bijections γ_1, γ_2, γ_3. For every element $0\in Q$ there exists a group $(Q;+,0)$, bijections α, β and an element $a\in Q$ such that $\alpha0=\beta0=0$ and

$$x\cdot y:=\alpha x+a+\beta y. \tag{1}$$

A group isotope $(Q;\cdot)$ is called *left linear (i.e., 2-linear) (right linear, i.e., 1-linear) over a group* $(Q;+,0)$ [2], if (1) holds and α (respectively β) is an automorphism of $(Q;+,0)$ and $(Q;\cdot)$ is called *linear*, if both α and β are automorphisms of the group. To compliment left and right linearity to make the set of all parastrophes of the left linearity over Abelian groups complete we need an additional notion: a group isotope $(Q;\cdot)$ is called *middle linear (i.e., 3-linear) over an Abelian group* $(Q;+,0)$, if (1) holds and $\beta^{-1}\alpha$ is an automorphism of $(Q;+,0)$.

Theorem 1. *Let an isotope of an Abelian group be i-linear, then its σ-parastrophe is* $i\sigma^{-1}$-*linear for all* $i\in\{1,2,3\}$ *and for all* $\sigma\in S_3$.

Theorem 2. *A quasigroup* $(Q;\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot})$ *satisfies the identity* $x\cdot(yu\overset{\ell}{\cdot}v)=(v\overset{r}{\cdot}ux)\cdot y$ *iff there exists an Abelian group* $(Q;+,0)$, *its permutations* α *and* β *with* $\alpha0=\beta0=0$ *and an element* $a\in Q$ *such that*

$$x\cdot y=\alpha x+a+\beta y,\qquad(\beta\alpha^{-1})^3=\iota,\qquad\beta\alpha^{-1}\in Aut(Q;+).$$

Each quasigroup from this variety is middle linear.

Theorem 3. *A quasigroup* $(Q;\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot})$ *satisfies the identity* $x\cdot(yu\overset{\ell}{\cdot}v)=u\cdot(yx\overset{\ell}{\cdot}v)$ *iff there exists an Abelian group* $(Q;+,0)$, *its permutations* α *and* β *with* $\alpha0=\beta0=0$ *and an element* $a\in Q$ *such that*

$$x\cdot y=\alpha x+a+\beta y,\qquad(\beta\alpha^{-1})^2=\iota,\qquad\beta\alpha^{-1}\in Aut(Q;+).$$

Each quasigroup from this variety is middle linear.

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On the Tate-Shafarevich groups of certain elliptic curves

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By using the methods elaborated by J. Thorne in the case of global field [1] we prove the following result describing the Tate-Shafarevich groups of certain elliptic curves defined over pseudoglobal fields of positive characteristic. By a pseudoglobal field K we mean an algebraic function field in one variable over a pseudofinite [2] constant field k.

Let p be a prime congruent to 1 modulo 4, and let q be a power of p. Consider the elliptic curve $E : t(t-1)y^2 = x(x-1)(x-t)$ over the field $\mathbb{F}_q(t)$. Let (K, E) be a Tate-Shafarevich group.

Theorem. *Let $l \neq p$ be an odd prime. There is an isomorphism $(K, E)[l^\infty] \cong (\mathbb{Z}[i] \otimes \mathbb{Q}_l/\mathbb{Z}_l)[(\pi/\overline{\pi})^f - 1]$, where $q = p^f$.*

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Finite graph exploration by two agents

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The problem of exploration of an environment giving by a finite graph is widely studied in the literature in various contexts [1, 2]. An algorithm of exploration of finite graph [3] by two agents is proposed.

The agent-researcher (AR) traverse on unknown connected undirected graph $G = (V, E)$ without loops and multiple edges [2]. It can read and change colors of graph elements and transfer information about its movements and colorings to the agent-experimenter.

The aim of the paper is to create an algorithm of functioning of these agents that leads to exploring of the graph.

Functions of agents:

1. agent-researcher (agent with limited memory, which moving on graph):
   - perceives marks of all elements in the neighborhood of the node;
   - moves on graph from node v to node u by edge (v, u);
   - can change color of nodes, edges and incidentors;
2. agent-experimenter (stationary agent with unlimited growing internal memory):
   - conveys, receives, identifies messages from AR;
   - builds a graph representation based on messages from AR.

Conclusion. The new algorithm with $O(n^2)$ time $O(n^2)$ space and $O(n^2)$ communication complexities that explores any finite undirected graph with n nodes is proposed. Agent-researcher uses two different marks. The method is based on depth-first traversal method.

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ON COEFFICIENTS OF TRANSITIVENESS OF POSETS OF SPECIAL TYPE

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For a quiver $Q = (Q_0, Q_1)$ with the set of vertices Q_0 and the set of arrows Q_1, P. Gabriel introduced the following quadratic form, called by him the quadratic Tits form of Q:

$$q_Q(z) = q_Q(z_1, \ldots, z_n) := \sum_{i \in Q_0} z_i^2 - \sum_{i \to j} z_i z_j,$$

where $i \to j$ runs through the set Q_1. He proved that the quiver Q has finite representation type over a field k iff its Tits form is positive. This Gabriel's result laid the foundations of a new direction in the representation theory. This quadratic form is naturally generalized to a finite poset $S \not\ni 0$:

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i.$$

Yu. A. Drozd showed that a poset S has finite representation type iff its Tits form is weakly positive (representations of posets were introduced by L. A. Nazarova and A. V. Roiter). For posets, in contrast to quivers, the sets of those with weakly positive and with positive Tits forms do not coincide. Therefore the investigations of posets with positive Tits form seems to be quite natural; they are analogs of the Dynkin diagrams. Posets of this type were classified in [1]. In this paper it is also introduced and classified the P-critical posets, which are analogs of the extended Dynkin diagrams. A poset S is called P-critical if its Tits quadratic form is not positive, but that of any proper subset of S is positive.

Let S be a finite poset and $S^2_< := \{(x, y) \mid x, y \in S, x < y\}$. If $(x, y) \in S^2_<$ and there is no z satisfying $x < z < y$, then we say that x and y are neighboring. We put $n_w = n_w(S) := |S^2_<|$ and denote by $n_e = n_e(S)$ the number of pairs of neighboring elements. On the language of the Hasse diagram $H(S)$, n_e is equal to the number of all its edges and n_w to the number of all its paths, up to parallelity, going bottom-up (two path is called parallel if they start and terminate at the same vertices). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w we call the coefficient of transitiveness of S. If $n_w = 0$ (then $n_e = 0$), we assume $k_t = 0$.

Recall that an element of a poset T is called nodal, if it is comparable with all elements of T. It follows from the results of [1] that any P-critical poset S is uniquely represented in the form $S = S_0^- 0 \cup S_1 \cup S_0^+$ where S_0^-, S_0^+ are chains (maybe empty), S_1 does not contain nodal elements and $S_0^- < S_1 < S_0^+$ ($X < Y$ means that $x < y$ for any $x \in X$, $y \in Y$). Then $S_0 = S_0^- \cup S_0^+$ is the set of all nodal elements of S.

