Dissertation

Titel der Dissertation

Group analysis and invariant parameterization in dynamic meteorology

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Abstract

This thesis is devoted to a study of symmetry properties of models in dynamic meteorology. In particular, we are interested in investigating the problem of finding parameterization schemes that lead to closed (averaged) systems of differential equations exhibiting the same symmetry properties as their original (unaveraged) counterparts.

In the first part of the thesis we develop tools for the group classification of classes of differential equations that should serve as the basis for invariant parameterization schemes. Thus, we introduce the first systematic theory of preliminary group classification and, more generally, completely revise the algebraic framework of group classification. This theoretical development is exemplified by studying the symmetry properties of a class of generalized diffusion equations and a class of generalized nonlinear wave equations. For the former class, we carry out the first complete preliminary group classification in the case of an infinite dimensional equivalence algebra. The group classification problem for the latter class could not be solved completely for more than 20 years. In total, the two examples investigated show the versatility of the enhanced group classification machinery. This is essential for the application to the invariant parameterization problem which in general leads to challenging group classification problems.

As a further tool for the construction of invariant parameterization schemes we use the moving frame method to describe algebras of differential invariants for transformation pseudogroups. To give a first example, we revise the computation of the algebra of differential invariants for the maximal Lie invariance group of the Korteweg–de Vries equation. Thus, for first time we find a functional basis of this algebra in an explicit form.

In the second part of the thesis we use the above theoretical tools to compute invariant parameterization schemes for the barotropic vorticity equation on the beta-plane. A problem of particular interest here is to derive invariant turbulence models for freely decaying geophysical turbulence on the beta-plane. We verify that invariant turbulence models are capable of reproducing the k^{-3} slope of the energy spectrum in the enstrophy inertial range, which is traditionally hard to find with conventional turbulence models. We also show for the example of the barotropic vorticity equation that it is possible to determine entire functional bases of algebras of differential invariants for infinite dimensional pseudogroups of transformations.

Using tools of group analysis, we construct a nontrivial point transformation reducing the primitive equations on the f-plane to the primitive equations in a resting reference frame. Finding such transformations is of practical relevance in the parameterization problem too as they allow one to simplify the initial system for which closure schemes are sought.

Another feature presented is the development of a new algebraic method for finding the complete point symmetry groups of differential equations. The method rests on a combination of the modern notion of megaideals and Hydon's method of factoring out internal automorphisms of Lie algebras. Unlike existing techniques, the new method can be used to determine complete point symmetry (pseudo)groups of differential equations admitting infinite dimensional maximal Lie invariance algebras. We show the effectiveness of the new approach by computing the complete point symmetry groups of the vorticity equation on the sphere and the primitive equations.

Zusammenfassung

In dieser Arbeit werden Symmetrieeigenschaften von Modellen der dynamischen Meteorologie studiert. Von besonderem Interesse ist die Frage der Konstruktion von physikalischen Parameterisierungsschemata, die zu geschlossenen (gemittelten) Systemen von Differentialgleichungen führen und dieselben Symmetrieeigenschaften wie die (ungemittelten) Originalgleichungen aufweisen.

Im ersten Teil dieser Arbeit entwickeln wir Methoden für die Gruppenklassifikation von Klassen von Differentialgleichungen die als Basis für die Konstruktion invarianter Parameterisierungsschemata dienen sollen. Wir formulieren zum ersten Mal eine systematische Theorie der *preliminary group classification* und erweitern diese zu einer vollständigen Neuformulierung der algebraischen Gruppenklassifikation. Zur Veranschaulichung der neuentwickelten Theorie klassifizieren wir die Symmetrieeigenschaften einer generalisierten Diffusionsgleichung und einer Klasse von nichtlinearen Wellengleichungen. Für die Diffusionsgleichungsklasse wird die erste vollständige *preliminary group classification* im Falle einer unendlich dimensionalen Äquivalenzalgebra durchgeführt. Das Klassifikationsproblem für die Wellengleichungsklasse war für mehr als 20 Jahre ungelöst. Beide Beispiele demonstrieren die Vielseitigkeit der verallgemeinerten Klassifikationstechniken, die ein zentraler Bestandteil für die Anwendung auf das invariante Parameterisierungsproblem sind. Letzteres kann im Allgemeinen als ein besonders kompliziertes Klassifikationsproblem aufgefasst werden.

Ein weiteres Werkzeug zur Konstruktion von invarianten Parameterisierungen ist die *moving frame* Methode, die zu einer vollständigen Beschreibung der Algebra von Differentialinvarianten von Transformationsgruppen verwendet werden kann. Als ein erstes Beispiel präsentieren wir eine ausführliche Beschreibung der Algebra der Differentialinvarianten der Korteweg–de Vries Gleichung. Für diese Algebra finden wir die erste Basis der Differentialinvarianten in expliziter Form.

Im zweiten Teil dieser Arbeit verwenden wir die entwickelten theoretischen Werkzeuge um invariante Parameterisierungsschemata der barotropen Vorticitygleichung auf der Beta-Ebene zu finden. Hier konzentrieren wir uns besonders auf die Konstruktion von invarianten Turbulenzmodellen um freie geophysikalische Turbulenz auf der Beta-Ebene zu simulieren. Wir zeigen, dass die entwickelten Schemata in der Lage sind die k^{-3}-Steigung des Energiespektrums im Enstrophieträgheitsbereich zu reproduzieren. Dieses Ergebnis ist schwer mit traditionellen Turbulenzmodellen zu erreichen. Zudem zeigen wir für das Beispiel der Vorticitygleichung dass es möglich ist die vollständige Basis der Algebra von Differentialinvarianten unendlich dimensionaler Transformationsgruppen zu bestimmen.

Ein weiteres Resultat das mittels Gruppenanalyse hergeleitet wird ist, dass die primitiven Gleichungen auf der f-Ebene auf die primitiven Gleichungen in einem ruhenden Koordinatensystem abgebildet werden können. Das Auffinden solcher nichttrivialen Punkttransformationen ist von großer praktischer Bedeutung für die Konstruktion von Parameterisierungsschemata da sie die Ausgangsgleichungen, für die Parameterisierungen gefunden werden sollen, vereinfachen.

Zudem entwickeln wir eine neue algebraische Methode zur Bestimmung der vollständigen Punktsymmetriegruppe von Differentialgleichungen. Die Methode kombiniert die moderne Begrifflichkeit von *megaideals* mit der Faktorisierung von inneren Automorphismen nach der Methode von Hydon. Im Gegensatz zu den existierenden Methoden erlaubt die neue algebraische Methode auch die Berechnung von Punktsymmetriegruppen von Differentialgleichungen die eine unendlich dimensionale maximale Lie-Invarianz Algebra besitzen. Wir demonstrieren die Methode mit der Berechnung der vollständigen Symmetriegruppen der Vorticitygleichung auf der Kugel und den primitiven Gleichungen.

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This thesis emerged from my employment in the projects P20632 *Classification problems of group analysis*, P25064 *Extended group analysis of differential equations* (P.I. D.Sc. Roman O. Popovych) and P23714 *Geometry of algebras of generalized functions* (P.I. Prof. Dr. Michael Grosser), funded by the Austrian Science Fund (FWF).

Contents

Chapter 1

Introduction

1.1 Motivation

The development of the theory of differential equations has its beginning with the study of classical mechanics as an attempt to mathematically formulate the motions of celestial bodies. Since then, differential equations have become essential and popular mathematical models to describe a variety of physical processes. This led to an increased interest in studying properties of differential equations and their solutions.

The construction of methods for solving differential equations that rest on their geometric properties was first initiated by the Norwegian mathematician Sophus Lie (1842–1899). Lie started a systematic endeavor to understand various techniques known at his time for the solution of ordinary differential equations. Introducing the concept of a continuous transformation group, Lie was able to show that most of these integration techniques are indeed special instances of the application of transformations that do not change the form of a differential equation – so called *symmetries*. Probably the most important observation made by Lie was that the study of continuous transformation groups can be replaced by the consideration of their infinitesimal counterparts, which are Lie algebras of vector fields. The significance of Lie's discovery for differential equations was that rather than studying continuous transformations of differential equations, which is largely a nonlinear problem, it suffices to study the infinitesimal action of these transformations on the differential equations of interest, which yields a linear problem [84, 85].

It is interesting to note here that after Lie's work on differential equation, the research on Lie groups took a very different direction, regarding them more as abstract entities in the interplay of group theory and differential geometry rather than a concept of central importance in the study of solutions of differential equations. Due to this, Lie's important concepts were later reconsidered by the Russian mathematician Lev V. Ovsiannikov in the study of *group analysis of differential equations* [112]. Ovsiannikov conducted extensive research devoted to symmetries of ordinary and partial differential equations and a multitude of new methods were developed and used in the years to follow. The main area of application of Lie's methods of group analysis soon turned out to be hydrodynamics [6]. The reason for this is that most governing equations of hydrodynamics are fully nonlinear and it is in general not possible to find their general solution. In fact, most of the physically relevant solutions of hydrodynamical equations were either discovered or rediscovered using symmetry methods. It is also not without irony that due to the long period of indifference towards Lie's original work, some of the results he established that were forgotten were reinvented decades later.

Particular attention was brought by Ovsiannikov to the case when a differential equation includes arbitrary constants or arbitrary functional parameters. These all are collectively referred to as *arbitrary elements*. Arbitrary elements in differential equations provide means to adjust a differential equation, serving as a model for a physical process, for example, to experimental data. This is why most differential equations in physics and engineering include such arbitrary elements. In the study of group analysis, such parameterized differential equations are known as *classes of differential equations*. A class of differential equations is challenging from the point of view of studying its symmetries because different values of the arbitrary elements can result in different symmetry properties. The problem of systematically finding those particular values of the arbitrary elements where differential equations from a given class admit additional symmetries compared to the case where arbitrary elements are truly generic is referred to as the *problem of group classification* [112, 127]. Historically, it was once again Lie who solved the first classification problems when studying the properties of the classes of second order ordinary differential equations and second order linear partial differential equations with two independent variables [85].

The problem of group classification is central for the present thesis, in which we are going to use classification techniques to find parameterization schemes with symmetry properties. The somewhat intuitive relation between parameterization schemes for averaged differential equations and group classification was first established in the work by Oberlack, who investigated the problem of finding subgrid-scale closure models for the LES-filtered Navier–Stokes equations [98]. Oberlack, following some earlier considerations by Speziale [141] on the narrower problem of preserving Galilean invariance in the filtered Navier–Stokes equations, systematically laid down the conditions on both the filter kernel and the subgrid-scale closure model that have to be satisfied in order for the filtered Navier–Stokes equations to preserve the entire point symmetry group of the original (unfiltered) Navier–Stokes equations. The idea that the maximal Lie invariance group of a system of differential equations should be preserved in the process of averaging and closing of the model was then recently extended using the language of group analysis and group classification [123].

The reasoning behind connecting the parameterization problem with the problem of group classification will be detailed now. Let us start with the parameterization problem as it arises in hydrodynamics and geophysical fluid mechanics first. In order to prepare a system of differential equations serving as a model for a physical system for a numerical integration, it is inevitable to choose the resolution with which the numerical model will be able to operate. Mathematically, this boils down to approximating the true solution of the initial model, which is an element of an infinite dimensional vector space, by a numerical solution which always lies in a finite dimensional vector space. The reason for this is of course that no computer can store or manipulate an infinite dimensional vector. There is thus inevitably an information loss associated with the process of discretization of a system of differential equations. This loss of information could be regarded as acceptable as long as the information in the numerical solution is self-contained, meaning that the evolution of the resolved numerical solution is only determined by the resolved part of the true solution in a sufficiently accurate way.

Unfortunately, in fluid mechanics this is not the case. Fluid mechanics in general and atmospheric phenomena in particular are to a great extent the results of the interaction between large and small scales. Since the 1950s there have been continuous attempts to increase the model resolution of numerical weather prediction models further in order to incorporate increasingly smaller scale features. However, to date it is still not possible to resolve all the dynamically relevant scales in an Earth simulation model. In other words, the information contained in the resolved part of the flow is not sufficient to advance this resolved part of the flow for all times. This is the famous closure problem of fluid dynamics: The resolved part of the flow interacts with the unresolved part and it is necessary to model the effects of the unresolved or *subgrid-scale* part on the resolved or *grid-scale* quantities. As by definition one does not have the entire information on what is happening below the grid scale, modeling these effects is always approximate. In geophysical fluid dynamics these approximations are know as *parameterization schemes* [143, 144].

The main difficulty in finding reasonable parameterization schemes is that small scale processes often cannot be observed well enough and are therefore difficult to measure and, accordingly, to model. Moreover, for several processes in the atmosphere our physical understanding is yet not sufficient to find a conclusive model for them. This concerns, for example, the important problem of turbulence modeling, which is a part of this thesis. Even with the further advancement of technology, e.g. through an improved observational network and increased computational power, it is questionable whether it will be possible to gather all the relevant dynamical information on the multitude of active scales in the atmosphere–ocean system. Hence, the parameterization problem will remain one of the most important problems to be tackled in numerical weather and climate prediction models.

Having established that parameterization schemes are, and most likely will always be, an important part of a numerical model for the atmosphere–ocean system, the question arises as to what strategies should be used to find consistent closure models for the unresolved parts of geophysical flows. The methodology that we propose here is that *a parameterization scheme should be constructed in such a manner that the closed system of differential equations possesses symmetry properties that are well connected to and derivable from the symmetries of the original system of differential equations.* By this we mean that the symmetry group of the original system of differential equations should to a large extend determine the form of any consistent parameterization scheme.

The reasoning behind requiring a parameterized model to preserve (a symmetry subgroup of) the symmetry group of the original system of differential equations is that such symmetries usually determine the physical essence of the laws of nature. To give an example, any classical mechanical model should be invariant under the action of the Galilean group (consisting of translations, rotations and Galilean boosts), which is a direct consequence of Newton's Laws of Motion [127]. Therefore, a parameterization for an unresolved process that is known to fit into the framework of classical mechanics should be invariant under the Galilean group as well. We show in this thesis that the requirement of preservation of important symmetry transformations as admitted by the equations of hydrodynamics (such as translations, dilations and rotations) often places surprisingly rigid restrictions on the form of a physical parameterization scheme. From the practical perspective of finding such parameterization schemes for physical processes this is most desirable, because usually there is a multitude of ways for constructing a closure for a particular process [143, 144]. This means that the requirement placed on parameterization schemes to preserve a particular symmetry group can be an efficient guide in finding consistent approximations for unresolved processes.

In turn, we exemplify below that several of the existing parameterization schemes that can be found in the literature *break* certain symmetries and it is interesting to note that these parameterizations are often physically flawed. It is therefore tempting to attribute the physical inconsistencies arising from such non-invariant parameterization schemes to the violation of first principles. On the contrary, we also show here that invariant parameterization schemes can yield physical results that are traditionally hard to achieve with non-invariant closure models.

In the light of what was said above, we are now in the position to formulate the relation between the parameterization problem and the group classification problem. Mathematically speaking, the averaging or filtering of a system of differential equations leads to a system of equations that describes the evolution of the grid-scale quantities. Unfortunately, the averaged system involves new unknown functions (physically representing the subgrid-scale terms) for which no evolution equations are available, i.e. this system is *under-determined*. To close the system one has to express the unresolved terms using the grid-scale quantities. That is, one seeks a functional expression that models the effects of the subgrid-scale processes on the grid-scale variables. Introducing such a generic parameterization ansatz always turns the unclosed averaged system of differential equations into a class of closed averaged differential equations [123], which reduces the problem of finding invariant parameterization schemes to a problem of group classification.

This intimate relation between invariant parameterization schemes and the problem of group classification is the reason why the bulk of this thesis is devoted to the development of methods of group classification. Group classification is at the heart of the study of invariant parameterization schemes and without efficient techniques for the former it is hard or impossible to find the latter. The first task of this thesis is therefore to significantly extend the arsenal of methods available for the study of symmetries of differential equations including arbitrary elements.

There are several classification strategies available for classes of differential equations, and what strategy to choose depends largely on the structure of the class under consideration. In general, one distinguishes between two general strategies, which are known as the *inverse* and the *direct* classification methods. Both ways have been used for the construction of invariant parameterization schemes in [123].

Inverse group classification enables one to realize the paradigm that a parameterization scheme should preserve a symmetry group of the original, unfiltered equations most straightforwardly. This is because inverse group classification is done by first selecting those subgroups with respect to which one requires a parameterization to be invariant and then constructs the parameterization accordingly. The key to this method is the construction of *differential invariants*, which are functions of the equation variables and their derivatives that are invariant under the group action [33, 105, 112]. Constructing the parameterization scheme out of these differential invariants always leads to closed systems of differential equations that are invariant under the given symmetry group.

A powerful technique that can be used to compute these differential invariants of a group action is the recently developed *equivariant moving frame* method, in the formulation by Fels and Olver [45, 46]. Moving frames enable one to establish a mapping between non-invariant and invariant functions through a procedure referred to as *invariantization*. As the construction of a moving frame is an entirely algorithmic task, it always allows one to find differential invariants, and hence invariant parameterization schemes, systematically. More specifically we show below that the invariantization technique can also be used to start with an existing, non-invariant parameterization scheme for a physical process and make it invariant by applying the moving frame to it. That is, existing parameterization schemes can be taken as the starting point for finding invariant closure models.

As what concerns the direct method of group classification, here one starts with a general class of systems of differential equations and attempts to find all particular values for the arbitrary elements for which symmetry extensions arise [112, 127]. The result of the direct group classification method is therefore a list of systems from the class (corresponding to particular arbitrary elements) with their associated symmetry groups. These systems are inequivalent in the sense that they cannot be mapped to each other by point transformations.

Interpreted in the framework of invariant parameterization all the systems from the class represent different parameterization models leading to different closed systems of differential equations. We regard this method as suitable e.g. for those physical processes for which we yet lack a precise understanding and therefore have no a priori guidance on how the parameterization should be constructed. The direct group classification method then provides a list of possible candidate parameterizations leading to closed differential equations with distinct symmetry properties that can hereafter be tested numerically. This way one can obtain the most suitable parameterization scheme for a particular process from a usually reasonable number of candidate parameterizations.

A caveat in the direct classification approach to invariant parameterization schemes is that the classes of differential equations arising are usually so general that it is not possible to solve the classification problem completely. That is, one is not able to find an exhaustive list of all equations with inequivalent symmetry properties. This problem is well known in the theory of group classification and the reason for the development of several specialized classification techniques for such classes of differential equations. Because complicated classes of differential equations are the rule in the study of symmetry-preserving parameterization schemes, we devoted a considerable part of the research reported below to the refinement and extension of group classification techniques. Indeed, we have revised the entire framework of the algebraic method of group classification and showed its effectiveness by solving some of the long-standing problems in the group classification of differential equations. This new framework then forms a mathematical basis for the construction of invariant parameterization schemes using the direct classification method.

To test the effectiveness of the theoretical work carried out in this thesis we compute invariant parameterization schemes for problems of *turbulence modeling*. Turbulence is ubiquitous in geophysical fluid mechanics and can be found on almost all spatial and temporal scales [48, 147]. Turbulence on the large scale in the atmosphere is particular in that it is quasi-two-dimensional. It is this so-called geostrophic turbulence we are interested in. Specifically, we use the methods of invariant parameterization to demonstrate that conventional turbulence models usually do not admit the entire maximal Lie invariance group of the incompressible Euler equations in a rotating reference frame. We then construct invariant turbulence models and use them to simulate the behavior of freely-decaying turbulence. The resulting turbulent energy spectra are similar to the theoretically predicted ones and thus serve as a proof-of-the-concept of the physical relevance of invariant parameterization schemes.

Before we detail the further organization of this thesis, we find it convenient to illustrate more explicitly the problem of conventional parameterization schemes when it comes to symmetry-preservation.

1.2 Symmetry properties of physical parameterization schemes

In this section we will analyze the symmetry properties of a class of existing parameterization schemes that has been used in meteorology. These parameterizations are relatively simple turbulence closures for the Reynolds averaged primitive equations modeling a dry atmospheric boundary layer.

1.2.1 Turbulence closure in the primitive equations

It was mentioned above that turbulence is ubiquitous in meteorology, effecting all the scales from the planetary boundary layer to the synoptic scale. The *turbulence closure problem*, which will play a central role in this thesis, is that for a statistical description of turbulence, a system of infinitely many differential equations would be needed.

For the sake of simplicity, we restrict ourselves to *first order closure models* for the incompressible primitive equations in an inertial reference frame. That is, we aim to describe the averaged wind, pressure and entropy fields using just the information contained in these mean fields. Although from the meteorological point of view this is certainly an overly simplified model, it serves the purpose to show how parameterization schemes can destroy the symmetry properties of the original governing equations of thermo-hydrodynamics. The issues identified below are the same as arise for more sophisticated parameterizations in more realistic models of geophysical fluid dynamics.

The initial system of primitive equations is the following

$$
\begin{aligned}
&u_t + uu_x + vu_y + wu_z + \rho^{-1}p_x = 0,\\
&v_t + uv_x + vv_y + wv_z + \rho^{-1}p_y = 0,\\
&w_t + uw_x + vw_y + ww_z + \rho^{-1}p_z + g = 0,\\
&u_x + v_y + w_z = 0,\\
&\theta_t + u\theta_x + v\theta_y + w\theta_z = 0.
\end{aligned} \tag{1.2.1}
$$

In this system, u, v and w are the components of the three-dimensional wind field, p is the atmospheric pressure and θ is the potential temperature, constituting a measure for the entropy of the system. The density ρ is assumed as constant (incompressibility) and thus can be scaled to $\rho = 1$; g is the magnitude of the Earth' gravitational acceleration. Throughout this thesis, the notation $u_t = \partial u/\partial t$, etc. is used to abbreviate partial derivatives.

Physically, the first three equations of the above system are the momentum equations, the fourth equation is the mass continuity equation and the last equation represents the first law of thermodynamics. We should like to stress that this model is a three-dimensional version of the model used in [144] to illustrate several eddy closure schemes.

We determine the maximal Lie invariance algebra of the above system using the computer algebra package `desolv` [153]. This algebra is spanned by the vector fields

$$
\begin{aligned}
&\mathcal{D}_1 = 2t\partial_t + x\partial_x + y\partial_y + z\partial_z - u\partial_u - v\partial_v - w\partial_w - (2p + 3gz)\partial_p,\\
&\mathcal{D}_2 = x\partial_x + y\partial_y + z\partial_z + u\partial_u + v\partial_v + w\partial_w + (2p + gz)\partial_p, \quad \partial_t,\\
&\mathcal{R}^x = -y\partial_x + x\partial_y - v\partial_u + u\partial_v, \quad \mathcal{R}^y = -z\partial_y + y\partial_z - w\partial_v + v\partial_w,\\
&\mathcal{R}^z = -x\partial_z + z\partial_x - u\partial_w + w\partial_u, \quad \mathcal{X}(f) = f(t)\partial_x + f'\partial_u - f''x\partial_p,\\
&\mathcal{Y}(g) = g(t)\partial_y + g'\partial_v - g''y\partial_p, \quad \mathcal{Z}(h) = h(t)\partial_z + h'\partial_w - h''z\partial_p,\\
&\mathcal{G}(\gamma) = \gamma(t)\partial_p, \quad \mathcal{H}(\delta) = \delta(\theta)\partial_\theta.
\end{aligned}
$$

The physical significance of these infinitesimal generators is the following. The vector fields $\mathcal{D}_1$ and $\mathcal{D}_2$ integrate to one-parameter scale transformations, ∂_t yields shifts in time, $\mathcal{R}^x$, $\mathcal{R}^y$ and $\mathcal{R}^z$ generate $SO(3)$, the three-dimensional rotational group, $\mathcal{X}(f)$, $\mathcal{Y}(g)$ and $\mathcal{Z}(h)$ are generalized Galilean boosts in the x-, y- and z-directions, $\mathcal{G}(\gamma)$ is a gauge transformation of the pressure and $\mathcal{H}(\delta)$ allows one to redefine the potential temperature.

We now average system (1.2.1) using the Reynolds average. To accomplish this, we first split the dependent variables according to $a = \bar{a} + a'$, where $\bar{a}$ is the mean of a and $a' = a - \bar{a}$ is the deviation of this mean. Then, by applying the averaging rules for products, $\overline{ab} = \bar{a}\bar{b} + \overline{a'b'}$ and using the continuity equation, the averaged form of the above system is

$$
\begin{aligned}
&\bar{u}_t + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y + \bar{w}\bar{u}_z + \bar{p}_x = -(\overline{u'u'})_x - (\overline{u'v'})_y - (\overline{u'w'})_z,\\
&\bar{v}_t + \bar{u}\bar{v}_x + \bar{v}\bar{v}_y + \bar{w}\bar{v}_z + \bar{p}_y = -(\overline{u'v'})_x - (\overline{v'v'})_y - (\overline{v'w'})_z,\\
&\bar{w}_t + \bar{u}\bar{w}_x + \bar{v}\bar{w}_y + \bar{w}\bar{w}_z + \bar{p}_z + g = -(\overline{u'w'})_x - (\overline{v'w'})_y - (\overline{w'w'})_z,\\
&\bar{u}_x + \bar{v}_y + \bar{w}_z = 0,\\
&\bar{\theta}_t + \bar{u}\bar{\theta}_x + \bar{v}\bar{\theta}_y + \bar{w}\bar{\theta}_z = -(\overline{u'\theta'})_x - (\overline{v'\theta'})_y - (\overline{w'\theta'})_z.
\end{aligned}
$$

The terms on the right-hand side of this system are those that need to be parameterized, as there are no explicit equations from which they could be determined in the present case. Physically, the subgrid-scale terms in the momentum equations are the momentum fluxes and the subgrid-scale terms in the thermodynamic equations are the heat fluxes. These fluxes play an important role, e.g. in the planetary boundary layer and in general cannot be omitted.

A standard parameterization for these fluxes that is employed in the atmospheric sciences is to relate these quantities to the *gradient* of the mean field, where the flux is directed down-gradient. The general form of such a down-gradient ansatz for the momentum flux is

$$
\overline{u_i'u_j'} = -k^{ij}\frac{\partial \bar{u}_i}{\partial x_j},
$$

where k^{ij} is a diffusivity parameter that usually depends on the independent variables (for more realistic parameterization schemes, this parameter additionally depends on quantities such as the turbulent kinetic energy, see also Chapter 2). For the heat flux, one similarly employs

$$
\overline{u_i'\theta'} = -k^{i}\frac{\partial \bar{\theta}}{\partial x_i},
$$

with k^i being the turbulent heat diffusion coefficient.

Introducing this closure in the above averaged, unclosed model leads to

$$
\begin{aligned}
&\bar{u}_t + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y + \bar{w}\bar{u}_z + \bar{p}_x = (k^{xx}u_x)_x + (k^{xy}u_y)_y + (k^{xz}u_z)_z,\\
&\bar{v}_t + \bar{u}\bar{v}_x + \bar{v}\bar{v}_y + \bar{w}\bar{v}_z + \bar{p}_y = (k^{yx}v_x)_x + (k^{yy}v_y)_y + (k^{yz}v_z)_z,\\
&\bar{w}_t + \bar{u}\bar{w}_x + \bar{v}\bar{w}_y + \bar{w}\bar{w}_z + \bar{p}_z + g = (k^{zx}w_x)_x + (k^{zy}w_y)_y + (k^{zz}w_z)_z,\\
&\bar{u}_x + \bar{v}_y + \bar{w}_z = 0,\\
&\bar{\theta}_t + \bar{u}\bar{\theta}_x + \bar{v}\bar{\theta}_y + \bar{w}\bar{\theta}_z = (k^{x}\theta_x)_x + (k^{y}\theta_y)_y + (k^{z}\theta_z)_z.
\end{aligned}
\tag{1.2.2}
$$

The only undetermined quantities in this system are now the parameterization parameters k^{ij} and k^i. The symmetry properties of the closed system of primitive equation therefore depend on the values of the arbitrary functions k^{ij} and k^i, which is an example for a group classification problem as will be encountered throughout the entire thesis.

We do not attempt to solve this classification problem exhaustively here as in order to do so we would have to specify the precise dependency of k^{ij} and k^i on the independent variables (and possibly also on other quantities, such as certain derivatives of the dependent variables). We only investigate the simplest case when $k^{ij} = \text{const}$, $k^i = \text{const}$ subsequently. Moreover, on physical grounds it is common to neglect the horizontal fluxes of horizontal momentum compared to the vertical fluxes of horizontal momentum; similarly, as the vertical exchange of heat is usually larger than the horizontal heat transfer (at least in the planetary boundary layer), we also only preserve the vertical heat flux; the fluxes of vertical momentum are usually small too, and are omitted therefore. That is, all the k^{ij} and k^i are zero except for k^{xz}, k^{yz} and k^z. Further assuming horizontal homogeneity, we set $k^{xz} = k^{yz}$.

In this case, the closed system (1.2.2) admits the following generators of infinitesimal symmetry transformations

$$\begin{aligned}
&\bar{\mathcal{D}}_1 = 2t\partial_t + x\partial_x + y\partial_y + z\partial_z - \bar{u}\partial_{\bar{u}} - \bar{v}\partial_{\bar{v}} - \bar{w}\partial_{\bar{w}} - (2\bar{p} + 3gz)\partial_{\bar{p}}, \quad \partial_t,\\
&\bar{\mathcal{R}}^x = -y\partial_x + x\partial_y - \bar{v}\partial_{\bar{u}} + \bar{u}\partial_{\bar{v}}, \quad \bar{\mathcal{X}}(f) = f(t)\partial_x + f'\partial_{\bar{u}} - f''x\partial_{\bar{p}},\\
&\bar{\mathcal{Y}}(g) = g(t)\partial_y + g'\partial_{\bar{v}} - g''y\partial_{\bar{p}}, \quad \bar{\mathcal{Z}} = h(t)\partial_z + h'\partial_{\bar{w}} - h''z\partial_{\bar{p}},\\
&\bar{\mathcal{G}}(\gamma) = \gamma(t)\partial_{\bar{p}}, \quad \partial_{\bar{\theta}}, \quad \bar{\theta}\partial_{\bar{\theta}}.
\end{aligned} \tag{1.2.3}$$

From these generators a few important observations can be made that apply also to the general parameterization problem. First, there are symmetries that are induced by the symmetries of the original primitive equations. In the present case, these are the symmetries associated with $\mathcal{D}_1$, ∂_t, $\mathcal{R}^x$, $\mathcal{X}(f)$, $\mathcal{Y}(g)$, $\mathcal{Z}(h)$, $\mathcal{G}(\gamma)$, $\mathcal{H}(1)$ and $\mathcal{H}(\theta)$. That is, the above parameterization scheme preserves a scale transformation, the time translation symmetries, horizontal rotations, all the generalized Galilean boosts and the gauging of the pressure. Arbitrary transformations of the potential temperature are not permitted any more, as from the general operator $\mathcal{H}(\delta)$ only two special cases are preserved. All the other symmetry transformations are lost, i.e. the closed parameterized model of the primitive equations admits fewer symmetries than the original model.

This can be desirable as well. For example in the above discussion on the values of the k^{ij} we explicitly assumed that the vertical fluxes of horizontal momentum can be neglected compared to the horizontal fluxes of horizontal momentum. This introduces an anisotropy in the system of equations which justifies that the parameterized system should not admit the rotational symmetries $\mathcal{R}^y$ and $\mathcal{R}^z$. That is, the *breaking* of certain symmetries can also occur on physical grounds.

The central problem is thus the following: Averaging of a system of differential equations usually perturbs the geometric structure of these equations when an inappropriate closure model is applied. On the other hand, the physics of an averaged model can be different from the physics of the original model and this should reflect in the alteration of the symmetries admitted. One thus needs efficient methods that allow one to select among all the symmetries admitted in a model, which of them should also be admitted in the parameterized model. *The methods related to the group classification of differential equations developed in this thesis can be used to systematically construct such parameterization schemes with prescribed symmetry properties.*

In conclusion, we would like to stress that it is straightforward to verify that other choices for k^{ij} and k^i (as e.g. given in the book [144], see also Chapter 2 below) would lead to closed systems of differential equations of the general form (1.2.2) that admit even fewer symmetries than in the case discussed above.

1.3 Structure of the thesis

The chapters of this thesis are based on paper that are either already published or presently under consideration for publication. We list below the bibliographic data for these papers along with a short summary.

Chapter 2 A. Bihlo, E.M. Dos Santos Cardoso-Bihlo and R.O. Popovych, 2014. Invariant and conservative parameterization schemes, in volume 2 of *Parameterization of Atmospheric Convection*, (R. S. Plant and J. I. Yano, Eds.), Imperial College Press, in press.

This chapter is our contribution to the two-volume book that emerged from the COST action ES0905 *Basic concepts for convection parameterization in weather forecast and climate models*. We provide an introduction and an overview about the methods for finding invariant and conservative physical parameterization schemes for the meteorological audience.

Chapter 3 *Published* as: E.M. Dos Santos Cardoso-Bihlo, A. Bihlo and R.O. Popovych, 2011. Enhanced preliminary group classification of a class of generalized diffusion equations. *Commun. Nonlinear Sci. Numer. Simulat.* **9** (16), 3622–3638, arXiv:1012.0297.

The method of preliminary group classification is rigorously defined, enhanced and related to the theory of group classification of differential equations. Typical weaknesses in papers on this method are discussed and strategies to overcome them are presented. The preliminary group classification of the class of generalized diffusion equations of the form $u_t = f(x,u)u_x^2 + g(x,u)u_{xx}$ is carried out. This includes a justification for applying this method to the given class, the simultaneous computation of the equivalence algebra and equivalence (pseudo)group, as well as the classification of inequivalent appropriate subalgebras of the whole infinite-dimensional equivalence algebra. The extensions of the kernel algebra, which are induced by such subalgebras, are exhaustively described. These results improve those recently published in *Commun. Nonlinear Sci. Numer. Simul.*.

Chapter 4 *Published* as: A. Bihlo, E.M. Dos Santos Cardoso-Bihlo and R.O. Popovych, 2012. Complete group classification of a class of nonlinear wave equations. *J. Math. Phys.* **53**, 123515 (32 pp), arXiv:1106.4801.

Preliminary group classification became a prominent tool in the symmetry analysis of differential equations due to the paper by Ibragimov, Torrisi and Valenti [J. Math. Phys. **32** (1991), 2988–2995]. In this paper the partial preliminary group classification of a class of nonlinear wave equations was carried out via the classification of one-dimensional Lie symmetry extensions related to a fixed finite-dimensional subalgebra of the infinite-dimensional equivalence algebra of the class under consideration. We implement the complete group classification of the same class up to both usual and general point equivalence using the algebraic method of group classification. This includes the complete preliminary group classification of the class and finding those Lie symmetry extensions which are not associated with subalgebras of the equivalence algebra. The complete preliminary group classification is based on listing all inequivalent subalgebras of the whole infinite-dimensional equivalence algebra whose projections are qualified as maximal extensions of the kernel algebra. The set of admissible point transformations of the class is exhaustively described in terms of the partition of the class into normalized subclasses.

Chapter 5 *Published* as: E.M. Dos Santos Cardoso-Bihlo and R.O. Popovych, 2013. Complete point symmetry group of vorticity equation on rotating sphere, *J. Engrg. Math.* **82**, 31–38, arXiv:1206.6919.

The complete point symmetry group of the barotropic vorticity equation on the sphere is determined. The method we use relies on the invariance of megaideals of the maximal Lie invariance algebra of a system of differential equations under automorphisms generated by the associated point symmetry group. A convenient set of megaideals is found for the maximal Lie invariance algebra of the spherical vorticity equation. We prove that there are only two independent (up to composition with continuous point symmetry transformations) discrete symmetries for this equation.

Chapter 6 *Published* as: E.M. Dos Santos Cardoso-Bihlo, 2012. Differential invariants for the Korteweg–de Vries equation. *Proceedings of the Sixth International Workshop "Group Analysis of Differential Equations and Integrable Systems" (Protaras, Cyprus, June 17–21, 2012)*, 71–79.

Differential invariants for the maximal Lie invariance group of the Korteweg–de Vries equation are computed using the moving frame method and compared with existing results. Closed forms of differential invariants of any order are presented for two sets of normalization conditions. Minimal bases of differential invariants associated with the chosen normalization conditions are given.

Chapter 7 *Preprint*: E.M. Dos Santos Cardoso-Bihlo and R.O. Popovych, 2014. On the ineffectiveness of constant rotation in the primitive equations.

The primitive equations are the main system of nonlinear partial differential equations on which modern weather and climate prediction models are based on. The Lie symmetries of the primitive equations are computed and the structure of the maximal Lie invariance algebra, which is infinite dimensional, is investigated. It is found that the maximal Lie invariance algebra for the case of a constant Coriolis force can be mapped to the case of vanishing Coriolis force. The same mapping allows one to transform the constantly rotating primitive equations to the equations in a resting reference frame. This mapping is used to obtain exact solutions for the rotating case from exact solutions from the nonrotating equations. Another main result of the paper is the computation of the complete point symmetry group of the primitive equations using the algebraic method.

Chapter 8 *Published* as: A. Bihlo, E.M. Dos Santos Cardoso-Bihlo and R.O. Popovych, 2014. Invariant parameterization and turbulence modeling on the beta-plane. *Physica D* **269**, 48–62, arXiv:1112.1917.

Invariant parameterization schemes for the eddy-vorticity flux in the barotropic vorticity equation on the beta-plane are constructed and then applied to turbulence modeling. This construction is realized by the exhaustive description of differential invariants for the maximal Lie invariance pseudogroup of this equation using the method of moving frames, which includes finding functional bases of differential invariants of arbitrary order, a minimal generating set of differential invariants and a basis of operators of invariant differentiation in an explicit form. Special attention is paid to the problem of two-dimensional turbulence on the beta-plane. It is shown that classical hyperdiffusion as used to initiate the energy–enstrophy cascades violates the symmetries of the vorticity equation. Invariant but nonlinear hyperdiffusion-like terms of new types are introduced and then used in the course of numerically integrating the vorticity equation and carrying out freely decaying turbulence tests. It is found that the invariant hyperdiffusion scheme is closely reproducing the theoretically predicted k^{-1} shape of enstrophy spectrum in the enstrophy inertial range. By presenting conservative invariant hyperdiffusion terms, we also demonstrate that the concepts of invariant and conservative parameterizations are consistent.

Chapter 2

Invariant and conservative parameterization schemes

2.1 Introduction

The idea of preserving geometric properties of differential equations such as symmetries and conservation laws was recently introduced in the parameterization problem. Both symmetries and conservation laws play a superior role in modern physics and mathematics. It is immediately obvious that if a parameterization scheme violates basic scaling properties (e.g. the terms in the parameterization do not have the same dimension as the terms that should be closed), it should be regarded as suspicious. However, the violation of symmetries in a closure model can be more subtle and can go unnoticed for a while. For example, it is now established that the Kuo convection scheme is not compatible with Galilean invariance because it equates a Galilean invariant quantity (the rain rate) with a non-invariant quantity (the moisture flux convergence).

Here we lay down systematic methods for the construction of parameterization schemes with prescribed symmetry properties. That is, the parameterization is constructed in such a manner that selected symmetries from the original system of differential equations will be preserved in the closed model as well. We thus make the first principle of symmetry-preservation a constructive requirement for the design of parameterization schemes.

We also discuss the role of conservation laws for constructing parameterization schemes. There are several processes in the atmospheric sciences that comply with basic conservation laws, such as energy, momentum and mass conservation. It is thus sensible to require preservation of these conservation laws also in the case when considering only the resolved part of the flow. That is, if a process is known to be conservative, then also the parameterization for this process should be conservative.

We should also like to stress that methods related to the ones to be introduced in this chapter for the construction of geometry-preserving parameterization schemes are already in use in meteorology, namely in the field of geometric numerical integration. This area is devoted to the design of discretization schemes that preserve the fundamental properties of the governing equations of hydro-thermodynamics numerically. The properties are, inter alia, mass conservation, energy conservation, axial angular momentum conservation, the absence of spurious Rossby modes and stability of the geostrophic balance, see [142] for a more complete list along with further explanations. Guaranteeing these properties on the discrete level requires a careful design of the dynamical core of a numerical model. The methods used to accomplish this goal are referred to as *mimetic discretization*, see [27, 145] for a more thorough exposition of this recent field.

Drawing a parallel to the field of mimetic discretization, it is fair to say that the methods to be introduced in this chapter could be dubbed *mimetic parameterization*. If used in a numerical model that features a mimetic dynamical core, one could guarantee the discrete preservation of symmetries and conservation laws of the governing equations on the level of the resolved part (using mimetic discretization) as well as on the parameterization of the unresolved part (using mimetic parameterization).

We organize this chapter in the following way: In Section 2.2 we introduce the necessary background material on symmetries and conservation laws. This introduction is by no means complete but should serve to illustrate the techniques that are required to construct parameterization schemes that preserve symmetries and conservation laws. In Sections 2.3 and 2.4 we use the material presented in Section 2.2 to introduce different methods for the construction of invariant and conservative parameterization schemes. We include mostly minimal examples in these sections that should serve as an illustration of the general theory. More realistic examples are discussed in Section 2.5. The chapter concludes with Section 2.6 which contains a short summary and discussion of further problems in the field of geometry-preserving parameterization schemes. Many of the original results reported in this chapter can be found in some more details in the papers [14, 16, 24, 123].

2.2 Symmetries and conservation laws

In this section we present some of the fundamental concepts of symmetries and conservation laws as formulated in the field of *group analysis of differential equations*. A more thorough presentation of the material covered here can be found in the textbooks [25, 26, 59, 101, 112].

2.2.1 Symmetries, invariants and group classification

In what follows, we denote by $\mathcal{L}\colon \Delta_l(x,u^{(n)})=0$, $l=1,\dots,L$, a system of differential equations, which is regarded as a function of the independent variables $x=(x^1,\dots,x^p)$, the dependent variables $u=(u^1,\dots,u^q)$, as well as all derivatives of u with respect to x up to order n. For the sake of brevity, all these derivatives (including u itself as derivative of order zero) are collected in the term $u^{(n)}$. The space of variables $z=(x,u)$ is denoted by M, the extended space of variables $z^{(n)}=(x,u^{(n)})$ is denoted by $M^{(n)}$.

Thus, within the local approach, derivatives of u with respect to x up to order n are just assumed as additional dependent variables in the extended space $M^{(n)}$. Smooth functions defined on domains in $M^{(n)}$ for some n, like Δ_l, are called *differential functions*. The *order* of a differential function F is defined to be equal to the highest order of derivatives involved in F.

For specific examples, we will use the simpler notation of independent variables as t, x, y, …instead of x^1, x^2, x^3, … .

Example 2.1. In the case of a single dependent variable u of two independent variables t and x (i.e. $p=2$ and $q=1$), $u^{(2)}$ is the tuple $(u;u_t,u_x;u_{tt},u_{tx},u_{xx})$, where here and in the following we use the shorthand notation $u_t=\partial u/\partial t$, $u_x=\partial u/\partial x$, $u_{tt}=\partial^2u/\partial t^2$, *etc.*

Definition 2.2. A point symmetry of the system $\mathcal{L}$ is a (non-degenerate) point transformation $\Gamma\colon\ \tilde x=X(x,u)$, $\tilde u=U(x,u)$ that maps the system $\mathcal{L}$ to itself. Equivalently, the transformation Γ maps any solution of $\mathcal{L}$ to another solution of $\mathcal{L}$.

The set of point symmetries of any system of differential equations admits the structure of a *group* with respect to the composition of transformations. That is, this set contains the composition of its elements, the identity transformation and the inverse of each element.

General point symmetries of systems of differential equations are usually hard to find, as they are determined by solving systems of nonlinear partial differential equations, which is often hopeless to do without additional mathematical machinery. This is why most of the literature contents itself by studying so-called continuous (or Lie) symmetries. These are point transformations that are parameterized by one or more *continuous* parameters and constitute a group G.

Example 2.3. The (viscous) Burgers equation

$$u_t + uu_x + u_{xx} = 0, \tag{2.2.1}$$

can be regarded as a major simplification of the Navier–Stokes equations. It admits, inter alia, the continuous symmetry transformation $(t, x, u) \mapsto (t, x + \varepsilon t, u + \varepsilon)$, where $\varepsilon \in \mathbb{R}$. It is readily checked that this so-called *Galilean transformation* leaves the Burgers equation invariant. On the other hand, the transformation $(t, x, u) \mapsto (t, -x, -u)$ is also a symmetry of the Burgers equation, but it is not an element of a one-parameter Lie symmetry group of the Burgers equation. This is an example for a *discrete symmetry*.

The main advantage of Lie symmetries over other point symmetries is that they can be found algorithmically using infinitesimal techniques, which always boils down to linear problems. For most purposes it suffices to consider the action of the group linearized around the identity element. Moreover, the techniques for finding the infinitesimal action of a group, encoded in the *infinitesimal generators* of the group transformation, are already implemented in most major computer algebra systems such as `Mathematica`, `Maple` or `Reduce`. What is more, for various important physical systems of differential equations, the Lie symmetries are already computed and can be found in standard handbooks [2, 5, 60].

As was said above, solving the determining equations for Lie symmetries of a system of differential equation (either by hand or by a computer algebra system) yields a set of infinitesimal generators, or vector fields, that jointly span the *maximal Lie invariance algebra* of the system under consideration. Recovering the finite group transformations from these vector fields is accomplished by solving a system of first-order ordinary differential equations.

Example 2.4. The maximal Lie invariance algebra of the Burgers equation (2.2.1) is spanned by the following vector fields:

$$\begin{aligned} &\partial_t, \quad \partial_x, \quad t\partial_x + \partial_u, \quad 2t\partial_t + x\partial_x - u\partial_u, \\ &t^2\partial_t + tx\partial_x + (x - tu)\partial_u. \end{aligned} \tag{2.2.2}$$

More generally, if $\tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \phi(t, x, u)\partial_u$ is a vector field on the space of variables (t, x, u), one can recover the associated one-parameter Lie symmetry by integrating the system of ordinary differential equations

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\varepsilon} = \tau(\tilde{t}, \tilde{x}, \tilde{u}), \quad \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\varepsilon} = \xi(\tilde{t}, \tilde{x}, \tilde{u}), \quad \frac{\mathrm{d}\tilde{u}}{\mathrm{d}\varepsilon} = \phi(\tilde{t}, \tilde{x}, \tilde{u}),$$

with the initial conditions $\tilde{t}|_{\varepsilon=0} = t$, $\tilde{x}|_{\varepsilon=0} = x$, $\tilde{u}|_{\varepsilon=0} = u$. The extension of this algorithm to the case of several unknown functions of more than two variables is straightforward.

Example 2.5. For the Burgers equation (2.2.1), the one-parameter Lie symmetry groups associated with the vector fields (2.2.2) consist of the transformations, which map (t, x, u) to

$$
\begin{aligned}
&(t+\varepsilon_1, x, u), \quad (t, x+\varepsilon_2, u), \quad (t, x+\varepsilon_3 t, u+\varepsilon_3), \\
&(e^{2\varepsilon_4}t, e^{\varepsilon_4}x, e^{-\varepsilon_4}u), \quad \left(\frac{t}{1-\varepsilon_5 t}, \frac{x}{1-\varepsilon_5 t}, u(1-\varepsilon_5 t)+\varepsilon_5 x\right),
\end{aligned}
\tag{2.2.3}
$$

where ε_i, $i = 1, \dots, 5$, are arbitrary constants. The physical significance of these symmetries is thus: (i) time translations, (ii) space translations, (iii) Galilean transformations, (iv) scale transformations, (v) inversions in time.

An important role of symmetries in the study of partial differential equations is that they allow finding solution ansatzes that enable one to reduce the number of independent variables occurring in a given system. This is done by computing the *invariants* of suitable symmetry subgroups of a given invariance group and considering these invariants as the only new variables in the system.

Definition 2.6. An *invariant* of a transformation group G acting on M is a function $f(z)$ that satisfies

$$f(g \cdot z) = f(z)$$

for all $z \in M$ and all $g \in G$ such that the action $g \cdot z$ of g on z is defined.

Thus, invariants are functions that do not change their value if their arguments are transformed.

Invariants can be found either using *infinitesimal techniques* or *moving frames* [45, 46]. The infinitesimal criterion of invariance of a function $f(z)$ under a group G is

$$\mathbf{v}f(z) = 0,$$

for each infinitesimal generator $\mathbf{v}$ of the group G.

Example 2.7. The function $f(t, x, u) = x - ut$ is invariant under Galilean transformations $(t, x, u) \mapsto (t, x+\varepsilon t, u+\varepsilon)$. The infinitesimal generator of these transformations is $\mathbf{v} = t\partial_x + \partial_u$ and thus the function f satisfies the equation $\mathbf{v}f := tf_x + f_u = 0$.

It is meaningful to extend the definition of invariance of a function to functions that also depend on derivatives of the dependent variables. This leads to the definition of *differential invariants*.

Definition 2.8. A differential invariant of a transformation group G acting on M is a differential function $f(x, u^{(n)})$ that satisfies

$$f(g^{(n)} \cdot z^{(n)}) = f(z^{(n)}),$$

for all $z^{(n)} \in M^{(n)}$ and all $g \in G$ such that the transformation of $z^{(n)}$ using the prolongation of g, denoted by $g^{(n)}$: $\tilde{z}^{(n)} = g^{(n)} \cdot z^{(n)}$ is defined.

In practice, the *prolongation of a group action* to the derivatives of the dependent variables is implemented by repeatedly using the chain rule.

Example 2.9. For the Galilean transformation $(\tilde{t},\tilde{x},\tilde{u}) = (t, x+\varepsilon t, u+\varepsilon)$, the partial derivatives with respect to t and x transform as $\partial_t = \tilde{t}_t\partial_{\tilde{t}} + \tilde{x}_t\partial_{\tilde{x}} = \partial_{\tilde{t}} + \epsilon\partial_{\tilde{x}}$ and $\partial_x = \tilde{t}_x\partial_{\tilde{t}} + \tilde{x}_x\partial_{\tilde{x}} = \partial_{\tilde{x}}$. Therefore, the transformed partial derivative operators are $\partial_{\tilde{t}} = \partial_t - \varepsilon\partial_x$ and $\partial_{\tilde{x}} = \partial_x$. With these derivative operators, it is now possible to determine the action of Galilean transformations on the various derivatives u_t, u_x, *etc.* In particular, we have

$$\tilde{u}_{\tilde{t}} = u_t - \varepsilon u_x, \quad \tilde{u}_{\tilde{x}} = u_x.$$

Example 2.10. The function $u_t + uu_x$ is a differential invariant of Galilean transformations. Indeed, $\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} = u_t + uu_x$.

Differential invariants can be found using an infinitesimal invariance criterion as well, see e.g. [112]. It is more convenient though to determine them using the method of equivariant moving frames in the Fels and Olver formulation. Because this method plays a superior role in the construction of invariant parameterization schemes, we introduce it shortly here. More in-depth information on moving frames and their applications can be found in the original papers [45, 46, 103, 107]. For the sake of simplicity, we also only consider the case when the group G is finite-dimensional.

Definition 2.11. Let G be a finite-dimensional Lie group acting on M. A *(right) moving frame* ρ is a smooth map $\rho\colon M \to G$ satisfying the equivariance property

$$\rho(g\cdot z) = \rho(z)g^{-1}$$

for all $z \in M$ and $g \in G$.

The motivation behind introducing the moving frame ρ is that it allows one to associate to a given function an invariant function. This is accomplished in a process called *invariantization*.

Definition 2.12. The invariantization of a function $f\colon M \to \mathbb{R}$ using the (right) moving frame ρ is the invariant function $\iota(f)$, which is defined as $\iota(f)(z) = f(\rho(z)\cdot z)$.

It is readily checked that $\iota(f)$ is indeed an invariant function:

$$\iota(f)(g\cdot z) = f(\rho(g\cdot z)g\cdot z) = f(\rho(z)g^{-1}g\cdot z) = f(\rho(z)\cdot z) = \iota(f)(z),$$

which is nothing but the definition of an invariant function: The value of $\iota(f)$ is not changed if its argument is transformed.

The invariantization of a non-invariant function is the key to one of the methods for the construction of invariant parameterization: One can start with a given parameterization that fails to be invariant and turn it into an invariant scheme by applying the proper moving frame to it.

The theorem on moving frames [45, 46] guarantees the existence of a moving frame provided that the action of G on M is *free* and *regular*. Without going into more details, both requirements are usually satisfied for the groups of interest in physics, although the freeness property often requires the construction of the moving frame on the space $M^{(n)}$ rather than on M. The following is a recipe of how one can find a moving frame through a simple *normalization procedure* [33]. Once again, the construction is entirely algorithmic and to date already implemented in `Maple`. For the sake of simplicity, we assume that G is an r-dimensional Lie group with $r < \infty$ acting on M although the assumption $r < \infty$ is not principal.

Algorithm 2.13. The construction of a moving frame via normalization.

1. Explicitly write down the transformation formulas for the action of the group G prolonged to $M^{(n)}$ for n sufficiently large:

$$Z^n = (X, U^{(n)}) = g^{(n)} \cdot (x, u^{(n)}).$$

2. Choose r normalization constants c_i and equate r of the above transformed variables to these constants, i.e.

$$Z_i^n = c_i, \qquad i = 1, \dots, r.$$

3. Solve the arising system of algebraic equations for the group parameters of G in terms of the coordinates $z^{(n)}$.

Example 2.14. We detail the construction of a moving frame for the subgroup G^1 of the maximal Lie invariance group G of the Burgers equation (2.2.1) that consists of translations in time and space, Galilean transformations and scale transformations. Including the inversion symmetry is possible as well but would complicate the resulting computations and formulas without adding substantial information for this introductory example.

If we combine the first four one-parameter symmetry transformations given in (2.2.3), we obtain

$$(\tilde{t}, \tilde{x}, \tilde{u}) = (e^{2\varepsilon_4}(t + \varepsilon_1), e^{\varepsilon_4}(x + \varepsilon_2 + \varepsilon_3 t), e^{-\varepsilon_4}(u + \varepsilon_3)), \tag{2.2.4}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 are arbitrary constants. Because there are four group parameters but only three variables in the above transformation formula, it is not possible to produce a sufficient number of equations. Stated in other words, the group G^1 is not free on the space $M = \{(t, x, u)\}$. This is why it is necessary to prolong (2.2.4) to the first derivatives of u. Using the chain rule as shown in Example 2.9 for Galilean transformations, for the subgroup G^1 we find

$$\tilde{u}_{\tilde{t}} = e^{-3\varepsilon_4}(u_t - \varepsilon_3 u_x), \quad \tilde{u}_{\tilde{x}} = e^{-2\varepsilon_4} u_x.$$

This is step (1) of Algorithm 2.13. The space $M^{(1)}$ is five-dimensional and thus it is possible to single out a hypersurface of dimension $5 - 4 = 1$ that allows us to solve for all four group parameters. This hypersurface is defined through the following four equations,

$$\tilde{t} = 0, \quad \tilde{x} = 0, \quad \tilde{u} = 0, \quad \tilde{u}_{\tilde{x}} = 1,$$

which accomplishes step (2) of Algorithm 2.13. Note that other normalization conditions could be chosen, which would lead to equivalent moving frames. Solving this system of four algebraic equations for the four group parameters accomplishes step (3) and we obtain the moving frame

$$\varepsilon_1 = -t, \quad \varepsilon_2 = -x + ut, \quad \varepsilon_3 = -u, \quad \varepsilon_4 = \frac{1}{2} \ln u_x.$$

This frame can now be used to invariantize any non-invariant function of the variables $(x, u^{(n)})$.

To practically realize the invariantization procedure we first transform the function to be invariantized using G and then substitute the computed moving frame for the appearing group parameters.

Example 2.15. We show here how the moving frame constructed in the previous example can be used to construct an invariant function starting with a non-invariant expression. Consider the function $f = u_{xx}$. Under the action of the subgroup G^1, this function is transformed to $\tilde{f} = e^{-3\varepsilon_4}u_{xx}$. Thus, f is not invariant under the action of G^1. Let us invariantize f by setting

$$\iota(f) = \tilde{u}_{\tilde{x}\tilde{x}}|_{g=\rho(z)} = \frac{u_{xx}}{\sqrt{u_x^3}},$$

As was shown above, the resulting function $\iota(f)$ now is G^1-invariant.

Moving frames can also be used to obtain the invariant representation of systems of differential equations. A moving frame of the group G can also be used to obtain the representation of a G-invariant system of differential equations in terms of differential invariants of G. This is the content of the so-called *replacement theorem* [33].

Example 2.16. To obtain the Burgers equation expressed in terms of differential invariants of the subgroup G^1, we invariantize it:

$$\iota(u_t + uu_x + u_{xx}) = \iota(u_t) + \iota(u)\iota(u_x) + \iota(u_{xx}) = \frac{u_t + uu_x + u_{xx}}{\sqrt{u_x^3}} = 0.$$

Of course, this expression is equivalent to the original Burgers equation.

It is often the case that differential equations contain certain constants or functions that are to be determined externally. The relevance of these constants and functions can be different but they are usually related to the physical properties of the model that is expressed using differential equations.

Example 2.17. The incompressible Euler equations in stream function form on the β-plane

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x = 0, \quad \zeta = \psi_{xx} + \psi_{yy}, \tag{2.2.5}$$

includes the β-parameter as constant. This parameter is externally determined and different choices for β lead to different dynamical properties of solutions for ψ.

We henceforth collectively refer to such constants or functions as *arbitrary elements*. Systems of differential equations that include arbitrary elements are called *classes of differential equations*. Studying symmetry properties of classes of differential equations is generally more complicated than to determine the symmetries of a system of differential equations that does not include arbitrary elements. The reason for this complication is that for different values of the arbitrary elements, the corresponding equations from the class usually admit different symmetry properties. Exhaustively describing the symmetry properties of such classes is the problem of *group classification*.

Example 2.18. For the vorticity equation (2.2.5) there are two essentially different cases, given by $\beta = 0$ and $\beta \neq 0$. The former case leads back to the f-plane form of the equation. It can be checked by direct computation that the symmetry group for the vorticity equation on the f-plane is wider than that for the β-plane equation, see e.g. [18]. This is understandable as the presence of the β-parameter adds an anisotropy to the original f-plane model.

There exist different techniques to solve group classification problems and which technique to use largely depends on the class of differential equations under consideration and, in particular, of the form of the arbitrary elements. If the arbitrary elements are constants or functions of a single variable only, then it is often possible to directly integrate the determining equations of Lie symmetries. For more complicated classes of differential equations (as typically arise in the study of invariant parameterization schemes) such a direct integration of the determining equations is generally hopeless. Other techniques that rely on Lie algebra classification have to be used.

We do not attempt to give an introduction to the various techniques here as this is too large of a subject to be suitable for this chapter. Rather, we will illustrate the most relevant techniques directly in the course of the invariant parameterization problem in Section 2.3. More background material on group classification can be found in [15, 63, 97, 123, 127].

2.2.2 Conservation laws

Let us turn our attention to *conservation laws* of systems of partial differential equations now. We give here the formal definition of a local conservation law [26, 101].

Definition 2.19. A *local conservation law* of the system $\mathcal{L}$ is a divergence expression that vanishes on the solutions of the system $\mathcal{L}$ (denoted by $|_{\mathcal{L}}$),

$$\mathrm{D}_i\Phi^i|_{\mathcal{L}} = (\mathrm{D}_1\Phi^1 + \dots + \mathrm{D}_p\Phi^p)|_{\mathcal{L}} = 0. \tag{2.2.6}$$

The p-tuple of differential functions $\Phi = (\Phi^i(x, u^{(m)}),\ i = 1, \dots, p)$ with some $m \in \mathbb{N}_0$, is called a *conserved vector* of the conservation law.

Here and in the following the operator D_i is the operator of total differentiation with respect to the variable x^i, $i = 1, \dots, p$. It has the coordinate expression $\mathrm{D}_i = \partial_{x^i} + u^\alpha_{J,i}\partial_{u^\alpha_J}$, where $u^\alpha_J = \partial^{|J|}u^\alpha/\partial(x^1)^{j_1}\cdots\partial(x^p)^{j_p}$, $u^\alpha_{J,i} = \partial u^\alpha_J/\partial x^i$, $\alpha = 1, \dots, q$, $J = (j_1, \dots, j_p)$ is a multi-index, $j_i \in \mathbb{N}_0$ and $|J| = j_1 + \dots + j_p$. We use the summation convection for repeated indices.

Example 2.20. The Burgers equation $u_t + uu_x + u_{xx} = 0$ can be brought into the form of a conservation law (2.2.6), namely

$$\mathrm{D}_t(u) + \mathrm{D}_x\left(\frac{1}{2}u^2 + u_x\right) = 0,$$

where the operators of total differentiation with respect to t and x are

$$\mathrm{D}_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots,$$
$$\mathrm{D}_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots.$$

It is clear from Definition 2.19 and illustrated in Example 2.20 that the conserved vector associated with a conservation law can depend not only on the independent variables x and the unknown functions u but also on the derivatives of u with respect to x up to *any* order m. The missing bound on m is the reason why exhaustively describing the space of conservation laws of a system of differential equations is in general a complicated problem. In fact, for most equations of hydrodynamics, the conservation laws known are typically of low order.

The following two definitions will prove important in Section 2.4 for the construction of conservative parameterization schemes using the direct classification approach.

Definition 2.21. A conserved vector Φ is called *trivial* if it is represented as the sum $\Phi = \hat{\Phi} + \check{\Phi}$, where the components of the p-tuple of differential functions $\hat{\Phi}$ vanish on the solutions of $\mathcal{L}$ and $\check{\Phi}$ is a *null divergence*, i.e. $\mathrm{D}_i\check{\Phi}^i = 0$ holds identically.

Definition 2.22. Two conserved vectors Φ and Φ' are called *equivalent* if their difference $\Phi - \Phi'$ is a trivial conserved vector.

Trivial conserved vectors satisfy the divergence condition embodied in Definition (2.19) in a trivial manner and thus do not contain any essential physical information. This is why it is important that the computation of conservation laws is carried out by taking into account the possibility of equivalence among conserved vectors.

Example 2.23. Consider again the Burgers equation in the form $u_t + uu_x + u_{xx} = 0$. The conserved vector $\hat{\Phi} = (0, u_t + uu_x + u_{xx})^{\mathrm{T}}$ is trivial as it clearly vanishes on solutions of the Burgers equation. Similarly, the conserved vector $\check{\Phi} = (u_x u_{xx}, -u_x u_{xt})^{\mathrm{T}}$ is trivial, as $\mathrm{D}_t(u_x u_{xx}) + \mathrm{D}_x(-u_x u_{xt}) = 0$ vanishes identically, independent of solutions of the Burgers equation.

There are different methods available for finding conservation laws, see e.g. [158] for an accessible review. Here we focus on the *multiplier approach* to conservation laws [26]. This approach uses a reformulation of the definition of a conservation law (2.2.6) in the form

$$\mathrm{D}_i \Phi^i(x, u^{(m)}) = \Lambda^l(x, u^{(s)}) \Delta_l(x, u^{(n)}), \tag{2.2.7}$$

where the tuple of differential functions $\Lambda = (\Lambda^l(x, u^{(s)}),\ l = 1, \dots, L)$ with some $s \in \mathbb{N}_0$ is called the *characteristic* of the conservation law with the conserved vector Φ, and components of Λ are called *conservation law multipliers*. In the case of solutions of the system of differential equations $\mathcal{L}$, the right-hand side of (2.2.7) vanishes and thus reduces to the original definition of a conservation law.

Example 2.24. We can bring the Burgers equation into the form (2.2.7) by noting that

$$\mathrm{D}_t(u) + \mathrm{D}_x\left(\frac{1}{2}u^2 + u_x\right) = 1 \cdot (u_t + uu_x + u_{xx}),$$

i.e. the multiplier associated with the conservative form of the Burgers equation is $\Lambda = 1$.

The use of the characteristic form (2.2.7) can aid in the computation of conservation laws. This is done using the *Euler operator* or *variational derivative*.

Definition 2.25. The *Euler operator* with respect to the dependent variable u^α is the differential operator given by

$$\mathsf{E}_{u^\alpha} = \partial_{u^\alpha} - \mathrm{D}_{i_1} \partial_{u^\alpha_{i_1}} + \mathrm{D}_{i_1} \mathrm{D}_{i_2} \partial_{u^\alpha_{i_1 i_2}} - \cdots = (-\mathrm{D})^J \partial_{u^\alpha_J}, \tag{2.2.8}$$

where $(-\mathrm{D})^J = (-\mathrm{D}_1)^{j_1} \dots (-\mathrm{D}_p)^{j_p}$.

The Euler operators have the property to annihilate any divergence expression $\mathrm{D}_i \Phi^i$. That is, applying them to the characteristic form of the conservation law (2.2.7) yields

$$\mathsf{E}_{u^\alpha}(\Lambda^l \Delta_l) \equiv 0, \quad \alpha = 1, \dots, q, \tag{2.2.9}$$

which leads to a system of determining equations for the conservation law multipliers $\Lambda = (\Lambda^l(x, u^{(s)}),\ l = 1, \dots, L)$. It is generally possible to split the system (2.2.9) with respect to derivatives that are not involved in Λ. The result of this splitting is an overdetermined system of linear partial differential equations. Solving this system leads to the conservation law multipliers for local conservation laws of system $\mathcal{L}$. From these multipliers one can then reconstruct the conserved vectors Φ using either integration by parts or a *homotopy formula* [3, 4, 26]. Once again, the construction is entirely algorithmic and implemented in various packages for computer algebra systems, such as, e.g., `GeM` for `Maple` [37].

Example 2.26. Instead of our running example, the Burgers equation (possessing only one conservation law, which is the equation itself in the conserved form), we consider the Korteweg–de Vries (KdV) equation,

$$u_t + uu_x + u_{xxx} = 0,$$

which models the propagation of waves on shallow-water surfaces. We aim to find conservation laws for this equation for multipliers depending only on t, x and u, i.e. $\Lambda = \Lambda(t, x, u)$, see [26] for more details. Thus, Eq. (2.2.7) reduces to

$$\mathrm{D}_t\rho + \mathrm{D}_x\Phi = \Lambda(t, x, u)(u_t + uu_x + u_{xxx}),$$

where ρ is called the *conserved density* and Φ is the *flux* of the conservation law. Applying the Euler operator $\mathsf{E}_u = \partial_u - \mathrm{D}_t\partial_{u_t} - \mathrm{D}_x\partial_{u_x} + \cdots$ to this equation leads to

$$\mathsf{E}_u(\Lambda(u_t + uu_x + u_{xxx})) = 0.$$

Expanding this equation we obtain

$$(\Lambda_t + u\Lambda_x + \Lambda_{xxx}) + 3\Lambda_{xxu}u_x + 3\Lambda_{xuu}u_x^2 +$$
$$\Lambda_{uuu}u_x^3 + 3\Lambda_{xu}u_{xx} + 3\Lambda_{uu}u_xu_{xx} = 0,$$

and as Λ only depends on t, x and u we can split this equation with respect to derivatives of u. This leads to the determining equations of conservation law multipliers, which are

$$\Lambda_t + u\Lambda_x + \Lambda_{xxx} = 0, \quad 3\Lambda_{xxu} = 0, \quad 3\Lambda_{xuu} = 0,$$
$$\Lambda_{uuu} = 0, \quad \Lambda_{xu} = 0, \quad 3\Lambda_{uu} = 0.$$

Solving this linear system, we obtain the parameterized family of multipliers

$$\Lambda = c_1 + c_2u + c_3(x - tu),$$

where c_1, c_2 and c_3 are arbitrary constants. The three independent conservation laws associated to this family are derived by considering the three possibilities $(c_1, c_2, c_3) = (1, 0, 0)$, $(c_1, c_2, c_3) = (0, 1, 0)$ and $(c_1, c_2, c_3) = (0, 0, 1)$. The multiplier form of conservation laws then reads, respectively:

$$\mathrm{D}_t\rho^1 + \mathrm{D}_x\Phi^1 = 1 \cdot (u_t + uu_x + u_{xxx}),$$
$$\mathrm{D}_t\rho^2 + \mathrm{D}_x\Phi^2 = u \cdot (u_t + uu_x + u_{xxx}),$$
$$\mathrm{D}_t\rho^3 + \mathrm{D}_x\Phi^3 = (x - ut) \cdot (u_t + uu_x + u_{xxx}).$$

From these equation, it is straightforward to recover the conserved densities and fluxes using integration by parts[1], yielding:

$$\mathrm{D}_t\rho^1 + \mathrm{D}_x\Phi^1 = \mathrm{D}_t u + \mathrm{D}_x\left(\frac{1}{2}u + u_{xx}\right),$$
$$\mathrm{D}_t\rho^2 + \mathrm{D}_x\Phi^2 = \mathrm{D}_t\left(\frac{1}{2}u^2\right) + \mathrm{D}_x\left(\frac{1}{3}u^3 + uu_{xx} - \frac{1}{2}u_x^2\right),$$
$$\mathrm{D}_t\rho^3 + \mathrm{D}_x\Phi^3 = \mathrm{D}_t\left(\frac{t}{2}u^2 - xu\right) + \mathrm{D}_x\left(u_x - \frac{t}{2}u_x^2 + (tu - x)u_{xx} - \frac{x}{2}u^2\right).$$

as three conservation laws of the KdV equation. Using some more elaborate machinery, it can be shown that the KdV equation has infinitely many conservation laws, see e.g. [101].

[1] Similar as in this case, the use of the homotopy formula as presented e.g. in [26, 101] can often be avoided using integration by parts to construct the conserved vector.

2.3 Invariant parameterization schemes

There are different methods for the construction of parameterization schemes with symmetry properties. All of them are based on an intimate relation between the problem of group classification and invariant parameterization. More specifically, in [123] we demonstrated that *any problem of finding invariant parameterization schemes is a group classification problem.* This statement provides a concise path to the construction of symmetry-preserving closure models.

We start with invariant *local* parameterization schemes. Finding *nonlocal* parameterization schemes that preserve symmetry properties of the initial model is a subject that is not well investigated so far. We will comment on this in the end of this contribution.

Given a system of differential equations $\mathcal{L}$: $\Delta_l(x, u^{(n)}) = 0$, $l = 1, \dots, L$, we start with the splitting of the dependent variables u into an averaged and a deviational part:

$$u = \bar{u} + u'.$$

The theory that we will outline below is independent of the form of averaging or filtering method that is used. Introducing this splitting into the system $\mathcal{L}$ and averaging the resulting expression for $\mathcal{L}$ leads to a system of the form

$$\tilde{\Delta}_l(x, \bar{u}^{(n)}, w) = 0, \quad l = 1, \dots, L, \tag{2.3.1}$$

where $\tilde{\Delta}_l$ are smooth functions of their arguments whose explicit form is determined by the original system of differential equations $\mathcal{L}$ and the averaging rule invoked. We collect all the averaged nonlinear combinations of terms that cannot be obtained by means of the $\bar{u}^{(n)}$ in the tuple $w = (w^1, \dots, w^k)$. Thus, the closure problem consists in finding good expressions for w in terms of $\bar{u}^{(n)}$.[2]

We close system (2.3.1) by establishing a functional relation between the unknown subgrid-scale terms w and the resolved grid-scale part $\bar{u}^{(n)}$ by setting

$$w^s = f^s(x, \bar{u}^{(r)}), \quad s = 1, \dots, k, \tag{2.3.2}$$

where $f = (f^1, \dots, f^k)$ are the parameterization functions that we need to determine. Introducing this expression in (2.3.1) we arrive at

$$\Delta_l^f(x, \bar{u}^{(n')}) := \tilde{\Delta}_l(x, \bar{u}^{(n)}, f(x, \bar{u}^{(r)})) = 0, \quad l = 1, \dots, L, \tag{2.3.3}$$

which now is a closed system of differential equations $\mathcal{L}_f$ that depends on the yet unspecified form of f. Here $n' = \max\{n, r\}$. In other words, it is a *class of differential equation.* As was said above, the problem of finding parameterization functions in (2.3.2) that lead to a closed system (2.3.3) preserving prescribed symmetry properties is thus solved as a group classification problem.

We look here into two principal ways of solving the group classification problem and hence the invariantization problem, which are *inverse* and *direct* group classification.

[2]We assume here that the local closure is of first order, i.e. that the unknown subgrid-scale quantities w can be determined by $\bar{u}^{(n)}$ only. More realistically, one would use higher-order parameterizations to close system (2.3.1). As the description of such higher-order local closure schemes would clutter the presentation we refrain from using them and outline the theory for first-order closures only. We will comment on invariant higher-order schemes in the end of this chapter, see also Example 2.35.

2.3.1 Invariant parameterization using inverse group classification

Inverse group classification is done by first fixing a transformation group and then looking for those differential equations which admit the selected group as its symmetry group [112].

This approach to the classification problem is particularly useful as it allows one to start with the maximal Lie invariance group of the original, unaveraged system of differential equations and to *impose* it on the resulting averaged and closed system. In other words, the parameterized system will admit the same symmetries as the original model. This requirement that the closed model should admit the same symmetries as the original unaveraged model was first advocated in [98] for LES subgrid-scale closure models for the Navier–Stokes equations.

There might be physical problems for which it would be overly restrictive to require a closed system of differential equations to admit exactly the same symmetries as the original unaveraged model. After all, an averaged model only captures the grid-scale part of the solution of the original model, hence it might be natural that part of the geometry is lost by the averaged model. Mathematically speaking, the associated problem then is to find a parameterization scheme that leads to a closed system of differential equations admitting a *subgroup* of the maximal Lie invariance group of the original model.

To realize invariant parameterization schemes using inverse group classification, it is sufficient to determine the (differential) invariants of the subgroup that one aims to preserve in the closed model. *The replacement theorem discussed in Section 2.2 implies that if we compose the parameterization scheme out of these invariants, it will lead to a system of differential equation with the invariance requested.*

A natural criterion of selecting which symmetries should be preserved in a subgrid-scale closure model is given by the *initial-boundary value problem* at hand. When above we speak about symmetries of differential equations, no relations to the joint consideration with initial-boundary value problems are implied, i.e. we assume the absence of such restraining conditions. Indeed, the maximal Lie invariance group of a system of differential equations is reduced once boundaries are considered [25]. This is quite natural as a symmetry transformation then not only has to leave invariant the given system but also these supplementary conditions. On the other hand, in a particular physical model the boundary conditions are usually an integral part of the problem to be studied. Hence, when constructing a parameterization scheme for such a model, it is natural to at most preserve those symmetries of the system of differential equations that are also compatible with the initial-boundary value problem to be studied.

Example 2.27. In order to illustrate the inverse group classification procedure, we construct invariant parameterization schemes for the Burgers equation. For the sake of simplicity, we invoke a Reynolds time filtering operation to get

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{u}_{xx} = -\frac{1}{2}(\overline{u'u'})_x =: w \tag{2.3.4}$$

as the corresponding averaged but unclosed model. The momentum flux term on the right-hand side is the subgrid-scale quantity that we want to close in a symmetry-preserving fashion.

The one-parameter transformations from the maximal Lie invariance group of the Burgers equation were given in Eq. (2.2.3). Let us now discuss which of those transformation should be preserved when closing Eq. (2.3.4). In doing this, we first have to fix the initial-boundary value problem that we are considering. Here we assume periodic boundary conditions in space direction, i.e. $u(t,L) = u(t,0)$ for a channel of length L, and an initial value problem in time $u(0,x) = u_0(x)$.

As it stands, it then seems that this initial-boundary value problem does not preserve the time translation symmetry given in Eq. (2.2.3), because fixing the initial time obviously does not allow time shifts any more. On the other hand, from the physical point of view, shifting time just changes the absolute initial time after which one is interested in the evolution of the system. In other words, shifting time maps the original initial value problem to *another* initial value problem of *the same kind*. That is, time translations do not alter the principal nature of the class of problems we are considering. Time translations therefore act as equivalence transformations in the *class of all initial value problems* for the Burgers equation.

The above discussion is crucial in that it enables one to relax the rather rigid condition of point transformations acting as symmetry transformations that leave invariant *one* fixed problem to equivalence transformation of a *class* of similar problems. Therefore, as long as a transformation maps a given problem to another problem from a joint class, it should be preserved by the parameterization scheme.

It is straightforward to check that the first four transformations from Eq. (2.2.3) map the class of initial-boundary value problems for the Burgers equation with periodic boundary conditions onto the same class. Only the last transformation does not satisfy this requirement, as it reverses the time direction and hence does not preserve the condition $t \geqslant t_0$, which has to be ruled out from physical grounds.

In Example 2.14 we determined the moving frame for the subgroup G^1 given by all transformations from the maximal Lie invariance group except for inversions, which we omit due to the above reason. We therefore can use the moving frame associated with the subgroup G^1 to find the required differential invariants out of which we construct the invariant parameterizations in order to close Eq. (2.3.4).

So as to determine the maximum order of differential invariants required, we have to select the general form of the parameterization ansatz (2.3.2) first. To keep things simple, we aim for parameterizations of the form

$$w = f(t, x, u, u_x, u_{xx})$$

subsequently. That is, we only need the differential invariants of order not higher than two.

We can obtain all required differential invariants by invariantizing the arguments of the above function f. That is, we compute $\iota(t)$, $\iota(x)$, $\iota(u)$, $\iota(u_x)$, $\iota(u_{xx})$. In fact, we already computed the required expressions in Examples 2.14 and 2.15. The invariantization of $\iota(t)$, $\iota(x)$, $\iota(u)$, $\iota(u_x)$ just reproduces the normalization conditions, i.e. $\iota(t) = 0$, $\iota(x) = 0$, $\iota(u) = 0$, $\iota(u_x) = 1$. This is always the case when invariantizing the normalization conditions used to construct a moving frame, which is why these invariants obtain a special name: *phantom invariants*. The only non-phantom invariant is $\iota(u_{xx})$ and it was computed in Example 2.15. Expressed in terms of the mean variables, it reads

$$\iota(\bar{u}_{xx}) = \frac{\bar{u}_{xx}}{\sqrt{\bar{u}_x^3}}.$$

Before we can make use of this differential invariant, it is important to note that the left hand side of the averaged Burgers equation (2.3.4) is not yet expressed in invariant form. This invariant form is obtained by also invariantizing the left hand side using the moving frame associated with G^1, which was done in Example 2.16. Thus, an invariant closure model for (2.3.4) is given as

$$\frac{\bar{u}_t + \bar{u}\bar{u}_x + \bar{u}_{xx}}{\sqrt{\bar{u}_x^3}} = f\left(\frac{\bar{u}_{xx}}{\sqrt{\bar{u}_x^3}}\right).$$

A very simple example for an invariant parameterization is to choose $f(z) = kz$, leading to

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{u}_{xx} = k\bar{u}_{xx}.$$

Physically, this boils down to adding a turbulent diffusion term to the viscous diffusion term already present (with viscosity coefficient being equal to -1) and thus a standard down-gradient parameterization for the momentum flux with constant turbulent viscosity. Of course, there is an infinite number of other possible parameterization schemes that is invariant under the subgroup G^1.

In the previous example we have discussed the typical steps required for the construction of invariant parameterization schemes. For the sake of convenience, we summarize these steps here once again:

Algorithm 2.28. Invariant parameterization via inverse classification.

1. Compute the maximal Lie invariance group of the system of differential equations of interest.
2. Choose an averaging rule and average the initial system of differential equations.
3. Define the functional form of the parameterization scheme (2.3.2) to be invoked.
4. Determine which symmetries of the initial model should be inherited by the averaged closed model.
5. Compute the moving frame associated with the symmetry subgroup selected in the previous step.
6. Compute a suitable set of differential invariants using this moving frame and assemble the required parameterization out of these invariants.

Concerning Step 1 of the above algorithm, we recall once again that for most models of physical interest, the computation of Lie symmetries is already accomplished [2, 5, 60]. More realistic examples will be considered in Section 2.5.

There is another way moving frames can be used to construct invariant parameterization schemes. The original idea was presented in [16] and it consists in *invariantizing* existing parameterization schemes. That is, rather than starting from scratch with the construction of a symmetry-preserving closure model, one takes an existing parameterization that violates certain symmetries and makes it invariant by applying the appropriate moving frame to it. We illustrate the construction with an example.

Example 2.29. Again, consider the famous KdV equation

$$u_t + uu_x + u_{xxx} = 0.$$

The maximal Lie invariance group G of this equation is four-dimensional and is generated by the one-parameter Lie symmetry transformations that map (t, x, u) to

$$\begin{aligned} &(t+\varepsilon_1, x, u), \quad (t, x+\varepsilon_2, u), \quad (t, x+\varepsilon_3 t, u+\varepsilon_3), \\ &(e^{3\varepsilon_4}t, e^{\varepsilon_4}x, e^{-2\varepsilon_4}u), \end{aligned} \tag{2.3.5}$$

where ε_1, ε_2, ε_3 and ε_4 are arbitrary constants. Let us now see what happens when we average the KdV equation employing the Reynolds rule and close the subgrid-scale term $\overline{u'u'}$ with a simple down-gradient ansatz using a constant diffusion parameter κ. This leads to the closed KdV equation

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{u}_{xxx} = \kappa \bar{u}_{xx}. \tag{2.3.6}$$

It is not hard to see that this equation is invariant under the first three point symmetry transformations of the KdV equation listed in (2.3.5) but is no longer scale invariant (the term u_{xx} obviously does not scale like u_{xxx}). We can, however, make the closure (2.3.6) invariant under the same symmetry group G as admitted by the original KdV equation by constructing a moving frame for the group G and invariantizing Eq. (2.3.6) subsequently. Using the recipe for the construction of a moving frame, we first determine the most general transformation from the maximal Lie invariance group G. From the one-parameter transformations (2.3.5) it follows that this most general transformation is

$$(\tilde{t}, \tilde{x}, \tilde{u}) = (e^{3\varepsilon_4}(t + \varepsilon_1), e^{\varepsilon_4}(x + \varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_3 t), e^{-2\varepsilon_4}(u + \varepsilon_3)). \tag{2.3.7}$$

Then we set up a system of normalization conditions that subsequently allows us to solve for the group parameters. One possibility for the normalization conditions is

$$\tilde{t} = 0, \quad \tilde{x} = 0, \quad \tilde{u} = 0, \quad \tilde{u}_{\tilde{x}} = 1,$$

where using the chain rule one can find from (2.3.7) that $\tilde{u}_{\tilde{x}} = e^{-3\varepsilon_4}u$. Solving these normalization conditions for the group parameters ε_1, ε_2, ε_3 and ε_4, we find

$$\varepsilon_1 = -t, \quad \varepsilon_2 = -x, \quad \varepsilon_3 = -u, \quad \varepsilon_4 = \frac{1}{3}\ln u_x,$$

to be a moving frame. See e.g. [42] for further details. Now applying the associated invariantization map (replacing u with $\bar{u}$) to model (2.3.6) leads to

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{u}_{xxx} = \kappa \sqrt[3]{\bar{u}_x}\,\bar{u}_{xx},$$

which now again admits the same maximal Lie invariance group G as the original KdV equation. Stated in another way, in order to preserve scale invariance using a down-gradient parametrization, a variable diffusion parameter has to be used. This is quite typical when invariantizing such parameterization schemes, see also [16].

2.3.2 Invariant parameterization using direct group classification

Direct group classification is done by starting with a class of differential equations and subsequently aiming to find those specific equations from this class that admit more symmetries than those admitted by all equations from the class [112].

In order to enable a systematic approach to this comprehensive task, it is important to point out that the classification is only carried out *up to point equivalence*. This means that if two systems of differential equations can be related to each other by a point transformation, then it is not necessary to include both systems in the final classification list as one can be obtained from the other. A point transformation that maps any system from a given class to another system of the same class is called an *equivalence transformation* of this class. Finding the equivalence transformations of a given class is therefore an important first step in the direct group classification procedure.

Example 2.30. Consider generalized Burgers equations of the form

$$u_t + uu_x + f(t,x)u_{xx} = 0. \tag{2.3.8}$$

This is a class of differential equations with a single arbitrary element, which is a function of both t and x. Meteorologically, the arbitrary element f can be regarded as a variable diffusion parameter in a simple-down gradient parameterization of a Reynolds averaged advection term. This is why it is important to impose the additional constraint $f \neq 0$ for class (2.3.8) as the diffusion cannot vanish for physical reasons. From the mathematical point of view, it is obvious that the inviscid Burgers equation, for which $f = 0$, is essentially different in structure from the other equations of the form (2.3.8). In particular, this is the only equation of order one among equations of the form (2.3.8). As we consider the usual group classification problem for class (2.3.8) without an explicit connection to parameterization, we omit the bar over u.

Consider the two equations

$$u_t + uu_x + xu_{xx} = 0, \quad u_t + uu_x + (x+c)u_{xx} = 0,$$

which are both elements of the above class. The point transformation $(t,x,u) \mapsto (t,x+c,u)$ maps the first equation to the second equation and hence is an example for an *equivalence transformation* in the above class of generalized Burgers equations. Note that this transformation is no symmetry transformation of either equation as it maps neither of the two equations back to itself.

Systematically solving the group classification problem for the class of equations of form (2.3.8) is not a simple task and involves considerably more machinery than was introduced in this chapter. We thus content ourselves with giving some of the results of the classification of class (2.3.8) and indicate the specific features of direct group classification when applied to the invariant parameterization problem.

Example 2.31. Table 2.1 contains some equations from class (2.3.8) that admit particular symmetry properties that have been obtained using the methods of direct group classification.

Table 2.1: Some cases of Lie symmetry extensions for the class (2.3.8)

Case	Infinitesimal generators	$f(t,x)$	ω
(i)	No symmetries	$f(t,x)$	
(ii)	∂_t	$h(\omega)$	$\omega = x$
(iii)	$\partial_x, \quad t\partial_x + \partial_u$	$h(\omega)$	$\omega = t$
(iv)	$t\partial_t + (t+x)\partial_x + \partial_u$	$th(\omega)$	$\omega = x/t - \ln t$
(v)	see Eq. (2.2.2)	1	

Let us now interpret the results given in Table 2.1 in the light of the parameterization problem. In fact, all of the cases listed in this table are representative for typical results that are obtained when using direct group classification to determine invariant parameterization schemes.

Case (i) represents the generic form of equations from the class. In the group classification literature, the symmetries that are admitted by any equation from the class is referred to as the *kernel* of maximal Lie invariance groups. In the present case, if f is completely arbitrary, then

Eq. (2.3.8) admits no continuous symmetry transformation. In a sense, this case represents the conventional approach to the parameterization problem, that is, no particular care is taken of whether or not the resulting parameterization admits symmetries; in other words, possible invariance characteristics of the parameterization scheme are not determined constructively before the fact.

Cases (ii) and (iii) represent physical forms of invariant parameterization schemes. In the Case (ii), equations from the class (2.3.8) with $f = h(x)$, $u_t + uu_x + h(x)u_{xx} = 0$, are invariant under time translations. Likewise, if $f = h(t)$ as in Case (iii), then the equations of the form $u_t + uu_x + h(t)u_{xx} = 0$ are invariant under spatial translations and Galilean transformations, irrespectively of the precise form of h. Both parameterizations make sense physically for certain values of h: in the first case the diffusion parameter is spatially dependent whereas in the second case it depends on time; for both cases, associated physical conditions could be formulated. The two cases are also typically in that a resulting closure scheme may still constitute a (narrower) class of differential equations. In the present case, there would be still the requirement to determine the form of h precisely. This could be done by incorporating other desirable properties into the parameterization scheme. Indeed, the situation where preserving symmetries in a parameterization only restricts the initial form of the closure scheme (here, the function $f(t, x)$) rather than giving one particular closure is very typical, see e.g. [123].

Case (iv) unfortunately represents a typical case as well, namely that of an unphysical parameterization. Although the requirement of preserving symmetries in a parameterization schemes is well grounded physically, of course not all combinations of symmetry transformations lead to a physical model. In Case (iv), the resulting 'parameterization' is invariant under a linear combination of a scaling and the Galilean transformation but the indicated form of the function f in this case does not give a physical ansatz for the diffusion parameter. As direct group classification always produces a list of equations that admit different symmetry properties, there is a high chance that several equations from this list are not physical. This is one of the disadvantages of the direct classification approach to the invariant parameterization problem.

It should not surprise that if $f = 1$ (Case (v)), we are led back to the original Burgers equation. Thus, the resulting equation from class (2.3.8) has the symmetries given in Example 2.4.

We also point out that the classification results in Table 2.1 are optimal in the sense that there is no point transformation that maps one particular equation to another equation from the table. That is, the classification is carried out *up to point equivalence*. From the physical point of view, the equations listed in Table 2.1 should therefore not be regarded as single parameterizations but rather as members of inequivalent classes of parameterizations. To give an example, we have already seen in Example 2.30 that the transformation $(t, x, u) \mapsto (t, x + c, u)$ maps one equation from the class (2.3.8) to another equation from the same class. One can then use the equivalence transformations from a class to map a given parameterization scheme to a new one. Although this new parameterization scheme will be mathematically equivalent to the original one (as it was obtained from applying a point transformation to the initial scheme) it might still be interesting from the physical point of view. For example, shifting x in Case (ii) leads to the equation $u_t + uu_x + h(x + c)u_{xx} = 0$. Shifting x can be helpful if the model has to be shifted with respect to the origin. Applying equivalence transformation to a given parameterization scheme can thus be a powerful way of further customizing the closure model to given physical restrictions.

The complete classification of this class of generalized Burgers equations is given in [117]. Other examples on group classification of various classes of differential equations might be found in [43, 63, 79, 91, 125, 151]. First examples of the use of group classification in the study of physical parameterization scheme are given in [16, 123].

2.4 Conservative parameterization schemes

In this section we introduce a few methods for the construction of parameterization schemes that lead to closed equations possessing nontrivial conservation laws. As was said above, conservation laws are important features of physical models and they play a distinct role in hydrodynamics and geophysical fluid dynamics. However, care has to be taken when conservation laws should be preserved in a subgrid-scale model.

As with the problem of invariant parameterization, conservative parameterization schemes can be found either using *inverse* or *direct* classification techniques. That is, comparable to the group classification problem for classes of differential equations, a classification problem for conservation laws should be solved. Both approaches are summarized below. For further details, see [14] and [24].

We start here with the description of what the inverse and the direct approach have in common. In both approaches, it is necessary to fix in the beginning the general functional form for the parameterization of the subgrid-scale terms. This is done in a *local fashion*, meaning that the unresolved terms at a point are represented by a function of the independent variables, the resolved unknown functions $\bar u$ and the derivatives of $\bar u$ up to a certain fixed order r at the same point only. In fact, the procedure is the same as outlined above for invariant parameterization schemes, which we repeat here for the sake of convenience. Starting with the averaged unclosed system

$$\tilde\Delta_l(x,\bar u^{(n)},w)=0,\quad l=1,\dots,L,$$

where w as before denotes the k-tuple of unresolved terms, and fixing the parameterization ansatz

$$w^s=f^s(x,\bar u^{(r)}),\quad s=1,\dots,k,$$

we arrive at the averaged and closed system

$$\Delta_l^f(x,\bar u^{(n')}):=\tilde\Delta_l(x,\bar u^{(n)},f(x,\bar u^{(r)}))=0,\quad l=1,\dots,L,\tag{2.4.1}$$

which is of course the same as system (2.3.3). Here $n'=\max\{n,r\}$. The task is now to specify f in such a manner that system (2.4.1) admits different nontrivial conservation laws.

Both the inverse and the direct classification approach to conservative parameterization schemes can be realized using the characteristic form of conservation laws. That is, if system (2.4.1) is to possess certain nontrivial conservation laws, then there must exist characteristics Λ and conserved vectors Φ, such that

$$\Lambda^l(x,\bar u^{(s)})\Delta_l^f(x,\bar u^{(n')})=\mathrm D_i\Phi^i(x,\bar u^{(m)}).\tag{2.4.2}$$

Applying the Euler operators $\mathsf E_{u^\alpha}$ to this equation leads to the system

$$\mathsf E_{u^\alpha}(\Lambda^l\Delta_l^f)=0,\quad i=1,\dots,q,\tag{2.4.3}$$

which is the starting point for both the inverse and the direct approach. The main difference in the two methods is whether one specifies the multipliers Λ initially (inverse approach) or not (direct approach).

2.4.1 Conservative parameterization via inverse classification of conservation laws

Similar as in solving the invariant parameterization problem using inverse group classification, in this approach one specifies the conservation laws from the initial model that the closed model should admit and constructs the closure scheme accordingly.

Specifically, one first determines the conservation laws that are admitted by the original unaveraged system of governing equations for which a parameterization has to be constructed. This is conveniently done using the multiplier approach and boils down to solving a linear system of partial differential equations. After this, one uses physical reasoning to determine which of the conservation laws of the original model should also be preserved in the parameterized model.

System (2.4.3) is then solved by treating Λ as the *known* functions. Indeed, while computation of the characteristics for the original, unaveraged system $\mathcal{L}$ yielded $\Lambda = \Lambda(x, u^{(s)})$, replacing $u^{(s)}$ with the mean part $\bar{u}^{(s)}$, we obtain the characteristics $\Lambda(x, \bar{u}^{(s)})$, that correspond to the same conservation law but now expressed for the mean part of u only. As $\Lambda(x, \bar{u}^{(s)})$ is then known in system (2.4.3), solving this system allows one to find the associated forms of the parameterization functions f.

Alternatively, one could determine the parameterization functions f directly from Eq. (2.4.2) using integration by parts. This is often a convenient alternative if the characteristics of conservation laws include arbitrary functions. An example for this method will be presented in Section 2.5.

2.4.2 Conservative parameterization via direct classification of conservation laws

A second possibility for constructing conservative parameterization schemes is by treating system (2.4.3) as a system for both Λ and f. In order to do this efficiently a classification problem for conservation laws has to be solved.

We recall here that point transformations mapping one equation from a class of differential equations to another equation from the same class are called *equivalence transformation*. The group formed by these equivalence transformations is denoted with $G^{\sim}$. The direct problem to conservative parameterization essentially uses the following definition:

Definition 2.32. Let $\mathcal{L}|_{\mathcal{S}}$ denote a class of differential equations and $\mathcal{L}_\theta$ and $\mathcal{L}_{\theta'}$ be two elements (i.e. equations) from this class. Let $\mathcal{L}_\theta$ and $\mathcal{L}_{\theta'}$ admit conservation laws with conserved vectors Φ and Φ', respectively. The pairs $(\mathcal{L}_\theta, \Phi)$ and $(\mathcal{L}_{\theta'}, \Phi')$ are called $G^{\sim}$*-equivalent* if there exists a point transformation $\Gamma \in G^{\sim}$ which transforms the system $\mathcal{L}_\theta$ to the system $\mathcal{L}_{\theta'}$ and transforms the conserved vector Φ to the conserved vector $\tilde{\Phi}$ such that $\tilde{\Phi}$ and Φ' are equivalent as specified in Definition 2.22.

The direct classification approach to conservative parameterization schemes proceeds by first determining those conservation laws that are admitted by all equations from the initial class (2.4.1). Then, those particular equations from the class (corresponding to particular forms of the parameterization functions f) are found for which more conservation laws are admitted than in the case of general f.

In order to make this approach computationally feasible, the classification is carried out only up to $G^{\sim}$-equivalence. That is, if a point transformation mapping one equation $\mathcal{L}_f$ from class (2.3.3) corresponding to one parameterization scheme to another equation $\mathcal{L}_{f'}$ from the same class corresponding to another parameterization scheme *and* the associated transformed conserved vectors of $\mathcal{L}_f$ and the conserved vectors of $\mathcal{L}_{f'}$ are equivalent, then $\mathcal{L}_f$ is essentially the same closed model as $\mathcal{L}_{f'}$. In other words, $\mathcal{L}_f$ and $\mathcal{L}_{f'}$ represent two different forms of a closed model admitting the same physical conservation laws rather than two different models. Taking into account $G^{\sim}$-equivalence is hence a crucial ingredient to optimize the computations of conservative parameterization schemes.

In practice, the conservative parameterization problem in the framework of the direct approach is solved by solving system (2.4.3) for both Λ and f upon splitting into various subcases corresponding to different (inequivalent) forms of f leading to equations from the class (2.4.1) that admit nontrivial characteristics of conservation laws Λ. See [14] for an example for this direct classification procedure.

It is important to stress that it is often the case that the classification problem (2.4.3) cannot be solved completely. The situation is again comparable to the usual group classification problem as arising in invariant parameterization. If the class is chosen to be very wide (i.e. the parameterization functions depend on several arguments), solving system (2.4.3) exhaustively in order to find all inequivalent conservatively parameterized models can be computationally impossible. Rather than attempting to find all inequivalent models it is then advised to concentrate on finding those that appear physically relevant.

The result of the direct classification approach to conservative parameterization schemes is then a list of inequivalent closed models that possess different conservation laws. These different conservative closed models can then be tested numerically to assess which of them describes an unresolved process is the most optimal way.

2.4.3 Conservative and invariant parameterization schemes

We have seen in Section 2.3 that the construction of invariant parameterization schemes leads in general not to a single parameterization but to a class of closure models that has to be narrowed down further (see also the examples below in Section 2.5 and the discussion in the final Section 2.6). Similarly, the closed systems of differential equations found using the methods of conservative parameterization are generally also classes rather than single equations.

It is therefore possible to combine the methods for finding invariant parameterization schemes with the techniques for constructing conservative parameterization models. The resulting closed differential equations then admit predefined symmetries and conservation laws, which are generally inherited from the original system of governing equations.

The construction of such *conservative invariant parameterization* schemes is desirable for several reasons. First of all, as was said above it restricts the freedom which is generally typical for both invariant and conservative parameterizations. While it is possible to narrow down a class of either invariant or conservative parameterization schemes using physical reasoning, it is helpful to have this initial class as specific as possible before constructing a particular parameterization scheme to be used operationally. On the other hand, as was advocated throughout this chapter, both symmetries and conservation laws are linked to the physics of a process that is described using differential equations and hence should be preserved even if it is not possible to explicitly resolve that process. It is therefore quite natural to construct parameterization schemes that share both some of the symmetries *and* conservation laws of the original system of governing equations.

A powerful technique for constructing invariant and conservative parameterization schemes rests on the famous *Noether theorem*. Noether's theorem states that to each symmetry of a

Lagrangian there corresponds a conservation law of the associated Euler–Lagrange equations, see e.g. [26, 101]. That is, if one preserves the Lagrangian structure in a parameterized model and there are symmetries associated with this Lagrangian then the parameterized model will automatically be conservative as well. See again [14] for a simple example.

The problem with this approach is that most models of fluid mechanics expressed in *Eulerian variables* are not Lagrangian and hence Noether's theorem is not applicable. For problems that are not Lagrangian, it is usually the best to directly combine the methods for conservative and invariant parameterization (either using direct or inverse classification) to obtain invariant conservative schemes.

Suppose that the system (2.4.1) has been parameterized in an invariant way by expressing the parameterization functions f using differential invariants of the symmetry group G associated with the original system $\Delta_l(x, u^{(n)}) = 0$, $l = 1, \ldots, L$. Let these differential invariants be denoted by $I^1, \ldots, I^N$. Thus, $f = f(I^1, \ldots, I^N)$, see Example 2.27. We then require that Λ, an L-tuple of differential functions of $\bar{u}$, to be the characteristic of a local conservation law of the system (2.4.1) for certain values of f. That is, $\Lambda^l \Delta_l^f$ is a total divergence for appropriately chosen f. Again using the property of the Euler operators to annihilate any total divergence, we have that

$$\mathsf{E}_{u^a}(\Lambda^l \Delta_l^f) = 0.$$

Splitting this system with respect to derivatives of u whenever possible, one obtains the determining equations for the parameterization functions f, which should be solved so as to obtain those specific forms for f (as functions of the differential invariants) that admit Λ as a conservation law multiplier. The resulting parameterization scheme is then both invariant and conservative.

Example 2.33. In [16] we gave an example for an invariant and conservative parameterization scheme for the barotropic vorticity equation on the beta-plane. More precisely, a closure for the divergence of the eddy vorticity flux of the form

$$\nabla \cdot (\overline{\mathbf{v}'\zeta'}) = \nu \nabla^2 \frac{\nabla^2 \bar{\zeta}^7}{\bar{\zeta}} = 7\nu \nabla^2 (\bar{\zeta}^5 \nabla^2 \bar{\zeta} + 6\bar{\zeta}^4 (\nabla \bar{\zeta})^2),$$

is invariant under the entire maximal Lie invariance group of the vorticity equation on the beta-plane and additionally conserves generalized circulation, momentum in x-direction and energy (see Example 2.36 for the mathematical expression of these conservation laws). This example also demonstrates that the requirement of preserving both symmetries and conservation laws can lead to quite specific closure models. If a conservative process is know to be invariant under a specific transformation group then the introduced methods of invariant and conservative parameterization can be an efficient way of constructing a consistent closure for this process.

2.5 Examples

We give three examples for the use of the methods introduced above in the study of physical parameterization schemes. The first example is devoted to the study of a simple boundary layer parameterization as presented in [144], for which we compute Lie symmetries. The second example is a higher-order parameterization for geostrophic eddies in the ocean. The third example is the barotropic vorticity equation for which we construct conservative parameterizations.

Example 2.34. In the classical textbook by R. Stull [144] simple first-order closure schemes for the Reynolds averaged governing equations of a horizontally homogeneous, dry boundary layer with no subsidence were considered. Specifically, the unclosed model reads

$$\begin{aligned} \bar{u}_t &= f(\bar{v} - v_{\rm g}) - (\overline{u'w'})_z, \\ \bar{v}_t &= -f(\bar{u} - u_{\rm g}) - (\overline{v'w'})_z, \\ \bar{\theta}_t &= -(\overline{w'\theta'})_z, \end{aligned} \tag{2.5.1a}$$

where $\mathbf{v} = (u, v, w)$ is the wind vector, which is split as $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$, and $\bar{w} = 0$, θ is the potential temperature split as $\theta = \bar{\theta} + \theta'$, and f is the Coriolis parameter. The geostrophic wind vector $\mathbf{v}_g = (u_{\rm g}, v_{\rm g}, 0)$ will be neglected since its components $u_{\rm g}$ and $v_{\rm g}$ can be set to zero by the obvious shift of the averaged horizontal wind components $\bar{u} - u_{\rm g} \to \bar{u}$, $\bar{v} - v_{\rm g} \to \bar{v}$. The dependent variables, $\bar{u}$, $\bar{v}$ and $\bar{\theta}$, are functions of t and z only. The closure model proposed is a simple down-gradient ansatz of the form

$$\overline{w'\gamma'} = -K_\gamma \frac{\partial \bar{\gamma}}{\partial z}, \tag{2.5.1b}$$

with K_γ being the respective eddy viscosity parameters. It was argued that in statically neutral conditions, the various parameters K_γ are proportional, which is the case considered here, i.e. $K := K_u = K_v = cK_\theta$, for $c = \text{const}$. As a result, setting $u_{\rm g} = v_{\rm g} = 0$, the general form of the closure model is

$$\bar{u}_t = f\bar{v} - K\bar{u}_{zz}, \quad \bar{v}_t = -f\bar{u} - K\bar{v}_{zz}, \quad \bar{\theta}_t = -cK\bar{\theta}_{zz}, \tag{2.5.2}$$

where the coefficient $K = K(z, \bar{\theta}, \bar{u}_z, \bar{v}_z, \bar{\theta}_z)$ is still an arbitrary function of its arguments that should be specified in order to complete the parametrization procedure. In [144] (p. 209, Table 6-4), examples for parameterizations of the eddy viscosity parameter were proposed. We now investigate the symmetry properties of the resulting closed model that were derived from the model (2.5.1) upon using different choices for K reported in Table 6-4.