Theorem. *Let S be a P-critical poset. Then the following conditions are equivalent*:

a) $k_t(S) \geqslant k_t(T)$ for any P-critical poset T;

b) $|S_0| \geqslant |T_0|$ for any P-critical poset T, and S_0^- or S_0^+ is empty.

These studies were carried out together with Prof. V. M. Bondarenko.

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On a generalization of the Albert's theorem

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A quasigroup is a groupoid $(Q, \cdot)$, such that the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions in Q, for every $a, b \in Q$. A loops is a quasigroup with a neutral element.

The multiplication groups $LM(Q, \cdot) =< L_x |\ x \in Q >$, $RM(Q, \cdot) =< R_x |\ x \in Q >$, $M(Q, \cdot) =< L_x, R_y |\ x, y \in Q >$, where $L_x(u) = x \cdot u,\ R_x(u) = u \cdot x,\ \forall x, u \in Q$, of a loop are important tools when studying the properties and the structure of the loops (see, for example, [2-5]).

A. Albert proved in [1] that (left, right) multiplication groups of isotopic loops are isomorphic. We consider the following generalizations of the multiplication groups:

$GLM(Q, \cdot) =< L_x^{(\cdot)}, I_y^{(\cdot)} |\ x, y \in Q >$,

$GRM(Q, \cdot) =< R_x^{(\cdot)}, I_y^{(\cdot)} |\ x, y \in Q >$,

$GM(Q, \cdot) =< L_x^{(\cdot)}, R_y^{(\cdot)},\ I_z^{(\cdot)} |\ x, y, z \in Q >$,

where $I_x^{(\cdot)}(y) = y \backslash x,\ \forall x, y \in Q$, are the middle translations, and describe the connections between the generalized multiplication groups of isostrophic loops. In particular, we prove the following theorem.

Theorem. *The generalized multiplication groups of isostrophic loops are isomorphic.*

Corollary. *The generalized multiplication groups of principal isostrophic loops coincide.*

We also consider the action of multiplication groups on isostrophic Bol loops.

The ASM grant 15.817.02.26*F* *has partly supported the research for the present work*

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On integrable representations for toroidal EALAs

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Toroidal extended affine Lie algebras (EALAs for short) are higher dimensional generalization of affine Kac-Moody algebras. Motivated by Chari-Pressely's loop module construction for affine Kac-Moody algebras, we construct a class of irreducible modules for nullity 2 toroidal EALAs, and we then classify all irreducible integrable modules with finite dimensional weight spaces and non-zero central charge for nullity 2 toroidal EALAs.

This is a joint work with F. Chen and Z. Li.

On solvable groups with restrictions on cofactors of subgroups from Fitting subgroup

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In this paper all groups are finite.

In works [1, 2] noticed that the structure of the groups depends on the subgroups of the Fitting subgroup.

Recall that the cofactor of the subgroup H of group G is the quotient group $H/\mathrm{Core}_G H$, where $\mathrm{Core}_G H = \cap_{g \in G} H^g$ is the maximal normal subgroup of G contained in H.

The structure of groups with given restrictions on the cofactors of subgroups were studied by many authors, see [3]–[5].

To formulate the main result, we introduce the following notation:

Let p be a prime number. For a natural number i, we denote by $p^k \top i$ that p^k divides i, but p^{k+1} does not divide i. For a group G and a prime number p, we let

$$cof_p^F(G) = \max\{i \mid p^i \top \, |H/\mathrm{Core}_G H| \,, \ H \leqslant F(G)\};$$

$$cof^F(G) = \max_p cof_p^F(G).$$

Here, $F(G)$ is the Fitting subgroup of G.

As a continuation, in this paper the dependence of the derived and the nilpotent length of groups on the structure of cofactors of subgroups from Fitting subgroup are found. The following theorem is true.

Theorem. *Let G be a solvable group. Then the derived length of quotient group $G/\Phi(G)$ and the nilpotent length of G are at most $4 + cof^F(G)$.*

For small values of $cof^F(G)$, the estimate of the nilpotent length is refined.

Corollary. *Let G be a solvable group and $cof^F(G) \leqslant 2$. Then the nilpotent length of G is at most 4.*

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On bm-prime numbers

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Let K be a commutative local ring with principle Jacobson radical $R \neq 0$, $R^2 = 0$, and let t be a non-zero element of R. We call an $n \times n$ matrix M over K bimonomial if it has the form $M(t,k,n) = \Phi \left(\begin{smallmatrix} I_k & 0 \\ 0 & tI_{n-k} \end{smallmatrix}\right)$ with $1 < k < n$, where Φ is the companion matrix to the polynomial $x^n - 1$ and I_s denotes the identity $s \times s$ matrix. A natural number $n > 1$ is said to be bm-prime if, for any K, t and k, the bimonomial matrix $M(t,k,n)$ is irreducible.

Theorem. *The only bm-prime numbers are* 2, 3 *and* 5.

These studies were carried out together with Prof. V. M. Bondarenko.

Remark to the references. The maiden name of the author is Dinis.

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On primitivity of group algebras of certain groups of finite rank

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Let R be a ring and let M, X and Y be R-modules. We say that X and Y are separated from M if X and Y have no non-zero isomorphic R-sections which are isomorphic to a submodule of M.

Let A be a normal subgroup of a group H, let k be a field and let M and W be kA-modules. Then the subgroup $Sep_{(H,A)}(M,W)$ of H generated by all elements $g \in H$ such that W and Wg are not separated from M is called the separator of W in H.

Let G be a locally abelian-by-polycyclic group of finite rank and let H be a finitely generated dense subgroup of G. It follows from Lemma 2.1.3 of [1] that H has an abelian normal torsion-free subgroup A such that the quotient group H/A is polycyclic and A has no infinite polycyclic G-sections. The pair $(H;A)$ will be called an important pair of G.

Let G be a locally abelian-by-polycyclic group of finite rank, let k be a field and let M be a kG-module. Let $0 \neq a \in M$ then the subgroup $Sep_G(a)$ generated by subgroups $Sep_{(H,A)}(akG, akA)$, where (H,A) runs through all important pairs of G, is called the separator of a in G.

Theorem 1. *Let G be a locally abelian-by-policyclyc group of finite Prufer rank let k be a field of characteristics zero and let M be a kG-module. Then there is an element $a \in M\backslash\{0\}$ such that $akG = akS \bigotimes_{kS} kG$, where $S = Sep_G(a)$ and either $r_0(S) < r_0(G)$ or for any finitely generated dense subgroup H of S there is an important pair (H,A) of S such that $C = C_A(akH)$ is an isolated subgroup of A and akH is $k(A/C)$-torsion-free.*

This theorem allows us to obtain the following result which generalizes Theorem 5.1 of [1] If a group Γ acts on a set A we say an element is (Γ)-orbital if its orbit is finite and write $\Delta_\Gamma(A)$ for the subset of such elements. The FC-centre of a group G, denoted by $\Delta(G)$, is just $\Delta_G(G)$, where the action of G on itself is by conjugation.