$\boldsymbol{K} =$ **const.** Before computing Lie symmetries of the system (2.5.2) in the case of a constant eddy diffusivity, we can set $f = 0$ by the use of the point transformation[3]

$$\bar{u}\cos(ft) - \bar{v}\sin(ft) \to \bar{u}, \quad \bar{u}\sin(ft) + \bar{v}\cos(ft) \to \bar{v}, \tag{2.5.3}$$

which transforms the above system to the system of three decoupled linear heat equations

$$\bar{u}_t = -K\bar{u}_{zz}, \quad \bar{v}_t = -K\bar{v}_{zz}, \quad \bar{\theta}_t = -cK\bar{\theta}_{zz}. \tag{2.5.4}$$

The linear heat equation is one of the most studied examples in the group analysis of differential equations. The symmetries of system (2.5.4) are thus readily inferred. They are generated by the vector fields

$$\begin{aligned} &\partial_t, \quad \partial_z, \quad 2t\partial_t + z\partial_z, \quad \bar{u}\partial_{\bar{u}}, \quad \bar{v}\partial_{\bar{v}}, \quad \bar{u}\partial_{\bar{v}}, \quad \bar{v}\partial_{\bar{u}}, \quad \bar{\theta}\partial_{\bar{\theta}}, \\ &2Kt\partial_z + z\left(\bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}} + c^{-1}\bar{\theta}\partial_{\bar{\theta}}\right), \\ &4Kt^2\partial_t + 4Ktz\partial_z + (z^2 - 2Kt)(\bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}) + (c^{-1}z^2 - 2Kt)\bar{\theta}\partial_{\bar{\theta}}, \end{aligned}$$

[3] A similar point transformation was used in [35] to set $f = 0$ in the shallow-water equations on the f-plane. More generally, such point transformations can also be found using symmetries which again indicates the important role that such transformations play for the study of differential equations and their applications. See e.g. [26] for further details on how to use symmetries to construct mappings that relate differential equations.

$$U(t,z)\partial_{\bar u},\quad V(t,z)\partial_{\bar v},\quad \Theta(t,z)\partial_{\bar\theta},$$

where U, V and Θ run through the solution sets of the first, second and third equation in system (2.5.4), respectively. Physically, these vector field generate the one-parameter transformations of (i)–(ii) time and space translations, (iii) scalings of the independent variables, (iv)–(vii) general linear transformations in the space of $\bar u$ and $\bar v$, (vii) scalings in $\bar\theta$, (ix) Galilean boosts, (x) inversions and (xi)–(xiii) the linear superposition principle.

These computations show that the simple down-gradient ansatz with constant diffusion parameter admits a wide Lie invariance algebra. This is not surprising, as setting K to a constant leads to a linear system of differential equations, which always admits an infinite number of symmetries.

$\boldsymbol{K = k^2 z^2\sqrt{\bar u_z^2+\bar v_z^2}}$**.** Here k is the von Kármán constant. Using the point transformation (2.5.3), we can again set $f=0$. The symmetries admitted by the closed model are then generated by the vector fields

$$t\partial_t+z\partial_z,\quad z\partial_z+\bar u\partial_{\bar u}+\bar v\partial_{\bar v},\quad \bar\theta\partial_{\bar\theta},\quad \partial_t,\quad \partial_{\bar u},\quad \partial_{\bar v},\quad \partial_{\bar\theta},\quad \bar u\partial_{\bar v}-\bar v\partial_{\bar u}.$$

The associated one-parameter Lie symmetry transformations are: (i)–(iii) scalings, (iv)–(vii) shifts and (viii) rotations.

$\boldsymbol{K = l^2\bar u_z^2,\ l = k(z+z_0)/(1+k(z+z_0)/\Lambda)}$**.** In this parameterization, k again denotes the von Kármán constant and Λ is a length scale. The infinitesimal generators of one-parameter Lie symmetry transformations for the closed model employing this parameterization are

$$2t\partial_t-\bar u\partial_{\bar u},\quad \bar v\partial_{\bar v},\quad \bar u\partial_{\bar v},\quad \bar\theta\partial_{\bar\theta},\quad \partial_t,\quad \partial_{\bar u},\quad \partial_{\bar v},\quad \partial_{\bar\theta}.$$

The corresponding finite symmetry transformations are (i)–(iii) scalings, (iv) the modification of v with adding a summand proportional to u and (v)–(viii) shifts.

In summing up, we should like to stress that in order to bring this problem into the proper form of a direct group classification problem, we would first have to define a class of differential equations with the arbitrary element being K regarded as a function of a suitable subset of the independent variables, the dependent variables as well as their derivatives. More precisely, to account for all possible forms given in Table 6-4 of [144], we would have to consider the class of equations with the arbitrary element $K = K(z,\bar\theta,\bar u_z,\bar v_z,\bar\theta_z)$. This would result in a very general class, the complete group classification of which is too cumbersome. We do not attempt to give a partial classification here, as the model is too idealized to be of practical use in the era of supercomputers. Still, this example should serve as an illustration how critical the choice of a parameterization scheme influences the symmetries admitted by the closed model.

Example 2.35. All invariant local parameterization schemes constructed in the literature so far were of first order. That is, for the parameterization of the unclosed terms only the resolved variables and their derivatives have been used. However, the construction of invariant parameterization schemes is not restricted to first order closure schemes as will be demonstrated in this example.

More specifically, we are interested in finding invariant parameterization schemes for geostrophic eddies in the ocean. The initial model consists of the incompressible Euler equations on the beta-plane (written in stream function form) and the energy equation, i.e.

$$\begin{aligned}&\zeta_t+\psi_x\zeta_y-\psi_y\zeta_x+\beta\psi_x=0,\qquad \text{or}\qquad \eta_t+\psi_x\eta_y-\psi_y\eta_x=0,\\ &E_t+\nabla\cdot(B\mathbf{v})=0,\end{aligned}\tag{2.5.5}$$

where $\eta = \zeta + \mathrm{f}_0 + \beta y = \psi_{xx} + \psi_{yy} + \mathrm{f}_0 + \beta y$ is the absolute vorticity, given as the sum of the relative vorticity ζ and the Coriolis parameter $f \approx \mathrm{f}_0 + \beta y$ using the β-plane approximation, $B = \phi + E$ is the Bernoulli function, given as the sum of the mass-specific potential energy and the mass-specific kinetic energy $E = \mathbf{v}^2/2 = (\nabla\psi)^2/2$. This model was recently considered in [90].

Note that the second equation of system (2.5.5) is not independent of the first equation as it is a conservation law of this equation. This observation is important as in constructing invariant parameterization schemes for the averaged (time-filtered) system associated with model (2.5.5) only the symmetries of the first equation in (2.5.5) are relevant.

This averaged model is given by

$$\begin{aligned} &\bar\eta_t + \bar\psi_x\bar\eta_y - \bar\psi_y\bar\eta_x = \nabla\cdot(\overline{\mathbf{v}'\eta'}),\\ &k_t - \mathbf{v}\cdot\mathbf{k}\times\overline{\eta'\mathbf{v}'} + \nabla\cdot\overline{B'\mathbf{v}'} = 0, \end{aligned} \tag{2.5.6}$$

where $k = \overline{\mathbf{v}'^2}/2 = \overline{(\nabla\psi')^2}/2$ is the turbulent kinetic energy and $\mathbf{k} = (0,0,1)^{\mathrm{T}}$ is the vertical unit vector.

System (2.5.6) includes three unknown terms that have to be parameterized, namely $\nabla\cdot(\overline{\mathbf{v}'\eta'})$, $\mathbf{v}\cdot\mathbf{k}\times\overline{\eta'\mathbf{v}'}$ and $\nabla\cdot\overline{B'\mathbf{v}'}$. Here we consider parameterization schemes of order one-and-a-half, that is, we will find possible functional relations between the unclosed terms and $\bar\psi$, $\bar\eta$, k as well as their derivatives. Note that there is a fundamental difference between the dependent variable $\bar\psi$ (and hence $\bar\zeta$) and k. Whereas $\bar\psi$ has an unaveraged counterpart ψ, there is no such counterpart for k. It is therefore necessary to find the prolongation of the symmetries of the original vorticity equation on the space spanned by (t,x,y,ψ) to the relevant space for the closed form of system (2.5.6), which is spanned by $(t,x,y,\bar\psi,k)$.

The symmetries of the barotropic vorticity equation were first computed in [68], see also [18] for a recent discussion. The most general transformation from the maximal Lie invariance group of the vorticity equation on the beta-plane is

$$\begin{aligned} &\tilde t = e^{\varepsilon_3}(t+\varepsilon_1), \quad \tilde x = e^{-\varepsilon_3}(x+f(t)), \quad \tilde y = e^{-\varepsilon_3}(y+\varepsilon_2),\\ &\tilde\psi = e^{-3\varepsilon_3}(\psi - f_t(t)y + g(t)), \end{aligned} \tag{2.5.7}$$

where f and g are arbitrary real-valued functions depending on t, and ε_1, ε_2 and ε_3 are arbitrary constants. So as to extend this transformation to the turbulent kinetic energy, it is necessary to investigate the transformation properties of $\nabla\psi'$. This is readily done by considering the splitting $\psi_x = \bar\psi_x + \psi'_x$ and by determining the transformation behavior of the right-hand side expression. Note that ψ_x transforms as $\tilde\psi_{\tilde x} = e^{-2\varepsilon_3}\psi_x$ which is a mere consequence of (2.5.7) and the use of the chain rule. Thus, we have

$$\tilde\psi_{\tilde x} = e^{-2\varepsilon_3}\psi_x = e^{-2\varepsilon_3}(\bar\psi_x + \psi'_x) = \widetilde{\bar\psi_{\tilde x}} + \widetilde{\psi'_{\tilde x}}$$

from which we find that $\widetilde{\bar\psi_{\tilde x}} = e^{-2\varepsilon_3}\bar\psi_x$ and $\widetilde{\psi'_{\tilde x}} = e^{-2\varepsilon_3}\psi'_x$. In a similar fashion, we note that

$$\tilde\psi_{\tilde y} = e^{-2\varepsilon_3}(\psi_y - f_t) = e^{-2\varepsilon_3}(\bar\psi_y + \psi'_y - f_t) = \widetilde{\bar\psi_{\tilde y}} + \widetilde{\psi'_{\tilde y}}.$$

and therefore $\widetilde{\bar\psi_{\tilde y}} = e^{-2\varepsilon_3}(\bar\psi_y - f_t)$ and $\widetilde{\psi'_{\tilde y}} = e^{-2\varepsilon_3}\psi'_y$ hold. From the transformation results for ψ'_x and ψ'_y it follows that the turbulent kinetic energy k transforms as

$$\tilde k = e^{-4\varepsilon_3}k.$$

We now aim to construct invariant parameterization schemes for the system (2.5.6) using the method of invariantization. For this, we need the moving frame that is associated to the maximal Lie invariance group of the vorticity equation on the beta-plane and is extended to k. Without giving the details of this computation, we should like to stress that due to the presence of the arbitrary functions f and g in the transformation (2.5.7) the symmetry group of the vorticity equation is *infinite dimensional*. Moving frames can be computed for infinite dimensional Lie groups as well, see e.g. [107]. This is done by specifying not only the group parameters ε_i and the arbitrary functions, but also the derivatives of these arbitrary functions up to any order.

For the vorticity equation on the beta-plane, the moving frame was constructed in [16]. It reads

$$\begin{aligned} &\varepsilon_1 = \ln\sqrt{|\psi_x|}, \quad \varepsilon_2 = -t, \quad \varepsilon_3 = -y, \quad f = -x, \\ &\frac{\mathrm{d}^{k+1} f}{\mathrm{d}t^{k+1}} = (\mathrm{D}_t - \psi_y \mathrm{D}_x)^k \psi_y, \quad \frac{\mathrm{d}^k g}{\mathrm{d}t^k} = -(\mathrm{D}_t - \psi_y \mathrm{D}_x)^k \psi, \end{aligned} \tag{2.5.8}$$

where $k = 0, 1, \ldots$, and D_t, D_x and D_y denote the total derivative operators with respect to t, x and y. This moving frame can now be used to invariantize any existing parameterization scheme for the averaged unclosed model (2.5.6). For this invariantization, the dependent variables as well as their derivatives in (2.5.8) should be regarded as mean quantities.

As an example, consider the model proposed in [90], which is

$$\begin{aligned} &\bar\eta_t + \bar\psi_x \bar\eta_y - \bar\psi_y \bar\eta_x = \nabla\cdot(\kappa\nabla\bar\eta) - A\nabla^4\bar\eta, \\ &k_t + \bar\psi_x k_y - \bar\psi_y k_x = -\kappa\nabla\bar\psi\cdot\nabla\bar\eta + \nabla\cdot(\lambda\nabla k) - rk \end{aligned} \tag{2.5.9}$$

where A is a constant biharmonic diffusion coefficient and κ, λ and r are the parameters of the closure model. For κ, the expression

$$\kappa = 2\gamma\mathcal{T}_{\mathrm{eddy}}k \tag{2.5.10}$$

was proposed, in which γ denotes a dimensionless constant and $\mathcal{T}_{\mathrm{eddy}}$ is the eddy turnover time-scale. The constant λ is the eddy energy diffusivity and r is an inverse time scale for the eddy energy decay. To simplify this system, we set $r = 0$ which is relevant for the case of freely-decaying turbulence in the ocean. The following consideration could of course be adapted for the case $r \neq 0$.

It is straightforward to check that, as it stands, system (2.5.9) preserves all Lie symmetries of the barotropic vorticity equation except for the scale invariance associated with the group parameter ε_3. Specifically, while the terms on the right hand side scale as $e^{-2\varepsilon_3}$, the term $A\nabla^4\bar\eta$ scales as $e^{3\varepsilon_3}$. That is, the constant A cannot be dimensionless.[4] This problem was extensively analyzed in [16] where it was shown that linear hyperdiffusion cannot preserve the scale invariance of the original vorticity equation. In order to recover this invariance, we can use the invariantization map (2.5.8) and apply it to the first equation in the closed system (2.5.9). This leads to

$$\bar\eta_t + \bar\psi_x \bar\eta_y - \bar\psi_y \bar\eta_x = \nabla\cdot(\kappa\nabla\bar\eta) - \tilde A\sqrt{|\bar\psi_x^5|}\nabla^4\bar\eta. \tag{2.5.11}$$

where $\tilde A$ is now truly dimensionless. For the case $\kappa = 0$ this model was successfully used in [16] to carry out freely decaying turbulence tests that yielded energy and enstrophy spectra in close accordance with the Batchelor–Kraichnan theory of two-dimensional turbulence, see e.g. [147].

[4] The term $\nabla\cdot(\kappa\nabla\bar\eta)$ scales properly as $e^{-2\varepsilon_3}$ provided that relation (2.5.10) is used and the eddy turnover time scales similar as t, i.e. $T_{\mathrm{eddy}} \sim e^{\varepsilon_3}$.

Turning to the energy equation in system (2.5.9) we note that the terms on the left-hand side scale as $e^{-5\varepsilon_3}$ and thus the constants κ and λ have to be chosen in such a manner that also the right hand side scales as $e^{-5\varepsilon_3}$; specifically, κ and λ should scale like $e^{-3\varepsilon_3}$. We already fixed the form of κ by using Eq. (2.5.10) and due to scaling the eddy turnover time as $T_{\rm eddy} \sim e^{\varepsilon_3}$, the first term on the right-hand side in the energy equation scales properly. Then, choosing λ to be of similar form as κ, i.e. $\lambda = 2\tilde{\gamma}\mathcal{T}_{\rm eddy}k$, for another dimensionless constant $\tilde{\gamma}$, it indeed scales as $e^{-3\varepsilon_3}$ as required.

We should also like to note that the invariantization of the vorticity equation in (2.5.9) leading to Eq. (2.5.11) is not unique. More specifically, it is always possible to recombine an invariant equation with other differential invariants in order to arrive at a new invariant equation. For example, the equation

$$\bar{\eta}_t + \bar{\psi}_x\bar{\eta}_y - \bar{\psi}_y\bar{\eta}_x = \nabla\cdot(\kappa\nabla\bar{\eta}) - \nabla^2(\tilde{A}\sqrt[4]{k^5}\nabla^2\bar{\eta}).$$

is also readily checked to be invariant under the maximal Lie invariance group of the original vorticity equation. In particular, the term $\nabla^2(\tilde{A}\sqrt[4]{k^5}\nabla^2\bar{\eta})$ also scales like $e^{-2\varepsilon_3}$. From the physical point of view, this parameterization of the eddy vorticity flux might be desirable as the hyperdiffusion-like term is now in conserved form.

In the same way other invariant equations could be constructed and tested numerically. In particular, with the moving frame (2.5.8) at hand it is straightforward to determine various differential invariants and to recombine them to subgrid-scale closure models for the three unclosed terms in system (2.5.6). This is a constructive way for finding all possible invariant parameterization schemes of order one-and-a-half for the model (2.5.5).

Example 2.36. We construct conservative parameterization schemes for the eddy vorticity flux in the barotropic vorticity equation on the f-plane, see [24] for more details. Extensions to the beta-plane equation or the barotropic ocean model discussed in the previous example can be readily realized.

The Reynolds averaged vorticity equation on the f-plane is

$$\bar{\zeta} + \bar{\psi}_x\bar{\zeta}_y - \bar{\psi}_y\bar{\zeta}_x = \nabla\cdot(\overline{\mathbf{v}'\zeta'}), \quad \bar{\zeta} = \bar{\psi}_{xx} + \bar{\psi}_{yy}. \tag{2.5.12}$$

The task is to find a parameterization for the eddy vorticity flux in such a manner that the closed vorticity equation admits some of the conservation laws of the original vorticity equation.

In the following, we focus on conservation laws of the vorticity equation associated with the characteristics

$$\Lambda^1 = h(t), \quad \Lambda^2 = f(t)x, \quad \Lambda^3 = g(t)y, \quad \Lambda^4 = -\psi.$$

Denoting the left hand side of the vorticity equation by V, $V = \zeta_t + \psi_x\zeta_y - \psi_y\zeta_x$, the corresponding conservation laws read

$$\begin{aligned}
hV &= (h\zeta)_t + (-h\psi_y\zeta - h_t\psi_x)_x + (h\psi_x\zeta - h_t\psi_y)_y,\\
fxV &= (fx\zeta)_t + (-fx\psi_y\zeta + f\psi_x\psi_y - f_tx\psi_x + f_t\psi)_x\\
&\quad + \left(fx\psi_x\zeta - \frac{f}{2}(\psi_x^2 - \psi_y^2) - f_tx\psi_y\right)_y,\\
gyV &= (gy\zeta)_t + \left(-gy\psi_y\zeta - \frac{g}{2}(\psi_x^2 - \psi_y^2) - g_ty\psi_x\right)_x\\
&\quad + (gy\psi_x\zeta - g\psi_x\psi_y - g_ty\psi_y + g_t\psi)_y,
\end{aligned}$$

$$-\psi V=\left(\frac{1}{2}(\nabla\psi)^2\right)_t+\left(-\psi\psi_{tx}-\frac{1}{2}\psi^2\zeta_y\right)_x+\left(-\psi\psi_{ty}+\frac{1}{2}\psi^2\zeta_x\right)_y.$$

Physically, these conservation laws correspond to generalizations of the conservation of (i) circulation, (ii)–(iii) the momenta in x- and y-direction and (iv) usual kinetic energy conservation.

In [24] we proved the statement that if a single differential equation $\mathcal{L}$: $\Delta(x,u^{(n)})=0$ admits characteristics of conservation laws of the form $h(x^1)+f^i(x^1)x^i$, with arbitrary functions $h=h(x^1)$ and $f^i=f^i(x^1)$, $i=2,\dots,p$, then the left hand side Δ of $\mathcal{L}$ can be represented as

$$\Delta=\sum_{2\leqslant i_1\leqslant i_2<p}\mathrm{D}_{i_1}\mathrm{D}_{i_2}F^{i_1i_2},$$

where the $F^{i_1i_2}$ are certain differential functions of u.

Specifying this statement for the vorticity equation, it is possible to re-express V as

$$\zeta_t+\psi_x\zeta_y-\psi_y\zeta_x=\mathrm{D}_x^2(\psi_t-\psi_x\psi_y)+\mathrm{D}_x\mathrm{D}_y(\psi_x^2-\psi_y^2)+\mathrm{D}_y^2(\psi_t+\psi_x\psi_y).$$

The validity of this representation is readily checked by direct expansion of the right hand side.

The same statement can then be used to find parameterizations for the eddy vorticity flux $\nabla\cdot(\overline{\mathbf{v}'\zeta'})$. Here we know that the eddy vorticity flux must be represented as $\nabla\cdot(\overline{\mathbf{v}'\zeta'})=\mathrm{D}_x^2F^{11}+\mathrm{D}_x\mathrm{D}_yF^{12}+\mathrm{D}_y^2F^{22}$ for arbitrary differential functions F^{11}, F^{12} and F^{22} of $\bar\psi$ in order to preserve generalized circulation and momenta also in the parameterized equation. To additionally ensure energy conservation in the parameterization for $\nabla\cdot(\overline{\mathbf{v}'\zeta'})$, it is sufficient to choose F^{11}, F^{12} and F^{22} such that $\bar\psi\nabla\cdot(\overline{\mathbf{v}'\zeta'})$ is a total divergence, i.e.

$$\bar\psi\nabla\cdot(\overline{\mathbf{v}'\zeta'})=\bar\psi(\mathrm{D}_x^2F^{11}+\mathrm{D}_x\mathrm{D}_yF^{12}+\mathrm{D}_y^2F^{22})=\operatorname{Div}H$$

for some conserved vector $H=(H^1,H^2,H^3)$. In what follows Div denotes the total divergence, $\operatorname{Div}H=\mathrm{D}_tH^1+\mathrm{D}_xH^2+\mathrm{D}_yH^3$. Using integration by parts the last equation becomes

$$\bar\psi_{xx}F^{11}+\bar\psi_{xy}F^{12}+\bar\psi_{yy}F^{22}=\operatorname{Div}Q,$$

where $Q=(Q^1,Q^2,Q^3)$ is another tuple of differential functions. This is a single inhomogeneous linear algebraic equation for the components F^{11}, F^{12} and F^{22}. The solution of this equation can be represented in a symmetric way as

$$\begin{aligned}F^{11}&=\bar\psi_{yy}P^2-\bar\psi_{xy}P^3+R^1,\\F^{12}&=\bar\psi_{xx}P^3-\bar\psi_{yy}P^1+R^2,\\F^{22}&=\bar\psi_{xy}P^1-\bar\psi_{xx}P^2+R^3,\end{aligned}$$

where P^i are arbitrary differential functions of $\bar\psi$, $i=1,2,3$, and the triple of differential functions R^i is a particular solution of the equation,

$$\bar\psi_{xx}R^1+\bar\psi_{xy}R^2+\bar\psi_{yy}R^3=\operatorname{Div}Q.$$

A simple particular solution satisfies the additional constraints $R^2=0$, $R^1=R^3$, which gives

$$R^2=0,\quad R^1=R^3=\frac{\operatorname{Div}Q}{\bar\zeta}.$$

The possible singularity in points where the vorticity vanishes can be compensated by ensuring that $\operatorname{Div}Q$ vanishes in the same points. For example, if $Q=\bar\zeta^2S$ for some triple S of differential functions of $\bar\psi$, then

$$\operatorname{Div}Q=\bar\zeta^2\operatorname{Div}S+2\bar\zeta(S^1\bar\zeta_t+S^2\bar\zeta_x+S^3\bar\zeta_y).$$

The substitution of the above solution into the expressions for F^{11}, F^{12} and F^{22} leads to the following assertion:

Proposition 2.37. *If the unclosed vorticity flux $\nabla\cdot\overline{\zeta'\mathbf{v}'}$ is parameterized in the Reynolds averaged vorticity equation by*

$$
\begin{aligned}
&\mathrm{D}_x^2(\bar\psi_{yy}P^2-\bar\psi_{xy}P^3)+\mathrm{D}_x\mathrm{D}_y(\bar\psi_{xx}P^3-\bar\psi_{yy}P^1)+\mathrm{D}_y^2(\bar\psi_{xy}P^1-\bar\psi_{xx}P^2)\\
&+(\mathrm{D}_x^2+\mathrm{D}_y^2)(\bar\zeta\,\mathrm{Div}\,S+2S^1\bar\zeta_t+2S^2\bar\zeta_x+2S^3\bar\zeta_y)
\end{aligned}
$$

for some differential functions P^i and S^i of $\bar\psi$, $i=1,2,3$, the resulting closed equation (2.5.12) *possesses the conservation laws associated with characteristics $h(t)$, $f(t)x$, $h(t)y$ and $\bar\psi$. That is, the closed equation will preserve generalized circulation, generalized momenta in x- and y-direction and energy.*

We point out that this parameterization, although being of particular form, includes the arbitrary differential functions P^i and Q^i that can be chosen freely. This means that there is no single parameterization scheme that preserves generalized circulation, momenta and kinetic energy. Rather, there is a general class of parameterization schemes which is compatible with the preservation of these conservation laws. As similar observation was made in the previous Example 2.35 were it was shown that there is not a single invariant parameterization scheme for eddies in a barotropic ocean. This means that other desirable physical properties can be included in the parameterizations which gives quite some freedom in constructing suitable closure models.

2.6 Conclusion and outlook

In this chapter we have discussed the use of tools from the group analysis of differential equations to systematically construct parameterization schemes with symmetry properties. Similar techniques can also be applied to find closure models that lead to closed systems of equations that admit nontrivial conservation laws. These methods are constructive in that they allow one during the design of a parameterization to control the geometric properties of the closed system of governing equations. This is in stark contrast to the conventional design of parameterization schemes for unresolved processes where the preservation of symmetries or conservation laws is often only determined after the fact and not constructively included in the closure design itself.

Due to the fundamental importance of symmetries and conservation laws in physics, it is expected that parameterization schemes that capture these essential properties of differential equations are prime candidates for a realistic modeling of subgrid-scale processes. A crucial fact that has been pointed out throughout this chapter is that the parameterization schemes obtained using group analysis tools are usually still classes of closure models. That is, other desirable properties can be included to narrow down the class to a specific subgrid-scale model. A schematic summary of the construction of invariant and conservative parameterization scheme is presented in Figure 2.6.

The field of geometry-preserving subgrid-scale modeling is still relatively recent and hence there are numerous uninvestigated problems. Most of the methods that have been introduced in this chapter are designed for the construction of *local* parameterization schemes. The reason for this is that point symmetries of differential equations are naturally local objects, i.e. they are applied on each point of a domain. This nature of point symmetries matches well with local parameterization schemes. On the other hand, group analysis of differential equations is applicable to integro-differential equations as well, see e.g. the review [62]. Despite not being a well-developed subject today, the group analysis of integro-differential equations provides a viable route to extending the theory of invariant parameterization schemes to non-local closure models. This will be of obvious importance for various processes in atmosphere–ocean dynamics.

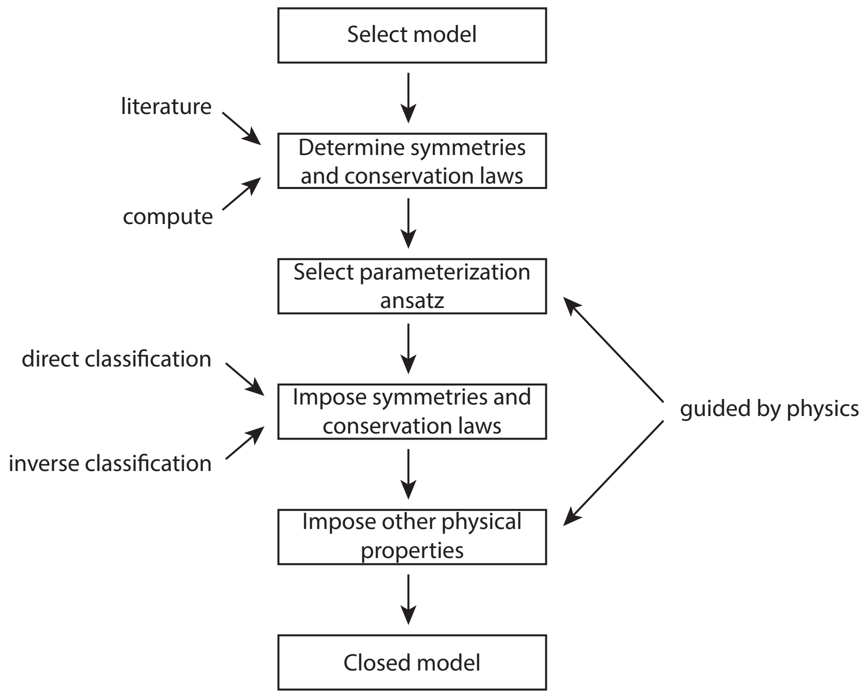

Figure 2.1: Schematic overview of the construction of invariant and conservation parameterization scheme.

Another problem that deserves particular attention is related to the construction of higher-order parameterization schemes. For the sake of simplicity, most of the material presented in this chapter focused on first-order closure models only. In Example 2.35 we have indicated that for the construction of such higher-order parameterizations still the symmetries of the original model are relevant, provided they are properly extended to those subgrid-scale flux variables, for which equations are attached to the averaged initial system. This prolongation of transformations to the proper subgrid-scale fluxes amounts to an extension of the space of dependent variables, which necessitates the computation of differential invariants for a wider set of variables. Still, the principal construction of parameterization schemes using this extended set of differential invariants does not change for higher-order parameterizations, although the practical realization becomes computationally more cumbersome due to the increased number of variables.

We finally indicate once again the relevance of conservative invariant parameterization schemes. Realistic processes of the atmosphere are usually linked to both symmetries and conservation laws of systems of differential equations and hence they should play a joint role in the construction of physical parameterization schemes. The attractive feature of conservative invariant parameterization is that it typically yields a more specific parameterization ansatz than closure models that only preserve either symmetries or conservation laws. The more specific a parameterization ansatz, the simpler it becomes to test a candidate parameterization in a numerical model for the Earth system.

Chapter 3

Enhanced preliminary group classification of a class of generalized diffusion equations

3.1 Introduction

Group classification of differential equations is an efficient tool for investigating symmetry properties of classes of differential equations. These are differential equations that include arbitrary constants or functions of the independent and dependent variables as well as of derivatives of the dependent variables up to a certain order. It is known for a long time that depending on the value of these arbitrary elements the resulting differential equations from the given class can have different Lie invariance groups. The first examples of group classification were presented by Sophus Lie for the class of second order linear partial differential equations [83] and the class of second order ordinary differential equations [85]. Later, Ovsiannikov began the rigorous development of the theory of group classification [112]. In short, the solution of the group classification problem consists in finding the kernel of Lie invariance groups (i.e. those Lie symmetries that are admitted for all values of the arbitrary elements) and all inequivalent extensions of Lie invariance groups with respect to the kernel group. The equivalence involved means the similarity of equations up to transformations from a certain equivalence group (e.g. usual, generalized or conditional equivalence), see [127] for more detailed information.

For classes of differential equations being of simple structure (e.g., ones parameterized only by constants or functions of the same single argument), the corresponding group classification problems can be completely solved via compatibility analysis and explicit integration of the determining equations for Lie symmetries depending on values of the arbitrary elements and up to the equivalence chosen. Complete group classification can also be carried out for classes of differential equations possessing the normalization property. The algebraic method of classification effectively works for such classes. See the next section and also [127, 150] for a more comprehensive review on different methods of group classifications.

In the situation where the class depends in a more complicated way on its arbitrary elements, it may happen that both the determining equations are too difficult to be directly solved and the application of the algebraic method does not give the exhaustive solution. In this case, however, at least a partial solution of the group classification problem, known as *preliminary group classification*, is possible. The basic idea of preliminary group classification is to study only those extensions of the kernel group that are induced by the transformations from the corresponding equivalence group. The problem of finding inequivalent cases of such Lie symmetry extensions then reduces to the classification of inequivalent subgroups (resp. algebras) of the equivalence group (resp. algebra). This approach was first described in [1] and became prominent due to the paper [63].

Despite the approach of preliminary group classification is rather common, it is not well developed up to now. The basic mechanisms were formulated in [63] as two propositions for the specific class of nonlinear wave equations of the form $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$ and were later adopted in other papers for respective classes of equations. In the present paper, we state a stronger version of these propositions for general classes of differential equations. Another weakness commonly observed is that, when the equivalence algebra $\mathfrak{g}^\sim$ of the class of equations under consideration is infinite dimensional, only Lie symmetry extensions induced by subalgebras of a finite-dimensional subalgebra $\mathfrak{g}^\sim_0$ of $\mathfrak{g}^\sim$ are investigated, without giving any sound justification for the choice of $\mathfrak{g}^\sim_0$. In fact, this restriction is needless as it is possible to classify subalgebras of infinite-dimensional algebras in much the same way as subalgebras of finite-dimensional algebras [9, 13, 18, 20, 49, 79, 88, 123, 127, 159]. It can even be simpler to classify low-dimensional subalgebras of the whole infinite-dimensional equivalence algebra $\mathfrak{g}^\sim$ as the adjoint action related to $\mathfrak{g}^\sim$ is more powerful and allows for greater simplification than the adjoint action corresponding to the finite-dimensional subalgebra $\mathfrak{g}^\sim_0$. One more common weakness in papers on the subject is that usually only extensions induced by one-dimensional subalgebras of equivalence algebras are studied. Moreover, these one-dimensional subalgebras (of a finite-dimensional subalgebra $\mathfrak{g}^\sim_0$ of $\mathfrak{g}^\sim$) are classified only with respect to the restricted equivalence relation which is generated by the adjoint representation of $\mathfrak{g}^\sim_0$. This leads to an overly large number of inequivalent subalgebras compared to the list of one-dimensional subalgebras that would be obtainable if the classification was done using the adjoint representation of the entire $\mathfrak{g}^\sim$.

In the present paper, we comprehensively carry out preliminary group classification for the class of $(1 + 1)$-dimensional second order quasilinear evolution equations of the general form

$$\Delta = u_t - f(x, u)u_x^2 - g(x, u)u_{xx} = 0, \tag{3.1.1}$$

where f and g are arbitrary smooth functions of x and u, and $g \neq 0$. The class (3.1.1) was considered in the recent paper [93] but results obtained therein are not correct. It is reviewed above that there are a number of typical weaknesses in papers on preliminary group classification, and results of [93] properly illustrate these weaknesses, cf. the first paragraphs of Sections 3.4 and 3.6 and Remark 3.27. This is why we aim to accurately present the revised preliminary group classification of the class (3.1.1) and to give all calculations in considerable detail.

The class (3.1.1) was considered in [93] as a class of generalized Burgers equations as it includes the *potential Burgers equation* as a particular element for the choice $f = g = 1$. This class also contains $(1 + 1)$-dimensional linear evolution equations, which correspond to the values $f = 0$ and g not depending on u. As a prominent example for a linear differential equation, one can recover the linear heat equation by choosing $f = 0$ and $g = 1$. An important subclass of the class (3.1.1) is the class of $(1+1)$-dimensional nonlinear diffusion equations of the general form $u_t = (F(u)u_x)_x$, where $F \neq 0$. It is singled from the class (3.1.1) by the constraints $g_x = 0$ and $f = g_u$. Moreover, any equation of the form (3.1.1) with $f_x = g_x = 0$ is reduced to a diffusion equation by a simple point transformation acting only on the dependent variable u. The solution of the group classification problem for this class by Ovsiannikov [111] (see also [1, 112]) gave rise to the development of modern group analysis.

The class (3.1.1) is included in the wider class of equations $u_t = F(t, x, u, u_x)u_{xx}+G(t, x, u, u_x)$, for which the complete group classification was carried out in [9] by a method similar to that applied in the present paper. The fact that this class is normalized (cf. Section 3.2) played a crucial role in the entire consideration in [9]. However, as this class is essentially wider than the class (3.1.1), the corresponding equivalence algebras are rather different. This is why the results of [9] cannot be directly used for deriving the group classification of the class (3.1.1).

It should also be stressed that it is not natural to exclude linear differential equations from the present consideration. In fact, there are equations in the class (3.1.1) which are linearized by point transformations from the equivalence group $G^{\sim}$ of this class. The most prominent example of such a transformation in the above class is the transformation of the potential Burgers equation to the linear heat equation by means of the point transformation $\tilde{u} = e^u$ [101, p. 122]. That is, u is a solution of the potential Burgers equation whenever $\tilde{u}$ is a solution of the linear heat equation. In the course of preliminary group classification of the class (3.1.1) we encounter other examples of linearizable equations. Furthermore, the equivalence algebra $\mathfrak{g}_0^{\sim}$ of the subclass of (3.1.1), which is compliment to the subclass of linear equations and, therefore, singled out by the constraint $f^2 + g_u^2 \neq 0$, is much narrower than the equivalence algebra $\mathfrak{g}^{\sim}$ of the entire class (3.1.1). More precisely, the algebra $\mathfrak{g}_0^{\sim}$ is singled out as a subalgebra of $\mathfrak{g}^{\sim}$ by the constraint $h_{uu} = 0$, cf. Theorem 3.11.

The further organization of this paper is the following. The subsequent Section 3.2 discusses the theory of preliminary group classification. We generalize and extend assertions presented in [63] and formulate them rigorously using the modern language of group analysis. In Section 3.3 we derive the determining equations for Lie point symmetries of equations from the class (3.1.1) and find the corresponding kernel of Lie invariance algebras. The equivalence algebra $\mathfrak{g}^{\sim}$ and the equivalence group $G^{\sim}$ of the class (3.1.1) is computed in Sections 3.4 and 3.5, respectively. Throughout the paper, by the equivalence group we mean the Lie pseudo-group of point equivalence transformations (i.e., local equivalence diffeomorphisms), cf. [109] and references therein for theory of pseudo-groups. The reason for carrying out preliminary group classification is elucidated. In Section 3.6, we classify inequivalent one- and two-dimensional subalgebras of the essential subalgebra of $\mathfrak{g}^{\sim}$. The corresponding inequivalent cases of symmetries extensions of the kernel algebra are presented in Section 3.7 and supplemented with three- and four-dimensional extensions via the classification of all appropriate subalgebras of $\mathfrak{g}^{\sim}$. The paper concludes with a short summary and further comments in Section 3.8.

3.2 Enhanced method of preliminary group classification

By now, the method of preliminary group classification was neither explained for general classes of differential equations nor properly related to the general group classification problem. This should be done first in this section before we study the preliminary group classification of (3.1.1). For this aim, we need a few notions of the theory of group classifications, which can be found in the recent paper [127].

The most essential notion concerns the formal definition of *classes of differential equations*. In general, a class (of systems) of differential equations is given by a system of l differential equations of the form $L(x, u_{(p)}, \theta(x, u_{(p)})) = 0$ in m dependent variables $u = (u^1, \dots, u^m)$ and n independent variables $x = (x_1, \dots, x_n)$, where $u_{(p)}$ denotes the set of u's and all their derivatives up to order p. The differential functions $\theta(x, u_{(p)}) = (\theta^1(x, u_{(p)}), \dots, \theta^k(x, u_{(p)}))$ denote a tuple of k arbitrary elements that parameterize the given class of differential equations. The tuple θ is usually constrained to satisfy a system $\mathcal{S}$ of auxiliary conditions, $S(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0$, in which x and $u_{(p)}$ are regarded as independent variables. The set of solutions of this auxiliary system will also be denoted by $\mathcal{S}$. In addition, this set can be further constrained by satisfying one or more nonvanishing conditions $\Sigma(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) \neq 0$. Putting together all these notions, we denote the class of differential equations with the arbitrary element running through the set $\mathcal{S}$ by $\mathcal{L}|_\mathcal{S}$. The single elements of this class are denoted by $\mathcal{L}_\theta$, respectively.

Specifically, for the class (3.1.1) we have $\theta = (f, g)$, and the arbitrary elements f and g depend only on x and u. Therefore, the associated auxiliary system $\mathcal{S}$ is formed by the equations

$$f_t = f_{u_t} = f_{u_x} = f_{u_{tt}} = f_{u_{tx}} = f_{u_{xx}} = 0,$$
$$g_t = g_{u_t} = g_{u_x} = g_{u_{tt}} = g_{u_{tx}} = g_{u_{xx}} = 0.$$

The auxiliary conditions $f_t = 0$ and $g_t = 0$ play a special role. All the other auxiliary conditions can be taken into account implicitly. The nonvanishing condition associated with the class (3.1.1) is $g \neq 0$, i.e., we have $\Sigma = g$. The condition $g \neq 0$ should be explicitly included in the definition of the class (3.1.1) since equations of the same form with $g = 0$ are of another (first) order, possess completely different transformational properties and are not related to equations with $g \neq 0$ by point or other reasonable transformations.

Having properly defined classes of differential equations, it remains to introduce the notion of admissible transformations and normalized classes of differential equations in order to explain the general strategy of (preliminary) group classification.

Definition 3.1. The set of admissible transformations in the class $\mathcal{L}|_\mathcal{S}$ is given by $\mathrm{T}(\mathcal{L}|_\mathcal{S}) = \{(\theta, \tilde\theta, \varphi) \mid \theta, \tilde\theta \in \mathcal{S}, \varphi \in \mathrm{T}(\theta, \tilde\theta)\}$, where $\mathrm{T}(\theta, \tilde\theta)$ denotes the set of point transformations that map the system $\mathcal{L}_\theta$ to the system $\mathcal{L}_{\tilde\theta}$.

The set of admissible transformations can be used for many issues related to the problem of group classification. It can be considered as an extension or generalization of the equivalence group of a class of differential equations. Indeed, the usual equivalence group $G^\sim$ of a class $\mathcal{L}|_\mathcal{S}$ is naturally embedded in the set of admissible transformations. In particular, it is given by the admissible transformations $(\theta, \Phi\theta, \Phi|_{(x,u)})$, where Φ is an equivalence transformation, i.e. $\forall\theta \in \mathcal{S}\colon \Phi\theta \in \mathcal{S}$. In this last tuple, $\Phi|_{(x,u)} \in \mathrm{T}(\theta, \Phi\theta)$ denotes the projection of Φ to the space of variables x, u. The maximal point symmetry group G_θ of the system $\mathcal{L}_\theta$ coincides with the set of admissible transformations from $\mathcal{L}_\theta$ to itself, i.e., $G_\theta = \mathrm{T}(\theta, \theta)$.

Important properties of classes of differential equations, relevant for the problem of group classification, are given by different kinds of normalization with respect to point (resp. contact) transformations [122, 127].

Definition 3.2. The class of differential equations $\mathcal{L}|_\mathcal{S}$ is *normalized* in the usual sense if any admissible transformation is induced by a transformation of the (usual) equivalence group, i.e. $\forall(\theta, \tilde\theta, \varphi)$, $\exists\Phi \in G^\sim\colon \tilde\theta = \Phi\theta$ and $\varphi = \Phi|_{(x,u)}$.

Denote by $\mathfrak{g}_\theta$ the maximal Lie invariance algebra of the equation $\mathcal{L}_\theta$. Using the above notations it is possible to obtain the general picture of the group classification problem. The first step in order to carry out group classification is the determination of the *kernel* $\mathfrak{g}^\cap$ (i.e., intersection) of maximal Lie invariance algebras of systems from the class $\mathcal{L}|_\mathcal{S}$. The kernel is found by deriving the determining equations of Lie symmetries and splitting with respect to both derivatives with respect to u and the arbitrary elements θ. This gives those part of the maximal Lie invariance algebra $\mathfrak{g}_\theta$ that is admitted for any value of θ. The subsequent step consists of determining the equivalence group $G^\sim$ (resp. the equivalence algebra $\mathfrak{g}^\sim$) of the class $\mathcal{L}|_\mathcal{S}$. The equivalence group $G^\sim$ is needed since it generates a natural equivalence relation on cases of symmetry extension of the kernel and hence they should be studied up to this equivalence. The final task is to describe all inequivalent cases of symmetry extension, i.e., values of θ for which $\mathfrak{g}_\theta \neq \mathfrak{g}^\cap$.

For the implementation of the above classification program, several special techniques have been developed. They either lead to the *complete group classification* or to a *preliminary group classification* of the given class.

Complete group classification is often possible for normalized classes of differential equations. For such classes, symmetry extensions of the kernel algebra can only be induced by transformations from the corresponding equivalence algebra. This reduces the group classification problem to the algebraic problem of classifying inequivalent subalgebras of the equivalence algebra. This is why we refer to this method as the algebraic method. Results on complete group classification of various classes of differential equations can be found, e.g. in [1, 9, 79, 123, 127, 128, 129, 159]. The equations studied in these papers all possess the normalization property.

Another method leading to the complete solution of the group classification problem consists of a compatibility analysis and direct integration of the determining equations of Lie symmetries of the given class [1, 2, 65, 94, 95, 96, 97, 112, 150]. It was indicated in the introduction that it is often only for rather simple classes that this method works.

Complete preliminary group classification employs essentially the same techniques that are used for complete group classification within the framework of the algebraic method. The main difference is that the underlying class does not possess the normalization property. This implies the existence of extensions of the kernel algebra that are not induced by subalgebras of the equivalence algebra. In turn, for normalized classes of differential equations the results of complete preliminary group classification and complete group classification coincide [122, 127].

In most papers on preliminary group classification only a partial solution of the corresponding problems is achieved since usually not the whole equivalence algebra is used for an investigation of cases of symmetry extensions. This is why we refer to this method as the method of *partial preliminary group classification*. It is the most incomplete and heuristic method of group classification, as there are often no obvious criteria which subalgebras of the equivalence algebra to single out for an investigations of symmetry extensions of the kernel algebra. Results on partial preliminary group classification are presented, e.g., in [1, 63, 93, 140].

On the side of complete group classification, the theoretical background was already settled [87, 112] and extended [122, 127]. It remains to detail the framework of preliminary group classification. In its essence, it rests on the following two propositions, which were first formulated without proof in [63] for the class of equations investigated. We present an enhanced version of these propositions for general classes of differential equations.

Proposition 3.3. *Let $\mathfrak{a}$ be a subalgebra of the equivalence algebra $\mathfrak{g}^\sim$ of the class $\mathcal{L}|_\mathcal{S}$, $\mathfrak{a} \subset \mathfrak{g}^\sim$, and let $\theta^0(x, u_{(r)}) \in \mathcal{S}$ be a value of the tuple of arbitrary elements θ for which the algebraic equation $\theta = \theta^0(x, u_{(r)})$ is invariant with respect to $\mathfrak{a}$. Then the differential equation $\mathcal{L}_{\theta^0}$ is invariant with respect to the projection of $\mathfrak{a}$ to the space of variables (x, u).*

Proof. Choose an arbitrary operator Q from $\mathfrak{a}$ and consider the one-parameter group G_1 generated by this operator. As the equation $\theta = \theta^0(x, u_{(r)})$ is invariant with respect to G_1, any transformation $\mathcal{T}$ from G_1 maps the corresponding equation $\mathcal{L}_{\theta^0}$ from the class $\mathcal{L}|_\mathcal{S}$ to itself. This means that the projection $\mathrm{P}\mathcal{T}$ of $\mathcal{T}$ to the space of variables (x, u) is a point symmetry of $\mathcal{L}_{\theta^0}$. Therefore, the projection $\mathrm{P}G_1$ of G_1 is a point symmetry group of $\mathcal{L}_{\theta^0}$ and its generator, which is the projection of the operator Q, belongs to the Lie invariance algebra of $\mathcal{L}_{\theta^0}$. □

Proposition 3.4. *Let $\mathcal{S}_i$ be the subset of $\mathcal{S}$ that consists of all arbitrary elements for which the corresponding algebraic equations are invariant with respect to the same subalgebra of the equivalence algebra $\mathfrak{g}^\sim$ and let $\mathfrak{a}_i$ be the maximal subalgebra of $\mathfrak{g}^\sim$ for which $\mathcal{S}_i$ satisfies this property, $i = 1, 2$. Then the subalgebras $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are equivalent with respect to the adjoint action of $G^\sim$ if and only if the subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ are mapped to each other by transformations from $G^\sim$.*

Proof. Assume that $\mathfrak{a}_2 = \mathcal{T}_*\mathfrak{a}_1$, where $\mathcal{T} \in G^\sim$ and $\mathcal{T}_*$ denotes the associated push-forward of vector fields. For $\theta^0 \in \mathcal{S}_1$ the algebraic equation $\theta = \theta^0$ is invariant with respect to $\mathfrak{a}_1$. Since $\mathcal{T}$ is an equivalence transformation, we also have that $\mathcal{T}\theta^0 \in \mathcal{S}$. By supposition, $\tilde{\theta} = \mathcal{T}\theta^0$ is invariant with respect to $\mathcal{T}_*\mathfrak{a}_1 = \mathfrak{a}_2$. This implies that $\mathcal{T}\theta^0 \in \mathcal{S}_2$ from which it can be concluded that $\mathcal{T}\mathcal{S}_1 \subset \mathcal{S}_2$. Similarly, for $\tilde{\theta}^0 \in \mathcal{S}_2$, the algebraic equation $\tilde{\theta} = \tilde{\theta}^0$ is invariant with respect to $\mathfrak{a}_2$ and $\mathcal{T}^{-1}\tilde{\theta}^0 \in \mathcal{S}$. As $\mathcal{T}_*^{-1}\mathfrak{a}_2 = \mathfrak{a}_1$, the algebraic equation $\theta = \mathcal{T}^{-1}\tilde{\theta}^0$ is invariant with respect to $\mathfrak{a}_1$, which implies that $\mathcal{T}^{-1}\tilde{\theta}^0 \in S_1$. From this last condition we obtain $\mathcal{T}\mathcal{S}_1 \supset \mathcal{S}_2$. It therefore can be concluded that there exists a bijection between $\mathcal{S}_1$ and $\mathcal{S}_2$, generated by a transformation from $G^\sim$.

Conversely, suppose that $\mathcal{S}^2 = \mathcal{T}\mathcal{S}^1$ for $\mathcal{T} \in G^\sim$. If $\theta = \theta^0$ is invariant with respect to $\mathfrak{a}_1$ then $\tilde{\theta} = \mathcal{T}\theta^0$ is invariant with respect to $\mathcal{T}_*\mathfrak{a}_1$. As θ^0 is arbitrary, this implies that $\mathcal{T}_*\mathfrak{a}_1 \subset \mathfrak{a}_2$. In a similar manner as in the previous paragraph, we can show that $\mathcal{T}_*^{-1}\mathfrak{a}_2 \subset \mathfrak{a}_1$ using the inverse transformation of $\mathcal{T}$. Then we have $\mathcal{T}_*\mathcal{T}_*^{-1}\mathfrak{a}_2 \subset \mathcal{T}_*\mathfrak{a}_1$ and thus $\mathfrak{a}_2 \subset \mathcal{T}_*\mathfrak{a}_1$. This is why $\mathfrak{a}_2 = \mathcal{T}_*\mathfrak{a}_1$ must hold, which completes the proof of the proposition. □

Roughly speaking, the first proposition defines the method of how to construct cases of symmetry extensions if the equivalence algebra of the class of differential equations to be investigated is already known. The second proposition then states that the problem of finding inequivalent cases of such symmetry extensions of the kernel algebra is reduced to the algebraic problem of the classification of subalgebras of the equivalence algebra.

Remark 3.5. Within the set $\mathcal{S}_i$ defined in Proposition 3.4, there is an equivalence relation generated by transformations from $G^\sim$ whose push-forwards to vector fields preserve the subalgebra $\mathfrak{a}_i$ of $\mathfrak{g}^\sim$. Such transformations form the normalizer of the subgroup of $G^\sim$ associated with $\mathfrak{a}_i$. This equivalence relation can be used to choose simple forms of representatives of the set $\mathcal{S}_i$.

This now completes the picture of the methods available for general group classification problems. It should be clear that these methods apply to different classes of differential equations. This is why it is essential to investigate properties of the given class *before* choosing a particular method of group classification. This is done in the present paper. It is shown in the subsequent sections that the class (3.1.1) is not normalized. Moreover, a compatibility analysis of the determining equations of Lie symmetries of this class is also an overly complicated task. This is why it cannot be expected to solve the complete group classification problem for (3.1.1) in a reasonable way. Still, the given class is adequate to be investigated using the method of complete preliminary group classification.

Recall that, as mentioned in the introduction, the class (3.1.1) is contained in the wider class of equations of the general form $u_t = F(t, x, u, u_x)u_{xx} + G(t, x, u, u_x)$, which is normalized and for which the group classification problem was solved in [9].

The following folklore assertion is true.

Proposition 3.6. *The kernel (common part) $G^\cap = \bigcap_{\theta\in\mathcal{S}} G_\theta$ of the maximal point symmetry groups G_θ, $\theta \in \mathcal{S}$, of systems from the class $\mathcal{L}|_\mathcal{S}$ is naturally embedded into the (usual) equivalence* *group $G^\sim$ of this class via trivial (identical) prolongation of the kernel transformations to the arbitrary elements. The associated subgroup $\hat G^\cap$ of $G^\sim$ is normal.*

Proof. Let $\mathcal T_0$ be an arbitrary element of $G^\cap$, i.e. $\mathcal T_0$ is a point symmetry transformation for any equation from the class $\mathcal L|_{\mathcal S}$. Denote by $\hat{\mathcal T}_0$ the trivial prolongation of $\mathcal T_0$ to the arbitrary elements θ, $\hat{\mathcal T}_0\theta=\theta$. The transformation $\hat{\mathcal T}_0$ obviously belongs to $G^\sim$, since it maps any equation from $\mathcal L|_{\mathcal S}$ to the same equation in the new variables and therefore saves the entire class $\mathcal L|_{\mathcal S}$.

Taking an arbitrary transformation $\mathcal T\in G^\sim$, consider the composition $\mathcal T^{-1}\hat{\mathcal T}_0\mathcal T$. In order to check that $\hat G^\cap$ is a normal subgroup of $G^\sim$, we should prove that this composition belongs to $\hat G^\cap$. We fix any $\theta\in\mathcal S$ and denote $\mathcal T\theta$ by $\tilde\theta$. Then $\hat{\mathcal T}_0\mathcal T\theta=\tilde\theta$ and hence $\mathcal T^{-1}\hat{\mathcal T}_0\mathcal T\theta=\theta$. This means that the projection $\mathrm P\mathcal T^{-1}\hat{\mathcal T}_0\mathcal T$ to the space of variables (x,u) is a point symmetry transformation of $\mathcal L_\theta$ for any $\theta\in\mathcal S$. In other words, the transformation $\mathrm P\mathcal T^{-1}\hat{\mathcal T}_0\mathcal T$ is an element of $G^\cap$. Therefore, $\mathcal T^{-1}\hat{\mathcal T}_0\mathcal T$, which is the trivial prolongation of $\mathrm P\mathcal T^{-1}\hat{\mathcal T}_0\mathcal T$ to the arbitrary elements, belongs to $\hat G^\cap$. □

Properties of $G^\cap$ described in Proposition 3.6 were first noted in different works by Ovsiannikov (see, e.g., [114] and [112, Section II.6.5]). Another formulation of this proposition is given in [87, p. 52], Proposition 3.3.9.

Corollary 3.7. *The trivial prolongation $\hat{\mathfrak g}^\cap$ of the kernel algebra $\mathfrak g^\cap$ to the arbitrary elements is an ideal in the equivalence algebra $\mathfrak g^\sim$.*

By definition, any element of the algebra $\hat{\mathfrak g}^\cap$ formally has the same form as the associated element from $\mathfrak g^\cap$, but in fact is a vector field in the different space augmented with the arbitrary elements.

Proof. Consider arbitrary vector fields $Q_0\in\mathfrak g^\cap$ and $Q\in\mathfrak g^\sim$. Denote the trivial prolongation of Q_0 to the arbitrary elements by $\hat Q_0$, so $\hat Q_0\in\hat{\mathfrak g}^\cap$. It is necessary only to prove that $\hat Q_0\in\mathfrak g^\sim$ and $[Q,\hat Q_0]\in\hat{\mathfrak g}^\cap$. Let $\hat G_0=\{\hat{\mathcal T}_0(\varepsilon)=\exp(\varepsilon\hat Q_0)\}$ and $G=\{\mathcal T(\varepsilon)=\exp(\varepsilon Q)\}$ be local one-parameter transformation group associated with $\hat Q_0$ and Q_0, respectively. As $\hat G_0$ is a subgroup of $G^\sim$, the vector field $\hat Q_0$ belongs to $\mathfrak g^\sim$.

For each sufficiently small ε define the composition $\tilde{\mathcal T}(\varepsilon)=\hat{\mathcal T}_0(-\sqrt\varepsilon)\mathcal T(-\sqrt\varepsilon)\hat{\mathcal T}_0(\sqrt\varepsilon)\mathcal T(\sqrt\varepsilon)$ and consider the vector field

$$\tilde Q=\frac{\mathrm d}{\mathrm d\varepsilon}\bigg|_{\varepsilon=0+}\tilde{\mathcal T}(\varepsilon),$$

which coincides with $[Q,\hat Q_0]$, see e.g. [101, Theorem 1.33]. As both $\mathcal T(-\sqrt\varepsilon)\hat{\mathcal T}_0(\sqrt\varepsilon)\mathcal T(\sqrt\varepsilon)$ and $\hat{\mathcal T}_0(-\sqrt\varepsilon)$ belong to $\hat G_0$ (cf. Proposition 3.6), the transformation $\tilde{\mathcal T}(\varepsilon)$ also is an element of $\hat G_0$. Therefore, $\tilde Q\in\mathfrak g^\sim$. □

As the kernel is included in the maximal Lie invariance algebra of any equation from the class, we should classify only subalgebras of the equivalence algebra that contain the ideal associated with the kernel.

Example 3.8. In general, the kernel $\mathfrak g^\cap$ is not necessarily an ideal of the maximal Lie invariance algebra $\mathfrak g_\theta$ for each $\theta\in\mathcal S$. Indeed, consider the class of $(1+1)$-dimensional nonlinear diffusion equations of the general form $u_t=(F(u)u_x)_x$, where $F\neq0$, cf. the introduction. The kernel of this class and the maximal Lie invariance algebra of the diffusion equation with $F=u^{-4/3}$ are

$\mathfrak{g}^\cap = \langle \partial_t,\, \partial_x,\, 2t\partial_t + x\partial_x \rangle$ and $\mathfrak{g}_1 = \langle \partial_t,\, \partial_x,\, 2t\partial_t + x\partial_x,\, 4t\partial_t + 3u\partial_u,\, x^2\partial_x - 3xu\partial_u \rangle$ [1, 111, 112], respectively. At the same time, the kernel $\mathfrak{g}^\cap$ is not an ideal of $\mathfrak{g}_1$, $[\mathfrak{g}^\cap, \mathfrak{g}_1] \not\subset \mathfrak{g}^\cap$, since

$$[\partial_x, x^2\partial_x - 3xu\partial_u] = 2x\partial_x - 3xu\partial_u \notin \mathfrak{g}^\cap.$$

Note that the class of diffusion equations is semi-normalized (see [122, 127] for the definition of semi-normalization) but not normalized in the usual sense.

Corollary 3.9. *If the class $\mathcal{L}|_\mathcal{S}$ is normalized in the usual sense, the kernel algebra $\mathfrak{g}^\cap$ is an ideal of the maximal Lie invariance algebra $\mathfrak{g}_\theta$ for each $\theta \in \mathcal{S}$.*

Proof. We fix an arbitrary element $\theta^0 \in \mathcal{S}$. Denote by $\hat{\mathfrak{g}}_{\theta^0}$ the maximal subalgebra of $\mathfrak{g}^\sim$ such that the algebraic equation $\theta = \theta^0(x, u_{(r)})$ is invariant with respect to it. This subalgebra necessarily contains the trivial prolongation $\hat{\mathfrak{g}}^\cap$ of the kernel algebra $\mathfrak{g}^\cap$ to the arbitrary elements. Thus, we have that $\hat{\mathfrak{g}}^\cap \subset \hat{\mathfrak{g}}_{\theta^0} \subset \mathfrak{g}^\sim$ and, in view of Corollary 3.7, $\hat{\mathfrak{g}}^\cap$ is an ideal in $\mathfrak{g}^\sim$. Therefore, $\hat{\mathfrak{g}}^\cap$ is an ideal in $\hat{\mathfrak{g}}_{\theta^0}$. As the class $\mathcal{L}|_\mathcal{S}$ is normalized in the usual sense, the projection of $\hat{\mathfrak{g}}_{\theta^0}$ to the space of the variables (x, u) coincides with the maximal Lie invariance algebra $\mathfrak{g}_{\theta^0}$ of the equation $\mathcal{L}_{\theta^0}$. By the construction, the projection of $\hat{\mathfrak{g}}^\cap$ to the space of the variables (x, u) coincides with $\mathfrak{g}^\cap$. Hence $\mathfrak{g}^\cap$ is an ideal in $\mathfrak{g}_{\theta^0}$. □

Often the equivalence algebra can be represented as a semi-direct sum of the ideal associated with the kernel and a certain subalgebra. To obtain preliminary group classification in this case, we in fact need to classify only inequivalent subalgebras of the complement of the kernel ideal. Projections of these subalgebras to the space of equation variables will give all possible inequivalent extensions of the kernel.

Example 3.10. We present a class of differential equations for which the above representation is not possible. This is the class of $(1+1)$-dimensional linear second order homogeneous evolution equations which has the general form

$$u_t = A(t,x)u_{xx} + B(t,x)u_x + C(t,x)u, \tag{3.2.1}$$

where $A = A(t,x)$, $B = B(t,x)$ and $C = C(t,x)$ are arbitrary smooth functions, $A \neq 0$. The kernel Lie algebra of class (3.2.1) is $\mathfrak{g}^\cap = \langle u\partial_u \rangle$. Its equivalence algebra $\mathfrak{g}^\sim$ is spanned by operators of the form

$$\tau\partial_t + \xi\partial_x + \eta^1 u\partial_u +$$
$$(2\xi_x - \tau_t)\partial_A + ((\xi_x - \tau_t)B - 2\eta^1_x A - \xi_t)\partial_B + (\eta^1_t - A\eta^1_{xx} - B\eta^1_x - C\xi_t)\partial_C,$$

where $\tau = \tau(t)$, $\xi = \xi(t,x)$ and $\eta^1 = \eta^1(t,x)$ are arbitrary smooth functions of their arguments. The kernel $\mathfrak{g}^\cap$ can be identified with the ideal of $\mathfrak{g}^\sim$, generated by the vector field $u\partial_u$, which is assumed now to act in the space of variables and arbitrary elements. Moreover, this vector field commutes with all elements of $\mathfrak{g}^\sim$. At the same time we have $[\partial_t, tu\partial_u] = u\partial_u$. Therefore the algebra $\mathfrak{g}^\sim$ cannot be represented as a semi-direct sum of $\mathfrak{g}^\cap$ and a subalgebra.

3.3 Determining equations of Lie symmetries

The method of computing Lie symmetries is classical and can be found in all textbooks on this subject, see, e.g. [25, 101, 112]. Owing to its algorithmic nature, it was implemented in a number of symbolic computation programs [31, 32, 55, 135]. For an equation $\Delta = 0$ from the class (3.1.1), the condition of infinitesimal invariance with respect to a vector field

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

has the form $Q^{(2)}\Delta|_{\Delta=0} = 0$, i.e.,

$$Q^{(2)}\Delta = \eta^t - \xi f_x u_x^2 - \eta f_u u_x^2 - 2fu_x\eta^x - \xi g_x u_{xx} - \eta g_u u_{xx} - g\eta^{xx} = 0 \tag{3.3.1}$$

wherever $\Delta = 0$. Here $Q^{(2)}$ is the second prolongation of the vector field Q,

$$Q^{(2)} = Q + \eta^t\partial_{u_t} + \eta^x\partial_{u_x} + \eta^{tt}\partial_{u_{tt}} + \eta^{tx}\partial_{u_{tx}} + \eta^{xx}\partial_{u_{xx}}, \tag{3.3.2}$$

where the coefficients can be determined by using the general prolongation formula. In (3.3.1) we only need the coefficients η^t, η^x and η^{xx}. They read [101, 112]

$$\begin{aligned}
\eta^t &= \mathrm{D}_t(\eta - \tau u_t - \xi u_x) + \tau u_{tt} + \xi u_{tx},\\
\eta^x &= \mathrm{D}_x(\eta - \tau u_t - \xi u_x) + \tau u_{tx} + \xi u_{xx},\\
\eta^{xx} &= \mathrm{D}_x^2(\eta - \tau u_t - \xi u_x) + \tau u_{txx} + \xi u_{xxx},
\end{aligned} \tag{3.3.3}$$

where D_t and D_x denote the operators of total differentiation with respect to t and x, respectively,

$$\mathrm{D}_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots, \qquad \mathrm{D}_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots.$$

Upon plugging the coefficients (3.3.3) into the infinitesimal invariance condition (3.3.1), we obtain the following equation

$$\begin{aligned}
&\mathrm{D}_t\eta - u_t\mathrm{D}_t\tau - u_x\mathrm{D}_t\xi - \xi f_x u_x^2 - \eta f_u u_x^2 - 2fu_x(\mathrm{D}_x\eta - u_t\mathrm{D}_x\tau - u_x\mathrm{D}_x\xi) - \xi g_x u_{xx} -\\
&\eta g_u u_{xx} - g(\mathrm{D}_x^2\eta - u_t\mathrm{D}_x^2\tau - u_x\mathrm{D}_x^2\xi - 2u_{tx}\mathrm{D}_x\tau - 2u_{xx}\mathrm{D}_x\xi) = 0.
\end{aligned} \tag{3.3.4}$$

In order to constrain this equation on the manifold of equations (3.1.1), we set $u_t = fu_x^2 + gu_{xx}$. Then, splitting (3.3.4) with respect to the various derivatives of u we obtain the following overdetermined system of determining equations of Lie symmetries:

$$\begin{aligned}
&u_xu_{xt}\colon && \tau_u = 0,\\
&u_{xt}\colon && \tau_x = 0,\\
&u_xu_{xx}\colon && \xi_u = 0,\\
&u_x^2\colon && f(\tau_t + \eta_u - 2\xi_x) + g\eta_{uu} + \xi f_x + \eta f_u = 0,\\
&u_{xx}\colon && g(\tau_t - 2\xi_x) + \xi g_x + \eta g_u = 0,\\
&u_x\colon && \xi_t + 2f\eta_x + g(2\eta_{xu} - \xi_{xx}) = 0,\\
&1\colon && \eta_t - g\eta_{xx} = 0.
\end{aligned} \tag{3.3.5}$$

As usual for classes of differential equations, the determining equations split into a part not involving the arbitrary elements and a part explicitly involving them (the *classifying part*). In the present case, the first three equations do not involve f and g and can therefore be integrated immediately. They give $\tau = \tau(t)$ and $\xi = \xi(t,x)$, i.e. the symmetry transformations are projectable and transformations of t only depend on t.

The remaining four equations form the system of classifying equations. In the case of arbitrariness of the functions f and g, we can further split system (3.3.5) with respect to derivatives of f and g. This yields the kernel of maximal Lie invariance algebras, which gives rise to those symmetry transformations that are admitted for all elements of the class of equations (3.1.1). Splitting yields

$$\xi = \eta = 0, \quad \tau_t = 0,$$

i.e. the *kernel algebra* $\mathfrak{g}^\cap$ is generated solely by the operator ∂_t, $\mathfrak{g}^\cap = \langle \partial_t \rangle$. That is, for arbitrary values of f and g, the only symmetry admitted by equations of the class (3.1.1) is the time translation symmetry $(t, x, u) \mapsto (t + \varepsilon, x, u)$, $\varepsilon \in \mathbb{R}$.

3.4 The equivalence algebra

In order to investigate inequivalent cases of symmetry extensions of the kernel algebra $\mathfrak{g}^\cap$, the equivalence algebra (group) must be computed. Unfortunately, the equivalence algebra presented in [93] is not correct. It can easily be checked that their operator $b(x)\partial_u$ does not generate an equivalence transformation for general values of b. Similarly, also their operator $a(x)\partial_x + 2fa(x)\partial_f + ga'(x)\partial_g$ cannot generate equivalence transformations for arbitrary values of a. The problem indeed is that their infinitesimal invariance condition for equivalence transformations is incorrect. This is why it is necessary to re-derive the equivalence algebra for the class of equations (3.1.1) here.

Theorem 3.11. *The equivalence algebra* $\mathfrak{g}^\sim$ *of the class of equations* (3.1.1) *is generated by the following operators,*

$$\begin{aligned} &\partial_t, \quad \partial_x, \quad \mathcal{D}^t = t\partial_t - f\partial_f - g\partial_g, \quad \mathcal{D}^x = x\partial_x + 2f\partial_f + 2g\partial_g, \\ &\mathcal{G}(h) = h\partial_u - (h_u f + h_{uu} g)\partial_f, \end{aligned} \tag{3.4.1}$$

where $h = h(u)$ *is an arbitrary smooth function of* u.

Proof. The proof is done using infinitesimal methods. We seek for operators of the form

$$Y = \tau\partial_t + \xi\partial_x + \eta\partial_u + \varphi\partial_f + \theta\partial_g$$

that generate continuous equivalence transformations, where τ, ξ and η are functions of the variables t, x and u, whereas φ and θ are regarded as functions of t, x, u, f and g. That is, we aim to determine the usual equivalence algebra rather than some generalized equivalence algebra [92, 127]. The class of equations (3.1.1) must be augmented with the auxiliary system

$$S_1 := f_t = 0, \quad S_2 := g_t = 0. \tag{3.4.2}$$

The complete auxiliary system should also include the conditions that the arbitrary elements f and g do not depend on nonzero order derivatives of u. However, these conditions already are implicitly taken into account by the supposition that the coefficients of Y does not involve these derivatives.

The joint invariance condition then reads

$$\tilde{Y}\Delta\big|_{\mathcal{M}} = 0, \quad \tilde{Y}S_1\big|_{\mathcal{M}} = 0, \quad \tilde{Y}S_2\big|_{\mathcal{M}} = 0, \tag{3.4.3}$$

where $\mathcal{M}$ denotes the joint system of the equations $\Delta = 0$, $S_1 = 0$ and $S_2 = 0$,

$$\tilde{Y} = Q^{(2)} + \varphi\partial_f + \theta\partial_g + \varphi^t\partial_{f_t} + \theta^t\partial_{g_t},$$

and $Q^{(2)}$ is defined by (3.3.2). The coefficients φ^t and θ^t can be obtained by the first prolongation considering (t,x,u) and (f,g) as independent and dependent variables, respectively,

$$\varphi^t = \tilde{\mathrm{D}}_t(\varphi - \tau f_t - \xi f_x - \eta f_u) + \tau f_{tt} + \xi f_{tx} + \eta f_{tu},$$
$$\theta^t = \tilde{\mathrm{D}}_t(\varphi - \tau g_t - \xi g_x - \eta g_u) + \tau g_{tt} + \xi g_{tx} + \eta g_{tu},$$

where $\tilde{\mathrm{D}}_t = \partial_t + f_t\partial_f + g_t\partial_g + \cdots$ is the corresponding operator of total differentiation with respect to t. In view of the auxiliary system (3.4.2), the total derivative operator reduces to the partial derivative, i.e. $\tilde{\mathrm{D}}_t = \partial_t$.

The second and the third conditions from (3.4.3) then imply that

$$\varphi_t - \xi_t f_x - \eta_t f_u = 0, \quad \theta_t - \xi_t g_x - \eta_t g_u = 0.$$

Since these equations should be satisfied for all values of the arbitrary elements f and g, we can split with respect to the derivatives f_x, f_u, g_x and g_u to obtain that

$$\varphi_t = \theta_t = \xi_t = \eta_t = 0.$$

It remains to investigate the first condition in (3.4.3). In detail, it reads

$$\eta^t - 2fu_x\eta^x - \varphi u_x^2 - g\eta^{xx} - \theta u_{xx} = 0,$$

or, after expanding,

$$\mathrm{D}_t\eta - u_t\mathrm{D}_t\tau - u_x\mathrm{D}_t\xi - 2fu_x(\mathrm{D}_x\eta - u_t\mathrm{D}_x\tau - u_x\mathrm{D}_x\xi) - \varphi u_x^2 -$$
$$g(\mathrm{D}_x^2\eta - u_t\mathrm{D}_x^2\tau - u_x\mathrm{D}_x^2\xi - 2u_{tx}\mathrm{D}_x\tau - 2u_{xx}\mathrm{D}_x\xi) - \theta u_{xx} = 0.$$

We now split this equation with respect to the derivatives of u similar as done in the course of deriving the determining equations of Lie symmetries. The splitting with respect to u_{tx} implies that $\tau = \tau(t)$. Splitting with respect to $u_x u_{xx}$, we derive that $\xi = \xi(x)$. These conditions already simplifies the above invariance condition substantially. Collecting coefficients of the remaining monomials of derivatives leads to

$$\begin{aligned}
u_{xx}\colon &\quad \theta = (2\xi_x - \tau_t)g,\\
u_x^2\colon &\quad \varphi = (2\xi_x - \tau_t - \eta_u)f - \eta_{uu}g,\\
u_x\colon &\quad 2\eta_x f + 2\eta_{xu}g - \xi_{xx}g = 0,\\
1\colon &\quad g\eta_{xx} = 0.
\end{aligned}$$

In view of $\varphi_t = \theta_t = \xi_t = \eta_t = 0$, the general solution of this system is

$$\tau = c_1t + c_2, \quad \xi = c_3x + c_4, \quad \eta = h(u),$$
$$\varphi = (2c_3 - c_1 - h_u)f - h_{uu}g, \quad \theta = (2c_3 - c_1)g,$$

where c_1, …, c_4 are arbitrary constants and h is an arbitrary smooth function of u.

This completes the proof of the theorem. □

The equivalence algebra $\mathfrak{g}^\sim$ can be represented in several ways, which are important for different purposes. The representation crucial for the present case is that $\mathfrak{g}^\sim = \langle\partial_t\rangle \Subset \langle\mathcal{D}^x, \mathcal{D}^t, \partial_x, \mathcal{G}(h)\rangle$, see Remark 3.15 for further details. Another natural representation is $\mathfrak{g}^\sim = \langle\partial_t \Subset \mathcal{D}^t\rangle \oplus \langle\partial_x \Subset \mathcal{D}^x\rangle \oplus \langle\mathcal{G}(h)\rangle$. This representation implies that $\mathfrak{g}^\sim$ is the direct sum of a finite-dimensional and an infinite-dimensional parts. This representation is helpful for the determination of the adjoint actions, see Section 3.7.

3.5 The equivalence group

In the previous section we have determined the equivalence algebra of the class (3.1.1) using infinitesimal techniques. In order to obtain the complete point equivalence group (including also discrete transformations), the direct method should be applied. For the sake of completeness and illustration we present the corresponding computations here.

Theorem 3.12. *The equivalence group* $G^{\sim}$ *of the class of equations* (3.1.1) *is formed by the transformations*

$$\tilde t = A_1 t + A_0, \quad \tilde x = B_1 x + B_0, \quad \tilde u = U(u),$$
$$\tilde f = \frac{B_1^2}{A_1 U_u}\left(f - \frac{U_{uu}}{U_u} g\right), \quad \tilde g = \frac{B_1^2}{A_1} g,$$

where $A_0, A_1, B_0, B_1 \in \mathbb{R}$, U *is an arbitrary smooth function of* u *and* $A_1 B_1 U_u \neq 0$.

Proof. We begin with a preliminary description of admissible transformations of the class (3.1.1). In other words, we derive determining equations for point transformations that map a fixed equation from the class (3.1.1) to an equation from the same class. As (3.1.1) defines a subclass of $(1+1)$-dimensional evolution equations, we at once know that the transformation component of t depends only on t, see e.g. [73, 88]. Moreover, each equation from the class (3.1.1) belongs to the class of second-order quasi-linear evolution equations having the form $u_t = F(t,x,u)u_{xx} + G(t,x,u,u_x)$. Hence in view of Lemma 1 of [65] the transformation component of x depends only on t and x. That is, the transformations of variables will be of the form $\tilde t = T(t)$, $\tilde x = X(t,x)$, $\tilde u = U(t,x,u)$, where $T_t X_x U_u \neq 0$. The transformed derivatives then read

$$\tilde u_{\tilde t} = \frac{1}{T_t}\left(\mathrm{D}_t U - \frac{X_t}{X_x}\mathrm{D}_x U\right), \quad \tilde u_{\tilde x} = \frac{1}{X_x}\mathrm{D}_x U, \quad \tilde u_{\tilde x\tilde x} = \left(\frac{1}{X_x}\mathrm{D}_x\right)^2 U.$$

Substituting these derivatives into the transformed form of (3.1.1) and taking into account the initial form (3.1.1), we obtain

$$\frac{1}{T_t}\left(U_t - \frac{X_t}{X_x}\mathrm{D}_x U\right) + \frac{U_u}{T_t}(f u_x^2 + g u_{xx}) = \tilde f \left(\frac{\mathrm{D}_x U}{X_x}\right)^2 + \tilde g \left(\frac{1}{X_x}\mathrm{D}_x\right)^2 U, \tag{3.5.1}$$

where $\tilde f = \tilde f(X,U)$ and $\tilde g = \tilde g(X,U)$. Splitting equation (3.5.1) with respect to u_{xx} and u_x yields

$$u_{xx}\colon \quad \tilde g = \frac{{X_x}^2}{T_t} g, \tag{3.5.2a}$$

$$u_x^2\colon \quad f\frac{U_u}{T_t} = \tilde f \frac{U_u^2}{X_x^2} + \tilde g \frac{U_{uu}}{X_x^2}, \tag{3.5.2b}$$

$$u_x\colon \quad -\frac{X_t U_u}{T_t X_x} = \frac{\tilde g}{X_x^2}\left(2U_{ux} - \frac{X_{xx}}{X_x}U_u\right) + 2\tilde f \frac{U_x U_u}{X_x^2}, \tag{3.5.2c}$$

$$1\colon \quad \frac{1}{T_t}\left(U_t - \frac{X_t}{X_x}U_x\right) = \tilde f \left(\frac{U_x}{X_x}\right)^2 + \frac{\tilde g}{X_x^2}\left(U_{xx} - \frac{X_{xx}}{X_x}U_x\right). \tag{3.5.2d}$$

Equation (3.5.2a) defines the transformation rule for the arbitrary element g. Substituting equation (3.5.2a) into (3.5.2b) leads to the transformation rule for the arbitrary element f,

$$\tilde f = \frac{X_x^2}{T_t U_u} f - \frac{X_x^2 U_{uu}}{T_t U_u^2} g.$$

In general, system (3.5.2) forms the determining equations for admissible transformations of the class (3.1.1). Unfortunately, this system is too difficult to be integrated since there are a lot of different cases of its solution depending on specific values of the arbitrary elements. However, this system allows to easily determine the equivalence group. For this aim, we can split equations (3.5.2c) and (3.5.2d) with respect to $\tilde{f}$ and $\tilde{g}$. This gives at once $X_t = X_{xx} = U_x = U_t = 0$ since $U_u \neq 0$. Furthermore, differentiating the first equation of system (3.5.2) with respect to t leads to the final restriction $T_{tt} = 0$. Solving these determining equations for equivalence transformations completes the proof of the theorem. □

Corollary 3.13. *A complete set of discrete equivalence transformations in the group $G^{\sim}$, which are independent up to their composition and composition with continuous transformations are exhausted by the three transformations of alternating signs*

$$
\begin{aligned}
I_t\colon &\quad (t,x,u,f,g) \mapsto (-t,x,u,-f,-g),\\
I_x\colon &\quad (t,x,u,f,g) \mapsto (t,-x,u,f,g),\\
I_u\colon &\quad (t,x,u,f,g) \mapsto (t,x,-u,-f,g).
\end{aligned}
$$

The equation (3.1.1) with the specific value $\theta_0 = (f,g) = (-4/3u^{-7/3}, u^{-4/3})$ admits the Lie symmetry operator $x^2\partial_x - 3xu\partial_u$. The transformations from the corresponding one-parameter transformation group belong to $\mathrm{T}(\theta_0,\theta_0)$. As the associated admissible transformations are not induced by elements of the equivalence group $G^{\sim}$ of the class (3.1.1), this class is not normalized. Similar assertions are true for the potential Burgers equations ($f = g = 1$), linear equations from the class (3.1.1) ($f = 0$, $g_u = 0$), etc. As system (3.5.2) is too complicated and the equivalence group $G^{\sim}$ is quite narrow in comparison with the class (3.1.1) (the transformations from $G^{\sim}$ are parameterized by four constants and only a singe function of one argument and, at the same time, the tuple of arbitrary elements consists of two functions of two arguments), this justifies why preliminary group classification is well suited for the class of equations (3.1.1).

Remark 3.14. It is not possible to simplify the general equation from the class (3.1.1) by equivalence transformations. The interesting particular case of simplification by equivalence transformations is given by equations of the form (3.1.1) with f proportional to g. If $f = cg$, where c is a nonzero constant, then the corresponding equation of the form (3.1.1) is mapped by the transformation

$$\tilde{t} = t,\ \ \tilde{x} = x,\ \ \tilde{u} = e^{cu} \tag{3.5.3}$$

to the equation of the same form with $\tilde{f} = 0$ and $\tilde{g} = g(\tilde{x}, c^{-1}\ln\tilde{u})$.

3.6 Classification of subalgebras

In order to carry out preliminary group classification, it is necessary to derive an optimal list of inequivalent subalgebras. In the existing literature on the subject, usually only subalgebras of a certain finite-dimensional subalgebra of the equivalence algebra are classified up to inner automorphisms of this subalgebra. This restriction is, however, not necessary in the present case, although this is done in [93]. Furthermore, it should be noted that in [93] an algebra was chosen for preliminary group classification, which is not related to the corresponding equivalence algebra that was derived.

To classify subalgebras of a Lie algebra of vector fields, it is necessary to know the adjoint action of the corresponding transformation (pseudo)group on this algebra. There exist two different methods for the computation of the adjoint action. The first method employs information on the structure of the Lie algebra and is more suitable in the finite-dimensional case although it also works for certain infinite-dimensional algebras [18, 20, 49, 123]. The adjoint action of a one-parameter Lie group generated by an element $\mathbf{v}$ of the Lie algebra on this algebra can be determined either from the Lie series

$$\mathbf{w}(\varepsilon) = \mathrm{Ad}(e^{\varepsilon\mathbf{v}})\mathbf{w}_0 := \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\{\mathbf{v}^n, \mathbf{w}_0\},$$

where $\{\mathbf{v}^0, \mathbf{w}_0\} := \mathbf{w}_0$, $\{\mathbf{v}^n, \mathbf{w}_0\} := (-1)^n[\mathbf{v}, \{\mathbf{v}^{n-1}, \mathbf{w}_0\}]$, or, by solving the Cauchy problem

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\varepsilon} = [\mathbf{w}, \mathbf{v}], \quad \mathbf{w}(0) = \mathbf{w}_0,$$

see [101] for more details.

Following the second method, it is necessary only to calculate the actions of push-forwards of the transformations from the (pseudo)group on generating vector fields of the algebra. While the first method involves only the abstract structure of Lie algebras and therefore at once gives results for the whole class of isomorphic algebras, the second method relies on the specific realization of the Lie algebra by vector fields. At the same time, the second method works more properly in the infinite-dimensional case.

We now derive the optimal lists of one- and two-dimensional subalgebras for the entire equivalence algebra $\mathfrak{g}^\sim$.

The nonzero commutation relations of generating elements (3.4.1) of $\mathfrak{g}^\sim$ are

$$[\partial_x, \mathcal{D}^x] = \partial_x, \quad [\partial_t, \mathcal{D}^t] = \partial_t, \quad [\mathcal{G}(h^1), \mathcal{G}(h^2)] = \mathcal{G}(h^1 h^2_u - h^2 h^1_u).$$

The nonidentical adjoint actions related to generating elements of $\mathfrak{g}^\sim$ and computed using the first method are

$$\begin{array}{lll} \mathrm{Ad}(e^{\varepsilon\partial_x})\mathcal{D}^x = \mathcal{D}^x - \varepsilon\partial_x, & \mathrm{Ad}(e^{\varepsilon\mathcal{D}^x})\partial_x = e^\varepsilon\partial_x, & \mathrm{Ad}(e^{\varepsilon\mathcal{G}(h^1)})\mathcal{G}(h^2) = \mathcal{G}(\tilde h^2), \\ \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{D}^t = \mathcal{D}^t - \varepsilon\partial_t, & \mathrm{Ad}(e^{\varepsilon\mathcal{D}^t})\partial_t = e^\varepsilon\partial_t, & \end{array}$$

where $\tilde h^2(u,\varepsilon) = h^2(H^1(u,-\varepsilon))/H^1_u(u,-\varepsilon)$ and $\{\tilde u = H^1(u,\varepsilon)\}$ is the one-parameter transformation group generated by the projection of the operator $\mathcal{G}(h^1)$ to the space of the variable u, i.e. $H^1_\varepsilon = h^1(H^1)$ and $H^1(u,0) = u$. Although the four adjoint actions related to the finite-dimensional part of $\mathfrak{g}^\sim$ are suitable to be applied to the classification, there arises an inconvenience with the adjoint action $\mathrm{Ad}(e^{\varepsilon\mathcal{G}(h^1)})$ owing to problems with proving the existence of the required function h^1.

This is why, in what follows we use the adjoint action of the entire equivalence group $G^\sim$ on the equivalence algebra $\mathfrak{g}^\sim$, calculated by the second method. Any transformation $\mathcal{T}$ from $G^\sim$ can be represented, for convenience, as a composition

$$\mathcal{T} = \mathfrak{T}^t(A_0)\mathfrak{T}^x(B_0)\mathfrak{D}^t(A_1)\mathfrak{D}^x(B_1)\mathfrak{G}(U),$$

cf. Theorem 3.12, where

$$\begin{array}{llllll} \mathfrak{T}^t(A_0)\colon & \tilde t = t + A_0, & \tilde x = x, & \tilde u = u, & \tilde g = g, & \tilde f = f, \\ \mathfrak{T}^x(B_0)\colon & \tilde t = t, & \tilde x = x + B_0, & \tilde u = u, & \tilde g = g, & \tilde f = f, \\ \mathfrak{D}^t(A_1)\colon & \tilde t = A_1 t, & \tilde x = x, & \tilde u = u, & \tilde g = A_1^{-1} g, & \tilde f = A_1^{-1} f, \\ \mathfrak{D}^x(B_1)\colon & \tilde t = t, & \tilde x = B_1 x, & \tilde u = u, & \tilde g = B_1^2 g, & \tilde f = B_1^2 f, \\ \mathfrak{G}(U)\colon & \tilde t = t, & \tilde x = x, & \tilde u = U(u), & \tilde g = g, & \tilde f = f/U_u - gU_{uu}/U_u^2 \end{array} \tag{3.6.1}$$

are translations with respect to t and x, scalings with respect to t and x and an arbitrary transformation of u, respectively, and $A_1B_1U_u\neq 0$. Transformations of each of the above kinds form a subgroup of $G^{\sim}$. The last three subgroups contain the discrete transformations I_t, I_x and I_u, respectively. Namely, $I_t=\mathcal{D}^t(-1)$, $I_x=\mathcal{D}^x(-1)$ and $I_u=\mathcal{G}(-u)$. As a result, additionally to avoiding the above problems with the existence of required values of functional parameters, in this way we at once include discrete equivalence transformations in the classification procedure.

The nonidentical actions of push-forwards of transformations (3.6.1) on generating elements of $\mathfrak{g}^{\sim}$ are exhausted by the followings:

$$\begin{aligned}&\mathcal{T}^x_*(B_0)\mathcal{D}^x=\mathcal{D}^x-B_0\partial_x, &&\mathcal{D}^x_*(B_1)\partial_x=B_1\partial_x, &&\mathcal{G}_*(U)\mathcal{G}(h)=\mathcal{G}(h(\tilde U)/\tilde U_u),\\ &\mathcal{T}^t_*(A_0)\mathcal{D}^t=\mathcal{D}^t-A_0\partial_t, &&\mathcal{D}^t_*(A_1)\partial_t=A_1\partial_t,\end{aligned}$$

where the function $\tilde U=\tilde U(u)$ is the inverse of U.

Remark 3.15. The kernel algebra generated by ∂_t is an ideal in the equivalence algebra $\mathfrak{g}^{\sim}$, which has the structure $\mathfrak{g}^{\sim}=\langle\partial_t\rangle\Subset\langle\mathcal{D}^x,\mathcal{D}^t,\partial_x,\mathcal{G}(h)\rangle$. Hence the classification of subalgebras of $\mathfrak{g}^{\sim}$ can be reduced to the classification of subalgebras of the algebra $\mathfrak{g}^{\sim}_{\rm ess}=\langle\mathcal{D}^x,\mathcal{D}^t,\partial_x,\mathcal{G}(h)\rangle$, which is the "essential" part of $\mathfrak{g}^{\sim}$. This will yield the possible Lie invariance algebra extensions of the kernel algebra obtainable by preliminary group classification. Moreover, the push-forwards of translations and scalings with respect to t should not be applied under the classification of subalgebras.

Theorem 3.16. *An optimal list of one-dimensional subalgebras of the algebra $\mathfrak{g}^{\sim}_{\rm ess}$ is exhausted by the algebras*

$$\langle\mathcal{D}^x+a\mathcal{D}^t-\mathcal{G}(\delta)\rangle,\quad \langle\mathcal{D}^t+\tilde\delta\partial_x-\mathcal{G}(\delta)\rangle,\quad \langle\partial_x-\mathcal{G}(\delta)\rangle,\quad \langle\mathcal{G}(1)\rangle, \tag{3.6.2}$$

where $a\in\mathbb{R}$ and $\delta,\tilde\delta\in\{0,1\}$.

Proof. We use the approach for the classification of subalgebras that is outlined in [101]. We start with the most general form of an element of the algebra $\mathfrak{g}^{\sim}_{\rm ess}$,

$$\mathbf{v}^1=a^1_1\mathcal{D}^x+a^1_2\mathcal{D}^t+a^1_3\partial_x+\mathcal{G}(h^1),$$

where the constants a^1_1, a^1_2, a^1_3 and the function $h^1=h^1(u)$ are arbitrary but fixed, and simplify it as much as possible by means of push-forwards of transformations from the equivalence group $G^{\sim}$. In the case $h^1\neq 0$ the function-parameter h^1 can be set to -1 by usage of $\mathcal{G}_*(U)$ with the inverse U to a solution $\tilde U=\tilde U(u)$ of the equation $\tilde U_u=-h^1(\tilde U)$. In other words, up to $G^{\sim}$-equivalence we can always assume that $-h^1=\delta\in\{0,1\}$.

If $a^1_1\neq 0$, we scale $\mathbf{v}^1$ to set $a^1_1=1$ and use the push-forward of a $\mathcal{T}^x(B_0)$ to set $a^1_3=0$. The notation $a=a^1_2$ leads to the first subalgebra in the list (3.6.2).

If $a^1_1=0$ and $a^1_2\neq 0$, we set $a^1_2=1$ by scaling $\mathbf{v}^1$ and use $\mathcal{D}^x_*(B_1)$ with certain B_1 to set $a^1_3=-\tilde\delta$, where $\tilde\delta\in\{0,1\}$. This gives the second listed subalgebra.

In the remaining case $a^1_1=a^1_2=0$ we obtain the two last subalgebras from the list (3.6.2) under the assumptions $a^1_3\neq 0$ and $a^1_3=0$, respectively, since the nonvanishing value of a^1_3 is set to be equal to 1 by scaling $\mathbf{v}^1$ and the condition $a^1_3=0$ necessarily implies that $h^1\neq 0$ and hence, up to $G^{\sim}$-equivalence, $h^1=1$. □

Theorem 3.17. *An optimal list of two-dimensional subalgebras of the algebra $\mathfrak{g}^{\sim}_{\rm ess}$ reads*

$$\begin{gathered}\langle\mathcal{D}^x-\mathcal{G}(\hat\delta),\mathcal{D}^t-\mathcal{G}(\delta)\rangle,\quad \langle\mathcal{D}^x+a\mathcal{D}^t+\mathcal{G}(u),\partial_x-\mathcal{G}(1)\rangle,\quad \langle\mathcal{D}^x+a\mathcal{D}^t-\mathcal{G}(\delta),\partial_x\rangle\\ \langle\mathcal{D}^t-\mathcal{G}(\delta),\partial_x-\mathcal{G}(\tilde\delta)\rangle,\quad \langle\mathcal{D}^x+a\mathcal{D}^t+b\mathcal{G}(u),\mathcal{G}(1)\rangle,\quad \langle\mathcal{D}^t-\delta\partial_x+b\mathcal{G}(u),\mathcal{G}(1)\rangle,\\ \langle\partial_x-\delta\mathcal{G}(u),\mathcal{G}(1)\rangle,\quad \langle\mathcal{G}(1),\mathcal{G}(u)\rangle,\end{gathered} \tag{3.6.3}$$

where a, b, δ, $\tilde\delta$ and $\hat\delta$ are constants, and we can assume that $\delta,\tilde\delta\in\{0,1\}$, $\hat\delta\in\mathbb R$ if $\delta=1$ and $\hat\delta\in\{0,1\}$ if $\delta=0$.

Proof. The proof of the above theorem is similar to those in the one-dimensional case, see a detailed explanation and other examples in [13, Chapter 7]. We start with two linearly independent copies of the most general element of $\mathfrak g^{\sim}$,

$$\mathbf v^1=a^1_1\mathcal D^x+a^1_2\mathcal D^t+a^1_3\partial_x+\mathcal G(h^1),$$
$$\mathbf v^2=a^2_1\mathcal D^x+a^2_2\mathcal D^t+a^2_3\partial_x+\mathcal G(h^2),$$

and simplify them as much as possible by means of adjoint actions and nondegenerate linear combining. The additional complication concerns taking into account that the elements $\mathbf v^1$ and $\mathbf v^2$ should form a basis of a Lie algebra, i.e., their commutator should lie in their span, $[\mathbf v^1,\mathbf v^2]\in\langle\mathbf v^1,\mathbf v^2\rangle$. Usually this places further restrictions on the admitted form of the elements.

To simply describe the conditions defining the different cases of the classification of two-dimensional subalgebras, we introduce the matrix notation

$$A_{\mu_1\cdots\mu_n}:=\begin{pmatrix}a^1_{\mu_1}&\cdots&a^1_{\mu_n}\\a^2_{\mu_1}&\cdots&a^2_{\mu_n}\end{pmatrix},$$

where $\mu_i\in\{1,2,3\}$ and $n\leqslant3$. In what follows, the right hand side of a matrix equation $A_{\mu_1\cdots\mu_n}=0$ or a matrix inequality $A_{\mu_1\cdots\mu_n}\neq0$ is the zero matrix of the appropriate dimension.

In the course of classification, we should investigate two principal cases.

1. $\operatorname{rank}(A_{123})=2$. This is the first cases which is partitioned into the three subcases

$$\text{(a)}\ \det A_{12}\neq0;\quad \text{(b)}\ \det A_{12}=0,\ \det A_{13}\neq0;\quad \text{(c)}\ \det A_{12}=0,\ \det A_{13}=0.$$

In the last subcase we necessarily have $\det A_{23}\neq0$. By means of a change of the basis we at first set $A_{12}=E$, $A_{13}=E$ and $A_{23}=E$, respectively. Here E is the 2×2 identity matrix. If the new h^2 is nonvanishing, we set $h^2=-1$ using $\mathcal G_*(U)$ with the inverse U to a solution of the equation $\tilde U_u=-h^2(\tilde U)$. In other words, up to $G^{\sim}$-equivalence we can always assume that $-h^2=\delta\in\{0,1\}$. We also set $h^1\in\{-1,0\}$ in a similar way if $h^2=0$. Specifically, in subcase (a) we further use the push-forward of $\mathcal T^x(a^1_3)$ to set $a^1_3=0$. As the resulting operators should commute, we derive that $a^2_3=0$ and $h^1_u=0$. This case hence leads to the first subalgebra from the list (3.6.3). In subcase (b) we re-denote a^1_2 by a. Under the assumptions made, the commutator $[\mathbf v^1,\mathbf v^2]$ equals $-\mathbf v_2$. Therefore, the condition $h_2=-1$ implies that $h^1_u=1$, i.e. we can set $h^1=u$ using a change of the basis and the push-forward of $\mathcal T^x(B_0)$ with certain B_0. This gives the second subalgebra from the list (3.6.3). If $h_2=0$, we obtain the third subalgebra. In subcase (c), the corresponding subalgebra is commutative and hence $h^1_u=0$. Applying a scaling of $\mathbf v_2$ and the push-forward of $\mathcal D^x(B_1)$ with certain B_1, we simultaneously set $h^1,h^2\in\{-1,0\}$ and hence construct the fourth listed subalgebra.

2. $\operatorname{rank}(A_{123})\leqslant1$. Up to a change of the basis, we can assume that $a^2_1=a^2_2=a^2_3=0$ and hence $h^2\neq0$, i.e., analogously to the previous case we can set $h^2=1$ by some $\mathcal G_*(U)$. Then up to a linear combining of $\mathbf v^1$ and $\mathbf v^2$ the commutation condition $[\mathbf v^1,\mathbf v^2]\in\langle\mathbf v^1,\mathbf v^2\rangle$ implies that $h^1_u=b=\text{const}$ and, therefore, we can set $h^1=bu$. The four last algebras from the list (3.6.3) represent the subcases

$$\text{(a)}\ a^1_1\neq0;\quad \text{(b)}\ a^1_1=0,a^1_2\neq0;\quad \text{(c)}\ a^1_1=0,a^1_2=0,a^1_3\neq0;\quad \text{(d)}\ a^1_1=a^1_2=a^1_3=0,$$

in which by a scaling of $\mathbf v_1$ we can set $a^1_1=1$, $a^1_2=1$, $a^1_3=1$ and $b=1$, respectively, For the basis elements to have the appropriate canonical form, we should additionally set $a^1_3=0$ by some $\mathcal{T}^x_*(B_0)$ and re-denote a^1_2 by a in subcase (a) and also set $a^1_3 \in \{-1, 0\}$ by some $\mathcal{D}^x_*(B_1)$ in subcase (b) and $b \in \{-1, 0\}$ by some $\mathcal{D}^x_*(B_1)$ and a scaling of $\mathbf{v}_1$ in subcase (c).

This completes the proof of the theorem. □

Note that all except the last subalgebras from the lists (3.6.2) and (3.6.3) represent parameterized classes of subalgebras rather than single subalgebras.

3.7 Preliminary group classification

Based on Proposition 3.3 and the above classification of subalgebras, we can obtain the extensions of the kernel algebra $\langle\partial_t\rangle$ within the class (3.1.1) by projections of inequivalent one- and two-dimensional subalgebras of the equivalence algebra $\mathfrak{g}^\sim$ to the space of variables (t, x, u). As a first step, for each of the subalgebras we solve the associated invariant surface condition for (f, g), namely, the system of equations $\xi f_x + \eta f_u = \varphi$, $\xi g_x + \eta g_u = \theta$, where the operator $\tau\partial_t + \xi\partial_x + \eta\partial_u + \varphi\partial_f + \theta\partial_g$ runs through a basis of the subalgebra.

In Tables 3.1 and 3.2 we collect the general solutions of the invariant surface condition for (f, g) (or, in other words, the entire subclass of the corresponding invariant equations), which is associated with the one- and two-dimensional subalgebras of $\mathfrak{g}^\sim$ listed in (3.6.2) and (3.6.3), respectively. In these tables, $\tilde f$ and $\tilde g$ are arbitrary functions of single arguments and c_1 and c_2 are arbitrary constants such that $\tilde g \neq 0$ and $c_2 \neq 0$.

Table 3.1: One-dimensional Lie symmetry extensions for class (3.1.1) related to $\mathfrak{g}^\sim$.

N	f	g	Additional operator
1	$\tilde f(u + \delta\ln\lvert x\rvert)x^{2-a}$	$\tilde g(u + \delta\ln\lvert x\rvert)x^{2-a}$	$at\partial_t + x\partial_x - \delta\partial_u$
2a	$\tilde f(u + \delta x)e^{-x}$	$\tilde g(u + \delta x)e^{-x}$	$t\partial_t + \partial_x - \delta\partial_u$
2b	$\tilde f(x)e^u$	$\tilde g(x)e^u$	$t\partial_t - \partial_u$
3	$\tilde f(u + \delta x)$	$\tilde g(u + \delta x)$	$\partial_x - \delta\partial_u$
4	$\tilde f(x)$	$\tilde g(x)$	∂_u

Table 3.2: Two-dimensional Lie symmetry extensions for class (3.1.1) related to $\mathfrak{g}^\sim$.

N	f	g	Additional operators
1	$c_1e^ux^{2+\tilde\delta}$	$c_2e^ux^{2+\tilde\delta}$	$x\partial_x - \tilde\delta\partial_u$, $t\partial_t - \partial_u$
2	$c_1\lvert u + x\rvert^{1-a}$	$c_2\lvert u + x\rvert^{2-a}$	$at\partial_t + x\partial_x + u\partial_u$, $\partial_x - \partial_u$
3a	$c_1e^{(2-a)u}$	$c_2e^{(2-a)u}$	$at\partial_t + x\partial_x - \partial_u$, ∂_x
3b	$\tilde f(u)$	$\tilde g(u)$	$2t\partial_t + x\partial_x$, ∂_x
4	$c_1e^{u+\tilde\delta x}$	$c_2e^{u+\tilde\delta x}$	$t\partial_t - \partial_u$, $\partial_x - \tilde\delta\partial_u$
5	$c_1\lvert x\rvert^{2-a-b}$	$c_2\lvert x\rvert^{2-a}$	$at\partial_t + x\partial_x + bu\partial_u$, ∂_u
6	$c_1e^{(1+b)x}$	c_2e^x	$t\partial_t - \partial_x + bu\partial_u$, ∂_u
7	$c_1e^{\delta x}$	c_2	$\partial_x - \delta u\partial_u$, ∂_u
8	0	$\tilde g(x)$	∂_u, $u\partial_u$

The second algebra from the list of one-dimensional subalgebras (3.6.2) is associated with a symmetry extension of an equation from the class (3.1.1) if and only if at least one of its parameters δ and $\tilde\delta$ does not vanish. In addition, to find the corresponding ansatzes for f and g it is necessary to consider different cases of values of the parameters. This is why this subalgebra leads to two cases (2a and 2b) of Table 3.1. Analogously, equations from the class (3.1.1) are invariant with respect to the projections of the first, fourth or sixth algebras from the list of two-dimensional subalgebras (3.6.3) if and only if $\delta\neq 0$, i.e., we can assume that $\delta=1$. For the third algebra we should have either $\delta\neq 0$ (then we can again assume that $\delta=1$) or $(\delta,a)=(0,2)$ that gives Cases 3a and 3b, respectively.

There are several reasons why Tables 3.1 and 3.2 do not give a proper classification result. We present these reasons in the form of the following series of remarks.

Remark 3.18. As the whole consideration is done up to $G^\sim$-equivalence, we should additionally factorize the general solutions of the invariant surface conditions for f and g with respect to this equivalence. Using transformations from $G^\sim$, in Table 3.2 we can set $c_2=1$ (by scaling of t and alternating its sign) and, in Cases $5_{b=0}$, $6_{b=0}$ and $7_{\delta=0}$, $c_1=0$ (by the transformation (3.5.3) with $c=c_1/c_2$, cf. Remark 3.14). For the other values of the parameters b and δ in these cases, the constant c_1 can be assumed, up to $G^\sim$-equivalence, to belong to $\{0,1\}$. If $a\neq 2$ in Case 3, we can scale the value $2-a$ to 1.

Remark 3.19. Extensions presented in Tables 3.1 and 3.2 are not necessarily maximal even for the general values of the parameter-functions $\tilde f$ and $\tilde g$ or the constant parameters c_1 and c_2. It lies in the nature of preliminary group classification that equations can admit operators which are not projections of operators of the equivalence algebra. For example, in the last case of Table 3.2 any corresponding equation is linear and therefore admits an infinite-dimensional Lie invariance algebra including also the operators of the form $\varphi(t,x)\partial_u$, where φ runs through the set of solutions of the equation under consideration. (Of course, for certain values of g this equation possesses an even wider Lie invariance algebra, cf. [83, 112].) A similar remark is true for Case $5_{b=0}$, (resp. Case $6_{b=0}$, resp. Case $7_{\delta=0}$) of Table 3.2 since each of the equations corresponding to this case is reduced by an equivalence transformation to the linear equation with $f=0$ and $g=|x|^{2-a}$ (resp. $g=e^x$, resp. $g=1$), cf. Remark 3.18.