Theorem 2. *Let G be a locally abelian-by-policyclyc group of finite Prufer rank and let k be a field of characteristics zero. The group algebra kG is primitive if and only if $\Delta(G) = 1$.*

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On the estimation of the number of indecomposable representations of given degree of a cyclic group

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Matrix representations of a finite group G over a field k are studied well enough. When $p = \operatorname{char} k$ does not divide $|G|$, the group G always has (up to equivalence) a finite number of indecomposable representations. When p divides $|G|$, the group G has a finite number of indecomposable representations if and only if its p-Sylow subgroup is cyclic. In both the cases, the number (up to equivalence) of irreducible representations of G is described in [1, 2]. The problem of the classification of all indecomposable representations is considered in [3].

Matrix representations of finite groups over rings are studied in general case much less. Concerning the papers on the number of indecomposable representations see, e.g., [4–9].

Let K denote a commutative principal ideal local ring (having an unity) with nilpotent maximal ideal $R = tK \neq 0$ and let its characteristic be equal to p^s (p is simple, $s \geqslant 1$). For a finite group G of order $|G| > 1$, we denote by $\operatorname{ind}_K(G, n)$ the number of nonequivalent indecomposable matrix K-representations of degree n of G.

Theorem 1. *Let $K_0 = K/R$ and R be nilpotent of degree $m \geqslant 2$. Then, for any $n > 1$ and for a cyclic p-group G of some order N depending on n (hence of greater order), $\operatorname{ind}_K(G, n) \geqslant (n-1)|K_0|$.*

Theorem 2. *Let the characteristic of K be p and $R = tK \neq 0$ with $t^2 = 0$. Then, for any cyclic p-group G and $n \geqslant |G|$, $\operatorname{ind}_K(G, n) \geqslant (|G| - 2)|K_0|$.*

These studies were carried out together with V. M. Bondarenko, J. Gildea and M. Salim.

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Wild and tame maps on free modules over commutative rings and graph based cryptography

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Affine Cremona group $C(K^n)$ contains all bijective polynomial transformations F_n of free module K^n such that their inverse ${F_n}^{-1}$ is also a polynomial map. All polynomial transformations of K^n form affine Cremona semigroup $S(K^n)$. We assume that element F_n of $S(K^n)$ is given in its standard form $x_i \to f_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, where $f_i \in K[x_1, x_2, \ldots, x_n]$ are presented via the list of monomial terms written in lexicographical order.

The pair of families of bijective multivariate maps of kind F_n and ${F_n}^{-1}$ on free module K^n over finite commutative ring K is a wild one if the degree of F_n is bounded from above by an independent constant d and degree of F^{-1} is bounded from below by c^n, $c > 1$. We refer to $(F_n, {F_n}^{-1})$ as a tame pair if $\deg(F_n)$ and $\deg({F_n}^{-1})$ are bounded from above by independent constants.

We say that $(F_n, {F_n}^{-1})$ is a pair with an invertible decomposition $F_n = {G^1}_n {G^2}_n \ldots {G^k}_n$ if the knowledge of G_i, $i = 1, 2, \ldots, k$ allows to compute the value of F^{n-1} in a given point $\mathrm{p} = (p_1, p_2, \ldots, p_n) \in K^n$ in a polynomial time $O(n^2)$.

Theorem 1. *For each commutative ring K affine Cremona group $C(K^n)$ contains a wild family of pairs F_n, ${F_n}^{-1}$ with an invertible decomposition.*

We say that the pair of families F_n, F'_n of nonbijective polynomial maps of free module K^n is a wild partially invertible pair if the composition $F_n F'_n$ leaves each element of K^{*n} unchanged, $\deg(F_n)$ is bounded by an independent constant and $\deg(F'_n)$ has an exponential size. If there is a decomposition ${G^1}_n {G^2}_n \ldots {G^k}_n$ of F_n which allows to compute the reimage of vector from $F(K^{*n})$ in time $O(n^2)$ we say that partially invertible wild pair has an invertible decomposition.

Theorem 2. *For each commutative ring K affine Cremona semigroup $S(K^n)$ contains a wild family of partially invertible pairs F_n, ${F_n}^{-1}$ with an invertible decomposition.*

Proofs of Theorem 1 and Theorem 2 are based on explicit constructions. In the cases of large commutative rings $K = F_q$ and $K = Z_m$ the constructions allow us to introduce new multivariate cryptosystems over plainspaces K^n and K^{*n}. Some of these public keys are based on the idea of hidden discrete logarithm problem related to secret Eulerian equations of kind $x^\alpha = a$, $(\alpha, |K^*|) = 1$ (see [1]).

We say that a family of transformations $F_n \in C(K^n)$ is stable of degree k, $k > 1$ if all elements ${F_n}^k \neq e$ are of the same degree k. Stable family of transformations is an example of a tame family.

Theorem 3. *For each commutative ring K and $k > 1$ affine Cremona groups $C(K^n))$ contain maps F_n which form a tame stable family of degree k.*

Explicit constructions of stable maps F_n satisfying conditions of theorem 3 allow to introduce new generalized Diffie Hellman multivariate key exchange protocols and generalized multivariate El Gamal cryptosystems.

The talk is dedicated to the 75-th anniversary of Volodymyr Vasyljovych Kirichenko whose fundamental results on Ring Theory and Representations of Algebras, his service for community of algebraists inspired me.

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r-DIVISORS ON $\mathbb{Z}$ AND $\mathbb{Z}[i]$

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We study the multiplicative functions $\tau^{(r)}(n)$ and $\sigma^{(r)}(n)$ defined on prime power number by the following equations

$$\tau^{(r)}(p^a) = \begin{cases} 1 & if \quad a \leqslant r, \\ 2 & if \quad a > r, \end{cases}$$

$$\sigma^{(r)}(p^a) = \begin{cases} p^a & if \quad a \leqslant r, \\ p^a + p^r & if \quad a > r. \end{cases}$$

These functions have been investigated in works of Minculete, Lelechenko etc.

The analogous functions can be defined on the ring of Gaussian integers.

In our talk we investigate the k-fold iterates of $\sigma^r(n)$ as $\sigma^r_k(n)$, $k = 2, 3, \ldots$. The second part devotes to asymptotic formulas for the summatory functions for $\tau^{(r)}$ and $\sigma^{(r)}$ of the special sequences over $\mathbb{Z}$ and $\mathbb{Z}[i]$.

Furthermore, we generalize the results of J.-M. De Koninck and I. Katai [1] on the iterates of the sum of unitary devisors over $\mathbb{N}$ in case of the sum of r-divisors over $\mathbb{Z}$ and $\mathbb{Z}[i]$.

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Congruent generators of pseudo-random numbers

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Uniform pseudorandom numbers (abbreviate, PRN's) in the interval $[0,1]$ are basic ingredients of any stochastic simulation. Their quality is of fundamental importance for the success of the simulation, since the typical stochastic simulation essentially depends on the structural and statistical properties of the producing PRN generators. In the cryptographical applications of PRN's the significant importance is of the availability of property of the unpredictability to generated sequence of PRN's. The classical and most frequently used method for generation of PRN's still is the linear congruential method. Unfortunately, its simple linear nature implies several undesirable regularities. Therefore, a variety of nonlinear methods for the generation of PRN's have been introduced as alternatives to linear methods. It is particularly interesting the nonlinear generators for producing the uniform PRN's, such as the inversive generators and its generalizations. Such generators were introduced and studied by Eichenauer, Lehn, Topuzoğlu, Niederreiter, Shparlinski etc. The standard form of such generator is

$$y_{n+1} \equiv f(y_n, \ldots, y_{n-k}) \pmod{p^m}$$

where k we call the order of generator.