Remark 3.20. What is more essential is that presented extensions are not maximal even among extensions related to subalgebras of $\mathfrak g^\sim$. In particular, Case $3_{\delta=0}$ of Table 3.1 coincides by the arbitrary elements with Case 3b of Table 3.2 and hence should be excluded from the extension list. Within Table 3.2, if $a=2$ the arbitrary elements in Cases 3a and $5_{b=0}$ coincide with those of Case $7_{\delta=0}$. Hence in Case $7_{\delta=0}$ we have the additional operator $2t\partial_t+x\partial_x$ induced by the operator $\mathcal D^x+2\mathcal D^t$. The algebra presented in Case $3_{a\neq2}$ is also not maximal, cf. Case 1 of Table 3.3. Cases 4, 6 and 7 admit additional extensions by the operator $u\partial_u$ if $c_1=0$ or $e^{-c_1u/c_2}\partial_u$ if $c_1\neq 0$ and $b=0$ (resp. $\delta=0$), owing to the connection of these cases with Case 8 via the transformation (3.5.3).

Remark 3.21. An effect of the lack of maximality of extensions is that under the simplification of the form of arbitrary elements by equivalence transformations the corresponding invariance algebra may be replaced a similar one. Thus, under setting $c_1=0$ in Cases $5_{b=0}$, $6_{b=0}$ and $7_{\delta=0}$ the basis element ∂_u is replaced by $u\partial_u$.

In order to complete the preliminary group classification of the class (3.1.1), we should at first construct the exhaustive list of $G^\sim$-inequivalent subalgebras of $\mathfrak g^\sim$ whose projections to the space of the variables (t, x, u) are Lie invariance algebras of equations from the class (3.1.1). For convenience such subalgebras will be called appropriate. Then we should study the problem whether these subalgebras are maximal among the subalgebras with the same property for a certain subclass of the class (3.1.1). The majority of one- and two-dimensional subalgebras of $\mathfrak{g}^{\sim}$ are appropriate. This is why for subalgebra dimensions one and two it is not too important whether all or only appropriate subalgebras are classified but this is not the case for greater dimensions. As the arbitrary elements f and g depend on two arguments, the condition that the associated projection is a Lie invariance algebra of an equation from the class (3.1.1) is a strong restriction for subalgebras of $\mathfrak{g}^{\sim}$ of dimension greater than two and even leads to the boundedness of dimension of such subalgebras.

Let $\mathfrak{g}_1^{\sim} = \langle \mathcal{D}^t, \mathcal{G}(h) \rangle$, where h runs through the set of smooth functions of u. For a subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\sim}$, we denote $\dim \mathfrak{s} \cap \mathfrak{g}_1^{\sim}$ by $m_{\mathfrak{s}}$.

Lemma 3.22. *$\mathcal{D}^t \notin \mathfrak{s}$ and $m_{\mathfrak{s}} \leqslant 2$ for any appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\sim}$.*

Proof. Let $\mathfrak{s}$ be an appropriate subalgebra of $\mathfrak{g}^{\sim}$. Then the system of invariant surface conditions associated with elements of $\mathfrak{s}$ should have a solution (f^0, g^0) with $g^0 \neq 0$. The invariant surface condition for g associated with the operator $\mathcal{D}^t$ is $g = 0$ that contradicts the auxiliary inequality $g \neq 0$. Hence $\mathcal{D}^t \notin \mathfrak{s}$.

In what follows, the indices i and j run from 1 to 3. Suppose that the subalgebra $\mathfrak{s}$ contains at least three linearly independent elements from $\mathfrak{g}_1^{\sim}$, $\mathbf{v}^i = \mathcal{G}(h^i) + a^i \mathcal{D}^t$. The corresponding invariant surface conditions for g form the system $h^i g_u + a^i g = 0$. We consider it as a homogenous system of linear algebraic equations with respect to (g_u, g). This system should have a nonzero solution since $g \neq 0$. Therefore $h^i a^j - h^j a^i = 0$. In view of the linear independence of $\mathbf{v}^1$, $\mathbf{v}^2$ and $\mathbf{v}^3$, this implies that all $a^i = 0$ and thus $g_u = 0$. Now we interpret the system of invariant surface conditions $h^i f_u + h^i_u f + h^i_{uu} g = 0$ for f as a homogenous system of linear algebraic equations with respect to (f_u, f, g). As $g \neq 0$, this system should possess a nonzero solution and hence the determinant of its matrix vanishes. At the same time, the determinant coincides with the Wronskian of the linearly independent functions h^1, h^2 and h^3, which is not equal to zero. The contradiction obtained implies that $m_{\mathfrak{s}} \leqslant 2$. □

Corollary 3.23. *Any appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\sim}$ is of dimension not greater than four.*

Proof. The projection of any element from $\mathfrak{s} \setminus \mathfrak{g}_1^{\sim}$ to $\langle \mathcal{D}^x, \partial_x \rangle$ should be nonzero. Therefore, $\dim \mathfrak{s} \leqslant \dim \langle \mathcal{D}^x, \partial_x \rangle + m_{\mathfrak{s}} = 4$. □

Corollary 3.24. *$\mathfrak{s} \cap \mathfrak{g}_1^{\sim} = \mathfrak{s} \cap \langle \mathcal{G}(h) \rangle$ for any appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\sim}$ with $m_{\mathfrak{s}} = 2$, where h runs through the set of smooth functions of u.*

Proof. As $m_{\mathfrak{s}} = 2$, the subalgebra $\mathfrak{s}$ contains two linearly independent elements from $\mathfrak{g}_1^{\sim}$, $\mathbf{v}^i = \mathcal{G}(h^i) + a^i \mathcal{D}^t$, $i = 1, 2$. Analogously to the proof of Lemma 3.22, we consider the system of the invariant surface conditions $h^i g_u + a^i g = 0$ for g associated with $\mathbf{v}^i$ as a homogenous system of linear algebraic equations with respect to (g_u, g), which has a nonzero solution since $g \neq 0$. Therefore, the determinant of its matrix equal zero, $h^1 a^2 - h^2 a^1 = 0$. In view of the linear independence of $\mathbf{v}^1$ and $\mathbf{v}^2$, this implies that $a^1 = a^2 = 0$. □

As we have classified all one- and two-dimensional subalgebras of $\mathfrak{g}^{\sim}$, it is enough to describe appropriate subalgebras only of dimensions greater than 2.

Theorem 3.25. *A complete list of $G^\sim$-inequivalent appropriate subalgebras of $\mathfrak{g}^\sim$ is exhausted by the following subalgebras:*

$$\langle \mathcal{D}^x + \mathcal{G}(2), \partial_x, \mathcal{D}^t - \mathcal{G}(1)\rangle, \quad \langle \mathcal{D}^x + 2\mathcal{D}^t + b\mathcal{G}(u), \partial_x, \mathcal{G}(1)\rangle,$$
$$\langle \mathcal{D}^x + a\mathcal{D}^t, \mathcal{G}(1), \mathcal{G}(u)\rangle, \quad \langle \partial_x - \delta\mathcal{D}^t, \mathcal{G}(1), \mathcal{G}(u)\rangle, \quad \langle \mathcal{D}^x + 2\mathcal{D}^t, \partial_x, \mathcal{G}(1), \mathcal{G}(u)\rangle,$$

where a, b and δ are constants and we can assume that $\delta \in \{0,1\}$.

Proof. Let $\mathfrak{s}$ be an appropriate subalgebra of $\mathfrak{g}^\sim$ and $\dim\mathfrak{s} \geqslant 3$. Then, $m_\mathfrak{s} > 0$. Hence we should consider only the cases $m_\mathfrak{s} = 1$ and $m_\mathfrak{s} = 2$.

The condition $m_\mathfrak{s} = 1$ means that the subalgebra $\mathfrak{s}$ contains exactly one operator of the form $\mathbf{v}^1 = \mathcal{G}(h^1) + a_2^1\mathcal{D}^t$, where $h^1 \neq 0$, in view of Lemma 3.22. By scaling of $\mathbf{v}^1$ we can set $a_2^1 = -1$ if $a_2^1 \neq 0$. Moreover, as the function-parameter h^1 does not vanish it can be set to 1 upon using $\mathcal{G}_*(U)$ with the inverse U to a solution $\tilde U = \tilde U(u)$ of the equation $\tilde U_u = h^1(\tilde U)$. As a result we have two $G^\sim$-inequivalent forms for $\mathbf{v}^1$: (i) $\mathbf{v}^1 = \mathcal{G}(1) - \mathcal{D}^t$, (ii) $\mathbf{v}^1 = \mathcal{G}(1)$. The conditions $\dim\mathfrak{s} \geqslant 3$ and $m_\mathfrak{s} = 1$ simultaneously imply that $\dim\mathfrak{s} = 3$. This is why we should have two more linearly independent operators of the form $\mathbf{v}^i = a_1^i\mathcal{D}^x + a_2^i\mathcal{D}^t + a_3^i\partial_x + \mathcal{G}(h^i)$, $i = 2, 3$, from $\mathfrak{s} \setminus \mathfrak{g}_1^\sim$ for which $\operatorname{rank}(a_1^i, a_3^i)_{i=2,3} = 2$, cf. the proof of Corollary 3.23. By linear combining of $\mathbf{v}^2$ and $\mathbf{v}^3$ we set $a_1^2 = a_3^3 = 1$ and $a_3^2 = a_1^3 = 0$.

In subcase (i) we additionally subtract $a_2^i\mathbf{v}^1$ from $\mathbf{v}^i$ to obtain $a_2^i = 0$ in the new operator $\mathbf{v}^i$, $i = 2, 3$. The simplified form of $\mathbf{v}^2$ and $\mathbf{v}^3$ is $\mathbf{v}^2 = \mathcal{D}^x + \mathcal{G}(h^2)$ and $\mathbf{v}^3 = \mathcal{D}^t + \mathcal{G}(h^3)$, respectively. As $\mathfrak{s}$ is a Lie algebra and $\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$ is a basis of $\mathfrak{s}$, any commutator of $\mathbf{v}$'s should lie in their linear span. This in particular implies that the operators $\mathbf{v}^2$ and $\mathbf{v}^3$ should commute with $\mathbf{v}^1$, which is equivalent to the conditions $h_u^2 = 0$ and $h_u^3 = 0$. Then, the commutator $[\mathbf{v}^2, \mathbf{v}^3]$ equals ∂_x, which should belong to $\mathfrak{s}$. Therefore, $h^3 = 0$. The complete system of invariant surface conditions associated with $\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$ has a solution with nonvanishing g if and only if $h^2 = 2$. As a result we obtain the first listed subalgebra.

Analogously, in subcase (ii) we have $[\mathbf{v}^1, \mathbf{v}^i] = \mathcal{G}(h_u^i) = b^i\mathbf{v}^1$, where $b^i = \text{const}$, $i = 2, 3$, i.e., up to linear combining of $\mathbf{v}^i$ with $\mathbf{v}^1$, $h^i = b^iu$. The condition $[\mathbf{v}^2, \mathbf{v}^3] = \partial_x \in \mathfrak{s}$ yields that $a_2^3 = b^3 = 0$. In order to provide the requested compatibility of the entire system of the associated invariant surface conditions with the inequality $g \neq 0$, we necessarily have $a_2^2 = 2$. Re-denoting $b^2 = b$, we recover the second subalgebra from the above list.

If $m_\mathfrak{s} = 2$, the subalgebra $\mathfrak{s}$ contains two linearly independent operators $\mathbf{v}^i = \mathcal{G}(h^i)$, $i = 1, 2$. Similarly to case 2d of the proof of Theorem 3.17, we can assume up to $G^\sim$-equivalence and a change of the basis in $\langle \mathbf{v}^1, \mathbf{v}^2\rangle$ that $h^1 = 1$ and $h^2 = u$. Consider any $\mathbf{v} = a_1\mathcal{D}^x + a_2\mathcal{D}^t + a_3\partial_x + \mathcal{G}(h)$ from the complement to $\langle \mathcal{G}(1), \mathcal{G}(u)\rangle$ in $\mathfrak{s}$. Lemma 3.22 implies that $(a_1, a_3) \neq (0, 0)$. Therefore, $[\mathbf{v}^i, \mathbf{v}] \in \langle \mathbf{v}^1, \mathbf{v}^2\rangle$. The last condition is equivalent to $h_u, uh_u - h \in \langle 1, u\rangle$. Consequently, we obtain $h \in \langle 1, u\rangle$. Hence, up to linear combining with elements from $\langle \mathcal{G}(1), \mathcal{G}(u)\rangle$, we can always assume that $h = 0$. In other words, the subalgebra $\mathfrak{s}$ can be represented as a direct sum of the algebra $\langle \mathcal{G}(1), \mathcal{G}(u)\rangle$ and a subalgebra of $\mathfrak{g}_2^\sim = \langle \mathcal{D}_x, \mathcal{D}_t, \partial_x\rangle$. $G^\sim$-inequivalent subalgebras of $\mathfrak{g}_2^\sim$ that do not contain the operator $\mathcal{D}^t$ are exhausted by the algebras $\langle \mathcal{D}^x + a\mathcal{D}^t\rangle$, $\langle \partial_x - \delta\mathcal{D}^t\rangle$ and $\langle \mathcal{D}^x + a\mathcal{D}^t, \partial_x\rangle$, cf. the proofs of Theorems 3.16 and 3.17. In the last subalgebra, owing to the required compatibility of the system of invariant surface conditions associated with $\mathfrak{s}$ we have $a = 2$.

This completes the proof of the theorem. □

The symmetry extensions induced by subalgebras from Theorem 3.25 are collected in Table 3.3, where a is an arbitrary constant, $a \neq 2$. Note that the extension induced by the second subalgebra is not maximal among extensions related to $\mathfrak{g}^{\sim}$. This is why we do not include it into Table 3.3. The general solution of the associated system of invariant surface conditions is $f = c_1$ and $g = c_2$, where c_1 and c_2 are arbitrary constants, $c_2 \neq 0$. Such values of arbitrary elements correspond to the potential Burgers equation or the linear heat equation if $c_1 \neq 0$ or $c_1 = 0$, respectively. The linear heat equation is given by Case 4 of Table 3.3 and the potential Burgers equation, which additionally possesses the Lie symmetry operator $e^{-c_1u/c_2}\partial_u$ induced by $\mathcal{G}(e^{-c_1u/c_2})$, is reduced to the same case by a transformation similar to (3.5.3), cf. Remark 3.20. Analogously, we should choose $\delta = 1$ in the fourth subalgebra for the associated extension to be maximal.

Table 3.3: Lie symmetry extensions for class (3.1.1) related to $\mathfrak{g}^{\sim}$ of dimension greater than two.

N	f	g	Additional operators
1	ce^u	e^u	$x\partial_x + 2\partial_u,\ \partial_x,\ t\partial_t - \partial_u$
2	0	$\lvert x\rvert^{2-a}$	$at\partial_t + x\partial_x,\ \partial_u,\ u\partial_u$
3	0	e^x	$t\partial_t - \partial_x,\ \partial_u,\ u\partial_u$
4	0	1	$2t\partial_t + x\partial_x,\ \partial_x,\ \partial_u,\ u\partial_u$

Summing up the whole consideration of the present paper, we prove the following theorem:

Theorem 3.26. *The complete preliminary group classification of class* (3.1.1) *is split into Tables 3.1–3.3, where* $\delta \neq 0$ *in Case 3 of Table 3.1 and in Table 3.2 we should globally set* $c_2 = 1$*, exclude Case 3a and assume that* $\tilde\delta \neq -2$ *in Case 1,* $\tilde\delta \neq 0$ *in Case 4,* $b \neq 0$ *in Cases 5 and 6, and* $\delta = 1$ *in Case 7.*

Remark 3.27. Table 3 from [93], summing up the partial preliminary group classification of the class (3.1.1) therein, is incorrect. Neither are all of the equations listed really invariant under the operators presented in the table, nor are these operators proper additional operators in view of the kernel $\langle\partial_t\rangle$. The main problem is that the basis element ∂_t of the kernel is involved by linear combining to these additional operators which, moreover, are not linearly independent.

The number of inequivalent cases to be investigated under the usage of the entire infinite-dimensional equivalence algebra $\mathfrak{g}^{\sim}$ is rather small. This is due to the greater effectiveness of the adjoint action of the whole equivalence group, which allows for stronger simplifications under classification of inequivalent subalgebras. By using only a finite-dimensional subalgebra of $\mathfrak{g}^{\sim}$ as usually done, the number of cases of extensions to be treated is generally greater. This is one more justification why it is favorable to use complete preliminary group classification rather than partial preliminary group classification.

3.8 Conclusion

The main aim of this paper is a careful explanation of the technique of preliminary group classification, its status in the picture of group classification, its benefits and its limitations. These points are those we consider to be mainly lacking so far. While preliminary group classification is generally attractive due to the relative simplicity of its algorithm, various of the results obtained by now using this approach have only little practical relevance, since they are presented without a detailed analysis of the class of differential equations. Moreover, in various papers only partial preliminary group classification was carried out, without indicating a sound physical justification for the chosen subalgebras of the respective equivalence algebras. Indeed, in some instances this choice might been motivated for the sake of pure mathematical convenience, which counteracts the initial aim of group classification of differential equations.

In the present paper we substantially enhance the existing framework of preliminary group classification. We show that it is possible and convenient to treat subalgebras of the entire equivalence algebra even in the case if this algebra is infinite dimensional. This is the principal difference compared to existing works on the subject of preliminary group classification, in which the problem is only partially solved by involving classification of subalgebras of a fixed finite-dimensional subalgebra of the equivalence algebra with respect to restricted adjoint actions. Furthermore, it is emphasized that only appropriate subalgebras satisfying certain properties should be classified.

The algorithm of *complete preliminary group classification* can be summed up as follows:

- Find the equivalence algebra $\mathfrak{g}^\sim$ and the equivalence group $G^\sim$ of the class $\mathcal{L}|_\mathcal{S}$ under consideration.
- Classify appropriate subalgebras of $\mathfrak{g}^\sim$ up to $G^\sim$-equivalence, each of which satisfies the properties below:
  - It contains the kernel algebra $\mathfrak{g}^\cap$ of $\mathcal{L}|_\mathcal{S}$.
  - The associated system of invariant surface conditions with respect to the arbitrary elements is compatible.
  - It is the maximal subalgebra among all subalgebras of $\mathfrak{g}^\sim$ that have the same set of solutions for the associated systems of invariant surface conditions.
- For each of the listed subalgebras, find the general solution of the associated system of invariant surface conditions with respect to the arbitrary elements.
- Simplify these solutions using transformations from $G^\sim$ whose push-forwards to vector fields preserve the corresponding subalgebras of $\mathfrak{g}^\sim$, i.e., these transformations lie in the normalizers of the corresponding subgroups of $G^\sim$.

The systematic approach of complete preliminary group classification is exemplified with the class of generalized diffusion equation (3.1.1) that was recently attempted to be investigated in [93] using symmetry tools. Owing to the number of inconveniences of [93], we regard the class (3.1.1) as well-suited to explain the methodology of preliminary group classification. We use both the framework of the infinitesimal and the direct methods to derive the equivalence algebra and the equivalence group of the class (3.1.1). In addition, the direct method also allows us to obtain the classifying equations of admissible transformations. Similar as the determining equations of Lie symmetries, these classifying equations of admissible transformations are too difficult to be solved directly, which at once limits the chance to obtain a complete group classification of the class (3.1.1) directly.

It is important to indicate once more that the extensions of the kernel algebra constructed in this paper by using preliminary group classification are not necessarily maximal. That is, there are various equations in the class (3.1.1) which have the maximal Lie invariance algebras wider than the associated subalgebras of the equivalence algebra. This observation is another way of proving that the class (3.1.1) is not normalized.

Chapter 4

Complete group classification of a class of nonlinear wave equations

4.1 Introduction

The method of preliminary group classification was first introduced in Ref. [1] and became well-known due to Ref. [63]. In the latter paper, a partial preliminary group classification was carried out for the class of equations of the form

$$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x), \tag{4.1.1}$$

where $f \neq 0$. The essence of the approach applied in [63] is given by the classification of one-dimensional extensions of the kernel algebra with respect to a fixed finite-dimensional subalgebra of the infinite-dimensional equivalence algebra of the class studied.

The symmetry analysis of the same class was continued in a number of papers. The interest in such studies is stimulated because equations of the form (4.1.1) are used as mathematical models of different continuous media. They arise, e.g., in the theory of elasticity, in particular, in the course of modeling hyperelastic homogeneous rods [66].

Thus, in [54] the partial preliminary group classification of the class (4.1.1) with respect to one-dimensional subalgebras of an infinite-dimensional subalgebra of the equivalence algebra was considered. Second-order differential invariants of the equivalence algebra were computed in [64]. Another direction of investigation for the class (4.1.1) was initiated in [61]. Instead of equations of the form (4.1.1), related systems of two equations where the first derivatives of u play the role of the dependent variables, were considered and mapped to the form $v_t = a(x, v)w_x$ and $w_t = b(x, v)v_x$. For the class of systems of this form, certain properties were investigated within the framework of symmetry analysis, including the computation of the equivalence and kernel algebras and the compatibility analysis of the determining equations for Lie symmetries. Upper bounds for the dimension of Lie symmetry extensions were established for the two cases which arose. This study was completed in [69] by exhaustive group classification of such systems using the algebraic method.

A comprehensive review of the literature on group analysis of different classes of $(1 + 1)$-dimensional wave equations was presented in [79]. Some of these classes are contained in the class (4.1.1) or intersect it nontrivially. In particular, the simple subclasses of (4.1.1) singled out by the constraints $f_x = g = 0$ and $f_x = g_x = 0$ were considered in [110] and [50], respectively. The class (4.1.1) has also a subclass in common with the class of nonlinear wave equations of the general form $u_{tt} = u_{xx} + F(t, x, u, u_x)$, whose Lie symmetries were exhaustively investigated in [79] using the algebraic method. The intersection obviously consists of equations of the form (4.1.1) with $f = 1$. Any equation of the form (4.1.1) is a potential equation for the wave equation of another form, $v_{tt} = (f(x, v)v_x + g(x, v))_x$, also called the nonlinear telegraph equation [26, 56].

Following the paper [63], several classes of differential equations were investigated within the framework of preliminary group classification. Given a class of differential equations, this approach in its essence rests on computing optimal lists of inequivalent subalgebras of the associated equivalence algebra and studying the Lie symmetry extensions induced by these subalgebras. While in the majority of papers on this subject, including Ref. [63], only symmetry extensions by means of inequivalent subalgebras of a fixed finite-dimensional subalgebra of a possibly infinite-dimensional equivalence algebra are considered, we have shown in [43] that this restriction is in fact not necessary. Stated in another way, there is no obstacle in studying extensions induced by subalgebras of the whole (infinite-dimensional) equivalence algebra. This is, what we have called the complete preliminary group classification as opposed to the various partial preliminary group classifications, which were carried out e.g. in [63, 140]. As an example, in [43] we solved the complete preliminary group classification problem for the class of nonlinear diffusion equations of the general form $u_t = f(x, u)u_x^2 + g(x, u)u_{xx}$.

Moreover, in case when the class is normalized (at least in the weak sense [123, 127]) the same approach gives at once the complete group classification, cf. Section 4.3. This fact was implicitly used in various instances. The most classical examples for this finding are Sophus Lie's classifications of second order ordinary differential equations [85] and of second order two-dimensional linear partial differential equations [83]. For numerous modern examples see, e.g. [9, 78, 79, 80, 126, 156, 159] and references therein. The technique of group classification explicitly based on the notion of normalized classes of differential equations was developed in [122, 123, 127] and then applied to different classes of Schrödinger equations, generalized vorticity equations, generalized Korteweg–de Vries equations, etc. All the above techniques can be interpreted as particular versions of the algebraic method.

The purpose of the present paper is to systematically carry out the preliminary group classification of the class of differential equations (4.1.1) in a similar fashion as in [43] and thereby to exhaustively solve the complete group classification problem for this class of nonlinear wave equations using the partition into normalized subclasses. The version of the algebraic method applied in the present paper differs from the Lahno–Zhdanov approach [9, 78, 79, 80, 159] as it does not involve the classification of low-dimensional Lie algebras but is rather based on the classification of all appropriate subalgebras of the corresponding equivalence algebra.

In order to guarantee the nonlinearity of equations of the form (4.1.1), we explicitly include the nonvanishing condition $(f_{u_x}, g_{u_xu_x}) \neq (0, 0)$ in the definition of the class to be studied. The reason why we are only concerned with the nonlinear case here is that nonlinear and linear equations of the form (4.1.1) are not mixed by point transformations (cf. Remark 4.29) and have quite different Lie symmetry properties. Moreover, linear wave equations of the form (4.1.1) were already well investigated within the framework of classical symmetry analysis in [25, 113]. Note that the consideration is local and all variables and functions take real values throughout the paper although the transition to the complex case needs only minor corrections.

The further organization of this paper is the following. The theoretical background of point transformations in classes of differential equations is reviewed in Section 4.2. This includes the definitions and properties of a class of differential equations, its subclasses, the set of admissible transformations, the usual equivalence group and algebra, different notions of normalized classes of differential equations, etc. Section 4.3 contains a concise description of the group classification problem together with a discussion on the theory of preliminary group classification and complete group classification with the algebraic method. The group analysis of the class (4.1.1) is started in Section 4.4 by studying the structure of the equivalence algebra of (4.1.1). The computation of the determining equations for admissible transformations of the class (4.1.1) and the equivalence group of this class by the direct method is given in Section 4.5 and Section 4.6, respectively. The determining equations of Lie symmetries of equations from the class (4.1.1) are analyzed in Section 4.7. We obtain the kernel algebra of this class and prove that the major subclass of the class (4.1.1) is weakly normalized with respect to the equivalence algebra of (4.1.1). The group classification of the complement of the subclass is also carried out. Completing the study of admissible transformations by the direct method, in Section 4.8 we partition the class (4.1.1) into two subclasses, which are, respectively, normalized and semi-normalized with respect to the equivalence group of the entire class (4.1.1). In this way we prove that the class (4.1.1) is semi-normalized. Both the equivalence algebra and the equivalence group are used in Section 4.9 to classify subalgebras of the equivalence algebra that may be used for preliminary group classification. The adjoint action of the equivalence group on the associated algebra is computed using pushforwards of vector fields, as it was recently proposed in [43]. The calculations completing the group classification of the class (4.1.1) and the corresponding list of inequivalent Lie symmetry extensions can be found in Section 4.10. In Section 4.11 we briefly sum up the results of the present paper and compare the techniques of partial preliminary group classification, complete preliminary group classification and complete group classification within the framework of the general algebraic method.

4.2 Point transformations in classes of differential equations

To make this paper self-contained, in this and in the next sections we restate some important notions from the theory of group classification. More information on this subject can be found, e.g. in [87, 112, 123, 127].

The central notion underlying the theory of group classification is an appropriate definition of a class of (systems of) differential equations. In practice, the structure and properties of a class of differential equations determine which methods of group classification (e.g. complete vs. preliminary, direct vs. algebraic) are the most appropriate for it. In short, the definition of a class of differential equations comprises two ingredients.

The first ingredient is a system of differential equations $\mathcal{L}_\theta$: $L(x,u_{(p)},\theta_{(q)}(x,u_{(p)}))=0$, parameterized by the tuple of arbitrary elements $\theta(x,u_{(p)})=(\theta^1(x,u_{(p)}),\dots,\theta^k(x,u_{(p)}))$, where $x=(x_1,\dots,x_n)$ is the tuple of independent variables and $u_{(p)}$ is the set of all dependent variables $u=(u^1,\dots,u^m)$ together with all derivatives of u with respect to x up to order p. The symbol $\theta_{(q)}$ stands for the set of partial derivatives of θ of order not greater than q with respect to the variables x and $u_{(p)}$.

The second ingredient concerns possible values of the tuple of arbitrary elements θ. This tuple is required to run through the solution set $\mathcal{S}$ of a joint system (also denoted by $\mathcal{S}$) of auxiliary differential equations $S(x,u_{(p)},\theta_{(q')}(x,u_{(p)}))=0$ and inequalities $\Sigma(x,u_{(p)},\theta_{(q')}(x,u_{(p)}))\neq 0$, in which both x and $u_{(p)}$ play the role of independent variables and S and Σ are tuples of smooth functions depending on x, $u_{(p)}$ and $\theta_{(q')}$. The nonvanishing conditions $\Sigma\neq 0$ might be essential to guarantee that each element of the class has some common properties with all the other elements of the same class, such as the same order p or the same linearity or nonlinearity properties. Thereby, these inequalities can be the crucial factor in order to solve the given group classification problem up to a certain stage. In spite of this, they are often omitted without good reason.

Definition 4.1. The set $\{\mathcal{L}_\theta \mid \theta \in \mathcal{S}\}$ denoted by $\mathcal{L}|_\mathcal{S}$ is called a *class of differential equations* defined by the parameterized form of systems $\mathcal{L}_\theta$ and by the set $\mathcal{S}$ of the arbitrary elements θ.

An additional problem in defining a class of differential equations is that the correspondence $\theta \to \mathcal{L}_\theta$ between arbitrary elements and systems (treated not as formal algebraic expressions but as real systems of differential equations or manifolds in the jet space $J^{(p)}$ which is the space of the variables $(x, u_{(p)})$) may not be injective. The values θ and $\tilde\theta$ of arbitrary elements are called *gauge-equivalent* if $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ are the same system of differential equations, i.e., their solution sets coincide. We formally consider $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ as different representations of the same system from $\mathcal{L}|_\mathcal{S}$. For the correspondence $\theta \to \mathcal{L}_\theta$ to be one-to-one in the presence of nontrivial gauge equivalence, the set $\mathcal{S}$ of arbitrary elements should be factorized with respect to the gauge equivalence relation by changing the representation of the class $\mathcal{L}|_\mathcal{S}$. If this is not convenient, the gauge equivalence should be carefully taken into account when carrying out symmetry analysis of the class $\mathcal{L}|_\mathcal{S}$ [65].

In the course of group classification of a complicated class of differential equations it is often helpful to consider subclasses of this class. A *subclass* is singled out from the class $\mathcal{L}|_\mathcal{S}$ by attaching additional equations or nonvanishing conditions to the auxiliary system $\mathcal{S}$.

Thus, for the class of equations of the general form (4.1.1) we have the single unknown function u of two independent variables t and x. The associated tuple of arbitrary elements consists of two functions f and g whose domains are contained in the second-order jet space, i.e., in the space of t, x and u together with all derivatives of u up to second order. The indicated dependence of f and g only on x and u_x requires that the arbitrary elements of this class satisfy the auxiliary system of differential equations

$$f_t = f_u = f_{u_t} = f_{u_{tt}} = f_{u_{tx}} = f_{u_{xx}} = 0,$$
$$g_t = g_u = g_{u_t} = g_{u_{tt}} = g_{u_{tx}} = g_{u_{xx}} = 0.$$

As we consider wave equations, we also impose the inequality $f \neq 0$. In the present paper we study the subclass of equations of the form (4.1.1) that consists only of truly nonlinear equations and, therefore, is singled out from the entire class of equations of the general form (4.1.1) by the additional nonvanishing condition $(f_{u_x}, g_{u_x u_x}) \neq (0, 0)$. It is the set of equations which is called the class (4.1.1) throughout the paper.

Several properties hold for subclasses of a class $\mathcal{L}|_\mathcal{S}$. The intersection of a finite number of subclasses of $\mathcal{L}|_\mathcal{S}$ is also a subclass of $\mathcal{L}|_\mathcal{S}$, which is defined by the union of the additional auxiliary systems associated with the intersecting sets. At the same time, the complement $\overline{\mathcal{L}|_{\mathcal{S}'}} = \mathcal{L}|_{\overline{\mathcal{S}'}}$ of the subclass $\mathcal{L}|_{\mathcal{S}'}$ in the class $\mathcal{L}|_\mathcal{S}$ is a subclass of $\mathcal{L}|_\mathcal{S}$ only in special cases, e.g., if the additional system of equations or the additional system of nonvanishing conditions is empty (cf. Remark 4.20). Namely, if the subset $\mathcal{S}'$ of arbitrary elements is singled out from $\mathcal{S}$ by the system $S'_1 = 0, \ldots, S'_{s'} = 0$ then the additional auxiliary condition for $\overline{\mathcal{S}'}$ is $|S'_1|^2 + \cdots + |S'_{s'}|^2 \neq 0$. If $\mathcal{S}'$ is defined by the inequalities $\Sigma'_1 \neq 0, \ldots, \Sigma'_{\sigma'} \neq 0$ then the additional auxiliary condition for $\overline{\mathcal{S}'}$ is $\Sigma'_1 \cdots \Sigma'_{\sigma'} = 0$.

A point transformation in a space is an invertible smooth mapping of an open domain in this space into the same domain. Given a class $\mathcal{L}|_\mathcal{S}$ of differential equations, point transformations related to $\mathcal{L}|_\mathcal{S}$ form objects of different structure.

Let $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ be elements of the class $\mathcal{L}|_\mathcal{S}$. By $\mathrm{T}(\theta, \tilde\theta)$ we denote the set of point transformations in the space of variables (x, u) mapping the system $\mathcal{L}_\theta$ to the system $\mathcal{L}_{\tilde\theta}$. Using this notation, the maximal point symmetry (pseudo)group G_θ of the system $\mathcal{L}_\theta$ coincides with $T(\theta, \theta)$.

If $\mathrm{T}(\theta,\tilde\theta)\neq\varnothing$, i.e. the systems $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ are similar with respect to point transformations, then $\mathrm{T}(\theta,\tilde\theta)=\varphi^0\circ G_\theta=G_{\tilde\theta}\circ\varphi^0$, where φ^0 is a fixed transformation from $\mathrm{T}(\theta,\tilde\theta)$.

Definition 4.2. The *equivalence groupoid* of the class $\mathcal{L}|_\mathcal{S}$ is called the set of admissible transformations of this class $\mathrm{T}=\mathrm{T}(\mathcal{L}|_\mathcal{S})=\{(\theta,\tilde\theta,\varphi)|\ \theta,\tilde\theta\in\mathcal{S},\varphi\in\mathrm{T}(\theta,\tilde\theta)\}$ naturally equipped with the groupoid structure.

If the number m of dependent variables is equal to one, one can consider more general contact transformations [112] in the same way instead of point transformations.

The notion of admissible transformations [122, 127] is a formalization of the notion of form-preserving [73, 74] or allowed [156] transformations. First descriptions of the sets of admissible transformations for nontrivial classes of differential equations were given by Kingston and Sophocleous [72] for a class of generalized Burgers equations and by Winternitz and Gazeau [156] for a class of variable-coefficient Korteweg–de Vries equations. An infinitesimal analogue of the notion of admissible transformations was proposed and studied by Borovskikh in [28]. In terms of equivalence groups and normalization properties of subclasses (see Definitions 4.6 and 4.7), the sets of admissible transformations were exhaustively described for a number of different classes of differential equations which are important in applications, such as nonlinear Schrödinger equations [126, 127], variable-coefficient diffusion–reaction equations [149, 150], generalized Korteweg–de Vries equations including variable-coefficient Korteweg–de Vries and modified Korteweg–de Vries equations [129], systems of $(1+1)$-dimensional second-order evolution equations [128], generalized vorticity equations [123], etc.

Definition 4.3. The *(usual) equivalence group* $G^\sim=G^\sim(\mathcal{L}|_\mathcal{S})$ of the class $\mathcal{L}|_\mathcal{S}$ is the (pseudo) group of point transformations in the space of $(x,u_{(p)},\theta)$ which are projectable to the space of $(x,u_{(p')})$ for any $0\le p'\le p$, are consistent with the contact structure on the space of $(x,u_{(p)})$ and preserve the set $\mathcal{S}$ of arbitrary elements.

Recall that a point transformation φ: $\tilde z=\varphi(z)$ in the space of the variables $z=(z_1,\dots,z_k)$ is called projectable to the space of the variables $z'=(z_{i_1},\dots,z_{i_{k'}})$, where $1\le i_1<\dots<i_{k'}\le k$, if the expressions for $\tilde z'$ depend only on z'. We denote the restriction of φ to the z'-space as $\varphi|_{z'}$: $\tilde z'=\varphi|_{z'}(z')$. A point transformation Φ in the space of $(x,u_{(p)},\theta)$, which is projectable to the space of $(x,u_{(p')})$ for any $0\le p'\le p$, is consistent with the contact structure on the space of $(x,u_{(p)})$ if $\Phi|_{(x,u_{(p)})}$ is the p-th order prolongation of $\Phi|_{(x,u)}$.

Each transformation Φ from the equivalence group $G^\sim$ (i.e., an equivalence transformation of the class $\mathcal{L}|_\mathcal{S}$) induces the family of admissible transformations of the form $(\theta,\Phi\theta,\Phi|_{(x,u)})$ parameterized by the arbitrary elements θ running through the entire set $\mathcal{S}$. Roughly speaking, $G^\sim$ is the set of admissible transformations which can be applied to any $\theta\in\mathcal{S}$.

There exist several generalizations of the notion of equivalence group in the literature, in which some of the restrictions on equivalence transformations (projectability or locality with respect to arbitrary elements) are weakened [65, 91, 127, 149, 150].

The common part $G^\cap=G^\cap(\mathcal{L}|_\mathcal{S})=\bigcap_{\theta\in\mathcal{S}}G_\theta$ of all G_θ, $\theta\in\mathcal{S}$, is called the *kernel of the maximal point symmetry groups* of systems from the class $\mathcal{L}|_\mathcal{S}$ [112]. The following assertion is true (see, e.g., [43, 127]).

Proposition 4.4. *The kernel group $G^\cap$ of the class $\mathcal{L}|_\mathcal{S}$ is naturally embedded in the (usual) equivalence group $G^\sim$ of this class by the trivial (identical) prolongation of the kernel transformations to the arbitrary elements. The associated subgroup $\hat G^\cap$ of $G^\sim$ is normal.*

Properties of $G^\cap$ described in Proposition 4.4 were first noted in different works by Ovsiannikov (see, e.g., [114] and [112, Section II.6.5]). Another formulation of this proposition was given in [87, p. 52], Proposition 3.3.9.

As the study of point transformations of differential equations usually involves cumbersome and sophisticated calculations, instead of finite point transformations one may consider their infinitesimal counterparts. This leads to the linearization of the related problem which essentially simplifies the whole consideration. In the framework of the infinitesimal approach, a (pseudo)group G of point transformations is replaced by the Lie algebra $\mathfrak{g}$ of vector fields on the same space, which are generators of one-parameter local subgroups of G.

In particular, the vector fields in the space of (x,u) generating one-parameter subgroups of the maximal point symmetry (pseudo)group G_θ of the system $\mathcal{L}_\theta$ form a Lie algebra $\mathfrak{g}_\theta$ called the *maximal Lie invariance algebra* of the system $\mathcal{L}_\theta$. Analogously to symmetry groups, the common part $\mathfrak{g}^\cap=\mathfrak{g}^\cap(\mathcal{L}|_\mathcal{S})=\bigcap_{\theta\in\mathcal{S}}\mathfrak{g}_\theta$ of all $\mathfrak{g}_\theta$, $\theta\in\mathcal{S}$, is called the *kernel of the maximal Lie invariance algebras* of systems from the class $\mathcal{L}|_\mathcal{S}$. It is the Lie algebra associated with the kernel group $G^\cap$.

The equivalence algebra $\mathfrak{g}^\sim$ is the Lie algebra formed by generators of one-parameter groups of equivalence transformations for the class $\mathcal{L}|_\mathcal{S}$. These generators are vector fields in the space of $(x,u_{(p)},\theta)$ which are projectable to the space of $(x,u_{(p')})$ for any $0\le p'\le p$ and whose projections to the space of $(x,u_{(p)})$ are the p-th order prolongations of the corresponding projections to the space of (x,u).

An infinitesimal analogue of Proposition 4.4 is the following assertion.

Corollary 4.5. *The trivial prolongation $\hat{\mathfrak{g}}^\cap$ of the kernel algebra $\mathfrak{g}^\cap$ to the arbitrary elements is an ideal in the equivalence algebra $\mathfrak{g}^\sim$.*

By definition, any element of the algebra $\hat{\mathfrak{g}}^\cap$ formally has the same form as the associated element from $\mathfrak{g}^\cap$ but in fact is a vector field on the different space augmented with the arbitrary elements.

It is convenient to characterize and estimate transformational properties of classes of differential equations in terms of normalization.

Definition 4.6. A class of differential equations $\mathcal{L}|_\mathcal{S}$ is *normalized* if its equivalence groupoid is induced by transformations of its equivalence group $G^\sim$, meaning that for any triple $(\theta,\tilde\theta,\varphi)$ from $\mathrm{T}(\mathcal{L}|_\mathcal{S})$ there exists a transformation Φ from $G^\sim$ such that $\tilde\theta=\Phi\theta$ and $\varphi=\Phi|_{(x,u)}$.

Definition 4.7. A class of differential equations $\mathcal{L}|_\mathcal{S}$ is called *semi-normalized* if its equivalence groupoid is induced by transformations from its equivalence group $G^\sim$ and the maximal point symmetry groups of its equations, meaning that for any triple $(\theta,\tilde\theta,\varphi)$ from $\mathrm{T}(\mathcal{L}|_\mathcal{S})$ there exist a transformation Φ from $G^\sim$ and a transformation $\tilde\varphi$ from the maximal point symmetry group G_θ of the system $\mathcal{L}_\theta$, such that $\tilde\theta=\Phi\theta$ and $\varphi=\Phi|_{(x,u)}\circ\tilde\varphi$.

In other words, a class of differential equations is semi-normalized if arbitrary similar systems from the class are related via transformations from the equivalence group of this class.

Normalized and semi-normalized classes of differential equations have a number of interesting properties which essentially simplify the study of such classes. In particular, if the class $\mathcal{L}|_\mathcal{S}$ is normalized in the usual sense, its kernel algebra $\mathfrak{g}^\cap$ is an ideal of the maximal Lie invariance algebra $\mathfrak{g}_\theta$ for each $\theta\in\mathcal{S}$. In general, this claim is not true even if the class is only semi-normalized. See Example 1 in [43].

The above notion of normalization (resp. semi-normalization) relies on finite admissible transformations. A weaker version of normalization is defined in infinitesimal terms [123].

Definition 4.8. A class of differential equation $\mathcal{L}|_{\mathcal{S}}$ is *weakly normalized* if the union and, therefore, the span of maximal Lie invariance algebras $\mathfrak{g}_\theta$ of all systems $\mathcal{L}_\theta$ from the class is contained in the projection of the equivalence algebra $\mathfrak{g}^\sim$ of the class to vector fields in the space of independent and dependent variables, i.e.

$$\bigcup_{\theta\in\mathcal{S}} \mathfrak{g}_\theta \subset \mathrm{P}\mathfrak{g}^\sim \quad (\text{or} \quad \langle \mathfrak{g}_\theta \mid \theta\in\mathcal{S}\rangle \subset \mathrm{P}\mathfrak{g}^\sim).$$

Here by P we denote the projection operator that acts on vector fields of the general form $Q = \xi^i(x,u)\partial_{x_i} + \eta^a(x,u)\partial_{u^a} + \varphi^s(x,u,\theta)\partial_{\theta^s}$ on the space of variables x, u and θ yielding vector fields of the form $\mathrm{P}Q = \xi^i\partial_{x_i} + \eta^a\partial_{u^a}$, which are defined on the space of variables x and u.

It is obvious that any normalized class of differential equations is both semi-normalized and weakly normalized.

In general, the normalization of a class of differential equations can be checked by computing the equivalence groupoid of the class and its equivalence group (e.g. using the direct method) and testing whether the condition from Definition 4.6 is satisfied. It is often convenient to begin with a normalized superclass and construct a hierarchy of normalized subclasses of the superclass or a simple chain of such nested subclasses each of which contain the class under consideration [127, 128, 129]. The weak normalization property in turn can be verified by finding the equivalence algebra of the class and an inspection of the determining equations for Lie symmetries of systems from the class (see the next section). As the computations needed to check for weak normalization involve the solution of linear partial differential equations (in contrast to the computations using the direct method of finding equivalence and admissible transformations), they can be realized in an algorithmic way even for quite cumbersome classes of multidimensional partial differential equations. At the same time, establishing the usual normalization property is more useful and allows one to obtain deeper results than using its weak infinitesimal analogue.

4.3 Algebraic method of group classification

Now that we have introduced necessary notions related to point transformations within classes of differential equations, we can go on with the general discussion of the framework of group classification in some more detail.

The solution of the Lie–Ovsiannikov group classification problem for a class $\mathcal{L}|_{\mathcal{S}}$ of differential equations includes the construction of the following elements:

- the equivalence group $G^\sim$ of the class $\mathcal{L}|_{\mathcal{S}}$,
- the kernel algebra $\mathfrak{g}^\cap = \mathfrak{g}^\cap(\mathcal{L}|_{\mathcal{S}}) = \bigcap_{\theta\in\mathcal{S}} \mathfrak{g}_\theta$ of the class $\mathcal{L}|_{\mathcal{S}}$, i.e., the intersection of the maximal Lie invariance algebras of systems from this class,
- an exhaustive list of $G^\sim$-equivalent extensions of the kernel algebra $\mathfrak{g}^\cap$ in the class $\mathcal{L}|_{\mathcal{S}}$, i.e., an exhaustive list of $G^\sim$-equivalent values of θ with the corresponding maximal Lie invariance algebras $\mathfrak{g}_\theta$ for which $\mathfrak{g}_\theta \neq \mathfrak{g}^\cap$.

More precisely, the classification list consists of pairs $(\mathcal{S}_\gamma, \{\mathfrak{g}_\theta, \theta\in\mathcal{S}_\gamma\})$, $\gamma\in\Gamma$. For each $\gamma\in\Gamma$ $\mathcal{L}|_{\mathcal{S}_\gamma}$ is a subclass of $\mathcal{L}|_{\mathcal{S}}$, $\mathfrak{g}_\theta \neq \mathfrak{g}^\cap$ for any $\theta\in\mathcal{S}_\gamma$ and the structures of the algebras $\mathfrak{g}_\theta$ are similar for all $\theta\in\mathcal{S}_\gamma$. In particular, the algebras $\mathfrak{g}_\theta$, $\theta\in\mathcal{S}_\gamma$, have the same dimension or display the same arbitrariness of algebra parameters in the infinite-dimensional case. Moreover, for any

$\theta \in \mathcal{S}$ with $\mathfrak{g}_\theta \neq \mathfrak{g}^\cap$ there exists $\gamma \in \Gamma$ such that $\theta \in \mathcal{S}_\gamma$ mod $G^\sim$. All elements from $\bigcup_{\gamma\in\Gamma} \mathcal{S}_\gamma$ are $G^\sim$-inequivalent. Note that in all examples of group classifications presented in the literature the set Γ was finite.

The procedure of group classification can be supplemented by deriving auxiliary systems of differential equations for the arbitrary elements, providing extensions of Lie symmetry, cf. Remark 4.20. In other words, for each $\gamma \in \Gamma$ one should explicitly describe the subset $\bar{\mathcal{S}}_\gamma$ of $\mathcal{S}$ which is the union of $G^\sim$-orbits of elements from $\mathcal{S}_\gamma$. Although this step is usually neglected, it may lead to nontrivial results (see, e.g., [29]).

If the class $\mathcal{L}|_\mathcal{S}$ is not semi-normalized, the classification list may include equations similar with respect to point transformations which do not belong to $G^\sim$. The knowledge of such *additional* equivalences allows one to substantially simplify the further symmetry analysis of the class $\mathcal{L}|_S$. Their construction can be considered as one further step of the algorithm of group classification [65, 125, 150]. Often it can be implemented using empiric tools, e.g., the fact that similar equations have similar maximal invariance algebras. A more systematic way is to describe the complete set of admissible transformations.

In practice, the *procedure of group classification* within the Lie–Ovsiannikov approach can be realized by implementing a few consecutive steps.

Given a class $\mathcal{L}|_\mathcal{S}$, it is convenient to start the procedure with the computation of the *equivalence algebra*. This can be done either using the infinitesimal method [1, 112] or by deriving the set of generators for the one-parameter groups of the equivalence group, provided that the latter is known. Computing the equivalence algebra independently from the equivalence group is important, as it gives a test and a tool for the calculation of the equivalence group. In particular, often only the connected component of unity of the equivalence group is found using the knowledge of the equivalence algebra. The equivalence algebra also plays a distinct role in the course of applying the algebraic method of group classification.

The most powerful tool for the construction of the *equivalence group*, which is the next step of the procedure, is the direct method involving finite point transformations. Such a construction can be understood as the final stage in the preliminary investigation of the equivalence groupoid of the class $\mathcal{L}|_\mathcal{S}$ and allows finding both continuous and discrete equivalence transformations. Due to involving finite point transformations the related calculations are cumbersome and lead to a nonlinear system of partial differential equations. An alternative approach in order to at least restrict the form of point equivalence transformations is based on the condition that any such transformation induces an automorphism of the equivalence algebra.

The *system of determining equations* for the coefficients of Lie symmetry operators of a system $\mathcal{L}_\theta$ from the class $\mathcal{L}|_\mathcal{S}$ follows from the infinitesimal invariance criterion [25, 101, 112], stating that

$$Q_{(p)}L(x,u_{(p)},\theta_{(q)}(x,u_{(p)}))\big|_{\mathcal{L}^p_\theta}=0$$

holds for any operator $Q = \xi^i(x,u)\partial_{x_i} + \eta^a(x,u)\partial_{u^a}$ from $\mathfrak{g}_\theta$, where the arbitrary elements θ play the role of parameters. In what follows we assume the summation convention for repeated indices. The indices i and a run from 1 to n and from 1 to m, respectively. $Q_{(p)}$ denotes the standard p-th prolongation of the operator Q,

$$Q_{(p)} := Q + \sum_{0<|\alpha|\leqslant p} \Big(D_1^{\alpha_1}\dots D_n^{\alpha_n}\big(\eta^a(x,u)-\xi^i(x,u)u^a_i\big)+\xi^i u^a_{\alpha,i}\Big)\partial_{u^a_\alpha}.$$

$D_i = \partial_i + u^a_{\alpha,i}\partial_{u^a_\alpha}$ is the operator of total differentiation with respect to the variable x_i. The tuple $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index, $\alpha_i \in \mathbb{N}\cup\{0\}$, $|\alpha|: = \alpha_1+\dots+\alpha_n$. The variable u^a_α on the jet space $J^{(p)}$ is identified with the derivative $\partial^{|\alpha|}u^a/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$, and $u^a_{\alpha,i} := \partial u^a_\alpha/\partial x_i$. Some determining equations do not involve the arbitrary elements and thus can be integrated immediately. The remaining determining equations explicitly depending on the arbitrary elements are referred to as the classifying equations.

Varying the arbitrary elements θ, we can split the determining equations with respect to different derivatives of θ. The additional splitting results in equations for those symmetries that are admitted for any value of the arbitrary elements and form the *kernel of the maximal Lie invariance algebras*.

The further analysis of the determining equations is usually much more intricate. The classifying equations are inspected for specific values of the arbitrary elements θ, which give extensions of the solution sets of the determining equations, associated with symmetry extensions of the kernel algebra. The sets of values found for θ should be factorized with respect to the equivalence relation requested. Still, it is the complexity of this analysis that led to the development of a great variety of specialized techniques of group classification, which are conventionally partitioned into two approaches.

The first method is the direct compatibility analysis and integration, up to the equivalence relation, of the determining equations depending on the values of the arbitrary elements. It is mostly suitable for classes with arbitrary elements that are constants or functions of single arguments. Algorithms of group classification that are realized in present day's computer algebra packages for the calculation of Lie symmetries are based on this method [5, 32, 55, 152, 157].

The other method is of algebraic nature. It is based on the following two properties: For each fixed value of the arbitrary elements the solution space of the determining equations is associated with a Lie algebra of vector fields. Additionally, if systems of differential equations are similar with respect to a point transformation then its pushforward relates the corresponding maximal Lie invariance algebras. This is why any version of the algebraic method of group classification existing in the literature involves, in some way, the classification of algebras of vector fields up to certain equivalence induced by point transformations. The key question is what set of vector fields should be classified and what kind of equivalence should be used.