The present report deals with two inversive congruential generators of second order determined by the recursions

(I) $y_{n+1} \equiv a y_n^{-1} y_{n-1}^{-1} + b + cF(n)y_1 \pmod{p^m}$,

(II) $y_{n+1} \equiv a y_n^{-1} y_{n-1}^{-1} + b y_n^{-1} + dF(n) \pmod{p^m}$,

where $(y_0, p) = (y_1, p) = 1$, $(a, p) = 1$, $b \equiv c \equiv d \equiv 0 \pmod{p}$, $F(n)$ is an integral valued function.

On Hartley sets and injectors of a finite group

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All groups considered are finite. For notation we refer to [1]. Let $\mathbb{P}$ be the set of all primes, $\pi \subseteq \mathbb{P}$, and $\pi^{'} = \mathbb{P} \backslash \pi$. We denote by $\mathfrak{E}_{p'}$ and $\mathfrak{N}_p$ the classes of all $p^{'}$-groups and all p-groups, respectively; $\mathfrak{G}^{\pi}$ denote the class of all π-soluble groups. A subgroup H of a group G is called a *Hall π-subgroup* if $|H|$ is a π-number and $|G{:}H|$ is a $\pi^{'}$-number.

Recall that a class of groups $\mathfrak{F}$ is called a *Fitting class* if $\mathfrak{F}$ is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. A *Fitting set* $\mathscr{F}$ of a group G is called the nonempty set of subgroups of G which is closed with respect to taking normal subgroups, their products, and conjugate subgroups. Note that if $\mathfrak{F}$ is a nonempty Fitting class, then the set of subgroups $\{H \leqslant G : H \in \mathfrak{F}\}$ of G is a Fitting set of G. It is denoted by $Tr_{\mathfrak{F}}(G)$ and referred to as the *trace of the Fitting class* $\mathfrak{F}$ in the group G. As is well known (see [1, Examples VIII.2.2]), to every Fitting class $\mathfrak{F}$ there corresponds its trace in the group G; however, the converse is false in general.

Let $\mathscr{F}$ be a Fitting set of G. A subgroup V of G is said to be $\mathscr{F}$-maximal if $V \in \mathscr{F}$ and $U = V$ whenever $V \leqslant U \leqslant G$ and $U \in \mathscr{F}$. Every group G has a unique maximal normal $\mathscr{F}$-subgroup, which is called the $\mathscr{F}$-radical of G and denoted by $G_{\mathscr{F}}$. A subgroup V of a group G is said to be an *$\mathscr{F}$-injector* of G [1] if $V \cap N$ is an $\mathscr{F}$-maximal subgroup of N for every subnormal subgroup N of G. For a Fitting set $\mathscr{F}$ of G and a nonempty Fitting class $\mathfrak{X}$, we call the set $\{H \leqslant G : H/H_{\mathscr{F}} \in \mathfrak{X}\}$ of subgroups of G the *product* of $\mathscr{F}$ and $\mathfrak{X}$ and denote it by $\mathscr{F} \circ \mathfrak{X}$.

For studying the structure of the Fitting classes, a local method was first proposed by Hartley [2]. A function $h{:}\mathbb{P} \to \{$*Fitting sets of* $G\}$ is called a *Hartley function* (or in brevity an *H-function*).

Definition. *Let $\varnothing \neq \pi \subseteq \mathbb{P}$ and h be an H-function of a group G. Let $HS(h) = \bigcap_{p \in \pi} h(p) \circ (\mathfrak{E}_{p'}\mathfrak{N}_p)$. A Fitting set $\mathscr{H}$ of G is called the Hartley set of G if $\mathscr{H} = HS(h)$ for some H-function h.*

It is proved

Theorem. *Support that $\mathscr{H}$ =HS(h) be a Hartley set of G, defined by an H-function h such that $h(p) = \mathscr{X}$ for all prime $p \in \pi$, where $\mathscr{X}$ is a nonempty Fitting set of G, and $G \in \mathscr{X} \circ \mathfrak{S}^{\pi}$. Then the following statements hold:*

(1) G possesses an $\mathscr{H}$-injector and any two $\mathscr{H}$-injectors are conjugate in G.

(2) Every $\mathscr{H}$-injector V of G is a subgroup of G of type $G_{\mathscr{X} \circ \mathfrak{S}_{\pi'}} L$, where L is the subgroup of G such that $L/G_{\mathscr{X}}$ is the $\mathfrak{N}_{\pi}$-injector of some Hall π-subgroup of $G/G_{\mathscr{X}}$.

This research is supported by the State Research Programme "Convergence" (2016–2020). Research of the second author is supported by the Belarusian Republican Foundation for Fundamental Research (F17M-064).

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K-theoretical aspect of elementary divisor ring

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The problem of matrix diagonalization is one of classical algebraic problems (the most comprehensive survey about the history, achievements and researchers can be found in [1]). The specific role in modern researches concerning elementary divisor rings is played by one of the K-theoretical invariants, namely the stable range. For example, the Hermite rings plays an important role in studying elementary divisor rings. In the case of commutative Bezout ring an Hermite ring is a ring of stable range 2 [1], and a commutative Bezout domain is an elementary divisor ring if it is a ring of Gelfand range 1 [2]. In the case of commutative Bezout rings the concept of dyadic range 1 [3] helps to solve the elementary division ring problem.

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On characteristic properties of semigroups of order 3

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Let $\mathcal{K}$ be a class of semigroups and $\mathcal{P}$ be some set of general properties of semigroups (that are preserved on going to anti-isomorphic semigroups). A subset Q of $\mathcal{P}$ is called characteristic for a semigroup $X \in \mathcal{K}$ if, up to isomorphism and anti-isomorphism, X is the only semigroup in $\mathcal{K}$, which satisfies all the properties from Q; if $Q = \{q_1, \dots, q_s\}$, then we also say that the properties $q_1, \dots, q_s$ are characteristic for X. The set of properties $\mathcal{P}$ is called char-complete for $\mathcal{K}$ if there exists a characteristic subset for each semigroup $X \in \mathcal{K}$.

We indicate a char-complete set of properties for the class of semigroups of order 3. It consists of seven elements.

These studies were carried out together with Prof. V. M. Bondarenko.

Minimal exponent matrices

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One of the most important classes, which appear in various questions of the ring theory and the image theory, is the class of the tiled orders. In terms of the abstract ring theory tiled order is primary Noetherian, semi-perfect and semi-distributive Noetherian ring with non-zero Jacobson radical. Exponent matrices appear in the theory of tiled orders. Each tiled order is completely determined by its exponent matrix and discrete valuation ring. Many of the properties of these rings are completely determined by their exponent matrix, such as quivers of rings. The exponent matrix quiver coincides with the tiled order quiver. In order to research the exponent matrices and their quivers there can be applied the combinatorial and geometric methods.