It is obvious that for each equation $\mathcal{L}_\theta$ from the class $\mathcal{L}|_\mathcal{S}$ its maximal Lie invariance algebra $\mathfrak{g}_\theta$ is contained in the union $\mathfrak{g}^\cup = \bigcup_{\theta\in\mathcal{S}}\mathfrak{g}_\theta$. The definition of $\mathfrak{g}^\cup$ implies that this set consists of vector fields for which the system of determining equations is consistent with respect to the arbitrary elements with the auxiliary system of the class $\mathcal{L}|_\mathcal{S}$. Therefore, the set $\mathfrak{g}^\cup$ can be obtained at the onset of group classification, independently from deriving the maximal Lie invariance algebras of equations from the class $\mathcal{L}|_\mathcal{S}$. As it is not convenient to select linear subspaces of the set $\mathfrak{g}^\cup$ in the general case, we can extend $\mathfrak{g}^\cup$ to its linear span $\mathfrak{g}^{\langle\rangle} = \langle\mathfrak{g}_\theta|\theta\in\mathcal{S}\rangle$, but fortunately we often have $\mathfrak{g}^\cup = \mathfrak{g}^{\langle\rangle}$. Via pushforwarding of vector fields, equivalence (resp. admissible) point transformations for the class $\mathcal{L}|_\mathcal{S}$ induce an equivalence relation on algebras contained in $\mathfrak{g}^\cup$. Such an algebra is called appropriate if it is the maximal Lie invariance algebra of an equation from the class $\mathcal{L}|_\mathcal{S}$. We should classify, up to the above equivalence relation, only appropriate algebras. They satisfy additional constraints. The simplest restriction for appropriate subalgebras is that each of them contains the kernel algebra $\mathfrak{g}^\cap$. The condition that the algebras are really maximal Lie invariance algebras for equations from the class $\mathcal{L}|_\mathcal{S}$ is more nontrivial to verify.

Substituting the basis elements of each appropriate algebra obtained in the course of the algebra classification into the determining equations gives a compatible system for values of the arbitrary elements associated with Lie symmetry extensions within the class $\mathcal{L}|_\mathcal{S}$. Solving the last system completes the group classification within the most general framework of the algebraic method. This whole construction is based on the following assertion:

Proposition 4.9. *Let $\mathcal{S}_i$ be the subset of $\mathcal{S}$ that consists of all arbitrary elements for which the corresponding equations from $\mathcal{L}|_\mathcal{S}$ are invariant with respect to the same algebra of vector fields, $i=1,2$. Then the algebras $\mathfrak{g}^\cap(\mathcal{L}|_{\mathcal{S}_1})$ and $\mathfrak{g}^\cap(\mathcal{L}|_{\mathcal{S}_2})$ are similar with respect to pushforwards of vector fields by transformations from $G^\sim$ (resp. point transformations) if and only if the subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ are mapped to each other by transformations from $G^\sim$ (resp. point transformations).*

If the class $\mathcal{L}|_\mathcal{S}$ is weakly normalized, the union $\mathfrak{g}^\cup$ (resp. the span $\mathfrak{g}^{\langle\rangle}$) is well agreed with $G^\sim$-equivalence. As a result, the algebraic method is appropriate for *complete group classification* of the class $\mathcal{L}|_\mathcal{S}$. This is not the case when the main part of $\mathfrak{g}^\cup$ does not lie in the projection $\mathrm{P}\mathfrak{g}^\sim$. Then the approach of *preliminary group classification* [1, 63] is relevant to give a partial solution of the group classification problem for the class $\mathcal{L}|_\mathcal{S}$ by the algebraic method. Preliminary group classification essentially rests on the following two propositions (they were first formulated in [63] in the particular case of the class (4.1.1); see [43] for their general formulation and proofs):

Proposition 4.10. *Let $\mathfrak{a}$ be a subalgebra of the equivalence algebra $\mathfrak{g}^\sim$ of the class $\mathcal{L}|_\mathcal{S}$, $\mathfrak{a}\subset\mathfrak{g}^\sim$, and let $\theta^0(x,u_{(r)})\in\mathcal{S}$ be a value of the tuple of arbitrary elements θ for which the algebraic equation $\theta=\theta^0(x,u_{(r)})$ is invariant with respect to $\mathfrak{a}$. Then the differential equation $\mathcal{L}_{\theta^0}$ is invariant with respect to the projection of $\mathfrak{a}$ to the space of variables (x,u).*

Proposition 4.11. *Let $\mathcal{S}_i$ be the subset of $\mathcal{S}$ that consists of tuples of arbitrary elements for which the corresponding algebraic equations are invariant with respect to the same subalgebra of the equivalence algebra $\mathfrak{g}^\sim$ and let $\mathfrak{a}_i$ be the maximal subalgebra of $\mathfrak{g}^\sim$ for which $\mathcal{S}_i$ satisfies this property, $i=1,2$. Then the subalgebras $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are equivalent with respect to the adjoint action of $G^\sim$ if and only if the subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ are mapped to each other by transformations from $G^\sim$.*

Roughly speaking, in the course of preliminary group classification of the class $\mathcal{L}|_\mathcal{S}$ we classify subalgebras of $\mathfrak{g}^\sim$ instead of algebras of vector fields contained in $\mathfrak{g}^\cup$. Then the objects to be classified (subalgebras of $\mathfrak{g}^\sim$) are well agreed with the equivalence relation used ($G^\sim$-equivalence). If a proper subalgebra $\mathfrak{s}$ of $\mathfrak{g}^\sim$ is fixed and then only subalgebras of $\mathfrak{s}$ instead of the entire algebra $\mathfrak{g}^\sim$ are classified up to the internal equivalence relation of subalgebras in $\mathfrak{s}$ and used within the framework of the algebraic method, we call this approach *partial preliminary group classification.*

In view of Definition 4.8 and Proposition 4.10 the following assertion is obvious.

Corollary 4.12. *For a class of differential equations that is weakly normalized, complete preliminary group classification and complete group classification coincide.*

In fact, only certain subalgebras of the equivalence algebra $\mathfrak{g}^\sim$ whose projections are contained in $\mathfrak{g}^\cup\cap\mathrm{P}\mathfrak{g}^\sim$ should be classified.

Definition 4.13. Within the framework of preliminary group classification, we call a subalgebra $\mathfrak{a}$ of the equivalence algebra $\mathfrak{g}^\sim$ *appropriate* if its projection $\mathrm{P}\mathfrak{a}$ to the space of system variables is maximal among Lie invariance algebras of a system from the class $\mathcal{L}|_\mathcal{S}$, which are induced by subalgebras of $\mathfrak{g}^\sim$.

Appropriate subalgebras of $\mathfrak{g}^\sim$ satisfy restrictions similar to those for appropriate algebras contained in $\mathfrak{g}^\cup$. As the kernel is included in the maximal Lie invariance algebra of any equation from the class, in view of Corollary 3.7 any appropriate subalgebra $\mathfrak{a}$ of $\mathfrak{g}^\sim$ should contain, as an ideal, the trivial prolongation $\hat{\mathfrak{g}}^\cap$ of the kernel algebra $\mathfrak{g}^\cap$ to the arbitrary elements. The condition that the projection $\mathrm{P}\mathfrak{a}$ of $\mathfrak{a}$ is a Lie invariance algebra of a system from $\mathcal{L}|_\mathcal{S}$ can be checked by two obviously equivalent ways: It is sufficient to prove that there exists a value $\theta^0(x, u_{(r)}) \in \mathcal{S}$ of the tuple of arbitrary elements θ for which the algebraic equation $\theta = \theta^0(x, u_{(r)})$ is invariant with respect to $\mathfrak{a}$. The other way is to study the consistency of the system $\mathrm{DE}_\mathfrak{a}$ with the auxiliary system of the class $\mathcal{L}|_\mathcal{S}$ with respect to the arbitrary elements. By $\mathrm{DE}_\mathfrak{a}$ we denote the system obtained by the substitution of the coefficients of each basis element of $\mathrm{P}\mathfrak{a}$ into the determining equations of the class $\mathcal{L}|_\mathcal{S}$. Simultaneously we should verify the maximality of the projection $\mathrm{P}\mathfrak{a}$ in the sense of Definition 4.13.

Remark 4.14. Often the equivalence algebra can be represented as a semi-direct sum of the ideal associated with the kernel algebra and a certain complementary subalgebra. To obtain preliminary group classification in this case, we in fact need to classify only inequivalent subalgebras of the complement. Projections of these subalgebras to the space of equation variables will give all possible inequivalent extensions of the kernel algebra. This was the case for the class of generalized diffusion equations studied in [43]. In the present paper, the situation will be different, see Remark 4.31.

The importance of semi-normalization of a class of differential equations for the optimal solution of the group classification problem for this class is connected with the following property of semi-normalized classes.

Proposition 4.15. *For a class of differential equations that is semi-normalized, the group classification up to equivalence generated by the corresponding equivalence group coincides with the group classification up to general point equivalence.*

In other words, for a semi-normalized class of differential equations there are no additional equivalence transformations between cases of Lie symmetry extensions which are inequivalent with respect to the corresponding equivalence group. This results in a clear representation of the final classification list. As normalized classes of differential equations are both semi-normalized and weakly normalized, it is especially convenient to carry out group classification in such classes by the algebraic method. This is why the normalization property can be used as a criterion for selecting classes of differential equations to be classified or for splitting such classes into subclasses which are appropriate for group classification.

4.4 Equivalence algebra

The equivalence algebra of the entire class of equation of the general form (4.1.1) was already computed in [63]. It coincides with the equivalence algebra of the class considered in the present paper, which consists of purely nonlinear equations of the above form. This is why here we only represent the generating elements of this algebra in a convenient form and refer the reader to [63] for more details. The equivalence algebra $\mathfrak{g}^\sim$ of the class (4.1.1) is generated by the vector fields

$$\begin{aligned}
&\mathcal{D}^u = u\partial_u + u_x\partial_{u_x} + g\partial_g, \quad \mathcal{D}^t = t\partial_t - 2f\partial_f - 2g\partial_g, \quad \mathcal{P}^t = \partial_t,\\
&\mathcal{D}(\varphi) = \varphi\partial_x - \varphi_x u_x\partial_{u_x} + 2\varphi_x f\partial_f + \varphi_{xx}u_x f\partial_g,\\
&\mathcal{G}(\psi) = \psi\partial_u + \psi_x\partial_{u_x} - \psi_{xx}f\partial_g, \quad \mathcal{F}^1 = t\partial_u, \quad \mathcal{F}^2 = t^2\partial_u + 2\partial_g,
\end{aligned} \tag{4.4.1}$$

where $\varphi = \varphi(x)$ and $\psi = \psi(x)$ run through the set of smooth functions of x. The nonvanishing commutation relations between the these vector fields are exhausted by

$$[\mathcal{G}(\psi), \mathcal{D}^u] = \mathcal{G}(\psi), \quad [\mathcal{F}^1, \mathcal{D}^u] = \mathcal{F}^1, \quad [\mathcal{F}^2, \mathcal{D}^u] = \mathcal{F}^2,$$

$$[\mathcal{D}^t,\mathcal{F}^1]=\mathcal{F}^1,\quad [\mathcal{D}^t,\mathcal{F}^2]=2\mathcal{F}^2,$$
$$[\mathcal{P}^t,\mathcal{D}^t]=\mathcal{P}^t,\quad [\mathcal{P}^t,\mathcal{F}^1]=\mathcal{G}(1),\quad [\mathcal{P}^t,\mathcal{F}^2]=2\mathcal{F}^1,$$
$$[\mathcal{D}(\varphi^1),\mathcal{D}(\varphi^2)]=\mathcal{D}(\varphi^1\varphi^2_x-\varphi^1_x\varphi^2),\quad [\mathcal{D}(\varphi),\mathcal{G}(\psi)]=\mathcal{G}(\varphi\psi_x).$$

In fact, in (4.4.1) we present only the projections of generating elements of $\mathfrak{g}^\sim$ to the space of (t,x,u,u_x,f,g) instead of the whole elements, which are vector fields in the space of $(t,x,u_{(2)},f,g)$. In each generating vector field the coefficients associated with derivatives of u can be computed by prolongation from the coefficients of ∂_t, ∂_x and ∂_u and, therefore, are not essential. However, it is necessary to include the terms with ∂_{u_x} in the representation of these vector fields in order to ensure proper commutation relations between them. Moreover, the derivative u_x is a significant argument of the parameter-functions f and g and hence the minimal space on which equivalence transformations can be correctly restricted is the space of the variables (t,x,u,u_x,f,g). This is why at least the projections to the same space should be given for vector fields from $\mathfrak{g}^\sim$.

The form (3.16) of the equivalence algebra given in [63] differs from (4.4.1). Namely, the operators $\mathcal{G}(1)$ and $\mathcal{G}(x)$ were singled out from the family $\{\mathcal{G}(\psi)\}$. In addition, we combined the operators from [63] in order to separate scalings with respect to u, which gives simpler commutation relations between generating vector fields.

4.5 Preliminary study of admissible transformations

The infinitesimal invariance criterion allows finding of all continuous equivalence transformations by means of solving a linear system of partial differential equations. In order to determine the complete point equivalence group (including both continuous and discrete equivalence transformations) and the set of admissible transformations, it is necessary to apply the direct method. We will start our consideration with a preliminary investigation of the set of admissible transformations, which will give relevant information also on the equivalence group of the class (4.1.1). That is, we directly seek for all point transformations

$$\tilde t=T(t,x,u),\quad \tilde x=X(t,x,u),\quad \tilde u=U(t,x,u),\tag{4.5.1}$$

for which the Jacobian $\mathrm{J}=\partial(T,X,U)/\partial(t,x,u)$ does not vanish, that map a fixed equation of the form (4.1.1) to an equation of the same form, $\tilde u_{\tilde t\tilde t}=\tilde f(\tilde x,\tilde u_{\tilde x})\tilde u_{\tilde x\tilde x}+\tilde g(\tilde x,\tilde u_{\tilde x})$. To carry out this transformation in practice, it is necessary to find the transformation rules for the various derivatives of $\tilde u$ with respect to $\tilde t$ and $\tilde x$. In order to obtain them we apply the total derivative operators D_t and D_x, respectively, to the expression $\tilde u(\tilde t,\tilde x)=U(t,x,u)$, assuming $\tilde t=T(t,x,u)$ and $\tilde x=X(t,x,u)$. This gives

$$\tilde u_{\tilde t}\mathrm{D}_tT+\tilde u_{\tilde x}\mathrm{D}_tX-\mathrm{D}_tU=0,$$
$$\tilde u_{\tilde t}\mathrm{D}_xT+\tilde u_{\tilde x}\mathrm{D}_xX-\mathrm{D}_xU=0,$$
$$\tilde u_{\tilde t\tilde t}(\mathrm{D}_tT)^2+2\tilde u_{\tilde t\tilde x}(\mathrm{D}_tX)(\mathrm{D}_tT)+\tilde u_{\tilde x\tilde x}(\mathrm{D}_tX)^2+\tilde u_{\tilde t}\mathrm{D}_t^2T+\tilde u_{\tilde x}\mathrm{D}_t^2X-\mathrm{D}_t^2U=0,$$
$$\tilde u_{\tilde t\tilde t}(\mathrm{D}_xT)^2+2\tilde u_{\tilde t\tilde x}(\mathrm{D}_xX)(\mathrm{D}_xT)+\tilde u_{\tilde x\tilde x}(\mathrm{D}_xX)^2+\tilde u_{\tilde t}\mathrm{D}_x^2T+\tilde u_{\tilde x}\mathrm{D}_x^2X-\mathrm{D}_x^2U=0.$$

Solving the last two equations for u_{tt} and u_{xx}, respectively, and substituting the results into (4.1.1), we obtain

$$\begin{aligned}&\tilde u_{\tilde t\tilde t}(\mathrm{D}_tT)^2+2\tilde u_{\tilde t\tilde x}(\mathrm{D}_tT)(\mathrm{D}_tX)+\tilde u_{\tilde x\tilde x}(\mathrm{D}_tX)^2+\tilde u_{\tilde t}V^tT+\tilde u_{\tilde x}V^tX-V^tU\\&=f\big(\tilde u_{\tilde t\tilde t}(\mathrm{D}_xT)^2+2\tilde u_{\tilde t\tilde x}(\mathrm{D}_xT)(\mathrm{D}_xX)+\tilde u_{\tilde x\tilde x}(\mathrm{D}_xX)^2+\tilde u_{\tilde t}V^xT+\tilde u_{\tilde x}V^xX-V^xU\big)\\&-g(\tilde u_{\tilde t}T_u+\tilde u_{\tilde x}X_u-U_u),\end{aligned}\tag{4.5.2}$$

where we use the notation $V^t = \partial_{tt} + 2u_t\partial_{tu} + {u_t}^2\partial_{uu}$ and $V^x = \partial_{xx} + 2u_x\partial_{xu} + {u_x}^2\partial_{uu}$ and additionally have to set $\tilde u_{\tilde t\tilde t} = \tilde f\tilde u_{\tilde x\tilde x} + \tilde g$ wherever it occurs. As the derivative $\tilde u_{\tilde t\tilde x}$ does not appear in the transformed form of equations from the class (4.1.1), the associated coefficient in (4.5.2) vanishes, i.e.

$$(T_t + T_u u_t)(X_t + X_u u_t) = f(T_x + T_u u_x)(X_x + X_u u_x). \tag{4.5.3}$$

Eq. (4.5.3) involves only original (untilded) variables and is a polynomial in u_t. Therefore, we can split it with respect to u_t by collecting the coefficients of different powers of this derivative. (Note that we cannot as directly split Eq. (4.5.3) with respect to the derivative u_x, which is an argument of the function f.) As a result, we derive that

$$u_t^2\colon \quad T_u X_u = 0, \tag{4.5.4}$$
$$u_t^1\colon \quad T_u X_t + T_t X_u = 0, \tag{4.5.5}$$
$$u_t^0\colon \quad T_t X_t = f(T_x X_x + (T_u X_x + T_x X_u)u_x). \tag{4.5.6}$$

Multiplying Eq. (4.5.5) by T_u (resp. X_u), we obtain, in view of Eq. (4.5.4), that $T_u X_t = 0$ (resp. $T_t X_u = 0$). We apply the trick with the multiplication by T_u (resp. X_u) also to Eq. (4.5.6) and take into account the equations $T_u X_u = 0$, $T_u X_t = 0$ and $T_t X_u = 0$ already derived and the inequality $f \neq 0$. This gives equations which involve no arbitrary elements and hence can be further split with respect to u_x. Therefore, these equations are equivalent to the equations $T_u X_x = 0$ and $X_u T_x = 0$, respectively. The system $T_u X_t = 0$, $T_u X_x = 0$, $T_u X_u = 0$ (resp. $X_u T_t = 0$, $X_u T_x = 0$, $X_u T_u = 0$) implies that $T_u = 0$ (resp. $X_u = 0$) since otherwise the Jacobian J of the point transformation (4.5.1) vanishes. The condition

$$T_u = X_u = 0$$

means that any admissible point transformation of the class (4.1.1) is fiber-preserving. In view of this condition, Eqs. (4.5.4) and (4.5.5) are identically satisfied and the remainder of Eq. (4.5.6) is

$$T_t X_t = f T_x X_x. \tag{4.5.7}$$

After substituting $\tilde u_{\tilde t\tilde t} = \tilde f\tilde u_{\tilde x\tilde x} + \tilde g$, we can also split (4.5.2) with respect to $\tilde u_{\tilde x\tilde x}$, which gives, in view of $T_u = X_u = 0$, the equation

$$\tilde f {T_t}^2 + {X_t}^2 = f(\tilde f {T_x}^2 + {X_x}^2). \tag{4.5.8}$$

Unfortunately, the direct splitting with respect to other derivatives in Eq. (4.5.2) is not possible. The remaining part of (4.5.2) therefore is

$$\begin{aligned} &\tilde g T_t^2 + \tilde u_{\tilde t} T_{tt} + \tilde u_{\tilde x} X_{tt} - (U_{tt} + 2U_{tu}u_t + U_{uu}u_t^2) \\ &= f(\tilde g T_x^2 + \tilde u_{\tilde t} T_{xx} + \tilde u_{\tilde x} X_{xx} - (U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2)) + gU_u. \end{aligned} \tag{4.5.9}$$

The additional condition to keep in mind is the nondegeneracy of transformations (4.5.1), which in view of the conditions $T_u = X_u = 0$ is reduced to the inequality $U_u(T_t X_x - T_x X_t) \neq 0$ and hence $(T_t X_x - T_x X_t) \neq 0$ and $U_u \neq 0$.

4.6 Equivalence group

At this point, we continue the consideration by computing the equivalence group as it is needed even for the analysis of the determining equations for coefficients of Lie symmetry operators and the exhaustive description of admissible transformations. In the case of equivalence transformations, the arbitrary elements f and g should be varied. Hence we can split Eqs. (4.5.7)–(4.5.9) also with respect to the arbitrary elements. Eq. (4.5.7) and the nondegeneracy constraint $T_tX_x - T_xX_t \neq 0$ imply that either $T_t = X_x = 0$ and $T_xX_t \neq 0$ or $T_x = X_t = 0$ and $T_tX_x \neq 0$.

For $T_t = X_x = 0$, Eq. (4.5.8) is simplified to $X_t^2 = f\tilde{f}T_x^2$. As the expression for the derivative u_x in the new variables is $u_x = (T_x\tilde{u}_{\tilde{t}} - U_x)/U_u$, i.e., it does not involve $\tilde{u}_{\tilde{x}}$, the equality $X_t^2 = f\tilde{f}T_x^2$ is split into the two equations $T_x = 0$ and $X_t = 0$, which contradict the nondegeneracy condition.

Therefore we necessarily have $T_x = X_t = 0$ and thus $T = T(t)$, $X = X(x)$, where $T_t \neq 0$ and $X_x \neq 0$. Then Eq. (4.5.8) is reduced to $\tilde{f}T_t^2 = fX_x^2$ and the differentiation of this equation with respect to t yields

$$2T_tT_{tt}\tilde{f} + T_t^2\frac{U_{tx} + U_{tu}u_x}{X_x}\tilde{f}_{\tilde{u}_{\tilde{x}}} = 0. \tag{4.6.1}$$

Since Eq. (4.6.1) holds for all $\tilde{f}$, we can split it and derive $T_{tt} = 0$, $U_{tx} = 0$ and $U_{tu} = 0$. Collecting coefficients of u_t^2 in Eq. (4.5.9) we moreover find that $U_{uu} = 0$. Taking all the constraints derived into account, Eq. (4.5.9) reads

$$\tilde{g}T_t^2 - U_{tt} = f\left(\frac{U_uu_x + U_x}{X_x}X_{xx} - U_{xx} - 2U_{xu}u_x\right) + gU_u.$$

Differentiating this equation with respect to u and t allows deriving that $U_{xu} = 0$ and $U_{ttt} = 0$.

Collecting all the restrictions derived up to now, any equivalence transformation must satisfy the following system of differential equations

$$T_u = T_x = T_{tt} = 0, \quad X_u = X_t = 0, \quad U_{uu} = U_{tu} = U_{xu} = U_{tx} = U_{ttt} = 0. \tag{4.6.2}$$

Integrating the above system in view of the nondegeneracy condition $\mathrm{J} \neq 0$, we proved the following theorem:

Theorem 4.16. *The equivalence group $G^{\sim}$ of the class* (4.1.1) *consists of the transformations*

$$\begin{aligned}&\tilde{t} = c_1t + c_0, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = c_2u + c_4t^2 + c_3t + \psi(x), \quad \tilde{u}_{\tilde{x}} = \frac{c_2u_x + \psi_x}{\varphi_x},\\ &\tilde{f} = \frac{\varphi_x^2}{c_1^2}f, \quad \tilde{g} = \frac{1}{c_1^2}\left(c_2g + \frac{c_2u_x + \psi_x}{\varphi_x}\varphi_{xx}f - \psi_{xx}f + 2c_4\right),\end{aligned} \tag{4.6.3}$$

where c_0, …, c_4 are arbitrary constants satisfying the condition $c_1c_2 \neq 0$ and φ and ψ run through the set of smooth functions of x, $\varphi_x \neq 0$.

After comparing the equivalence algebra $\mathfrak{g}^{\sim}$ and the equivalence group $G^{\sim}$, the following corollary is evident:

Corollary 4.17. *The class of equations* (4.1.1) *admits three independent discrete equivalence transformations, which are given by $(t,x,u,f,g) \mapsto (-t,x,u,f,g)$, $(t,x,u,f,g) \mapsto (t,-x,u,f,g)$ and $(t,x,u,f,g) \mapsto (t,x,-u,f,-g)$.*

Theorem 4.16 implies that any transformation $\mathcal{T}$ from $G^{\sim}$ can be represented as the composition

$$\mathcal{T} = \mathcal{D}^t(c_1)\mathcal{P}^t(c_0)\mathcal{D}(\varphi)\mathcal{D}^u(c_2)\mathcal{F}^1(c_4)\mathcal{F}^2(c_3)\mathcal{G}(\psi),$$

where the above parameterized equivalence transformations are

$$\begin{array}{llllllll}
\mathcal{P}^t(c_0)\colon & \tilde t = t + c_0, & \tilde x = x, & \tilde u = u, & \tilde u_{\tilde x} = u_x, & \tilde f = f, & \tilde g = g,\\
\mathcal{D}^t(c_1)\colon & \tilde t = c_1 t, & \tilde x = x, & \tilde u = u, & \tilde u_{\tilde x} = u_x, & \tilde f = c_1^{-2} f, & \tilde g = c_1^{-2} g,\\
\mathcal{D}(\varphi)\colon & \tilde t = t, & \tilde x = \varphi, & \tilde u = u, & \tilde u_{\tilde x} = u_x/\varphi_x, & \tilde f = \varphi_x^2 f, & \tilde g = g + \varphi_{xx} u_x f/\varphi_x,\\
\mathcal{D}^u(c_2)\colon & \tilde t = t, & \tilde x = x, & \tilde u = c_2 u, & \tilde u_{\tilde x} = c_2 u_x, & \tilde f = f, & \tilde g = c_2 g,\\
\mathcal{F}^1(c_3)\colon & \tilde t = t, & \tilde x = x, & \tilde u = u + c_3 t, & \tilde u_{\tilde x} = u_x, & \tilde f = f, & \tilde g = g,\\
\mathcal{F}^2(c_4)\colon & \tilde t = t, & \tilde x = x, & \tilde u = u + c_4 t^2, & \tilde u_{\tilde x} = u_x, & \tilde f = f, & \tilde g = g + 2c_4,\\
\mathcal{G}(\psi)\colon & \tilde t = t, & \tilde x = x, & \tilde u = u + \psi, & \tilde u_{\tilde x} = u_x + \psi_x, & \tilde f = f, & \tilde g = g - \psi_{xx} f,
\end{array}$$

and the nondegeneracy requires that $c_1 c_2 \varphi_x \neq 0$. These transformations are shifts and scalings in t, arbitrary transformations in x, scalings of u, gauging transformations of u with square polynomials in t and arbitrary functions of x.

4.7 Determining equations for Lie symmetries

Suppose that a vector field $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ belongs to the maximal Lie invariance algebra $\mathfrak{g}^{\max}$ of an equation $\mathcal{L}$: $L = 0$ from the class (4.1.1), i.e. it is the generator of a one-parameter Lie symmetry group of the equation $\mathcal{L}$. The criterion for infinitesimal invariance of $\mathcal{L}$ with respect to Q is implemented using the second prolongation of Q, which reads

$$Q_{(2)} = Q + \eta^t\partial_{u_t} + \eta^x\partial_{u_x} + \eta^{tt}\partial_{u_x} + \eta^{tx}\partial_{u_{tx}} + \eta^{xx}\partial_{u_{xx}}.$$

The coefficients η^t, η^x, η^{tt}, η^{xx} in $Q_{(2)}$ can be determined from the general prolongation formula for vector fields, see, e.g. [25, 101, 112]. Using the second prolongation of Q, the infinitesimal invariance criterion reads $Q_{(2)}L|_{L=0} = 0$, where the notation $|_{L=0}$ means that the condition $Q_{(2)}L$ is required to hold only on equations from the class (4.1.1). Applying the infinitesimal invariance condition to the class (4.1.1) then yields

$$\eta^{tt} - (\xi f_x + \eta^x f_{u_x})u_{xx} - f\eta^{xx} - \xi g_x - \eta^x g_{u_x} = 0 \qquad \text{for} \qquad u_{tt} = f u_{xx} + g, \tag{4.7.1}$$

where

$$\begin{aligned}
\eta^x &= \mathrm{D}_x(\eta - \tau u_t - \xi u_x) + \tau u_{tx} + \xi u_{xx},\\
\eta^{xx} &= \mathrm{D}_x^2(\eta - \tau u_t - \xi u_x) + \tau u_{txx} + \xi u_{xxx},\\
\eta^{tt} &= \mathrm{D}_t^2(\eta - \tau u_t - \xi u_x) + \tau u_{ttt} + \xi u_{ttx},
\end{aligned}$$

$\mathrm{D}_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots$ and $\mathrm{D}_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots$ are the operators of total differentiation with respect to t and x, respectively. Expanding the infinitesimal invariance condition (4.7.1) we obtain

$$\begin{aligned}
&\mathrm{D}_t^2\eta - u_t\mathrm{D}_t^2\tau - u_x\mathrm{D}_t^2\xi - 2u_{tt}\mathrm{D}_t\tau - 2u_{tx}\mathrm{D}_t\xi\\
&= f(\mathrm{D}_x^2\eta - u_t\mathrm{D}_x^2\tau - u_x\mathrm{D}_x^2\xi - 2u_{tx}\mathrm{D}_x\tau - 2u_{xx}\mathrm{D}_x\xi)\\
&+ (\xi f_x + (\mathrm{D}_x\eta - u_t\mathrm{D}_x\tau - u_x\mathrm{D}_x\xi)f_{u_x})u_{xx} + \xi g_x + (\mathrm{D}_x\eta - u_t\mathrm{D}_x\tau - u_x\mathrm{D}_x\xi)g_{u_x},
\end{aligned} \tag{4.7.2}$$

where we have to substitute $u_{tt} = fu_{xx} + g$. The above equation can be split with respect to the derivatives u_{tx}, u_{xx} and u_t. Collecting the coefficients of $u_{tx}u_t$, u_{tx} and $u_{xx}u_t$, we produce $\xi_u = 0$, $\xi_t = f(\tau_x + \tau_u u_x)$, $2f\tau_u = (\tau_x + \tau_u u_x)f_{u_x}$. Supposing that $\xi_t = 0$, the second equation immediately implies that $\tau_u = 0$. Otherwise, for $\xi_t \neq 0$ we can solve the second equation for f and substitute the obtained expression into the third equation. After simplification we get that $\xi_t\tau_u = 0$, i.e. $\tau_u = 0$. Therefore, we always have

$$\xi_u = 0, \quad \tau_u = 0, \quad \xi_t = f\tau_x, \quad \tau_x f_{u_x} = 0.$$

Further splitting of Eq. (4.7.2) and taking into account the above restrictions gives

$$\eta_{uu} = 0, \tag{4.7.3a}$$

$$2(\tau_t - \xi_x)f + \xi f_x + (\eta_x + (\eta_u - \xi_x)u_x)f_{u_x} = 0, \tag{4.7.3b}$$

$$2\eta_{tu} - \tau_{tt} + \tau_{xx}f + \tau_x g_{u_x} = 0, \tag{4.7.3c}$$

$$\begin{aligned}&\eta_{tt} - \xi_{tt}u_x - (\eta_{xx} + (2\eta_{ux} - \xi_{xx})u_x)f + (\eta_u - 2\tau_t)g \\ &\quad - \xi g_x - (\eta_x + (\eta_u - \xi_x)u_x)g_{u_x} = 0.\end{aligned} \tag{4.7.3d}$$

The equations $\xi_u = 0$, $\tau_u = 0$ and $\eta_{uu} = 0$ neither involve the arbitrary elements f or g nor their derivatives. This is why they can be integrated immediately and give restricting conditions on Lie symmetries valid for all equations of the form (4.1.1). In particular, we have $\eta = \eta^1(t,x)u + \eta^0(t,x)$.

In order to derive the kernel algebra of the class (4.1.1), we further split the classifying part of the determining equations (4.7.3) with respect to the arbitrary elements and their derivatives. This immediately gives that the kernel algebra is

$$\mathfrak{g}^\cap = \langle \partial_t, \partial_u, t\partial_u \rangle,$$

which is a realization of the three-dimensional (nilpotent) Heisenberg algebra. Consequently, the Lie symmetries admitted by each equation from the class (4.1.1) are exhausted by transformations of the form $(t,x,u) \mapsto (t + \varepsilon_1, x, u + \varepsilon_2 + \varepsilon_3 t)$, where ε_1, ε_2 and ε_3 are arbitrary constants.

Up to this point the nonlinearity of the equations under consideration was of no importance. Only the general form (4.1.1) was essential. Now we start to exploit the nonlinearity condition $(f_{u_x}, g_{u_x u_x}) \neq (0,0)$, which is included in the definition of the class (4.1.1).

First assume that $f_{u_x} = 0$ and therefore $g_{u_x u_x} \neq 0$. Differentiating Eq. (4.7.3c) with respect to u_x, we immediately find that $\tau_x = 0$. In view of the equation $\xi_t = f\tau_x$ we also have $\xi_t = 0$. Upon differentiating Eq. (4.7.3b) with respect to t we obtain $\tau_{tt} = 0$. Eq. (4.7.3c) then implies $\eta_{tu} = 0$. Finally, we differentiate Eq. (4.7.3d) with respect to u and u_x (resp. t and u_x, resp. t). This gives $\eta_{xu} = 0$ (resp. $\eta_{xt} = 0$, resp. $\eta_{ttt} = 0$).

Now we assume that $f_{u_x} \neq 0$. In this case, the equation $\xi_t = f\tau_x$ can be split to yield $\xi_t = \tau_x = 0$. Eq. (4.7.3c) thus implies $2\eta_{tu} = \tau_{tt}$. Differentiating Eq. (4.7.3b) with respect to u, we obtain $\eta_{xu} = 0$. The differentiation of Eq. (4.7.3d) with respect to u then yields $\eta_{ttu} = 0$ and hence $\tau_{ttt} = 0$ as $2\eta_{tu} = \tau_{tt}$. Differentiating Eq. (4.7.3b) twice with respect to t leads to $\eta_{ttx} = 0$.

Collecting the results from the above two cases, for the class (4.1.1), whose definition includes the condition $(f_{u_x}, g_{u_x u_x}) \neq (0,0)$, we always have

$$\tau_u = \tau_x = \xi_u = \xi_t = \eta_{uu} = \eta_{xu} = \eta_{ttx} = \tau_{ttt} = 0, \quad 2\eta_{tu} = \tau_{tt}. \tag{4.7.4}$$

Hence only Eqs. (4.7.3b) and (4.7.3d) essentially involve arbitrary elements and are really classifying determining equations for the class (4.1.1). They must be solved up to the equivalence relation induced by transformations from $G^{\sim}$. Also note that for $f_{u_x} \neq 0$ and $\tau_{tt} = 0$ we find by differentiating both these equations with respect to t that $\eta_{tx} = 0$ and $\eta_{ttt} = 0$.

This completes the proof of the following proposition:

Proposition 4.18. *For each equation from the class* (4.1.1)*, any symmetry operator Q with $\tau_{tt} = 0$ lies in the projection of the algebra $\mathfrak{g}^{\sim}$ to the space of equation variables, i.e. $Q \in \mathrm{P}\mathfrak{g}^{\sim}$.*

It thus remains to investigate the case in which $f_{u_x} \neq 0$ and the corresponding maximal Lie invariance algebra $\mathfrak{g}^{\max} = \mathfrak{g}^{\max}(f,g)$ contains a vector field $\breve{Q}$ with $\tau_{tt} \neq 0$. In view of system (4.7.4) the general form of vector fields from $\mathfrak{g}^{\max}$ is

$$Q = (a_2t^2 + a_1t + a_0)\partial_t + \xi(x)\partial_x + ((a_2t + b_1)u + \eta^0(t,x))\partial_u,$$

where the constants a_0, a_1, a_2 and b_1 and the functions $\xi = \xi(x)$ and $\eta^0 = \eta^0(t,x)$, where $\eta^0_{ttx} = 0$, are additionally constrained in such a way that the coefficients $\tau = a_2t^2 + a_1t + a_0$, $\xi = \xi(x)$ and $\eta = (t + b_1)u + \eta^0(t,x)$ also satisfy Eqs. (4.7.3b) and (4.7.3d). For convenience, we will mark the values of coefficients and parameters corresponding to the vector field $\breve{Q}$ by breve. As $\breve{a}_2 \neq 0$, by scaling of $\breve{Q}$ we can set $\breve{a}_2 = 1$. As the vector field ∂_t belongs to $\mathfrak{g}^{\cap}$, it necessarily is in $\mathfrak{g}^{\max}$ for any f and g. Therefore, the algebra $\mathfrak{g}^{\max}$ contains also the commutator $[\partial_t, \breve{Q}]$. Linearly combining $\breve{Q}$ with ∂_t and $\tilde{Q} = [\partial_t, \breve{Q}]$ we can also set $\breve{a}_0 = \breve{a}_1 = 0$. Hence $\tilde{Q} = 2t\partial_t + (u + \breve{\eta}^0_t)\partial_u$.

Substituting the coefficients of $\tilde{Q}$ into Eq. (4.7.3b), we obtain the equation $4f + (u_x + \beta)f_{u_x} = 0$, where $\beta = \breve{\eta}^0_{tx}$ should be a smooth function depending at most on x since $\eta^0_{ttx} = 0$ for any operator Q from $\mathfrak{g}^{\max}$. The general solution of this equation is $f = \alpha(x)(u_x + \beta(x))^{-4}$, where α is an arbitrary function of x. Using transformations from the equivalence group $G^{\sim}$ we can simplify f and set $\alpha = \pm 1$ and $\beta = 0$. If we plug the form $f = \pm u_x^{-4}$ into Eq. (4.7.3b), we obtain that $\tau_t - \xi_x = 2(\eta_u - \xi_x) + 2\eta_x u_x^{-1}$ for an arbitrary operator from $\mathfrak{g}^{\max}$. From this condition, we can immediately conclude that $\eta_x = 0$ and $\xi_x = 2b_1$, i.e. $\xi = 2b_1x + b_0$ for some constant b_0.

As $\breve{\eta}^0_x = 0$, the substitution of the coefficients of $\tilde{Q}$ into Eq. (4.7.3d) gives the equation $u_xg_{u_x} + 3g = \breve{\eta}^0_{ttt}$ with separated variables. Both the sides of this equation are equal to a constant which can be set to zero by a transformation $\mathcal{F}^2(c_4)$ from $G^{\sim}$. The equation $u_xg_{u_x} + 3g = 0$ is equivalent to the representation $g = \mu(x)u_x^{-3}$, where μ is a smooth function of x. Then Eq. (4.7.3d) takes the form $\eta^0_{tt} - 2b_1\mu u_x^{-3} - (2b_1x + b_0)\mu_x u_x^{-3} = 0$ and the subsequent splitting with respect to u_x implies that $\eta^0_{tt} = 0$ and $(\mu(2b_1x + b_0))_x = 0$. We now distinguish the following cases for values of b_0 and b_1 depending on the value of μ:

0. μ is arbitrary. In this case $b_1 = b_0 = 0$.

1. μ is a nonzero constant. Then b_0 is arbitrary and $b_1 = 0$. Using an equivalence transformation, we can scale μ to one.

2. $\mu = \nu x^{-1} \bmod G^{\sim}$, where ν is a nonzero constant. (A constant summand of x can be set equal to 0 by a shift of x.) For this value of μ we have $b_0 = 0$ and b_1 is arbitrary.

3. $\mu = 0$. This implies that b_1 and b_2 are arbitrary.

We denote by $\mathcal{K}$ the subclass of equations from the class (4.1.1), which are $G^{\sim}$-equivalent to equations with $f = \pm u_x^{-4}$ and $g = \mu(x)u_x^{-3}$ and by $\bar{\mathcal{K}}$ the complement of this subclass in the class (4.1.1). The above consideration shows that only equations from the subclass $\mathcal{K}$ admit Lie symmetry operators that are not contained in $\mathrm{P}\mathfrak{g}^{\sim}$. In other words, the following theorem is true:

Theorem 4.19. *The subclass $\bar{\mathcal{K}}$ of the class* (4.1.1) *that is singled out by the condition*

$$(f,g) \neq (\pm u_x^{-4}, \mu(x)u_x^{-3}) \bmod G^{\sim},$$

where $\mu(x)$ is an arbitrary function of x is weakly normalized.

Remark 4.20. The sets $\mathcal{K}$ and $\bar{\mathcal{K}}$ of equations are really subclasses of the class (4.1.1) since the condition $(f,g) = (\pm u_x^{-4}, \mu(x)u_x^{-3}) \bmod G^{\sim}$ and its negation are equivalent to systems of equations and/or inequalities with respect to the arbitrary elements f and g. Indeed, by acting on the arbitrary elements $f = \pm u_x^{-4}$ and $g = \mu(x)u_x^{-3}$ with transformations from $G^{\sim}$ and eliminating the involved group parameters and the parameter-function μ, we arrive at a system of differential equations in f and g characterizing the subclass $\mathcal{K}$. Namely, the subclass $\mathcal{K}$ is singled out from the class (4.1.1) by the system

$$V_{u_x} = 1, \quad W_{xu_x}(V^3)_{u_x} - W_{u_x}(V^3)_{xu_x} = 0, \quad W_{u_xu_x}(V^3)_{u_x} - W_{u_x}(V^3)_{u_xu_x} = 0,$$

where $V = -4f/f_{u_x}$ and $W = V^3(g + fV_x + f_xV/2)$. This implies that the subclass $\bar{\mathcal{K}}$, as the complement of $\mathcal{K}$, is defined by the inequality

$$\big(V_{u_x} - 1\big)^2 + \big(W_{xu_x}(V^3)_{u_x} - W_{u_x}(V^3)_{xu_x}\big)^2 + \big(W_{u_xu_x}(V^3)_{u_x} - W_{u_x}(V^3)_{u_xu_x}\big)^2 \neq 0.$$

The above Cases 0–3 represent the complete group classification of equations from the subclass $\mathcal{K}$ up to $G^{\sim}$-equivalence. Recall that by the definition of the subclass $\mathcal{K}$ any equation from this subclass is $G^{\sim}$-equivalent to an equation with $f = \pm u_x^{-4}$ and $g = \mu(x)u_x^{-3}$.

Lemma 4.21. *A complete list of $G^{\sim}$-inequivalent Lie symmetry extensions for equations of the general form*

$$u_{tt} = \pm u_x^{-4}u_{xx} + \mu(x)u_x^{-3}, \tag{4.7.5}$$

where μ traverses the set of smooth functions depending on x, is exhausted by the following cases:

$$\begin{array}{lll}
0. & \textit{arbitrary } \mu: & \mathfrak{g}_1^{\cap} = \mathfrak{g}^{\cap} + \langle t^2\partial_t + tu\partial_u,\ 2t\partial_t + u\partial_u\rangle,\\
1. & \mu = 1: & \mathfrak{g}^{\max} = \mathfrak{g}_1^{\cap} + \langle \partial_x\rangle,\\
2. & \mu = \nu x^{-1},\ \nu \neq 0: & \mathfrak{g}^{\max} = \mathfrak{g}_1^{\cap} + \langle 2x\partial_x + u\partial_u\rangle,\\
3. & \mu = 0: & \mathfrak{g}^{\max} = \mathfrak{g}_1^{\cap} + \langle \partial_x,\ 2x\partial_x + u\partial_u\rangle.
\end{array} \tag{4.7.6}$$

Remark 4.22. We can use equations of the general form

$$u_{tt} = \theta(x)u_x^{-4}u_{xx} \tag{4.7.7}$$

as canonical representatives of elements from the class $\mathcal{K}$ instead of (4.7.5). Indeed, each equation from the subclass (4.7.5) is mapped to an equation from the subclass (4.7.7) by the transformation $\mathcal{D}(\varphi)$, where $\varphi_{xx} \pm \mu\varphi_x = 0$ and $\theta(\tilde{x}) = \pm(\varphi_x(x))^{-2}$. (Here and in what follows all $\pm$ and $\mp$ are consistent with those from Lemma 4.21.) In other words, we construct a point transformation mapping [150] between the subclasses (4.7.5) and (4.7.7) which is generated by a family of equivalence transformations parameterized by the arbitrary element μ. Hence, mapping and rearrangement of the classification list (4.7.6) lead to the equivalent list based on the representative form (4.7.7):

$$\begin{array}{lll}
0. & \text{arbitrary } \theta: & \mathfrak{g}_1^{\cap} = \mathfrak{g}^{\cap} + \langle t^2\partial_t + tu\partial_u,\ 2t\partial_t + u\partial_u\rangle,\\
1. & \theta = \pm e^{2x}: & \mathfrak{g}^{\max} = \mathfrak{g}_1^{\cap} + \langle 2\partial_x + u\partial_u\rangle,\\
2. & \theta = \pm|x|^{2p},\ p \neq 0: & \mathfrak{g}^{\max} = \mathfrak{g}_1^{\cap} + \langle 2x\partial_x + (p+1)u\partial_u\rangle,\\
3. & \theta = \pm 1: & \mathfrak{g}^{\max} = \mathfrak{g}_1^{\cap} + \langle \partial_x,\ 2x\partial_x + u\partial_u\rangle.
\end{array} \tag{4.7.8}$$

Cases 0, 1, $2|_{\nu=\pm1}$, $2|_{\nu\neq\pm1}$ and 3 of the list (4.7.6) are mapped to Cases 0, $2|_{p=-1}$, 1, $2|_{p=\nu/(\nu\mp1)}$ and 3 of the list (4.7.8), respectively. Each of the classification lists has certain advantages.

Thus, the form (4.7.7) is more compact than (4.7.5). At the same time, basis elements of the algebras presented in the list (4.7.6) do not depend, in contrast to Case 2 of (4.7.8), on equation parameters. The equation associated with Case 1 of the list (4.7.6) does not explicitly involve the independent variable x, as opposed to its image given in Case $2|_{p=-1}$ of the list (4.7.8), for which the arbitrary element θ equals $\pm x^{-2}$.

Remark 4.23. Due to Theorem 4.19, to complete the group classification of the class (4.1.1) it is enough to investigate symmetry extensions induced by subalgebras of the equivalence algebra $\mathfrak{g}^{\sim}$. The corresponding Lie symmetry generators satisfy the following simplified determining equations:

$$
\begin{aligned}
&\tau_u = \tau_x = \tau_{tt} = \xi_u = \xi_t = \eta_{uu} = \eta_{xu} = \eta_{tx} = \eta_{tu} = \eta_{ttt} = 0,\\
&\xi f_x + ((\eta_u - \xi_x)u_x + \eta_x)f_{u_x} = 2(\xi_x - \tau_t)f,\\
&\xi g_x + ((\eta_u - \xi_x)u_x + \eta_x)g_{u_x} = (\eta_u - 2\tau_t)g + (\xi_{xx}u_x - \eta_{xx})f + \eta_{tt}.
\end{aligned}
\tag{4.7.9}
$$

Remark 4.24. Lemma 4.21 obviously implies that the entire class (4.1.1) is not weakly normalized. This can also be proved by the direct computation of the union $\mathfrak{g}^{\cup}$ of the maximal Lie invariance algebras of equations from the class (4.1.1) without the study of the subclass structure. The set $\mathfrak{g}^{\cup}$ consists of vector fields of the form $\tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ for which the whole system of determining equations and the nonvanishing condition $(f_{u_x}, g_{u_xu_x}) \neq (0,0)$ are consistent with respect to the functions $f = f(x,u_x)$ and $g = g(x,u_x)$. The consistency condition is the joint system of (4.7.4) and

$$
\begin{aligned}
&\eta_{tx}(\eta_u - \xi_x) = \eta_x\eta_{tu} + \xi\eta_{txx},\\
&\eta_{tu}(\xi_{xx} + \eta_{xx}) = \eta_{txx}(\eta_u - 2\tau_t + \xi),\\
&\eta_{ttt}(\eta_u - 2\tau_t) = \eta_{tt}(\eta_{tu} - 2\tau_{tt}).
\end{aligned}
\tag{4.7.10}
$$

It is clear that $\mathfrak{g}^{\cup}$ is not contained in the projection $\mathrm{P}\mathfrak{g}^{\sim}$ of the equivalence algebra $\mathfrak{g}^{\sim}$, which is associated with the solution set of the system given in the first row of (4.7.9).

4.8 Equivalence groupoid

Since we have established the equivalence group of the class (4.1.1), we can now describe the equivalence groupoid of this class in terms of its normalized subclasses. Theorem 4.19 and its proof give us hints on feasible ways for the classification of admissible transformations.

First we assume that $f_{u_x} = 0$ and therefore $g_{u_xu_x} \neq 0$. We differentiate Eq. (4.5.9) with respect to $\tilde u_{\tilde t}$ and $\tilde u_{\tilde x}$ and take into account Eq. (4.5.7). From the equation $g_{u_xu_x}T_xX_xU_u^{-2} = 0$ obtained and the inequality $g_{u_xu_x} \neq 0$ we can conclude that $T_xX_x = 0$. Then Eq. (4.5.7) also implies $T_tX_t = 0$.

Suppose that $T_t = 0$. Consequently, in view of the nondegeneracy condition of point transformations we have $T_x \neq 0$ and $X_t \neq 0$ and therefore $X_x = 0$. The expressions of u_t and u_x via $\tilde u_{\tilde t}$ and $\tilde u_{\tilde x}$ take the form $u_t = (X_t\tilde u_{\tilde x} - U_t)/U_u$ and $u_x = (T_x\tilde u_{\tilde t} - U_x)/U_u$, i.e., the expression of u_t (resp. u_x) does not involve $\tilde u_{\tilde t}$ (resp. $\tilde u_{\tilde x}$). We differentiate Eq. (4.5.9) twice with respect to $\tilde u_{\tilde x}$ and once with respect to u. In view of the supposition $T_t = 0$, this gives $(U_{uu}/U_u^2)_u = 0$. Then we differentiate Eq. (4.5.9) twice with respect to $\tilde u_{\tilde t}$:

$$
g_{u_xu_x} = 2f\frac{U_{uu}}{U_u}.
\tag{4.8.1}
$$

The subsequent differentiation of Eq. (4.8.1) with respect to u gives the equation $(U_{uu}/U_u)_u = 0$, which together with $(U_{uu}/U_u^2)_u = 0$ implies that $U_{uu} = 0$. Then Eq. (4.8.1) is reduced to $g_{u_x u_x} = 0$ and therefore leads to a contradiction.

This is why we necessarily have $T_t X_x \neq 0$ and consequently $X_t = T_x = 0$. In view of Eq. (4.5.8), we also obtain the equation $\tilde{f} = f X_x^2/T_t^2$ from which we can conclude, by differentiation with respect to t that $T_{tt} = 0$. It is also evident that $\tilde{f}_{\tilde{u}_{\tilde{x}}} = 0$.

Owing to the restrictions derived so far, it is now possible to split Eq. (4.5.9) with respect to u_t. The coefficient of u_t^2 gives $U_{uu} = 0$ and that of u_t leads to $U_{tu} = 0$. The rest of Eq. (4.5.9) is

$$\tilde{g}T_t^2 - U_{tt} = f(\tilde{u}_{\tilde{x}} X_{xx} - U_{xx} - 2U_{xu}u_x) + gU_u.$$

This obviously implies that $\tilde{g}_{\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{x}}} \neq 0$ since $g_{u_x u_x} \neq 0$. We successively differentiate the above rest with respect to three combinations of variables, $(u, \tilde{u}_{\tilde{x}})$, $(t, \tilde{u}_{\tilde{x}})$ and t, which gives $U_{xu} = 0$, $U_{tx} = 0$ and $U_{ttt} = 0$, respectively.