Theorem 1. *The sum of elements of the exponent matrix with a unit quiver not more than* $C_{n+1}^3 = \frac{(n+1)n(n-1)}{6}$.

Theorem 2. *If the admissible quiver* $Q = Q(E)$ *is simple cycle or has loops in each vertexes then the sum of elements of the exponent matrix is* pC_n^2 *, where* p *is the weight of cycle.*

Theorem 3. *The sum of elements of the exponent matrix* $\mathcal{E}=(\alpha_{ij})= \mathcal{E}(\omega^*, Q)$ *is not more then* $\frac{n^2(n-1)}{2}$ *for any admissible quiver* Q *with loops in each vertexes and the weight function* $\omega^*(\sigma_{ij}) = 1$.

Proposition 1. *The exponent matrix of rigid quiver is the minimum exponent matrix.*

Proposition 2. *Weight reduction simple cycle quiver, could increase the amount the sum of elements of matrix of quiver.*

We found limit for the sum of elements of the exponent matrix with a unit quiver and limit for sum of elements of the minimal exponent matrix with a quiver with loops in each vertexes. It is proved that a rigid quiver obtained from a minimal exponent matrix.

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On acts with the insertion-of-factor-property and two-sided subacts

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Let S be a monoid with zero.

Let $Act - S$ be a category of unitary and centered right acts over monoid S.

Subact B of the right act A is said to have the insertion-of-factor-property (IFP) if whenever $sa \in B$ for $s \in S$, $a \in A$, we have $sSa \subseteq B$.

An act A has IFP if the zero subact has IFP.

Subact B of the right act A is called two-sided subact if all right subacts of the subact B have IFP.

For all $a \in A$ define the set $Ann(a) = \{(s,t) \in S \times S \mid as = at\}$. Then $Ann(a)$ is called right annihilator of element a. $Ann(a)$ is right congruence on act A. Zero component of this congruence is called right annihilator ideal of element $a \in A$ [3].

Theorem 1. *Let $A \in Act - S$. Then the following conditions are equivalent:*

(i) every subact of the act A has IFP;

(ii) every finitely generated subact of the act A has IFP;

(iii) every cyclic subact of the act A has IFP;

(iv) for every subact B of the act A the Rees factor act A/B has the property: right annihilator ideal of every element of subact A/B is two-sided ideal in S.

The act A is called two-sided act if it satisfies the conditions of Theorem 1.

Any subact of two-sided act is two-sided subact.

Theorem 2. *Let A, B, $C \in Act-S$. If A is two-sided subact of the act B and B is two-sided subact of the act C, then A is two-sided subact of the act C.*

Theorem 3. *The two-sided subacts of any act form a complete lattice.*

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About finding all factorizations of symmetric matrices over polynomial and quasipolynomial rings with involution

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Let $\mathbf{K}$ be a polynomial ring $\mathbf{C}[x]$ or quasipolynomial ring $\mathbf{C}[x, x^{-1}] = \{f(x) = \sum_{i=-l}^{m} a_i x^i,\ a_i \in \mathbf{C}\}$ with involution ∇ [1, 2]. The involution ∇ extended to the matrix ring $M_n(\mathbf{K})$ as follows:

$$A(x)^{\nabla} = \|a_{ij}(x)\|^{\nabla} = \|a_{ji}(x)^{\nabla}\|.$$

A matrix $A(x)$ is called symmetric if $A(x) = A(x)^{\nabla}$. A factorization of symmetric matrix $A(x)$ is called its decomposition

$$A(x) = B(x)C(x)B(x)^{\nabla}. \tag{1}$$

Let's define the Smith form of matrix $A(x)$ as S_A

$$S_A = P(x)A(x)Q(x) = diag(\epsilon_1(x), \ldots, \epsilon_n(x)). \tag{2}$$

In the articles [1, 2] were founded necessary and sufficient conditions of the existence of factorization (1), where $B(x)$ is an unital matrix with the Smith form $\Phi(x) = diag(\varphi_1(x), \ldots, \varphi_n(x))$ and $C(x)$ is a nonsingular symmetric matrix.

Theorem 1. *In factorization* (1) *the unital factor* $B(x)$ *is unique with the Smith form* $\Phi(x)$ *iff the Smith form matrix* $A(x)$ *is equal to the product of the Smith forms its multipliers.*

While constructing factorization (1) methods, which are discussed in the articles [1, 2] selected from (2) matrix $P(x)$ doesn't not always give a possibility to find all divisors of $B(x)$ with the Smith form $\Phi(x)$.

With the use of results of [3] we obtain the following theorem.

Theorem 2. *For a symmetric matrix matrix* $A(x)$ *with the Smith form* S_A *any divisor* $B(x)$ *with the Smith form* $\Phi(x)$ *in factorization* (1) *can be chosen from the set* $P^{-1}(x)\Phi(x) \in GL_n(\mathbf{K})$ *iff*

$$\frac{\varphi_i(x)}{(\varphi_i(x), \epsilon_j(x))} = \left(\frac{\varphi_i(x)}{(\varphi_j(x)}, \frac{\epsilon_i(x)}{\epsilon_j(x)}\right)$$

for $i = 2, \ldots, n,\ j = 1, \ldots, n,\ i > j$.

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Matrix bimodule problems

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It is a joint work with Yu. A. Drozd and Xu Yunge.

We define matrix bimodule problems and establish their relations to boxes. Then we construct reduction algorithms for this kind of matrix problems and describe an exact structure in the representation categories. The final aim of these new constructions is to prove the "*tame = almost AR-homogeneus*" conjecture.

On free commutative trioids

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Following [1], a trioid is a nonempty set T equipped with three binary associative operations $\dashv$, $\vdash$, and $\perp$ satisfying the following eight axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \quad (x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \quad (x \dashv y) \dashv z = x \dashv (y \perp z),$$
$$(x \perp y) \dashv z = x \perp (y \dashv z), \quad (x \dashv y) \perp z = x \perp (y \vdash z),$$
$$(x \vdash y) \perp z = x \vdash (y \perp z), \quad (x \perp y) \vdash z = x \vdash (y \vdash z).$$

A trioid $(T, \dashv, \vdash, \perp)$ is called commutative [2] if semigroups $(T, \dashv)$, $(T, \vdash)$ and $(T, \perp)$ are commutative. A trioid which is free in the variety of commutative trioids will be called a free commutative trioid. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that $(T, \dashv, \vdash, \perp)/\rho$ is a commutative trioid, we say that ρ is a commutative congruence. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that two operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and it is a dimonoid (see, e.g., [3]), we say that ρ is a dimonoid congruence. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide, we say that ρ is a semigroup congruence.

As usual, $\mathbb{N}$ denotes the set of all positive integers. Let X be an arbitrary nonempty set and ω an arbitrary word in the alphabet X. The length of ω will be denoted by l_ω. Let further $F^\star[X]$ be the free commutative semigroup on X, Ω the free monoid on the three-element set $\{a, b, c\}$ and $\theta \in \Omega$ the empty word. By definition, the length l_θ of θ is equal to 0 and $u^0 = \theta$ for any $u \in \Omega \backslash \{\theta\}$. For all $u_1, u_2 \in \Omega$ let

$$f_\dashv(u_1, u_2) = a, \quad f_\vdash(u_1, u_2) = \begin{cases} b, & u_1 = u_2 = \theta, \\ a & \mathit{otherwise}, \end{cases}$$

$$f_\perp(u_1, u_2) = \begin{cases} c, & u_1 = c^k, u_2 = c^p, k, p \in \mathbb{N} \cup \{0\}, \\ a & \mathit{otherwise}. \end{cases}$$

By $\overline{\Omega}$ denote the subset $\{y^k \mid y \in \{a, c\}, k \in \mathbb{N} \cup \{0\}\} \cup \{b\}$ of Ω. Define operations $\dashv$, $\vdash$, and $\perp$ on $A = \{(w, u) \in F^\star[X] \times \overline{\Omega} \mid l_w - l_u = 1\}$ by

$$(w_1, u_1) * (w_2, u_2) = (w_1 w_2, f_*(u_1, u_2)^{l_{u_1} + l_{u_2} + 1})$$

for all $(w_1, u_1), (w_2, u_2) \in A$ and $* \in \{\dashv, \vdash, \perp\}$. The algebra $(A, \dashv, \vdash, \perp)$ will be denoted by $FCT(X)$.

Theorem. $FCT(X)$ *is the free commutative trioid.*

Moreover, we give examples of commutative trioids, study separately free commutative trioids of rank 1 and establish that the automorphism group of $FCT(X)$ is isomorphic to the symmetric group on X. We also characterize the least commutative congruence, the least (commutative) dimonoid congruences and the least semigroup congruence on a free trioid.

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Representations of ordered trioids by binary relations

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Trioids were introduced by J.-L. Loday and M. O. Ronco [1] during the study of ternary planar trees. A nonempty set T with three binary associative operations $\dashv$, $\vdash$ and $\perp$ is called *a trioid* if for all $x, y, z \in T$ the following conditions hold:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \;\; (x \vdash y) \dashv z = x \vdash (y \dashv z),$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \;\; (x \dashv y) \dashv z = x \dashv (y \perp z),$$

$$(x \perp y) \dashv z = x \perp (y \dashv z), \;\; (x \dashv y) \perp z = x \perp (y \vdash z),$$

$$(x \vdash y) \perp z = x \vdash (y \perp z), \;\; (x \perp y) \vdash z = x \vdash (y \vdash z).$$

Let $(T, \dashv, \vdash, \perp)$ be an arbitrary trioid, $\leqslant$ an order relation on T which is stable with respect to each of operations $\dashv, \vdash$ and $\perp$. In this case, $(T, \dashv, \vdash, \perp, \leqslant)$ will be called an ordered trioid.

We study representations of ordered trioids by binary relations.

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Об ассоциативности и полуассоциативности одной полиадической операции

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Напомним, что l-арную операцию $[\ \ ]$ l-арного группоида $< A, [\ \ ] >$ называют *ассоциативной*, если в нем для любого $i = 2, ..., l$ выполняется тождество

$$[[x_1...x_l]x_{l+1}...x_{2l-1}] = [x_1...x_{i-1}[x_i...x_{i+l-1}]x_{i+l}...x_{2l-1}].$$

Если указанное тождество выполняется для $i = l$, то l-арную операцию $[\ \ ]$ и l-арный группоид $< A, [\ \ ] >$ называют *полуассоциативными.*

Пусть A – полугруппа, $k \geqslant 2$, $l \geqslant 2$, σ – подстановка из $\mathbf{S}_k$. Определим [1] на A^k l-арную операцию $[\ \ ]_{l,\sigma,k}$ следующим образом: если $\mathbf{x}_i = (x_{i1}, x_{i2}, ..., x_{ik}) \in A^k, i = 1, 2, ..., l$, то

$$[\mathbf{x}_1...\mathbf{x}_l]_{l,\sigma,k} = (y_1, ..., y_k), y_j = x_{1j}x_{2\sigma(j)}...x_{l\sigma^{l-1}(j)}, j = 1, ..., k.$$

Частные случаи этой l-арной операции изучал Э. Пост в [2]. В качестве полугруппы A он рассматривал либо симметрическую группу, либо полную линейную группу над полем комплексных чисел. При этом арность полиадической операции и число k были связаны равенством $l = k + 1$, а роль подстановки σ в обоих случаях играл цикл $(12...k)$.

В [1] доказано, что *если подстановка σ удовлетворяет условию $\sigma^l = \sigma$, то l-арная операция $[\ \ ]_{l,\sigma,k}$ является ассоциативной.*

Теорема 1. *Пусть полугруппа A обладает идемпотентом a и отличным от него элементом b таким, что $ab \neq a$. Тогда следующие утверждения равносильны:*

1) *l-арная операция $[\ \ ]_{l,\sigma,k}$ является ассоциативной;*

2) *l-арная операция $[\ \ ]_{l,\sigma,k}$ является полуассоциативной;*

3) *подстановка σ^{l-1} – тождественная.*

Следствие 1. *Пусть A – регулярная полугруппа, в которой для некоторого ее идемпотента a и отличного от него элемента b верно $ab \neq a$. Тогда утверждения* 1)–3) *из теоремы 1 равносильны.*

Следствие 2. *Пусть неодноэлементная полугруппа A с левым сокращением обладает идемпотентом. Тогда утверждения* 1)–3) *из теоремы* 1 *равносильны.*

Теорема 2. *Пусть полугруппа A обладает идемпотентом a и отличным от него элементом b таким, что $ab \neq a$; σ – подстановка из $\mathbf{S}_k$ порядка $d \geqslant 2$. Тогда:*

1) *l-арная операция $[\ \ ]_{l,\sigma,k}$ является ассоциативной тогда и только тогда, когда*

$$l \in \{td + 1 | t = 1, 2, ..., \};$$

2) *l-арная операция $[\ \ ]_{l,\sigma,k}$ не является ассоциативной тогда и только тогда, когда*

$$l \in \{td + r | t = 0, 1, 2, ..., ; r = 2, ..., d\}.$$

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О тождествах ассоциативности в полиадических группоидах

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Напомним, что n-арную операцию η n-арного группоида $< A, \eta >$ называют ассоциативной, если в нем для любого $i = 2, ..., n$ выполняется тождество

$$\eta(\eta(x_1...x_n)x_{n+1}...x_{2n-1})) = \eta(x_1...x_{i-1}\eta(x_i...x_{i+n-1})x_{i+n}...x_{2n-1}).$$

Понятно, что следствиями указанных выше $n-1$ тождеств, определяющих ассоциативность n-арной операции η, являются следующие тождества

$$\eta(x_1...x_{i-1}\eta(x_i...x_{i+n-1})x_{i+n}...x_{2n-1}) = \eta(x_1...x_{j-1}\eta(x_j...x_{j+n-1})x_{j+n}...x_{2n-1})$$

для любых $i, j \in \{1, 2, ..., n\}$.