Summing up, for the components T, X and U of admissible transformations of any equation with $f_{u_x} = 0$ and $g_{u_x u_x} \neq 0$ within the class (4.1.1) we derive the same system of determining equations as in the case of equivalence transformations, cf. (4.6.2). Moreover, the conditions $f_{u_x} = 0$ and $g_{u_x u_x} \neq 0$ are saved by the admissible transformations. In this way, we have established the following theorem:

Theorem 4.25. *The subclass of the class* (4.1.1) *which is singled out by the constraints* $f_{u_x} = 0$ *and* $g_{u_x u_x} \neq 0$ *is preserved by admissible point transformations within the class* (4.1.1). *This subclass is normalized and its equivalence group coincides with the equivalence group* $G^{\sim}$ *of the entire class* (4.1.1).

It now remains to study the case $f_{u_x} \neq 0$. Eq. (4.5.7) immediately implies that $T_t X_t = T_x X_x = 0$.

Supposing $T_x \neq 0$, we obtain that $X_x = 0$, $X_t \neq 0$ and hence $T_t = 0$. In view of these conditions Eq. (4.5.8) is reduced to $X_t^2 = \tilde{f} f T_x^2$. Differentiating the last equation with respect to u_x leads to the equation $\tilde{f} f_{u_x} T_x^2 = 0$, which is equivalent to the equation $T_x = 0$, contradicting the initial supposition.

Therefore, we have $T_x = 0$, $T_t \neq 0$, $X_t = 0$ and $X_x \neq 0$ and Eq. (4.5.8) reads

$$\tilde{f}T_t^2 = fX_x^2. \tag{4.8.2}$$

As the transformation rules for the first derivatives are simplified to

$$\tilde{u}_{\tilde{t}} = \frac{U_t + U_u u_t}{T_t}, \quad \tilde{u}_{\tilde{x}} = \frac{U_x + U_u u_x}{X_x},$$

we can conclude from Eq. (4.8.2) that $\tilde{f}_{\tilde{u}_{\tilde{x}}} = 0$ if and only if $f_{u_x} = 0$.

Differentiating Eq. (4.8.2) with respect to u gives $(U_{xu}+U_{uu}u_x)\tilde{f}_{\tilde{u}_{\tilde{x}}} = 0$, and therefore $U_{uu} = 0$ and $U_{xu} = 0$. Differentiating Eq. (4.8.2) with respect to t results in the equation

$$\left(\frac{U_{tu}}{X_x}\frac{X_x\tilde{u}_{\tilde{x}} - U_x}{U_u} + \frac{U_{xt}}{X_x}\right)\tilde{f}_{\tilde{u}_{\tilde{x}}} + 2\tilde{f}\frac{T_{tt}}{T_t} = 0. \tag{4.8.3}$$

Taking into account the simplifications obtained so far, we represent Eq. (4.5.9) in the reduced form

$$\tilde{g}T_t^2 + \frac{U_t + U_u u_t}{T_t}T_{tt} - U_{tt} - 2U_{tu}u_t = f\left(\frac{U_x + U_u u_x}{X_x}X_{xx} - U_{xx}\right) + gU_u.$$

The last equation can be split with respect to u_t, giving the equations

$$\frac{U_u}{T_t}T_{tt} = 2U_{tu}, \quad \tilde{g}T_t^2 + \frac{U_t}{T_t}T_{tt} - U_{tt} = f\left(\frac{U_x + U_u u_x}{X_x}X_{xx} - U_{xx}\right) + gU_u. \tag{4.8.4}$$

We now distinguish the two cases $T_{tt} = 0$ and $T_{tt} \neq 0$.

In the case of $T_{tt} = 0$, the first of the above equations implies $U_{tu} = 0$. The corresponding form of Eq. (4.8.3) then leads to $U_{tx} = 0$. Differentiating the second equation of (4.8.4) with respect to t yields $U_{ttt} = 0$. Collecting all the results for this case implies that the transformation belongs to the equivalence group $G^{\sim}$.

We now investigate the case of $T_{tt} \neq 0$. Then, we solve the first equation of (4.8.4) with respect to U_{tu}/U_u and plug the resulting expression into Eq. (4.8.3). This yields

$$\left(\frac{T_{tt}}{2T_t}\left(\tilde{u}_{\tilde{x}} - \frac{U_x}{X_x}\right) + \frac{U_{xt}}{X_x}\right)\tilde{f}_{\tilde{u}_{\tilde{x}}} + 2\frac{T_{tt}}{T_t}\tilde{f} = 0, \quad \text{or,} \quad \left(\tilde{u}_{\tilde{x}} + \frac{U_{xt}}{X_x}\frac{2T_t}{T_{tt}} - \frac{U_x}{X_x}\right)\tilde{f}_{\tilde{u}_{\tilde{x}}} + 4\tilde{f} = 0.$$

The difference of the second and third terms in the bracket can be encapsulated as a function of x (or, equivalently, $\tilde{x}$), i.e. we can write $(\tilde{u}_{\tilde{x}} + \tilde{\beta}(\tilde{x}))\tilde{f}_{\tilde{u}_{\tilde{x}}} + 4\tilde{f} = 0$. This implies that

$$\tilde{f} = \frac{\tilde{\alpha}(\tilde{x})}{(\tilde{u}_{\tilde{x}} + \tilde{\beta}(\tilde{x}))^4} \tag{4.8.5}$$

for some smooth function $\tilde{\alpha} = \tilde{\alpha}(\tilde{x})$. We now differentiate the second equation of (4.8.4) with respect to u, which gives

$$U_{ttu} = \frac{U_{ut}T_{tt}}{T_t} = \frac{1}{2}\left(\frac{U_u T_{tt}}{T_t}\right)_t,$$

where the second equality holds upon differentiating the first equation in (4.8.4) with respect to t. This implies that

$$\frac{U_{tu}T_{tt}}{T_t} - U_u\left(\frac{T_{tt}}{T_t}\right)_t = 0,$$

which is equivalent to $(U_u T_t/T_{tt})_t = 0$. Integrating this equation gives an expression for U_u: $U_u = \varkappa T_{tt}/T_t$, where $\varkappa$ is a constant. We substitute the expression for U_u into the first equation of (4.8.4) to obtain $2T_{ttt}T_t - 3T_{tt}^2 = 0$. The general solution of the last equation is

$$T = \frac{a_1 t + a_0}{a_3 t + a_2},$$

where a_i, $i = 0, \ldots, 3$, are constants with $a_1 a_2 - a_0 a_3 \neq 0$ which are determined up to a common nonvanishing multiplier. As $T_{tt} \neq 0$, we moreover have $a_3 \neq 0$ and can assume $a_3 = 1$ due to the indeterminacy up to a constant multiplier. Then we successively gauge a_2, a_0 and a_1 to 0, 1 and 0 by a shift of t, a scaling of t and a shift of $\tilde{t}$, respectively. All the above transformations belong to the group $G^{\sim}$. In other words, $T = 1/t \bmod G^{\sim}$. Plugging the expression obtained for T into the equation $U_u = \varkappa T_{tt}/T_t$ allows deriving that $U_u = \hat{q}/t$, where $\hat{q}$ is a nonzero constant.

Combining Eq. (4.8.2) with the expression for $\tilde{f}$ established in Eq. (4.8.5) yields

$$f = \frac{{T_t}^2}{{X_x}^2}\frac{\tilde{\alpha}(X)X_x^4}{(U_u u_x + U_x + \tilde{\beta}(X)X_x)^4} = \frac{\alpha(x)}{(u_x + \beta(x))^4},$$

where $\alpha(x) := {T_t}^2{X_x}^2\tilde{\alpha}(X)/U_u^4$ and $\beta(x) := (X_x\tilde{\beta}(X) + U_x)/U_u$. Furthermore, upon using transformations from the equivalence group $G^{\sim}$, we can set $\tilde{\beta} = \beta = 0$, which consequently implies that $U_x=0$. By means of equivalence transformations, we can also set $\alpha,\tilde\alpha\in\{-1,1\}$ and as the multiplier relating α and $\tilde\alpha$ is strictly positive, we have that $\tilde\alpha=\alpha$. Since the transformation component X only depends on x, it also follows from ${T_t}^2{X_x}^2/U_u^4=1$ that $X_x=\mathrm{const}$. Due to scalings and translations of x, which belong to $G^\sim$, we can choose $X=x$. Therefore, ${T_t}^2/U_u^4=1$. As $T=1/t$ and thus $U_u=\hat q/t$, this means that $\hat q=1$, i.e., $U_u=1/t$ and hence $U=u/t+U^0(t)$ and $\tilde u_{\tilde x}=u_x/t$. Here $U^0=U^0(t)$ is a smooth function arising after integration with respect to u and depending only on x in view of the condition $U_x=0$.

The remaining part of Eq. (4.5.9) can be represented as

$$\frac{\tilde g}{t^3}-2U^0_t-tU^0_{tt}=g, \tag{4.8.6}$$

where $U=u/t+U^0(t)$. The differentiation of Eq. (4.8.6) with respect to t yields $\tilde g_{\tilde u_{\tilde x}}\tilde u_{\tilde x}+3\tilde g+t^4(tU^0_{tt}+2U^0_t)_t=0$. The first two terms do not depend on t and the last summand depends only on t. Thus, we can separate variables and set $t^4(tU^0_{tt}+2U^0_t)_t=-3\tilde\varkappa=\mathrm{const}$, where the factor of -3 was introduced for the sake of convenience. Integration of this equation yields $tU^0_{tt}+2U^0_t=\tilde\varkappa/t^3+\varkappa$, where $\varkappa=\mathrm{const}$. The general solution of this equation is $U^0=\hat\varkappa/(2t^2)-\varkappa t/2-\sigma_1/t+\sigma_0$, where $\sigma_1,\sigma_2=\mathrm{const}$. We also have $\tilde g_{\tilde u_{\tilde x}}\tilde u_{\tilde x}+3\tilde g=\tilde\varkappa$, which upon integration leads to $\tilde g=\tilde\mu(\tilde x)\tilde u_{\tilde x}^{-3}+\tilde\varkappa$. Plugging these results into Eq. (4.8.6) gives

$$g=\frac{\tilde\mu(X)}{u_x^3}+\frac{\tilde\varkappa}{t^3}-(tU^0_{tt}+2U^0_t)=\frac{\tilde\mu(X)}{u_x^3}+\varkappa.$$

Using equivalence transformations, we can put $\hat\varkappa=\varkappa=0$. This is why we have $\tilde f=\delta\tilde u_{\tilde x}^{-4}$, $f=\delta u_x^{-4}$, $\tilde g=\delta\tilde u_{\tilde x}^{-3}$ and $g=\delta u_x^{-3}$, where $\delta=\pm1$. That is, the equivalence transformations for this case reduce to symmetry transformations.

Owing to the above computations, we can formulate the following theorem:

Theorem 4.26. *The subclass $\mathcal K$ of the class* (4.1.1)*, that consists of equations $G^\sim$-equivalent to equations of the form* (4.7.5)*, is semi-normalized with respect to $G^\sim$. Any admissible transformation in this subclass is generated by $G^\sim$ or is represented as a composition of the transformations $(\theta_1,\theta_2,\mathcal T_1)$, $(\theta_2,\theta_2,\mathcal T_2)$ and $(\theta_2,\theta_3,\mathcal T_3)$, where $\theta_1=(f,g)$, $\theta_2=(\pm u_x^{-4},\mu u_x^{-3})$, $\theta_3=(\tilde f,\tilde g)$ and $\mathcal T_1$, $\mathcal T_3$ are equivalence transformations and $\mathcal T_2=1/t$ is a symmetry transformation of $\mathcal L_{\theta_2}$. The complement $\bar{\mathcal K}$ of $\mathcal K$ in the class* (4.1.1) *(as well as the complement of $\mathcal K$ in the subclass singled out from the class* (4.1.1) *by the condition $f_{u_x}\neq0$) is normalized with respect to $G^\sim$. The usual equivalence group of the subclass $\bar{\mathcal K}$ coincides with $G^\sim$.*

Corollary 4.27. *The entire class* (4.1.1) *is semi-normalized. Hence the group classification of the class* (4.1.1) *up to $G^\sim$-equivalence coincides with the group classification of this class up to general point equivalence.*

Remark 4.28. It can be proved using the above consideration that the class (4.7.5) is normalized. The equivalence group $G^\sim_1$ of this class consists of the transformations of the general form

$$\tilde t=\frac{a_1t+a_0}{a_3t+a_2},\quad \tilde x=b_1t+b_0,\quad \tilde u=\frac{\pm\sqrt{|b_1A|}\,u+b_3t+b_2}{a_3t+a_2},\quad \tilde\mu=\frac{\mu}{b_1},$$

where a_i, $i=0,\dots,3$, are constants with $A=a_1a_2-a_0a_3\neq0$ which are determined up to a common nonvanishing multiplier and b_i, $i=0,\dots,3$, are arbitrary constants with $b_1\neq0$. The group $G^\sim_1$ can be represented as the product of its two subgroups. The first subgroup is the ideal associated with the kernel group of the class (4.7.5) and formed by the transformations from $G_1^\sim$ with $b_1 = 1$ and $b_0 = 0$. The second subgroup corresponds to the subgroup of $G^\sim$ whose elements save equations of the form (4.7.5) and consists of the transformations from $G_1^\sim$ with $a_3 = 1$ and $a_2 = 0$. This is why the list presented in Lemma 4.21 is an exhaustive list of Lie symmetry extensions in the class (4.7.5) up to both $G_1^\sim$-equivalence and general point equivalence.

Remark 4.29. It follows from the above consideration that the entire class of equations of the general form (4.1.1) is partitioned into three subclasses associated with the additional constraints $f_{u_x} \neq 0$, $f_{u_x} = 0$ and $g_{u_xu_x} \neq 0$, and $f_{u_x} = g_{u_xu_x} = 0$, respectively. Equations from different subclasses of this partition are not mapped to each other by point transformations. This is the main reason why it is natural to separate nonlinear equations of the form (4.1.1) from linear ones, which are well studied and form the last subclass.

Remark 4.30. In order to simplify the calculations, we could have used Theorem 4.4b of Ref. [73], describing form-preserving transformations between $(1+1)$-dimensional second-order partial differential equations of the quite general form $u_{tt} = H(t, x, u, u_x, u_{xx})$, where $H_{u_{xx}} \neq 0$. This theorem directly implies the simplest constraints $T_u = T_x = X_u = X_t = 0$ for admissible transformations of the class (4.1.1), in view of which the coefficients of any Lie symmetry operator $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u$ of each equation from the class (4.1.1) satisfy the determining equations $\tau_u = \tau_x = \xi_u = \xi_t = 0$. A partial repetition of computations in the present paper was necessary in order to find the appropriate partition of the class (4.1.1) into subclasses.

4.9 Classification of appropriate subalgebras

In order to classify subalgebras of the equivalence algebra $\mathfrak{g}^\sim$, we should describe the adjoint action of the equivalence group $G^\sim$, which consists of transformations of the form (4.6.3), on the vector fields (4.4.1) generating $\mathfrak{g}^\sim$. The description will be implemented by the direct computation of actions of transformations from $G^\sim$ on elements of $\mathfrak{g}^\sim$ via pushforwards of vector fields by these transformations [43], which differs from the way presented e.g. in [101]. In other words, the usual transformation rule of vector fields under point transformations will be used. This method properly works for infinite-dimensional Lie algebras.

Employing elementary equivalence transformations (cf. the end of Section 4.6), we can compute the nonidentical adjoint actions using the respective pushforwards. This yields

$$
\begin{aligned}
&\mathcal{F}^2_*(c_4)\mathcal{D}^t = \mathcal{D}^t + 2c_4\mathcal{F}^2, && \mathcal{D}^t_*(c_1)\mathcal{F}^2 = c_1^{-2}\mathcal{F}^2,\\
&\mathcal{G}_*(\psi)\mathcal{D}^u = \mathcal{D}^u - \mathcal{G}(\psi), && \mathcal{D}^u_*(c_2)\mathcal{G}(\psi) = c_2\mathcal{G}(\psi),\\
&\mathcal{F}^2_*(c_4)\mathcal{D}^u = \mathcal{D}^u - c_4\mathcal{F}^2, && \mathcal{D}^u_*(c_2)\mathcal{F}^2 = c_2\mathcal{F}^2,\\
&\mathcal{G}_*(\psi)\mathcal{D}(\varphi) = \mathcal{D}(\varphi) + \mathcal{G}(\varphi\psi_x), && \mathcal{D}_*(\theta)\mathcal{G}(\psi) = \mathcal{G}(\psi(\hat\theta)),\\
&\mathcal{D}_*(\theta)\mathcal{D}(\varphi) = \mathcal{D}(\varphi(\hat\theta)/\hat\theta_x),
\end{aligned}
$$

where $\hat\theta = \hat\theta(x)$ is the inverse of the function θ. It should be stressed that there are more nonidentical adjoint actions of transformations from $G^\sim$ on generating vector fields of $\mathfrak{g}^\sim$ than listed above, namely those related with actions on the trivial prolongation $\hat{\mathfrak{g}}^\cap$ of the kernel algebra $\mathfrak{g}^\cap$ to the arbitrary elements, which is an ideal in $\mathfrak{g}^\sim$, and those involving $\mathcal{P}^t_*(c_0)$ and $\mathcal{F}^1_*(c_3)$. These adjoint actions, however, do not yield simplifications in the course of classification of extensions of the kernel algebra.

We will only classify appropriate subalgebras of $\mathfrak{g}^\sim$. Any appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}^\sim$ should contain $\hat{\mathfrak{g}}^\cap = \langle \mathcal{P}^t, \mathcal{F}^1, \mathcal{G}(1)\rangle$. For the class (4.1.1) we have two specific representations of $\mathfrak{s}$, which are given by $\mathfrak{s} = \hat{\mathfrak{g}}^\cap + \langle Q^1, \dots, Q^k\rangle = \langle \mathcal{P}^t, \mathcal{F}^1\rangle + \langle \mathcal{G}(1), Q^1, \dots, Q^k\rangle$, where "+" denotes the direct sum of vector spaces, $\hat{\mathfrak{g}}^\cap$ is an ideal of $\mathfrak{s}$ (since it is an ideal of the entire $\mathfrak{g}^\sim$) and $\langle \mathcal{G}(1), Q^1, \dots, Q^k\rangle$ is a subalgebra of $\mathfrak{s}$. Q^1, ..., Q^k are basis elements from the complement of $\hat{\mathfrak{g}}^\cap$ in $\mathfrak{s}$ and their projections to the space of equation variables yield a proper Lie symmetry extension of $\mathfrak{g}^\cap$ in the class (4.1.1).

Remark 4.31. The double representation of appropriate subalgebras is related to the representation of the whole algebra $\mathfrak{g}^\sim$ in the form $\mathfrak{g}^\sim = \hat{\mathfrak{g}}^\cap \dot{+} \bar{\mathfrak{g}}$, where $\hat{\mathfrak{g}}^\cap$ and $\bar{\mathfrak{g}} = \langle \mathcal{D}^u, \mathcal{D}^t, \mathcal{D}(\varphi), \mathcal{G}(\psi), \mathcal{F}^2\rangle$ are an ideal and a subalgebra of $\mathfrak{g}^\sim$ but the sum is not direct even in the sense of vector spaces since $\hat{\mathfrak{g}}^\cap \cap \bar{\mathfrak{g}} = \langle \mathcal{G}(1)\rangle$. Unfortunately, the equivalence algebra $\mathfrak{g}^\sim$ does not possess the representation as a semi-direct sum of the ideal $\hat{\mathfrak{g}}^\cap$ associated with the kernel algebra and a certain subalgebra, which additionally complicates the group classification of the class (4.1.1).

This is why we should classify only subalgebras of $\mathfrak{g}^\sim$ which are contained in $\bar{\mathfrak{g}}$ and contain $\langle \mathcal{G}(1)\rangle$. The classification should be carried out up to $G_0^\sim$-equivalence, where $G_0^\sim$ is a subgroup of $G^\sim$ formed by the transformations (4.6.3) with $c_0 = c_3 = 0$. In fact, we will present the classification results in terms of extensions of $\hat{\mathfrak{g}}^\cap$ excluding $\mathcal{G}(1)$ from the corresponding bases.

The determining equations for Lie symmetries of equations from the class (4.1.1) impose more restrictions on appropriate subalgebras.

Lemma 4.32. $\mathfrak{s} \cap \langle \mathcal{D}^u, \mathcal{G}(\psi), \mathcal{F}^2\rangle = \mathfrak{s} \cap \langle \mathcal{D}^t, \mathcal{F}^2\rangle = \{0\}$ *for any appropriate subalgebra* $\mathfrak{s}$.

Proof. Suppose that an appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}^\sim$ contains an operator $Q = b\mathcal{D}^u + \mathcal{G}(\psi) + c\mathcal{F}^2$, where at least one of the constants b and c or the derivative ψ_x of the function $\psi = \psi(x)$ does not vanish. Then the operator $\mathrm{P}Q$ is a Lie symmetry operator for an equation from the class (4.1.1). Substituting the coefficients of operator Q into the determining equations (4.7.9) implies the following conditions on the arbitrary elements f and g:

$$(bu_x + \psi_x) f_{u_x} = 0, \quad (bu_x + \psi_x) g_{u_x} = 2c - \psi_{xx} f + bg.$$

For both the cases $b \neq 0$ and $\psi_x \neq 0$ it follows that $f_{u_x} = 0$ and $g_{u_x u_x} = 0$, which contradicts the definition of the class (4.1.1). The case $b = 0$, $\psi_x = 0$ and $c \neq 0$ leads to a contradiction. Therefore, any appropriate subalgebra does not contain an operator of the form considered.

Analogously, an operator $\mathcal{D}^t + c\mathcal{F}^2$, where c is an arbitrary constant, gives the condition $f = 0$, which is also inconsistent with the definition of the class (4.1.1). □

Lemma 4.33. $\dim\big(\mathfrak{s} \cap \langle \mathcal{D}(\varphi), \mathcal{G}(\psi), \mathcal{F}^2\rangle\big) \leqslant 2$ *for any appropriate subalgebra* $\mathfrak{s}$.

Proof. Suppose that $\mathfrak{s}$ is an appropriate subalgebra of $\mathfrak{g}^\sim$ and $\dim\big(\mathfrak{s} \cap \langle \mathcal{D}(\varphi), \mathcal{G}(\psi), \mathcal{F}^2\rangle\big) \geqslant 2$. This means that the subalgebra $\mathfrak{s}$ contains (at least) two operators $Q^i = \mathcal{D}(\varphi^i) + \mathcal{G}(\psi^i) + c_i\mathcal{F}^2$, where the functions φ^i, $i = 1, 2$, should be linearly independent in view of Lemma 4.32. In other words, the projections $\mathrm{P}Q^i$ of Q^i simultaneously are Lie symmetry operators of an equation from the class (4.1.1). By W we denote the Wronskian of the functions φ^1 and φ^2, $W = \varphi^1\varphi^2_x - \varphi^2\varphi^1_x$. $W \neq 0$ as the functions φ^1 and φ^2 are linearly independent.

Plugging the coefficients of $\mathrm{P}Q^i$ into the first classifying equation from the system (4.7.9) gives two equations with respect to f only,

$$(\varphi^i_x u_x - \psi^i_x) f_{u_x} - \varphi^i f_x + 2\varphi^i_x f = 0. \tag{4.9.1}$$

We multiply the equation corresponding to $i=1$ by φ^2 and subtract it from the equation for $i=2$ multiplied by φ^1. Dividing the resulting equation by W, we obtain the ordinary differential equation $(u_x+\beta)f_{u_x}+2f=0$, where $\beta=\beta(x):=(\varphi^1\psi^2_x-\varphi^2\psi^1_x)/W$ and the variable x plays the role of a parameter. It is possible to set $\beta=0$ by means of an equivalence transformation, $\mathcal{G}(-\beta)$. Indeed, this transformation preserves the form of the operators Q^i, only changing the values of the functional parameters ψ^i. In particular, it does not affect the linear independency of the functions φ^i. The integration of the above equation for $\beta=0$ yields that $f=\alpha u_x^{-2}$, where $\alpha=\alpha(x)$ is a nonvanishing function of x. In view of the derived form of f, splitting of equations (4.9.1) with respect to u_x leads to $\varphi^i\alpha_x=0$ and $\psi^i_x\alpha=0$, i.e. $\alpha_x=0$ and $\psi^i_x=0$. As $\mathcal{G}(1)\in\mathfrak{s}$, we can assume, up to linear combining of elements of $\mathfrak{s}$ that $\psi^i=0$. The constant α can be scaled to $\alpha=\pm1$ by an equivalence transformation.

In a similar manner, consider the last equation from system (4.7.9), taking into account the restrictions set on parameter-functions and the form of f. For each Q^i, this classifying equation gives an equation with respect to g,

$$\varphi^i_x u_x g_{u_x}-\varphi^i g_x=-\varphi^i_{xx}\alpha u_x^{-1}-2c_i. \tag{4.9.2}$$

Again, we multiply the equation corresponding to $i=1$ with φ^2 and subtract it from the equation for $i=2$ multiplied by φ^1, divide the resulting equation by W and thereby obtain that $g_{u_x}=-\mu^2u_x^{-2}+\mu^1u_x^{-1}$, where $\mu^2=\mu^2(x):=\alpha W_x/W$ and $\mu^1=\mu^1(x):=2(c_1\varphi^2-c_2\varphi^1)/W$. Integration with respect to u_x directly gives $g=\mu^2u_x^{-1}+\mu^1\ln|u_x|+\mu^0$, where $\mu^0=\mu^0(x)$ is a smooth function of x. The parameter-function μ^2 can be set to zero by the equivalence transformation $\mathcal{D}(\zeta)$, where the function $\zeta=\zeta(x)$ is a solution of the equation $\alpha\zeta_{xx}+\mu^2\zeta_x=0$. Substituting the derived form of g into equations (4.9.2) and splitting with respect to u_x, we find that $\mu^1_x=0$, $\varphi^i_{xx}=0$, $\varphi^i\mu^0_x=\varphi^i_x\mu^1+2c_i$. Therefore, μ^1 is a constant and the functions φ^1 and φ^2 can be set to 1 and x, respectively, upon linear combining of Q^i. Then, we have $\mu^0_x=2c_1$, $x\mu^0_x=2c_2+\mu^1$, i.e. $c_1=0$, $c_2=-\mu^1/2$ and μ^0 is a constant that can be set to zero by the equivalence transformation $\mathcal{F}(-\mu^0/2)$.

Summing up, we have proved that any equation from the class (4.1.1) admitting (at least) two linearly independent operators $\mathrm{P}Q^i$ is $G^\sim$-equivalent to an equation of the form

$$u_{tt}=\pm u_x^{-2}u_{xx}+\mu^1\ln|u_x|,$$

where $\mu^1=\mathrm{const}$. However, the determining equations (4.7.9) in this case yield $\eta_x=0$, $\eta_u=\tau_t$, $\xi_{xx}=0$, $\mu^1\eta_u=0$, $\eta_{tt}=\mu^1(\tau_t-\xi_x)$. This obviously implies that the number of such operators Q^i cannot exceed two. □

Corollary 4.34. *There are two $G^\sim$-inequivalent cases of Lie symmetry extensions in the class (4.1.1) involve two linearly independent operators of the form $\mathrm{P}Q^i$, where $Q^i=\mathcal{D}(\varphi^i)+\mathcal{G}(\psi^i)+c_i\mathcal{F}^2$,*

1. $u_{tt}=\pm u_x^{-2}u_{xx}+2\ln|u_x|$: $\quad\mathfrak{g}^{\max}=\mathfrak{g}^\cap+\langle\mathrm{P}\mathcal{D}(1),\mathrm{P}\mathcal{D}(x)-\mathrm{P}\mathcal{F}^2\rangle$,
2. $u_{tt}=\pm u_x^{-2}u_{xx}$: $\quad\mathfrak{g}^{\max}=\mathfrak{g}^\cap+\langle\mathrm{P}\mathcal{D}(1),\mathrm{P}\mathcal{D}(x),\mathrm{P}\mathcal{D}^t+\mathrm{P}\mathcal{D}^u\rangle$.

Proof. For $\mu^1\neq0$, we have that $\eta_u=\tau_t=0$, $\xi_{xx}=0$ and, after scaling of μ^1 to two by an equivalence transformation, $\eta_{tt}=-2\xi_x$. This directly gives the first case. If $\mu^1=0$, we obviously recover the second case. □

Now that we have computed the essential adjoint actions and classified all appropriate subalgebras in Corollary 4.34 for which $\dim\big(\mathfrak{s}\cap\langle\mathcal{D}(\varphi),\mathcal{G}(\psi),\mathcal{F}^2\rangle\big)=2$, we go on with the computation of inequivalent appropriate extensions of $\hat{\mathfrak{g}}^{\cap}$, which contain at most one linearly independent operator of the form $\mathcal{D}(\varphi)+\mathcal{G}(\psi)+c\mathcal{F}^2$, where $\varphi=\varphi(x)$ is a nonvanishing function. In view of Lemma 4.32 it is obvious that the dimension of such extensions cannot be greater than three. Here we select candidates for such extensions using only restrictions on appropriate subalgebras presented in Lemma 4.32. As there exist specific restrictions for two- and three-dimensional extensions, we will make an additional selection of appropriate extensions from the set of candidates directly in the course of the construction of invariant equations.

The result of the classification is formulated in the subsequent lemmas.

Lemma 4.35. *A complete list of $G^{\sim}$-inequivalent appropriate one-dimensional extensions of $\hat{\mathfrak{g}}^{\cap}$ in $\mathfrak{g}^{\sim}$ is given by*

$$\begin{gathered}\langle\mathcal{D}^u+\tfrac{1}{2}\mathcal{D}^t+\mathcal{D}(\varepsilon)+\mathcal{F}^2\rangle,\quad \langle\mathcal{D}^u-p\mathcal{D}^t+\mathcal{D}(\varepsilon)\rangle,\quad \langle\mathcal{D}^t-\mathcal{D}(1)\rangle,\\ \langle\mathcal{D}^t-\mathcal{G}(x)\rangle,\quad \langle\mathcal{D}(1)+\varepsilon\mathcal{F}^2\rangle,\end{gathered} \tag{4.9.3}$$

where $\varepsilon\in\{0,1\}$ and p is an arbitrary constant.

Proof. The classification of the appropriate one-dimensional extensions can be carried out effectively by simplifying a general element of the linear span $\langle\mathcal{D}^u,\mathcal{D}^t,\mathcal{D}(\varphi),\mathcal{G}(\psi),\mathcal{F}^2\rangle$,

$$Q=a_1\mathcal{D}^u+a_2\mathcal{D}^t+\mathcal{D}(\varphi)+\mathcal{G}(\psi)+a_4\mathcal{F}^2,$$

using pushforwards of transformations from $G^{\sim}$. For this aim, it is necessary to distinguish multiple cases, subject to which of the constants a_i or functions φ, ψ are nonzero. We note in the beginning that owing to the pushforward $\mathcal{D}_*(\theta)$ we can always set $\varphi=a_3=\mathrm{const}$.

For $a_1\neq 0$ we can scale the vector field Q to achieve $a_1=1$. Using the pushforward by a suitable transformation $\mathcal{G}(\chi)$, we can set $\psi=0$. The further possibilities for simplification depend crucially on the value of a_2. For $a_2=1/2$, the sum $\mathcal{D}^u+a_2\mathcal{D}^t$ is invariant under the pushforward $\mathcal{F}^2_*(c_4)$ and therefore it is not possible to set $a_4=0$. The actions of the pushforwards of the transformations $\mathcal{D}(x)$ and $\mathcal{D}^u$ allow scaling of a_3 and a_4 to $\{0,1\}$. If $a_4=1$, by denoting $a_3=\varepsilon$ we obtain the first case from the list (4.9.3).

For $a_2\neq 1/2$ we can use the pushforward $\mathcal{F}^2_*(c_4)$ to additionally set $a_4=0$, which gives, jointly with the case $a_2=1/2$ and $a_4=0$, the second extension listed, where a_2 is denoted by $-p$.

If $a_1=0$ and $a_2\neq 0$, we scale $a_2=1$ and can use the pushforward $\mathcal{F}^2_*(c_4)$ to set $a_4=0$. For $a_3\neq 0$, we can scale $a_3=-1$ by means of the action of $\mathcal{D}_*(x)$ and additionally put $\psi=0$ upon using the pushforward of the transformation $\mathcal{G}(\chi)$. If $a_3=0$, we have $\psi_x\neq 0$ in view of Lemma 4.32 and hence we can use the action of $\mathcal{D}_*(\theta)$ to set $\psi=-x$. This gives the third and the fourth case of the list (4.9.3), respectively.

In case of $a_1=a_2=0$ but $a_3\neq 0$, we can set $a_3=1$ and use the pushforward $\mathcal{G}_*(\psi)$ for a certain ψ to arrive at $\psi=0$. The action of $\mathcal{D}^u_*(c_2)$ on the resulting vector field allows us to scale the coefficient a_4 so that we have $a_4\in\{0,1\}$, which yields the fifth element of the above list of one-dimensional inequivalent extensions.

In view of Lemma 4.32, the case $a_1=a_2=a_3=0$ is not appropriate. □

Lemma 4.36. *Up to $G^{\sim}$-equivalence, any appropriate two-dimensional extension of $\hat{\mathfrak{g}}^{\cap}$ in $\mathfrak{g}^{\sim}$, which contains at most one linearly independent operator of the form $\mathcal{D}(\varphi)+\mathcal{G}(\psi)+c\mathcal{F}^2$, belongs to the following list:*

$$\begin{gathered}\langle\mathcal{D}^u+\mathcal{D}(1),\ \mathcal{D}^t+\mathcal{D}(b)\rangle,\quad \langle\mathcal{D}^u+\mathcal{D}(1),\ \mathcal{D}^t+\mathcal{G}(e^x)\rangle,\\ \langle a_1\mathcal{D}^u+a_2\mathcal{D}^t+a_3\mathcal{D}(x)+\varepsilon_0\mathcal{G}(x)+\varepsilon_1\mathcal{F}^2,\ \mathcal{D}(1)+\varepsilon_2\mathcal{F}^2\rangle,\end{gathered} \tag{4.9.4}$$

where b, a_1, a_2, a_3, ε_0, ε_1 and ε_2 are constants with $b\neq 0$, $(a_1,a_2)\neq(0,0)$, $(a_2,a_3)\neq(0,0)$, $(a_1,a_3,\varepsilon_0)\neq(0,0,0)$ and $(a_1-2a_2-a_3)\varepsilon_2=0$. Due to scaling of the first basis element and $G^\sim$-equivalence we can also assume that one of the a's equals 1, $(a_1-a_3)\varepsilon_0=0$, $(a_1-2a_2)\varepsilon_1=0$, $\varepsilon_0,\varepsilon_1\in\{0,1\}$ and $\varepsilon_2\in\{-1,0,1\}$ or, if $\varepsilon_0=0$, $\varepsilon_2\in\{0,1\}$.

Proof. The general strategy is to take two arbitrary linearly independent elements Q^1 and Q^2 from the linear span $\langle\mathcal{D}^u,\mathcal{D}^t,\mathcal{D}(\varphi),\mathcal{G}(\psi),\mathcal{F}^2\rangle$ such that $\mathfrak{s}=\hat{\mathfrak{g}}^\cap+\langle Q^1,Q^2\rangle$ is a five-dimensional subalgebra of $\mathfrak{g}^\sim$ satisfying the restriction on elements of the form $\mathcal{D}(\varphi)+\mathcal{G}(\psi)+c\mathcal{F}^2$ and Lemma 4.32 and simplify Q^1 and Q^2 as much as possible by linear combining of elements of $\mathfrak{s}$ and pushforwards with transformations from $G^\sim$. The proof is split into two parts.

First, we consider possible extensions not involving operators of the form $\mathcal{D}(\varphi)+\mathcal{G}(\psi)+c\mathcal{F}^2$. In view of this additional restriction and Lemma 4.32, up to linear combining, we can take the elements Q^1 and Q^2 in the initial form

$$Q^1=\mathcal{D}^u+\mathcal{D}(\varphi^1)+\mathcal{G}(\psi^1)+c_1\mathcal{F}^2,\quad Q^2=\mathcal{D}^t+\mathcal{D}(\varphi^2)+\mathcal{G}(\psi^2)+c_2\mathcal{F}^2,$$

where $\varphi^1\neq 0$. We set $\varphi^1=1$, $\psi^1=0$ and $c_1=0$ using $\mathcal{D}_*(\theta)$, $\mathcal{G}_*(\chi)$ with suitable functions θ and χ and $\mathcal{F}^2_*(c_1)$, respectively, i.e. $Q^1=\mathcal{D}^u+\mathcal{D}(1)$. As the subalgebra $\mathfrak{s}$ is closed with respect to the Lie bracket of vector fields, we have $[Q^1,Q^2]=\mathcal{D}(\varphi^2_x)+\mathcal{G}(\psi^2_x-\psi^2)-c_2\mathcal{F}^2\in\langle\mathcal{G}(1)\rangle$ and hence $\varphi^2_x=0$, $c_2=0$ and $\psi^2_x-\psi^2=\text{const}$. Integrating the equations for φ^2 and ψ^2 gives that $\varphi^2=b$ and $\psi^2=d_1e^x+d_0$ for some constants b, d_1 and d_0. The constant d_0 can be always set to zero by linear combining with the operator $\mathcal{G}(1)$ belonging to $\hat{\mathfrak{g}}^\cap$. The further simplification of Q^2 depends on the value of b. If $b\neq0$, the pushforward of $\mathcal{G}(-d_1b^{-1}e^x)$ does not change Q^1 and leads to $d_1=0$. If $b=0$, the parameter d_1 is nonzero in view of Lemma 4.32 and, therefore, can be scaled to 1 by $\mathcal{D}^u_*(d_1^{-1})$. As a result, we obtain the first two elements of the list (4.9.4).

Now we investigate the case $\dim\big(\mathfrak{s}\cap\langle\mathcal{D}(\varphi),\mathcal{G}(\psi),\mathcal{F}^2\rangle\big)=1$. Then basis vector fields of the extension of $\hat{\mathfrak{g}}^\cap$ can be chosen in the form

$$Q^1=a_1\mathcal{D}^u+a_2\mathcal{D}^t+\mathcal{D}(\varphi^1)+\mathcal{G}(\psi^1)+c_1\mathcal{F}^2,\quad Q^2=\mathcal{D}(\varphi^2)+\mathcal{G}(\psi^2)+c_2\mathcal{F}^2,$$

where $(a_1,a_2)\neq(0,0)$ and $\varphi^2\neq0$. We set $\varphi^2=1$ and $\psi^2=0$ using $\mathcal{D}_*(\theta)$ and $\mathcal{G}_*(\chi)$ with suitably chosen functions θ and χ, respectively. As $\mathfrak{s}$ is a Lie algebra, we have that $[Q^2,Q^1]=\mathcal{D}(\varphi^1_x)+\mathcal{G}(\psi^1_x)+(a_1-2a_2)c_2\mathcal{F}^2=a_3Q_2+\mathcal{G}(c_0)$ for some constants a_3 and c_0. Therefore, $\varphi^1_x=a_3$, $(a_1-2a_2-a_3)c_2=0$ and $\psi^1_x=c_0$. Up to combing Q^1 with Q^2 and $\mathcal{G}(1)$ we obtain that $\varphi^1=a_3x$ and $\psi^1=c_0x$. Up to $G^\sim$-equivalence we can assume that $(a_1-a_3)c_0=0$ and $(a_1-2a_2)c_1=0$. Indeed, if $a_1-2a_2\neq0$, we can set $c_1=0$ using $\mathcal{F}^2_*(\tilde c_1)$ with an appropriately chosen constant $\tilde c_1$. To set $c_0=0$ in the case $a_1-a_3\neq0$, we act on $\mathfrak{s}$ by $\mathcal{G}_*(\tilde c_0x)$ with an appropriately chosen constant $\tilde c_0$ and linearly combine the vector field Q^2 with $\mathcal{G}(1)$. Using pushforwards of scalings of variables and alternating their signs, we can independently scale the constant parameters c_0, c_1 and c_2 and change signs of c_1 and, simultaneously, c_0 and c_2. Additionally we can multiply the whole vector field Q^1 by a nonvanishing constant to scale one of the nonvanishing a's to one. The conditions $(a_2,a_3)\neq(0,0)$ and $(a_1,a_3,c_0)\neq(0,0,0)$ follow from Lemma 4.32. After denoting c's by ε's, this yields the third case of the list (4.9.4) and thereby completes the proof of the lemma. □

Lemma 4.37. *Up to $G^\sim$-equivalence, any appropriate three-dimensional extension of $\hat{\mathfrak{g}}^\cap$ in $\mathfrak{g}^\sim$, which contains at most one linearly independent operator of the form $\mathcal{D}(\varphi)+\mathcal{G}(\psi)+c\mathcal{F}^2$, has one of the following forms:*

$$\langle\mathcal{D}^u+p_1\mathcal{D}(x),\ \mathcal{D}^t+p_2\mathcal{D}(x),\ \mathcal{D}(1)+\varepsilon\mathcal{F}^2\rangle,\quad \langle\mathcal{D}^u+\mathcal{D}(x)+d\mathcal{G}(x),\ \mathcal{D}^t-\mathcal{G}(x),\ \mathcal{D}(1)\rangle,\qquad(4.9.5)$$

where p_1, p_2 and d are constants, $\varepsilon\in\{0,1\}$, $p_1p_2\neq0$ and $\varepsilon(p_1-1)=\varepsilon(p_2+2)=0$.

Proof. In view of Lemma 4.32, any three-dimensional appropriate extension of $\hat{\mathfrak g}^\cap$, which contains at most one linearly independent operator of the form $\mathcal D(\varphi)+\mathcal G(\psi)+c\mathcal F^2$, is spanned by the vector fields $Q^1=\mathcal D^u+\mathcal D(\varphi^1)+\mathcal G(\psi^1)+c_1\mathcal F^2$, $Q^2=\mathcal D^t+\mathcal D(\varphi^2)+\mathcal G(\psi^2)+c_2\mathcal F^2$ and $Q^3=\mathcal D(\varphi^3)+\mathcal G(\psi^3)+c_3\mathcal F^2$, where φ^i and ψ^i are smooth functions of x, c_i are constants, $\varphi^1\varphi^3\neq 0$ and $(\varphi^2,\psi^2)\neq(0,0)$.

Using $\mathcal F^2_*(c_1)$, $\mathcal D_*(\theta)$ and $\mathcal G_*(\chi)$ with suitably chosen functions θ and χ of x and, if $c_3\neq 0$, $\mathcal D^u_*(c_3^{-1})$, we set $c_1=0$, $\varphi^3=1$, $\psi^3=0$ and $c_3=\varepsilon\in\{0,1\}$. The commutation relations of Q^3 with Q^1 and Q^2 are

$$[Q^3,Q^1]=\mathcal D(\varphi^1_x)+\mathcal G(\psi^1_x)+c_3\mathcal F^2=p_1Q^3+d_1\mathcal G(1),$$
$$[Q^3,Q^2]=\mathcal D(\varphi^2_x)+\mathcal G(\psi^2_x)-2c_3\mathcal F^2=p_2Q^3+d_2\mathcal G(1)$$

for some constants p_i and d_i, $i=1,2$. These commutation relations imply the conditions $\varphi^i_x=p_i$, $\psi^i_x=d_i$ and $(p_1-1)\varepsilon=(p_2+2)\varepsilon=0$. Therefore, up to combining Q^i with Q^3 and $\mathcal G(1)$ we obtain $\varphi^i=p_ix$ and $\psi^i=d_ix$. Then the commutation relation

$$[Q^2,Q^1]=(d_2+p_2d_1-p_1d_2)\mathcal G(x)+c_2\mathcal F^2=0$$

yields $c_2=0$ and $p_2d_1=(p_1-1)d_2$. If $p_1\neq 1$, we can set $d_1=0$ using $\mathcal G_*(-(p_1-1)^{-1}d_1x)$ and then the equality $p_2d_1=(p_1-1)d_2$ is reduced to $d_2=0$. Analogously, in the case $p_2\neq 0$ we can set $d_2=0$ using $\mathcal G_*(-p_2^{-1}d_2x)$ and then the equality $p_2d_1=(p_1-1)d_2$ is equivalent to $d_1=0$. Summing up, we always have $d_1=d_2=0 \bmod G^\sim$ if $(p_1,p_2)\neq(1,0)$. This gives the first extension in (4.9.5). Otherwise, $p_1=1$, $p_2=0$ and hence $\varepsilon=0$ and $d_2\neq 0$. Setting $d_2=-1$ by $\mathcal D^u_*(-d_2^{-1})$, we obtain the second extension in (4.9.5). This completes the proof. □

4.10 Result of group classification

To describe equations from the class (4.1.1) whose Lie invariance algebras contain the projection $\mathrm P\mathfrak s$ of a certain appropriate subalgebra $\mathfrak s$ of $\mathfrak g^\sim$ to the variable space, we can use two equivalent ways, which lead to the same system of partial differential equations in the arbitrary elements f and g: For each basis element $\mathcal Q$ of $\mathfrak s$ we should either substitute the coefficients of $\mathrm P\mathcal Q$ into the last two equations of system (4.7.9) or write the condition of invariance of the functions f and g with respect to $\mathcal Q$. Then we should solve the joint system of the equations derived. Simultaneously we should check whether the projection $\mathrm P\mathfrak s$ is really the maximal Lie invariance algebra for obtained values of the arbitrary elements f and g.

All the candidates for one-dimensional appropriate extensions listed in Lemma 4.35 are really appropriate. For each representative of the list we have an uncoupled system of two equations in f and g, which is easily solved. As a result, we obtain the following list of equations from the class (4.1.1) that admit one-dimensional Lie symmetry extensions of $\mathfrak g^\cap$ related to $\mathfrak g^\sim$:

1.1. $\mathcal D^u+\frac12\mathcal D^t+\mathcal D(\varepsilon)+\mathcal F^2$: $\quad u_{tt}=\hat f(\omega)u_x^{-1}u_{xx}+\hat g(\omega)+2\ln|u_x|$,
1.2. $\mathcal D^u-p\mathcal D^t+\mathcal D(\varepsilon)$: $\quad u_{tt}=|u_x|^{2p}(\hat f(\omega)u_{xx}+\hat g(\omega)u_x)$,
1.3. $\mathcal D^t-\mathcal D(1)$: $\quad u_{tt}=e^{2x}(\hat f(u_x)u_{xx}+\hat g(u_x))$,
1.4. $\mathcal D^t-\mathcal G(x)$: $\quad u_{tt}=e^{2u_x}(\hat f(x)u_{xx}+\hat g(x))$,
1.5. $\mathcal D(1)+\varepsilon\mathcal F^2$: $\quad u_{tt}=\hat f(u_x)u_{xx}+\hat g(u_x)+2\varepsilon x$,

where $\omega=x-\varepsilon\ln|u_x|$, $\varepsilon\in\{0,1\}$ and p is an arbitrary constant. Here and in what follows, in each case we present only vector fields which extend the basis $\{\mathcal P^t,\mathcal G(1),\mathcal F^1\}$ of $\hat{\mathfrak g}^\cap$ into a basis of the corresponding subalgebra of $\mathfrak g^\sim$.

Calculations of the two-dimensional extensions are more complicated. We first present the result of the calculations and then give some explanations.

2.1. $\mathcal{D}^u-\mathcal{D}(p),\ \mathcal{D}^t-\mathcal{D}(1),\ p\neq 0,:\quad u_{tt}=\pm e^{2x}|u_x|^{2p}(u_{xx}+\nu u_x),$

2.2. $\mathcal{D}^u+\mathcal{D}(x),\ \mathcal{D}^t-\mathcal{G}(x):\quad u_{tt}=\pm e^{2u_x}(x^2u_{xx}+\nu x),$

2.3. $\mathcal{D}^u+\mathcal{D}^t+\mathcal{D}(x),\ \mathcal{D}(1):\quad u_{tt}=\hat f(u_x)u_{xx},$

2.4. $\mathcal{D}^u+\mathcal{D}^t+\mathcal{D}(x)-\mathcal{G}(x),\ \mathcal{D}(1):\quad u_{tt}=\pm u_{xx}+e^{u_x},$

2.5. $2\mathcal{D}^u+\mathcal{D}^t+2\mathcal{D}(x)+\mathcal{G}(x)+\mathcal{F}^2,\ \mathcal{D}(1):\quad u_{tt}=\pm e^{2u_x}u_{xx}+2u_x,$

2.6. $\mathcal{D}^u+\mathcal{D}(x)+\mathcal{G}(x),\ \mathcal{D}(1)+\varepsilon_2\mathcal{F}^2,\ \varepsilon_2\in\{-1,0,1\}:\quad u_{tt}=\pm e^{2u_x}u_{xx}+e^{u_x}+2\varepsilon_2 x,$

2.7. $(2-q)\mathcal{D}^u+(1-q)\mathcal{D}^t+(2-q)\mathcal{D}(x)+\mathcal{G}(x),\ \mathcal{D}(1),\ q\neq 0,1:$
$u_{tt}=\pm e^{2u_x}u_{xx}+e^{qu_x},$

2.8. $(2+2p-q)\mathcal{D}^u+(1+p-q)\mathcal{D}^t+(1+2p-q)\mathcal{D}(x),\ \mathcal{D}(1),\ q\neq 0:$
$u_{tt}=\pm|u_x|^{2p}u_{xx}+|u_x|^q,$

2.9. $(3+2p)\mathcal{D}^u+\mathcal{D}^t+(1+2p)\mathcal{D}(x),\ \mathcal{D}(1)+\mathcal{F}^2:$
$u_{tt}=\pm|u_x|^{2p}u_{xx}+\varepsilon_3|u_x|^{p+1/2}+2x,\ \varepsilon_3\in\{0,1\},$

2.10. $2(1+p)\mathcal{D}^u+(1+p)\mathcal{D}^t+(1+2p)\mathcal{D}(x)+\mathcal{F}^2,\ \mathcal{D}(1):\quad u_{tt}=\pm|u_x|^{2p}u_{xx}+2\ln|u_x|,$

2.11. $2\mathcal{D}^u+\mathcal{D}^t+2\mathcal{F}^2,\ \mathcal{D}(1)+\mathcal{F}^2:\quad u_{tt}=\pm u_x^{-1}u_{xx}+2\ln|u_x|+2x.$

Nontrivial constraints for constant parameters which are imposed by the maximality condition for the corresponding extensions are discussed in detail after Theorem 4.38.

In Cases 2.1 and 2.2, ν is an arbitrary constant. These cases correspond to the first and second spans from Lemma 4.36, respectively. For the associated invariant equations to have a simpler form, these span are replaced by the equivalent spans $\langle\mathcal{D}^u-\mathcal{D}(p),\,\mathcal{D}^t-\mathcal{D}(1)\rangle$, where $p=-b^{-1}$, and $\langle\mathcal{D}^u+\mathcal{D}(x),\,\mathcal{D}^t-\mathcal{G}(x)\rangle$, respectively. Note that we can always set a constant multiplier of the arbitrary element f to ±1, e.g., by scaling of t.

The third span from Lemma 4.36 in fact represents a multiparameter series of candidates for appropriate extensions, which is partitioned into the Cases 2.3–2.11 in the course of the construction of invariant equations. Not all values of the series parameters give appropriate extensions. Additional constraints for parameters follow from the consistency conditions of the associated system in the arbitrary elements,

$$\begin{aligned} &f_x=0, &&((a_1-a_3)u_x+\varepsilon_0)f_{u_x}=2(a_3-a_2)f,\\ &g_x=2\varepsilon_2, &&((a_1-a_3)u_x+\varepsilon_0)g_{u_x}=(a_1-2a_2)g-2\varepsilon_2a_3x+2\varepsilon_1,\end{aligned}$$

with the inequality $f\neq 0$ and the requirement that the dimension of extensions does not exceed two.

The above partitioning is carried out in the following way.

If $a_1=a_3=a_2$, the common value of the a's is nonzero and we can set it to 1 by scaling of the first basis elements of the span. We also have that $\varepsilon_1=0 \bmod G^{\sim}$ and $\varepsilon_2=0$. Depending on either $\varepsilon_0=0$ or $\varepsilon_0=1$ (which is replaced by the equivalent value $\varepsilon_0=-1$) we obtain the Cases 2.3 and 2.4, respectively.