Теорема 1. *Пусть A – полугруппа с единицей, содержащая более одного элемента, $s \geqslant 1$, определим на A^3 $3s$-арную операцию*

$$\eta(\mathbf{x}_1\mathbf{x}_2...\mathbf{x}_{3s}) = \eta((x_{11}, x_{12}, x_{13})(x_{21}, x_{22}, x_{23})...(x_{(3s)1}, x_{(3s)2}, x_{(3s)3})) =$$

$$= (x_{11}x_{22}x_{33}...x_{(3s-2)1}x_{(3s-1)2}x_{(3s)3},$$

$$x_{12}x_{23}x_{31}...x_{(3s-2)2}x_{(3s-1)3}x_{(3s)1},$$

$$x_{13}x_{21}x_{32}...x_{(3s-2)3}x_{(3s-1)1}x_{(3s)2}).$$

Тогда в $< A^3, \eta >$ для любых $i, j \in \{1, 2, ..., 3s\}, i \neq j$ не выполняется тождество

$$\eta(\mathbf{x}_1...\mathbf{x}_{i-1}\eta(\mathbf{x}_i...\mathbf{x}_{3s+i-1})\mathbf{x}_{3s+i}...\mathbf{x}_{6s+1}) = \eta(\mathbf{x}_1...\mathbf{x}_{j-1}\eta(\mathbf{x}_j...\mathbf{x}_{3s+j-1})\mathbf{x}_{3s+j}...\mathbf{x}_{6s+1}).$$

Теорема 2. *Пусть A – полугруппа с единицей, содержащая более одного элемента, $s \geqslant 1$, определим на A^3 $3s$-арную операцию*

$$\mu(\mathbf{x}_1\mathbf{x}_2...\mathbf{x}_{3s}) = \mu((x_{11}, x_{12}, x_{13})(x_{21}, x_{22}, x_{23})...(x_{(3s)1}, x_{(3s)2}, x_{(3s)3})) =$$

$$= (x_{11}x_{23}x_{32}...x_{(3s-2)1}x_{(3s-1)3}x_{(3s)2},$$

$$x_{12}x_{21}x_{33}...x_{(3s-2)2}x_{(3s-1)1}x_{(3s)3},$$

$$x_{13}x_{22}x_{31}...x_{(3s-2)3}x_{(3s-1)2}x_{(3s)1}).$$

Тогда в $< A^3, \mu >$ для любых $i, j \in \{1, 2, ..., 3s\}, i \neq j$ не выполняется тождество

$$\mu(\mathbf{x}_1...\mathbf{x}_{i-1}\mu(\mathbf{x}_i...\mathbf{x}_{3s+i-1})\mathbf{x}_{3s+i}...\mathbf{x}_{6s+1}) = \mu(\mathbf{x}_1...\mathbf{x}_{j-1}\mu(\mathbf{x}_j...\mathbf{x}_{3s+j-1})\mathbf{x}_{3s+j}...\mathbf{x}_{5s+1}).$$

Если в теоремах 1 и 2 положить $s = 1$, то операции η и μ примут вид

$$\eta(\mathbf{xyz}) = \eta((x_1, x_2, x_3)(y_1, y_2, y_3)(z_1, z_2, z_3)) = (x_1y_2z_3, x_2y_3z_1, x_3y_1z_2),$$

$$\mu(\mathbf{xyz}) = \mu((x_1, x_2, x_3)(y_1, y_2, y_3)(z_1, z_2, z_3)) = (x_1y_3z_2, x_2y_1z_3, x_3y_2z_1).$$

Квазитождества нильпотентных йордановых луп

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Одна из основных теорем по теории квазимногообразий является теорема Ольшанского (см. [1]), согласно которой, конечная группа имеет конечный базис квазитождеств тогда и только тогда, когда все ее силовские подгруппы абелевы. В доказательстве этой теоремы, в случае когда конечная группа содержит некоммутативную нильпотентную подгруппу, автор показал что все квазитождества этой группы не имеют базис квазитождеств от конечного числа переменных. Аналогичные результаты и для других нильпотентных алгебр были получены и в других работ (например, [2–6]). В данной работе показано что подобные примеры существуют и в классе йордановых нильпотентных луп.

Лупа называется йордановой если в ней выполняются тождества

$$x \cdot y = y \cdot x, \ xx \cdot yx = (xx \cdot y)x.$$

Поскольку йордановая лупа коммутативна, то ее правое и левое деления совпадают, поэтому предполагается что сигнатура йордановых луп состоит из 2-х бинарных функциональных символов. В этой сигнатуры для любого простого числа p в классе йордановых луп с тождеством $x^p = 1$ построена неассоциативная и нильпотентная йордановая лупа L с наименьшим порядком. Для этих луп L доказана

Теорема. *Йордановая лупа L не имеет базис квазитождеств от конечного числа переменных, т.е. ее аксиоматический квазиранг бесконечен.*

Замечание. Наименьшая неассоциативная и нильпотентная йордановая лупа с экспонентом простое число p порождается двумя элементами и имеет p^3 элементы. В частности, наименьшая нильпотентная йордановая лупа с бесконечным аксиоматическим квазирангом состоит из 8 элементов.

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Спектри матриць суміжностей сагайдака $Q(A)$ та дводольного графа $\overline{PQ(A)}$ нетерового слабопервинного напівдосконалого 2-кільця A

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При дослідженні класичних алгебричних структур (груп, напівгруп, кілець) буває корисним використовувати певні допоміжні конструкції алгебро-комбінаторного характеру, такі як граф, сагайдак, схеми Кириченка або схеми Динкіна тощо.

У 1972 р. Ґабріель [4] (у явному вигляді) і Кругляк [3] (у неявному вигляді) незалежно запропонували зіставляти кожній скінченновимірній алгебрі над алгебраїчно замкненим полем орграф спеціального типу. Тоді ж Ґабріель запропонував називати скінченні орграфи сагайдаками, і в теорії зображень, а згодом і в теорії кілець цей термін прижився.

Згодом Кириченко В. В. та його учні ввели різні типи сагайдаків (зокрема, так звані похідні сагайдаки) для напівдосконалих кілець та для деяких інших класів кілець та алгебр [1, 2]. Сагайдаки зберігають важливу інформацію про будову кільця A та можуть служити мірою його комбінаторної складності.

Ми розглядаємо спектри матриць суміжностей сагайдака $Q(A)$ та дводольного графа $\overline{PQ(A)}$ нетерового слабопервинного напівдосконалого 2-кільця A.

Твердження 1. Власні числа матриці суміжності AB–сагайдака $\overline{PQ(A)}$ за модулем не перевищують 2.

Твердження 2. Власні числа матриці суміжності сагайдака $Q(A)$ є підмножиною спектра матриці суміжності графа $\overline{PQ(A)}$.

Твердження 3. Якщо λ є власним числом матриці суміжності сагайдака $Q(A)$, то всі спряжені із ним числа належать до спектра графа $\overline{PQ(A)}$.

Теорема 1. *Серед власних чисел матриці суміжності сагайдака $Q(A)$ існує λ, яке за модулем дорівнює 2, тоді й тільки тоді, коли напівстепені входу та виходу кожної вершини сагайдака дорівнюють 2. Серед власних чисел матриці суміжності дводольного графа $\overline{PQ(A)}$ існує λ, яке дорівнює -2, тоді й тільки тоді, коли степені входу та виходу кожної вершини графа дорівнюють 2.*

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Структура і мінімальна система твірних силовських 2-підгруп знакозмінної групи та їх централізатори

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Ми досліджуємо системи твірних силовських 2-підгруп $Syl_2 A_n$ знакозмінної групи A_n і властивості деяких підгруп силовських 2-підгруп симетричної та знакозмінних груп відповідно.

Нехай $X = \{0, 1\}$ та $X^{[k]}$ — скінченне бінарне k-рівневе дерево ($k \in \mathbb{N}$). Помітимо кожну вершину з $X^{[k]}$ символом 0 чи 1 залежно від наявності вершинної перестановки в ній. Отримане таким чином вершинно-розмічене регулярне кореневе дерево є портретом автоморфізма з $AutX^{[k]}$. Група $Syl_2 A_{2^k}$ ізоморфна підгрупі групи $AutX^{[k]}$ [1, 2]. Автоморфізм з $AutX^{[k]}$ належить $Syl_2 A_{2^k}$ тоді і лише тоді, коли на передостанньому, тобто $k-1$-ому рівні, кількість міток 1 — парна [1, 2].

В роботі [1] автором досліджено структуру силовської 2-підгрупи групи A_{2^k}, вона виявилась наступною $Syl_2 A_{2^k} \simeq (\wr_{i=1}^{k-1} C_2) \ltimes (C_2)^{2^{k-1}-1}$. Також там в лемі 3 показано існування системи твірних групи A_{2^k} з k елементів. Мінімальність цієї k елементної системи для $Syl_2 A_{2^k}$ показано в роботах автора [2, 3].

Нехай $n = 2^{k_0} + 2^{k_1} + \ldots + 2^{k_m}$, де $0 \leqslant k_0 < k_1 < \ldots < k_m$ і $m \geqslant 0$.

Теорема 1. *Якщо $m > 0$, то довільна мінімальна система твірних для $Syl_2 A_n$ має $\sum_{i=0}^{m} k_i - 1$ елементів.*

Нехай $Syl_2 S_n$ це силовська 2-підгрупа симетричної групи S_n.

В роботах [2, 3] розглянуто підгрупу Фраттіні групи $Syl_2 A_{2^k}$, в зв'язку з її будовою розглянемо структуру і властивості комутанта групи $Syl_2 A_{2^k}$.

Теорема 2. *Комутант групи $Syl_2 A_{2^k}, k > 2$ збігається з множиною всіх комутаторів елементів з $Syl_2 A_{2^k}$.*

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Деякі приклади напівгруп відповідностей

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Нехай G – універсальна алгебра. Якщо підалгебру з $G \times G$ розглядати як бінарне відношення на G, то множина $S(G)$ всіх підалгебр з $G \times G$ утворює напівгрупу відносно деморганівського добутку відношень, яка називається, *напівгрупою відповідностей* алгебри G.

Напівгрупи відповідностей вивчались, зокрема, в [1–3]. У даному повідомлені досліджується декілька прикладів напівгруп відповідностей.

1. Нехай G — напівгрупа з нулем 0 і нульовим множенням (тобто $ab = 0$ для довільних $a, b \in G$). Тоді $G \times G$ також є напівгрупою з нульовим множенням (нулем буде елемент $(0, 0)$). Тому піднапівгрупами в $G \times G$ (тобто елементами напівгрупи $S(G)$) будуть усі підмножини з $G \times G$, які містять $(0, 0)$. Зокрема, якщо $|G| = n$, то $|S(G)| = 2^{n^2-1}$.

2. Нехай G — напівгрупа лівих нулів (тобто $ab = a$ для всіх $a, b \in G$). Оскільки $(a, b)(c, d) = (ac, bd) = (a, b)$, то кожна підмножина з $G \times G$ буде підалгеброю. Таким чином, напівгрупа відповідностей $S(G)$ збігається з множиною всіх підмножин множини G, а тому є ізоморфною напівгрупі $\mathfrak{B}(G)$ усіх бінарних відношень на напівгрупі G.

Цей факт у певному сенсі можна обернути:

Твердження 1. Напівгрупа відповідностей $S(G)$ напівгрупи G збігається з напівгрупою $\mathfrak{B}(G)$ усіх бінарних відношень на множині G тоді і тільки тоді, коли G — напівгрупа лівих (правих) нулів.

3. Нехай $G = (N, \circ)$, де $N = \{1, 2, \ldots, n\}$, а дія $\circ$ визначається так: $a \circ b = \max\,(a, b)$. Тоді із рівності $(a_1, a_2) \circ (b_1, b_2) = (\max\,(a_1, b_1), \max\,(a_2, b_2))$ випливає, що підмножина $H \subseteq G \times G$ буде піднапівгрупою тоді і тільки тоді, коли для довільних $(a_1, a_2), (b_1, b_2) \in G \times G$ із $a_1 \geqslant b_1$ і $b_2 \geqslant a_2$ випливає, що $(a_1, b_2) \in H$.

Твердження 2. Нехай $0 \leqslant k \leqslant n$, $\{b_1, b_2, \ldots, b_k\} \subseteq N$, $b_1 < b_2 < \ldots < b_k$, $A_1, A_2, \ldots, A_k \subseteq N$, $a'_i = \max A_i$, $(i = 1, 2, \ldots, k)$, $a''_i = \min A_i$, $(i = 1, 2, \ldots, k)$, $a'_1 \leqslant a'_2 \leqslant \ldots \leqslant a'_k$ та для кожного j $(1 < j \leqslant k)$ виконується умова

$$A_j \supseteq [a''_j, a'_j] \cap \left(\bigcup_{i<j} A_i \right).$$

Тоді множина $C = \bigcup_{i=1}^{k} (A_i, b_i)$ буде елементом напівгрупи $S(G)$ і кожен елемент із $S(G)$ можна отримати в такий спосіб.

Наслідок 1. Нехай $G = (\{1, 2, \ldots, n\}, \circ)$, де $a \circ b = \max(a, b)$. Тоді порядок напівгрупи відповідностей $S(G)$ задовольняє нерівність

$$|S(G)| \geqslant 1 + \sum_{k=1}^{n} \binom{n}{k} \sum_{1 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_k \leqslant n} a_1 \cdot a_2 \cdot \ldots \cdot a_k.$$

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11th International Algebraic Conference in Ukraine dedicated to the 75th anniversary of V. V. Kirichenko

July 3 – 7, 2017

Kyiv, Ukraine

ABSTRACTS

Kyiv — 2017

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11-а Міжнародна алгебраїчна конференція в Україні, присвячена 75-річчю В. В. Кириченка

3 – 7 липня 2017 р.

Київ, Україна

ТЕЗИ ДОПОВІДЕЙ

Київ — 2017

Комп'ютерна верстка та підготовка оригінал-макета
І. Ю. Раєвська, М. Ю. Раєвська

Відповідальні за випуск
А. П. Петравчук, І. Ю. Раєвська, М. Ю. Раєвська

Підп. до друку 16.06.2017. Формат 60x84/8. Папір офс. Офс. друк. Обл. вид. арк. 19,9. Ум. друк. арк. 18,5. Зам. 42. Тираж 160 пр.

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