If $a_1=a_3\neq a_2$, scaling the first basis elements of the span allows us to set $a_3-a_2=1$. The parameter ε_0 should be nonzero since otherwise $f=0$. Therefore, $\varepsilon_0=1 \bmod G^{\sim}$. The conditions $a_2=1$, $a_2=0$ and $a_2\neq 0,1$ lead to the Cases 2.5, 2.6 and 2.7, respectively. In the last case we denote $1-a_2$ by q and hence $q\neq 0,1$. In Case 2.5 the parameter ε_1 is nonzero since otherwise the dimension of the extension is greater than two.

Let $a_1 \neq a_3$. Then $\varepsilon_0 = 0 \bmod G^\sim$ and by scaling the first basis elements of the span we can also set $a_1 - a_3 = 1$. Introducing the notation $p = a_3 - a_2$ and $q = a_1 - 2a_2$, we obtain that $a_1 = 2+2p-q$, $a_2 = 1+p-q$ and $a_3 = 1+2p-q$. The further partition depends on values of ε_2, q and ε_1. For $\varepsilon_2 = 0$ the dimension of extension is not greater than two only if either $q \neq 0$ and then $\varepsilon_1 = 0 \bmod G^\sim$ (Case 2.8) or $q = 0$ and $\varepsilon_1 \neq 0$ and then $\varepsilon_1 = 1 \bmod G^\sim$ (Case 2.10). The condition $\varepsilon_2 = 1$ implies that $q = p + 1/2$. If additionally either $q = \varepsilon_1 = 0$ or $q \neq 0$ (and then $\varepsilon_1 = 0 \bmod G^\sim$), we have Case 2.9. Case 2.11 corresponds to the additional constraints $q = 0$ and $\varepsilon_1 \neq 0$ (i.e. $\varepsilon_1 = 1 \bmod G^\sim$).

Consider the candidates for three-dimensional appropriate extensions listed in Lemma 4.37. The compatibility of the associated systems in the arbitrary elements, supplemented with the inequality $f \neq 0$, implies $p_1+p_2 = 1$ and $d = 0$ for the first and the second span of Lemma 4.37, respectively. The general solutions of these systems up to $G^\sim$-equivalence are $(f, g) = (\pm|u_x|^{2p}, 0)$ and $(f, g) = (\pm e^{2u_x}, 0)$. This gives the following cases of Lie symmetry extensions:

3.1. $(1 + p)\mathcal{D}^u + p\mathcal{D}(x),\ (1 + p)\mathcal{D}^t + \mathcal{D}(x),\ \mathcal{D}(1),\ p \neq -2, -1, 0$: $\quad u_{tt} = \pm|u_x|^{2p}u_{xx}$,
3.2. $\mathcal{D}^u + \mathcal{D}(x) + \mathcal{G}(x),\ \mathcal{D}^t - \mathcal{G}(x),\ \mathcal{D}(1)$: $\quad u_{tt} = \pm e^{2u_x}u_{xx}$.

Special cases of Lie symmetry extensions in the class (4.1.1) are presented before this section. More precisely, all inequivalent equations whose maximal Lie invariance algebras are not contained in the projection of the equivalence algebra $\mathfrak{g}^\sim$ to the variable space are listed in Lemma 4.21. Equations from the class (4.1.1) which are invariant with respect to two linearly independent operators of the form $\mathrm{P}Q^i$, where $Q^i = \mathcal{D}(\varphi^i) + \mathcal{G}(\psi^i) + c_i\mathcal{F}^2$, are described in Corollary 4.34. For convenience, we collect the derived cases in a single table and formulate the final result of group classification for the class (4.1.1) as a theorem. Recall that $G^\sim$-equivalence coincides with the general point equivalence within the class (4.1.1), cf. Corollary 4.27.

Theorem 4.38. *All $G^\sim$-inequivalent (resp. point-inequivalent) cases of Lie symmetry extensions of the kernel algebra $\mathfrak{g}^\cap = \langle\partial_t, \partial_u, t\partial_u\rangle$ for the class* (4.1.1) *are exhausted by the cases presented in Table 4.1.*

In each case of Table 4.1 we present only vector fields which extend the basis $\{\partial_t, \partial_u, t\partial_u\}$ of $\mathfrak{g}^\cap$ into a basis of the corresponding Lie invariance algebra. The spans of $\mathfrak{g}^\cap$ and the vector fields given in Cases 1–6 and 9 of Table 4.1 are the maximal Lie invariance algebras of the corresponding equations for the general values of the associated parameter-functions $\hat f$ and $\hat g$, but for certain values of $\hat f$ and $\hat g$ additional extensions are possible, which are equivalent to other cases of Table 4.1.

In the course of collecting cases of Lie symmetry extensions in Table 4.1, they are arranged properly. In particular, Cases 1 and 2 of Corollary 4.34 are merged with Cases 2.10 and 3.1 into Cases 16 and 20 of Table 4.1, respectively. As the value $p = -1$ is singular for the basis of Case 2.10, the bases of Case 2.10 and Case 2 of Corollary 4.34 are changed in order to be agreed. Case 2.6 with $\varepsilon_2 = 0$ is not included in Case 12 of Table 4.1 since it is united with Case 2.7 in Case 13 of this table.

Within the algebraic approach used for the group classification of the class (4.1.1), the construction of Lie invariance algebras precedes the construction of associated invariant equations. This is why the simplification of the form of bases of Lie symmetry extensions, in a certain sense, dominates in Table 4.1. The form of invariant equations can be slightly simplified if simultaneous minor complication of bases of the corresponding Lie invariance algebras are permitted. In particular, multipliers equal to two can be removed from arbitrary elements by equivalence transformations or re-denoting the parameter p.

Table 4.1: Lie symmetry extensions of the kernel algebra $\mathfrak{g}^{\cap} = \langle \partial_t, \partial_u, t\partial_u \rangle$ for the class (4.1.1)

N	f	g	Basis of extension
		One-dimensional extensions	
1	$\hat f(x-\varepsilon\ln\|u_x\|)u_x^{-1}$	$\hat g(x-\varepsilon\ln\|u_x\|)+2\ln\|u_x\|$	$t\partial_t+2\varepsilon\partial_x+2(u+t^2)\partial_u$
2	$\hat f(x-\varepsilon\ln\|u_x\|)\|u_x\|^{2p}$	$\hat g(x-\varepsilon\ln\|u_x\|)\|u_x\|^{2p}u_x$	$-pt\partial_t+\varepsilon\partial_x+u\partial_u$
3	$\hat f(u_x)e^{2x}$	$\hat g(u_x)e^{2x}$	$t\partial_t-\partial_x$
4	$\hat f(x)e^{2u_x}$	$\hat g(x)e^{2u_x}$	$t\partial_t-x\partial_u$
5	$\hat f(u_x)$	$\hat g(u_x)+2\varepsilon x$	$\partial_x+\varepsilon t^2\partial_u$
		Two-dimensional extensions	
6	δu_x^{-4}	$\hat g(x)u_x^{-3}$	$t^2\partial_t+tu\partial_u,\ 2t\partial_t+u\partial_u$
7	$\delta e^{2x}\|u_x\|^{2p},\ p\neq 0,-2$	$\nu e^{2x}\|u_x\|^{2p}u_x,\ \nu(p+1)\neq\delta$	$p\partial_x-u\partial_u,\ t\partial_t-\partial_x$
8	$\delta x^2e^{2u_x}$	$\nu xe^{2u_x},\ \nu\neq\delta$	$x\partial_x+u\partial_u,\ t\partial_t-x\partial_u$
9	$\hat f(u_x)$	0	$\partial_x,\ t\partial_t+x\partial_x+u\partial_u$
10	δ	e^{u_x}	$\partial_x,\ t\partial_t+x\partial_x+(u-x)\partial_u$
11	δe^{2u_x}	$2u_x$	$\partial_x,\ t\partial_t+2x\partial_x+(2u+x+t^2)\partial_u$
12	δe^{2u_x}	$e^{u_x}+2\varepsilon_2x,\ \varepsilon_2\in\{-1,1\}$	$x\partial_x+(u+x)\partial_u,\ \partial_x+\varepsilon_2t^2\partial_u$
13	δe^{2u_x}	$e^{qu_x},\ q\neq 0$	$\partial_x,\ (1-q)t\partial_t+(2-q)x\partial_x+((2-q)u+x)\partial_u$
14	$\delta\|u_x\|^{2p}$	$\|u_x\|^q,\quad *)$	$\partial_x,\ (1+p-q)t\partial_t+(1+2p-q)x\partial_x+(2+2p-q)u\partial_u$
15	$\delta\|u_x\|^{2p}$	$\varepsilon\|u_x\|^{p+1/2}+2x$	$\partial_x+t^2\partial_u,\ t\partial_t+(1+2p)x\partial_x+(3+2p)u\partial_u$
16	$\delta\|u_x\|^{2p}$	$2\ln\|u_x\|$	$\partial_x,\ (1+p)t\partial_t+(1+2p)x\partial_x+(2(1+p)u+t^2)\partial_u$
17	δu_x^{-1}	$2\ln\|u_x\|+2x$	$\partial_x+t^2\partial_u,\ t\partial_t+2(u+t^2)\partial_u$
		Three-dimensional extensions	
18	δu_x^{-4}	u_x^{-3}	$t^2\partial_t+tu\partial_u,\ 2t\partial_t+u\partial_u,\ \partial_x$
19	δu_x^{-4}	$\nu x^{-1}u_x^{-3},\ \nu\neq 0$	$t^2\partial_t+tu\partial_u,\ 2t\partial_t+u\partial_u,\ 2x\partial_x+u\partial_u$
20	$\delta\|u_x\|^{2p},\ p\neq -2,0$	0	$\partial_x,\ t\partial_t+x\partial_x+u\partial_u,\ pt\partial_t-u\partial_u$
21	δe^{2u_x}	0	$\partial_x,\ t\partial_t+x\partial_x+u\partial_u,\ t\partial_t-x\partial_u$
		Four-dimensional extensions	
22	δu_x^{-4}	0	$t^2\partial_t+tu\partial_u,\ 2t\partial_t+u\partial_u,\ \partial_x,\ 2x\partial_x+u\partial_u$

Here $\delta=\pm1 \bmod G^{\sim}$ and $\varepsilon\in\{0,1\} \bmod G^{\sim}$. In Case 15 $\varepsilon=0 \bmod G^{\sim}$ if $p=-1/2$.
*) $q\neq 0$, $(p,q)\neq(-1,-1),(-2,-3)$ in Case 14.

Note that the unique inequivalent case of Lie symmetry extension for which the corresponding Lie invariance algebra is of maximal dimension possible for equations from the class (4.1.1) and equal to seven, Case 22, is not associated with a subalgebra of the equivalence algebra $\mathfrak{g}^{\sim}$.

Now we discuss nontrivial constraints for constant parameters which are imposed by the maximality condition for the corresponding extensions.

The equation $u_{tt}=e^{2x}|u_x|^{2p}(\delta u_{xx}+\nu u_x)$ corresponding to Case 7 for general values of parameters is linear if $p=0$. If $p\neq -1$, it is reduced by the transformation $\tilde t=|p+1|^{-p-1}t$, $\tilde x=e^{-x/(p+1)}$, $\tilde u=u$ to the equation $\tilde u_{\tilde t\tilde t}=|\tilde u_{\tilde x}|^{2p}(\delta\tilde u_{\tilde x\tilde x}+\tilde\nu\tilde x^{-1}\tilde u_{\tilde x})$ with $\tilde\nu=\delta-\nu(p+1)$, which coincides with the equation of Case 19 (resp. 20, resp. 22) if $p\neq -2$ and $\tilde\nu\neq 0$ (resp. $p=-2$ and $\tilde\nu\neq 0$, resp. $p=-2$ and $\tilde\nu=0$).

The equation $u_{tt}=e^{2u_x}(\delta x^2u_{xx}+\nu xu_x)$ corresponding to Case 8 is similar with respect to the transformation $\tilde t=t$, $\tilde x=x$, $\tilde u=u+x\ln|x|-x$ to the equation $\tilde u_{\tilde t\tilde t}=e^{2\tilde u_{\tilde x}}(\delta\tilde u_{\tilde x\tilde x}+(\nu-\delta)\tilde x^{-1})$ which coincides with the equation of Case 21 if $\nu=\delta$.

Any equation from the class (4.1.1) is a potential equation for the equation of the form

$$v_{tt} = (f(x,v)v_x + g(x,v))_x \tag{4.10.1}$$

with the same value of the arbitrary elements f and g, where the argument u_x is replaced by v. Indeed, Eq. (4.10.1) possesses two inequivalent characteristics of conservation laws, $\lambda^1 = 1$ and $\lambda^2 = t$. The potential systems constructed with the simplest conserved vectors associated with these characteristics is

$$w^1_x = v_t, \quad w^1_t = f(x,v)v_x + g(x,v), \tag{4.10.2}$$

$$w^2_x = tv_t - v, \quad w^2_t = tf(x,v)v_x + tg(x,v). \tag{4.10.3}$$

We denote $tw^1 - w^2$ by u. In terms of the dependent variables v, w^1 and u, the joint potential system (4.10.2), (4.10.3) takes the form $u_x = v$, $u_t = w^1$, $w^1_t = f(x,v)v_x + g(x,v)$ which is a potential system for system (4.10.2), i.e., it is formally a second-level potential system of Eq. (4.10.1). Hence u is a second-level potential for this equation. Excluding v and w^1 from the last system, we obtain Eq. (4.1.1). In order to derive Eq. (4.10.1) from Eq. (4.1.1), we should take the total derivative of Eq. (4.1.1) with respect to x and replace u_x by v. As the coefficients of any Lie symmetry operator $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u$ of Eq. (4.1.1) satisfy the determining equations $\tau_u = \xi_u = \eta_{uu} = \eta_{xu} = 0$, the coefficient of ∂_v in the prolongation of this operator to v according to the equality $v = u_x$ is equal to $\eta_x + (\eta_u - \xi_x)u_x$ and hence does not depend on u. Therefore, Lie symmetries of Eq. (4.1.1) induce no purely potential symmetries of Eq. (4.10.1).

We checked cases from Table 4.1 using the package `DESOLV` [32, 152] for symbolic calculations of Lie symmetries, whenever it was possible.

4.11 Conclusion

The results of this paper and those existing in the literature on the symmetry analysis of differential equations allow us to compare different approaches to the group classification of differential equations (partial preliminary group classification, complete preliminary group classification and complete group classification) within the framework of the algebraic method. Given a class $\mathcal{L}|_\mathcal{S}$ of (systems of) differential equations with the equivalence group $G^\sim$ and the equivalence algebra $\mathfrak{g}^\sim$, the application of each of the above approaches involves, in some way, the classification of certain subalgebras of $\mathfrak{g}^\sim$. The essential point is *which* subalgebras of $\mathfrak{g}^\sim$ should be classified and *what* equivalence relation should be used in the course of the classification.

Within the approach of *partial preliminary group classification*, a proper subalgebra $\mathfrak{s}$ of $\mathfrak{g}^\sim$ is fixed and then solely subalgebras of $\mathfrak{s}$ are classified. This approach may be useful only if the subalgebra $\mathfrak{s}$ is relevant from the physical or another point of view. Hence the choice of such a subalgebra $\mathfrak{s}$ should be strongly justified which, unfortunately, is often ignored in the existing literature on that subject. The differences arising from the consideration of the subalgebra $\mathfrak{s}$ instead of the whole algebra $\mathfrak{g}^\sim$ are especially significant in the case when $\mathfrak{g}^\sim$ is an infinite-dimensional algebra whereas $\mathfrak{s}$ is a finite-dimensional subalgebra. An seeming advantage of replacing $\mathfrak{g}^\sim$ by $\mathfrak{s}$ is that in general finite-dimensional algebras are much simpler objects than infinite-dimensional ones. At the same time, partial preliminary group classification has a few essential weaknesses most of which are related to the following fact: As the fixed subalgebra $\mathfrak{s}$ of $\mathfrak{g}^\sim$ is usually not invariant under the adjoint action of the equivalence group $G^\sim$, this group does not generate a well-defined equivalence relation on subalgebras of $\mathfrak{s}$. This is a reason why subalgebras of $\mathfrak{s}$ are classified up to the weaker internal equivalence on $\mathfrak{s}$, which is induced by the adjoint action of the continuous transformation group associated with $\mathfrak{s}$, instead of $G^{\sim}$-equivalence.

The exhaustive classification of subalgebras up to internal equivalence is a cumbersome algebraic problem, possessing no algorithmic solution even for finite-dimensional algebras. In order to simplify it, only one-dimensional subalgebras are usually classified which crucially increases the incompleteness of results obtained in the framework of partial preliminary group classification. Although the number of classification cases remains quite large, many of them can be neglected up to $G^{\sim}$-equivalence, not to mention general point equivalence. The presence of equivalent cases unnecessarily complicates both the solution of the group classification problem and further applications of classification results, e.g., the construction of exact solutions of systems from the class $\mathcal{L}|_{\mathcal{S}}$.

Complete preliminary group classification of the class $\mathcal{L}|_{\mathcal{S}}$ is based on the classification of subalgebras of the entire equivalence algebra $\mathfrak{g}^{\sim}$ up to $G^{\sim}$-equivalence. As both the objects, $\mathfrak{g}^{\sim}$ and $G^{\sim}$, are directly related to the class $\mathcal{L}|_{\mathcal{S}}$ and well consistent with each other, this approach appears to be quite natural. For weakly normalized classes of differential equations, it gives an exhaustive classification list. Moreover, complete preliminary group classification is always a necessary step in the complete group classification within the framework of the algebraic method. It is obvious that complete preliminary group classification gives a list which is closer to exhausting all possible Lie symmetry extensions than any list obtained by a partial preliminary group classification. At the same time, due to the usage of $G^{\sim}$-equivalence, which is stronger than the internal equivalence on a subalgebra of $\mathfrak{g}^{\sim}$, the former list can contain even fewer cases than the latter one. For example, 33 cases of one-dimensional extensions of the kernel algebra were constructed for the class (4.1.1) in [63] in the course of the partial preliminary group classification involving a ten-dimensional subalgebra of the equivalence algebra of this class. All these cases are $G^{\sim}$-equivalent to particular subcases of Cases 1–5 from Table 4.1 of the present paper.

The approach of complete preliminary group classification can be optimized by the selection of appropriate subalgebras of $\mathfrak{g}^{\sim}$. The projection of each appropriate subalgebra to the space of system variables is maximal among Lie invariance algebras of a system from the class $\mathcal{L}|_{\mathcal{S}}$, which are induced by subalgebras of $\mathfrak{g}^{\sim}$. The simplest common property of appropriate subalgebras is that they contain the kernel algebra. Other criteria for the selection of appropriate subalgebras, including bounds for dimensions of extensions or additional extensions, are derived by examination the determining equations for Lie symmetries of systems from the class $\mathcal{L}|_{\mathcal{S}}$. In a certain sense, this approach combines the algebraic method of group classification with the direct method based on the study of compatibility and the integration of the determining equations up to $G^{\sim}$-equivalence. The use of the optimized technique often allows one to reduce the classification problem to the classification of certain low-dimensional subalgebras of the equivalence algebra, even if the equivalence algebra is infinite-dimensional and there exist infinite-dimensional extensions of the kernel. The calculations required are not too cumbersome. Thus, the minimal computations which are necessary for the complete preliminary group classification of the class (4.1.1) are those given in the first parts of Sections 4.4 and 4.7 and the entire Sections 4.5, 4.6, 4.9 and 4.10. These computations yield the majority of inequivalent cases of Lie symmetry extensions for the class (4.1.1), which are presented in Table 4.1 (the exceptions are only Cases 6, 18, 19 and 22).

There exist two ways to apply the algebraic method to *complete group classification*. The first way is to reduce the complete group classification to the preliminary group classification. The reduction can be realized, e.g., by proving that the class $\mathcal{L}|_{\mathcal{S}}$ is weakly normalized or by partitioning this class into weakly normalized subclasses and other subclasses which can be easily classified using the direct method. Although the partition into subclasses usually involves cumbersome and sophisticated computations, it is an effective tool of group analysis since it accurately adapts the classification procedure to the structure of the class $\mathcal{L}|_{\mathcal{S}}$. This is the way used in the present paper. The class (4.1.1) is partitioned into two subclasses possessing the same equivalence group as the whole class (4.1.1). One of the subclasses is normalized, the other is semi-normalized and mapped by equivalence transformations onto its subclass (4.7.5) of structure suitable for application of the direct method. The group classification of the subclass (4.7.5) has been obtained in the course of the partitioning which results in only four special cases of Lie symmetry extension (Cases 6, 18, 19 and 22 from Table 4.1) that are not related to subalgebras of $\mathfrak{g}^{\sim}$. The second way is to directly classify $G^{\sim}$-inequivalent appropriate algebras contained in the span $\mathfrak{g}^{\langle\rangle} = \langle \mathfrak{g}_\theta | \theta \in \mathcal{S} \rangle$ of maximal Lie invariance algebras, $\mathfrak{g}_\theta$, of all systems from the class $\mathcal{L}|_{\mathcal{S}}$. This way works properly only if the class $\mathcal{L}|_{\mathcal{S}}$ possesses certain properties, e.g., if the maximal Lie invariance algebra $\mathfrak{g}_\theta$ is of low dimension for any $\theta \in \mathcal{S}$ [69] or if the class $\mathcal{L}|_{\mathcal{S}}$ is at least weakly normalized or partitioned into weakly normalized subclasses, although this property is usually not explicitly checked [9, 79, 80, 159]. An explanation for the above observation is that $G^{\sim}$-equivalence is not appropriate in the course of classification of subalgebras contained in $\mathfrak{g}^{\langle\rangle}$ if $\mathfrak{g}^{\langle\rangle}$ is strongly inconsistent with the equivalence algebra $\mathfrak{g}^{\sim}$ (e.g., much wider than the projection $\mathrm{P}\mathfrak{g}^{\sim}$ of $\mathfrak{g}^{\sim}$ to the space of system variables).

Due to the above partition of the class (4.1.1) we have obtained essentially stronger results than the solution of the usual group classification problem by Lie–Ovsiannikov for this class. The partition exhaustively describes the equivalence groupoid of the class (4.1.1). Moreover, the fact that the whole class (4.1.1) is semi-normalized guarantees that there are no additional point equivalence transformations between cases of Lie symmetry extensions presented in Table 4.1, i.e., the same table gives the complete group classification of the class (4.1.1) with respect to general point equivalence.

The extension and clarification of the group classification toolbox is by no means a purely mathematical problem. For example, methods from group classification have the potential to provide a unifying framework to construct invariant local closure or parameterization schemes for averaged nonlinear differential equations [16, 98, 123, 132]. As finding appropriate closure ansatzes for averaged differential equations is at the basis of any numerical model of (geophysical) fluid dynamical systems, it is immediately clear that group classification can play a crucial role in the construction of different computational codes for such systems. The classes of differential equations arising in the course of the parameterization problem are usually much wider and have more complicated structure than the classes studied in conventional group classification. It generally cannot be expected to completely solve the group classification problems for such classes using existing methods. Hence, the development of new tools for the group classification of differential equations together with the improvement of well-known approaches remains an attractive and challenging research problem. Especially for complex classification problems, the whole framework of the algebraic method as described and extended in the present paper seems to be most appealing.

Chapter 5

Complete point symmetry group of the vorticity equation on a rotating sphere

5.1 Introduction

Lie symmetries of the inviscid barotropic vorticity equation on the rotating sphere (sBVE) were computed in [18] and used in [18, 23] in order to derive a point transformation mapping the equation in a rotating reference frame to the equation in the rest frame and to construct exact solutions. Therein it was also indicated that besides an infinite-dimensional maximal Lie invariance group the barotropic vorticity equation on the sphere admits two independent (up to composition with each other and with continuous transformations) discrete symmetries, which merely alternate signs of two pairs of variables, (the time, the longitude) and (the latitude, the stream function), respectively. However, no systematic derivation of discrete symmetries or, more generally, the complete point symmetry group of the sBVE was given in the literature up to now.

With the present paper, we aim to complete the description of point symmetries admitted by the sBVE by computing the complete point symmetry group G_Ω of this equation. We simplify the computation within the framework of the direct method via combining it with an advanced version of the algebraic approach originally proposed in [58, 59], essentially modified in [21] and then developed in [15, 20]. As a result, we prove that in fact the group G_Ω is generated by Lie symmetry transformations of the sBVE and the above two discrete transformations.

The sBVE is an appropriate equation to demonstrate advantages of the enhanced version of the algebraic approach. In principle, the group G_Ω might be computed using merely the direct method based on the definition of the set of transformations to be found and the prolongation of finite transformations to derivatives by the chain rule. This approach is widely applied in the literature for finding complete point symmetry groups of single systems of differential equations or equivalence groups and equivalence groupoids (i.e., sets of all admissible point transformations) of classes of such systems, see, e.g., [72, 73, 74, 123, 127, 149, 150] and references therein. At the same time, the sBVE is a third-order nonlinear partial differential equation for a single scalar function of three independent variables, and one of the independent variables explicitly appears in the equation. Hence the mere application of the direct method for the computation of the group G_Ω is too cumbersome. Moreover, the sBVE admits an infinite-dimensional maximal Lie invariance algebra and therefore the version of the algebraic method proposed in [58, 59] is not applicable here, because it relies on the computation of automorphism matrices, which properly works only in the finite-dimensional case. From the physical point of view, the sBVE is of superior importance as it is capable of describing qualitatively the large-scale behavior of the flow in the middle of the troposphere. Due to its relevance for the larger atmospheric scales, it is especially convenient to consider the vorticity equation in spherical coordinates.

Our paper is organized in the following way: In Section 5.2 we recall the known point symmetries of the sBVE. In Section 5.3 we describe a method that can be applied to determine the complete point symmetry group of a system of differential equations possessing a nontrivial Lie invariance algebra. This method involves the notion of megaideals of Lie algebras and properly works even for systems whose maximal Lie invariance algebras are infinite dimensional. Section 5.4 is central. After determining a convenient set of megaideals for the maximal Lie invariance algebra of the vorticity equation on the sphere, we derive a maximal system of constraints for elements of the group G_Ω, which are related to properties of the adjoint action of G_Ω on its Lie algebra, and then complete the computation of G_Ω by the direct method using the constraints derived within the framework of the algebraic approach. We briefly sum up our results in the conclusion.

5.2 The model

Introducing the stream function in the system of nonlinear incompressible Euler equations in a single thin atmospheric layer on the sphere leads to a single third-order nonlinear partial differential equation for the stream function, which is referred to as the barotropic vorticity equation on the sphere. It reads

$$\zeta_t + \psi_\lambda\zeta_\mu - \psi_\mu\zeta_\lambda + 2\Omega\psi_\lambda = 0, \qquad \zeta := \frac{1}{1-\mu^2}\,\psi_{\lambda\lambda} + ((1-\mu^2)\psi_\mu)_\mu, \tag{5.2.1}$$

where λ and φ are the longitude and latitude, respectively, and $\mu = \sin\varphi$, ψ is the stream function generating an incompressible two-dimensional flow on the sphere, which is related to the vorticity ζ by means of the Laplacian on the sphere and Ω is the constant angular velocity of the rotating sphere. The derived latitudinal variable μ runs from -1 (South Pole) to 1 (North Pole). For convenience, we assume the mean radius of the Earth to be scaled to one.

It was shown in [18, 23] that Eq. (5.2.1) admits the infinite-dimensional maximal Lie invariance algebra, which is denoted as $\mathcal{S}^\infty_\Omega$. This algebra is generated by the vector fields

$$\begin{aligned}
&\mathcal{D} = t\partial_t - (\psi - \Omega\mu)\partial_\psi - \Omega t\partial_\lambda, \qquad \mathcal{P} = \partial_t, \qquad \mathcal{Z}(g) = g(t)\partial_\psi, \qquad \mathcal{J}_1 = \partial_\lambda,\\
&\mathcal{J}_2 = \mu\frac{\sin(\lambda+\Omega t)}{\sqrt{1-\mu^2}}\partial_\lambda + \frac{\cos(\lambda+\Omega t)}{\sqrt{1-\mu^2}}\left((1-\mu^2)\partial_\mu + \Omega\partial_\psi\right),\\
&\mathcal{J}_3 = \mu\frac{\cos(\lambda+\Omega t)}{\sqrt{1-\mu^2}}\partial_\lambda - \frac{\sin(\lambda+\Omega t)}{\sqrt{1-\mu^2}}\left((1-\mu^2)\partial_\mu + \Omega\partial_\psi\right),
\end{aligned}$$

where the parameter-function g traverses the set of smooth functions of t. The structure of the algebra $\mathcal{S}^\infty_\Omega$ is $\mathfrak{so}(3) \oplus (\mathfrak{g}_2 \Subset \langle\mathcal{Z}(g)\rangle)$, where the three-dimensional orthogonal algebra $\mathfrak{so}(3)$ is realized by the vector fields $\mathcal{J}_i$, $i = 1, 2, 3$, $\mathfrak{g}_2 = \langle\mathcal{D}, \mathcal{P}\rangle$ is a realization of the two-dimensional non-Abelian algebra and $\langle\mathcal{Z}(g)\rangle$ is an infinite-dimensional Abelian ideal in $\mathcal{S}^\infty_\Omega$.

An important property of the above family of Lie invariance algebras parameterized by the angular velocity Ω is that it is not singular with respect to the parameter Ω at $\Omega = 0$, i.e. it includes the case of the rest reference frame, and it is natural to denote the maximal Lie invariance algebra of Eq. (5.2.1) with $\Omega = 0$ by $\mathcal{S}_0^\infty$. It was shown in [18, 23] that Ω can be set to zero in the algebra $\mathcal{S}_\Omega^\infty$ by means of the transformation

$$\tilde{t} = t, \quad \tilde{\mu} = \mu, \quad \tilde{\lambda} = \lambda + \Omega t, \quad \tilde{\psi} = \psi - \Omega\mu. \tag{5.2.2}$$

The same transformation also allows one to set Ω to zero in the vorticity equation (5.2.1). Note that the transformation (5.2.2) was originally derived in [116], where it was used to transform the vorticity equation into a reference frame with vanishing angular momentum. The transformation (5.2.2) can also be found by noting that the algebras $\mathcal{S}_\Omega^\infty$ and $\mathcal{S}_0^\infty$ are isomorphic and by constructing the mapping relating these two algebras. For our purpose this transformation is especially convenient as it leads to a simplified form of both the vorticity equation (5.2.1) and the maximal Lie invariance algebra $\mathcal{S}_\Omega^\infty$ (i.e. we can work with $\mathcal{S}_0^\infty$). Moreover, setting Ω to zero makes clear the physical meaning of basis elements of the algebra $\mathcal{S}_\Omega^\infty$. If $\Omega = 0$, the vector field $\mathcal{D}$ generates simultaneous scalings in t and ψ and the vector field $\mathcal{P}$ is associated with time translations. The elements of the form $\mathcal{Z}(g)$ are the infinitesimal counterparts of gauging of the stream function up to a summand being a smooth function of t. The vector fields $\mathcal{J}_i$, $i = 1, 2, 3$, generate rotations represented in angular coordinates.

It was claimed in [18, 23] that in addition to continuous symmetries generated by elements from the maximal Lie invariance algebra $\mathcal{S}_\Omega^\infty$ (or, equivalently, $\mathcal{S}_0^\infty$), there are two discrete symmetries admitted by Eq. (5.2.1) and these are given by the changes of the signs, $(t, \lambda, \mu, \psi) \mapsto (-t, -\lambda, \mu, \psi)$ and $(t, \lambda, \mu, \psi) \mapsto (t, \lambda, -\mu, -\psi)$, respectively. While it is straightforward to check by direct substitution that these transformations are indeed symmetries of (5.2.1), it is more elaborate to derive them directly from the invariance criterion. This was not done in [18, 23]. It is even harder to prove that there are no other independent (up to composition with each other and with continuous symmetry transformations) discrete symmetries than these two mirror symmetries. It is the purpose of this paper to show by determining the complete point symmetry group of the sBVE that there are indeed only these two discrete symmetries.

5.3 How to find the complete point symmetry group via the algebraic method

It is considerably more difficult to find discrete point symmetries of a system of differential equations than its Lie symmetries. The reason for this is that the powerful infinitesimal symmetry criterion is only applicable for transformations depending on continuous group parameters [25, 59, 101]. This is also the reason why to date no existing computer algebra package, such as [55, 135, 153] can be used for this purpose, because all such packages rely on the integration of the infinitesimal determining equations, which by definition only exist for Lie symmetries and are linear. Finding discrete point symmetries or the complete point symmetry group of a system of differential equations therefore has in fact to be done by hand, using computer programs only for related routine calculations in interactive mode.

Before we start the computation of the complete point symmetry group G_0 for Eq. (5.2.1) with $\Omega = 0$, let us recall about the general method proposed in [21], which in some sense can be seen as a refinement of the technique suggested in [58, 59] by involving the notion of megaideals [124].

Namely, we use the following property: *Given a system of differential equations $\mathcal{L}$, for any transformation $\mathcal{T}$ from the maximal point symmetry (pseudo)group G of the system $\mathcal{L}$ the linear mapping $\mathcal{T}_*\colon \mathfrak{g} \to \mathfrak{g}$ generated by $\mathcal{T}$ on the maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{L}$*

via push-forwarding of vector fields in the space of system variables is an automorphism of $\mathfrak{g}$*,* $\mathcal{T}_* \in \mathrm{Aut}(\mathfrak{g})$*, and hence it preserves all megaideals of* $\mathfrak{g}$*.*

The correspondence $\mathcal{T} \to \mathcal{T}_*$ defines a representation of G on $\mathfrak{g}$, which is often unfaithful. In particular, this is the case if the group G (resp. the algebra $\mathfrak{g}$) has a nontrivial center. The representation image G_* is a subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{g})$, which may be smaller than the entire group $\mathrm{Aut}(\mathfrak{g})$. The continuous point symmetries of the system $\mathcal{L}$ generate, via finite compositions, a connected normal subgroup U of G and induce mappings on $\mathfrak{g}$ which can be considered as internal automorphisms of $\mathfrak{g}$ and which generate a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$. We denote this subgroup by $\mathrm{Int}(\mathfrak{g})$. Elements of the factor group G/U (more precisely, their representatives in G) are interpreted as "discrete symmetries" of the system $\mathcal{L}$ which are independent up to composing with continuous symmetries of $\mathcal{L}$. As the representations of G and U on $\mathfrak{g}$ via push-forwarding of vector fields are not necessarily faithful, there is no assurance on the existence of a bijection between the factor groups G/U and $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$. The rigorous consideration gives rise to a number of difficult problems which concern the relation of algebras of vector fields and (pseudo)groups of transformations in the infinite-dimensional case and which are out of the subject of this paper. At the same time, these problems can be easily solved in particular cases, e.g., related to models of fluid dynamics and meteorology.

If the algebra $\mathfrak{g}$ is not low dimensional, the computation of $\mathrm{Aut}(\mathfrak{g})$ itself may be a complicated problem. Moreover, the group $\mathrm{Aut}(\mathfrak{g})$ may be much wider than G_*, especially if the algebra $\mathfrak{g}$ is infinite dimensional. If this is the case, in the course of the construction of $\mathrm{Aut}(\mathfrak{g})$ we will spend efforts for finding elements from $\mathrm{Aut}(\mathfrak{g}) \setminus G_*$, which are in fact needless for determining G_*. To avoid such needless computations, instead of the condition $G_* \subseteq \mathrm{Aut}(\mathfrak{g})$ we can use the weaker condition that $G_*\mathfrak{i} \subseteq \mathfrak{i}$ if $\mathfrak{i}$ is a megaideal of $\mathfrak{g}$.

In general, a *megaideal* $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ that is invariant under any mapping from the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ [21, 124], i.e., $\mathfrak{T}z = z$ for any $z \in \mathfrak{i}$ and any $\mathfrak{T} \in \mathrm{Aut}(\mathfrak{g})$. Every megaideal of $\mathfrak{g}$ is an ideal and a characteristic ideal of $\mathfrak{g}$. A set of megaideals of $\mathfrak{g}$ can be computed without knowing of $\mathrm{Aut}(\mathfrak{g})$. Both the improper subalgebras of $\mathfrak{g}$ (the zero subspace and $\mathfrak{g}$ itself) are (improper) megaideals of $\mathfrak{g}$. If $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are megaideals of $\mathfrak{g}$ then so are $\mathfrak{i}_1 + \mathfrak{i}_2$, $\mathfrak{i}_1 \cap \mathfrak{i}_2$ and $[\mathfrak{i}_1, \mathfrak{i}_2]$, i.e., sums, intersections and Lie products of megaideals are again megaideals. If $\mathfrak{i}_2$ is a megaideal of $\mathfrak{i}_1$ and $\mathfrak{i}_1$ is a megaideal of $\mathfrak{g}$ then $\mathfrak{i}_2$ is a megaideal of $\mathfrak{g}$, i.e., megaideals of megaideals are also megaideals. All elements of the derived, upper and lower central series of $\mathfrak{g}$, including the center and the derivative of $\mathfrak{g}$, as well as the radical and nil-radical of $\mathfrak{g}$ are its megaideals. In order to have a sufficient store of megaideals, we need one more way for finding new megaideals from known ones.

Proposition 5.1. *If* $\mathfrak{i}_0$*,* $\mathfrak{i}_1$ *and* $\mathfrak{i}_2$ *are megaideals of* $\mathfrak{g}$ *then the set* $\mathfrak{s}$ *of elements from* $\mathfrak{i}_0$ *whose commutators with arbitrary elements from* $\mathfrak{i}_1$ *belong to* $\mathfrak{i}_2$ *is also a megaideal of* $\mathfrak{g}$*.*

Proof. It is obvious that $\mathfrak{s}$ is a linear subspace of $\mathfrak{g}$. Consider an element $z_0 \in \mathfrak{i}_0$ such that $[z_0, z_1] \in \mathfrak{i}_2$ for arbitrary $z_1 \in \mathfrak{i}_1$. Then for arbitrary $\mathfrak{T} \in \mathrm{Aut}(\mathfrak{g})$ and arbitrary $z_1 \in \mathfrak{i}_1$ we have $[\mathfrak{T}z_0, z_1] = [\mathfrak{T}z_0, \mathfrak{T}\mathfrak{T}^{-1}z_1] = \mathfrak{T}[z_0, \mathfrak{T}^{-1}z_1] \in \mathfrak{i}_2$ as $\mathfrak{T}^{-1}z_1 \in \mathfrak{i}_1$, and hence $[z_0, \mathfrak{T}^{-1}z_1] \in \mathfrak{i}_2$. This means that $\mathfrak{T}z_0 \in \mathfrak{s}$, i.e., $\mathfrak{s}$ is a megaideal of $\mathfrak{g}$. □

As the megaideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are necessarily usual ideals and hence $[\mathfrak{i}_0, \mathfrak{i}_1] \subseteq \mathfrak{i}_0 \cap \mathfrak{i}_1$, it in fact suffices to consider the case when $\mathfrak{i}_2$ is contained in $\mathfrak{i}_0 \cap \mathfrak{i}_1$. If $\mathfrak{i}_0 \cap \mathfrak{i}_1 = \{0\}$, the megaideal $\mathfrak{s}$ coincides with $\mathfrak{i}_0$. A particular case of Proposition 5.1 with $\mathfrak{i}_0 = \mathfrak{i}_1 = \mathfrak{g}$ and $\mathfrak{i}_2 = \{0\}$ implies that the centralizer of every megaideal is a megaideal.

Once a convenient set of megaideals is found, one should try to obtain maximal restrictions on the form of a point symmetry transformation $\mathcal{T}$, which are feasible using the algebraic approach. If all derived restrictions on $\mathcal{T}$ are taken into account, one has to substitute the restricted form of a general point transformation into the initial system of differential equations and proceeds the computation of the complete point symmetry group within the framework of the direct method.

5.4 Computation of the complete point symmetry group

We fix the value $\Omega = 0$. It is straightforward to compute the following megaideals of $\mathfrak{g} = \mathcal{S}_0^\infty$:

$$\begin{aligned}&\mathfrak{g}' = \langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{P}, \mathcal{Z}(g)\rangle, \quad \mathfrak{g}'' = \langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{Z}(g)\rangle, \quad \mathfrak{g}''' = \langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\rangle,\\ &\mathrm{C}_\mathfrak{g}(\mathfrak{g}') = \mathrm{Z}_{\mathfrak{g}'} = \langle \mathcal{Z}(1)\rangle, \quad \mathrm{C}_\mathfrak{g}(\mathfrak{g}'') = \mathrm{Z}_{\mathfrak{g}''} = \langle \mathcal{Z}(g)\rangle, \quad \mathrm{C}_\mathfrak{g}(\mathfrak{g}''') = \langle \mathcal{D}, \mathcal{P}, \mathcal{Z}(g)\rangle,\\ &(\mathrm{C}_\mathfrak{g}(\mathfrak{g}'''))' = \langle \mathcal{P}, \mathcal{Z}(g)\rangle,\end{aligned}$$

where $\mathfrak{a}'$, $\mathrm{Z}_\mathfrak{a}$ and $\mathrm{C}_\mathfrak{a}(\mathfrak{b})$ denote the derivative and the center of a Lie algebra $\mathfrak{a}$ and the centralizer of a subalgebra $\mathfrak{b}$ in $\mathfrak{a}$, respectively.

Now we apply Proposition 5.1 to the case $\mathfrak{i}_0 = \mathfrak{i}_1 = (\mathrm{C}_\mathfrak{g}(\mathfrak{g}'''))'$ and vary $\mathfrak{i}_2$. If $\mathfrak{i}_2 = \langle \mathcal{Z}(1)\rangle$ we obtain $\mathfrak{s} = \langle \mathcal{Z}(1), \mathcal{Z}(t)\rangle$ and hence this is a megaideal. We reassign the last $\mathfrak{s}$ as $\mathfrak{i}_2$ and iterate this procedure, which results in the series of megaideals

$$\langle \mathcal{Z}(1), \mathcal{Z}(t), \dots, \mathcal{Z}(t^n)\rangle,\ n \in \mathbb{N}_0.$$

Megaideals of $\mathcal{S}_0^\infty$ which are sums of other megaideals are not essential for the computation of the complete point symmetry group G_0 of the vorticity equation (5.2.1) for $\Omega = 0$ by the algebraic method since they give weaker constraints for components of point symmetry transformations than their summands. Even if a megaideal $\mathfrak{i}$ is not a sum of other megaideals, the condition $G_*\mathfrak{i} \subseteq \mathfrak{i}$ may imply only constraints which are consequences of constraints derived in the course of the consideration of other megaideals. In order to simplify the computation, we choose a minimal set of megaideals which allow us to easily obtain a maximal set of constraints available within the algebraic framework. We selected such megaideals from the above list:

$$\langle \mathcal{Z}(1)\rangle, \quad \langle \mathcal{Z}(1), \mathcal{Z}(t)\rangle, \quad \langle \mathcal{P}, \mathcal{Z}(g)\rangle, \quad \langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\rangle. \tag{5.4.1}$$

The general form of a point transformation that can be applied to the vorticity equation on the sphere (5.2.1) with $\Omega = 0$ is

$$\mathcal{T}\colon \quad (\tilde t, \tilde\lambda, \tilde\mu, \tilde\psi) = (T, \Lambda, \mathrm{M}, \Psi),$$

where T, Λ, M and Ψ are regarded as functions of t, λ, μ and ψ, whose joint Jacobian J does not vanish. To derive a constrained form of $\mathcal{T}$, we use the selected four megaideals (5.4.1) of $\mathcal{S}_0^\infty$. For the transformation $\mathcal{T}$ to be qualified as a point symmetry of the vorticity equation on the sphere, its counterpart $\mathcal{T}_*$ push-forwarding vector fields should preserve each of these megaideals. Moreover, for any megaideal $\mathfrak{m}$ of $\mathfrak{g}$ the mapping induced by $\mathcal{T}$ on $\mathfrak{m}$ is an automorphism of $\mathfrak{m}$. This property is convenient to use for finite-dimensional megaideals. Thus, the megaideal $\langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\rangle$ is isomorphic to the algebra $\mathfrak{so}(3)$, whose automorphism group is exhausted by internal automorphisms and hence isomorphic to the special orthogonal group SO(3).

As a result, we obtain the conditions

$$\mathcal{T}_*\mathcal{Z}(1) = T_\psi\partial_{\tilde t} + \Lambda_\psi\partial_{\tilde\lambda} + \mathrm{M}_\psi\partial_{\tilde\mu} + \Psi_\psi\partial_{\tilde\psi} = c\tilde{\mathcal{Z}}(1), \tag{5.4.2a}$$

$$\mathcal{T}_*\mathcal{Z}(t) = t(T_\psi\partial_{\tilde t} + \Lambda_\psi\partial_{\tilde\lambda} + \mathrm{M}_\psi\partial_{\tilde\mu} + \Psi_\psi\partial_{\tilde\psi}) = d_1\tilde{\mathcal{Z}}(\tilde t) + d_0\tilde{\mathcal{Z}}(1), \tag{5.4.2b}$$

$$\mathcal{T}_*\mathcal{P} = T_t\partial_{\tilde t} + \Lambda_t\partial_{\tilde\lambda} + \mathrm{M}_t\partial_{\tilde\mu} + \Psi_t\partial_{\tilde\psi} = a_1\tilde{\mathcal{P}} + \tilde{\mathcal{Z}}(\tilde g), \tag{5.4.2c}$$

$$\mathcal{T}_*\mathcal{J}_i = \sum_{j=1}^{3} b_{ij}\tilde{\mathcal{J}}_j, \quad i = 1,2,3, \tag{5.4.2d}$$

where $\tilde g$ is a smooth function of $\tilde t$ which is determined, as the constant parameters c, d_0, d_1, a_1, a_2, a_3 and b_{ij}, by $\mathcal{T}_*$ and the vector field from the corresponding left-hand side, (b_{ij}) is a special orthogonal matrix, and $i, j = 1, 2, 3$.

We will derive constraints on $\mathcal{T}_*$ by sequentially equating the coefficients of vector fields in the conditions (5.4.2a)–(5.4.2d) and by taking into account the constraints obtained in previous steps.

Thus, condition (5.4.2a) directly implies that $T_\psi = \Lambda_\psi = \mathrm{M}_\psi = 0$ and $\Psi_\psi = c$. Then the last value is nonzero since the Jacobian J does not vanish. The equation $ct = d_1\tilde t + d_0$ derived from condition (5.4.2b) gives that $d_1 \neq 0$ and hence the t-component of the transformation $\mathcal{T}$ depends only on t and the dependence is affine, $\tilde t = T(t) = cd_1^{-1}t - d_0d_1^{-1}$. Condition (5.4.2c) is split into the equations $T_t = a_1$ (and hence $a_1 = cd_1^{-1} \neq 0$), $\Lambda_t = \mathrm{M}_t = 0$ and $\Psi_t = \tilde g$. Collecting coefficients of $\partial_{\tilde\psi}$ in condition (5.4.2d), we obtain that $\Psi_\lambda = \Psi_\mu = 0$. The integration and arrangement of all the above equations for the components of $\mathcal{T}$ results in the representation

$$T = a_1t + a_0, \quad \Lambda = \Lambda(\lambda,\mu), \quad \mathrm{M} = \mathrm{M}(\lambda,\mu), \quad \Psi = c\psi + f(t),$$

where a_1, a_0 and c are arbitrary constants with $a_1c \neq 0$, f is an arbitrary smooth function of t, the pair of the smooth functions Λ and M has nonvanishing Jacobian and additionally satisfies equations implied by condition (5.4.2d). Up to internal automorphisms of the algebra $\mathcal{S}_0^\infty$ which are generated by the rotation operators $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$, we can set the matrix (b_{ij}) to be the unit matrix. Then we obtain the following system of equations with respect to the functions Λ and M:

$$\mathcal{J}_1\Lambda = 1, \quad \mathcal{J}_2\Lambda = \frac{M}{\sqrt{1-M^2}}\sin\Lambda, \quad \mathcal{J}_3\Lambda = \frac{M}{\sqrt{1-M^2}}\cos\Lambda, \tag{5.4.3a}$$

$$\mathcal{J}_1M = 0, \quad \mathcal{J}_2M = \sqrt{1-M^2}\cos\Lambda, \quad \mathcal{J}_3M = -\sqrt{1-M^2}\sin\Lambda, \tag{5.4.3b}$$

The equations $\mathcal{J}_1\Lambda = 1$ and $\mathcal{J}_1M = 0$ imply that $\Lambda = \lambda + \Upsilon(\mu)$ and $M = M(\mu)$. We substitute these expressions into the last two equations of (5.4.3b) and split them with respect to λ. This gives the conditions $\sqrt{1-M^2}\sin\Upsilon = 0$ and $\sqrt{1-\mu^2}M_\mu = \sqrt{1-M^2}\cos\Upsilon$. As $M_\mu \neq 0$, we have that $\sin\Upsilon = 0$, i.e. $\Upsilon = \pi k$, where $k \in \mathbb{Z}$. The same procedure applied to the last two equations of (5.4.3a) results in the condition

$$\frac{\mu}{\sqrt{1-\mu^2}} = \frac{(-1)^kM}{\sqrt{1-M^2}},$$

which is equivalent to $M = (-1)^k\mu$. Then the equation $\sqrt{1-\mu^2}M_\mu = \sqrt{1-M^2}\cos\Upsilon$ is identically satisfied.

There are no more constraints which can be derived within the framework of the algebraic method. The further consideration is based on the direct calculation of transformed derivatives, which is quite easy since the expressions for the transformation components have already been specified. Thus, the transformed left-hand side of the vorticity equation (5.2.1) with $\Omega = 0$,

$$\tilde\zeta_{\tilde t} + (\tilde\psi_{\tilde\lambda}\tilde\zeta_{\tilde\mu} - \tilde\psi_{\tilde\mu}\tilde\zeta_{\tilde\lambda}) = \frac{c}{a_1}\zeta_t + (-1)^kc^2(\psi_\lambda\zeta_\mu - \psi_\mu\zeta_\lambda),$$

identically vanishes for each solution of (5.2.1) if and only if $c = (-1)^k/a_1$. This means that up to rotations, which are generated by vector fields from $\langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \rangle$, any transformation from the group G_0 takes the form

$$\tilde{t} = a_1 t + a_0, \quad \tilde{\lambda} = \lambda, \quad \tilde{\mu} = \varepsilon\mu, \quad \tilde{\psi} = \frac{\varepsilon}{a_1}\psi + f(t),$$

where a_0 and a_1 are arbitrary constants with $a_1 \neq 0$, $\varepsilon = \pm 1$ and f is an arbitrary smooth function of t. (We neglect the shift of λ by πk as it is a rotation associated with $\mathcal{J}_1$ and denote $(-1)^k$ by ε.) A transformation of the above form belongs to the connected component of the unity in G_0 if and only if $a_1 > 0$ and $\varepsilon = 1$. Therefore, there are only two discrete transformations in G_0 which are independent up to combinations with each other and with continuous transformations. These are, e.g., the transformations with $(a_1, \varepsilon) = (-1, 1)$ and $(a_1, \varepsilon) = (1, -1)$, where in both the cases we set $a_0 = 0$ and $f = 0$, which merely alternate the signs of the variables $\{t, \psi\}$ and $\{\mu, \psi\}$, respectively.

The transformation which alternates the signs of the variables $\{\lambda, \mu\}$ is in fact not a discrete symmetry of the vorticity equation (5.2.1) for $\Omega = 0$ as it is the rotation by the angle π with respect to the axis corresponding to $\lambda = 0$ and $\mu = 0$. The above symmetry transformations alternating signs of different sets of variables can be combined in order to obtain other pairs of simple discrete transformations which are independent of each other up to continuous transformations. An example of such a pair is given by the transformations merely alternating the signs of the variables $\{t, \lambda\}$ and $\{\mu, \psi\}$, respectively. These transformations coincide with those stated in [18, 23]. This completes the description of the complete point symmetry group of the barotropic vorticity equation on the sphere with $\Omega = 0$.

By use of the transformation (5.2.2) the above discrete transformations can also be transferred to discrete symmetries of the vorticity equation on a constantly rotating sphere.

Summing up the above consideration, we obtain the following assertion.

Theorem 5.2. *The complete point symmetry group of the barotropic vorticity equation on the sphere* (5.2.1) *is generated by one-parameter groups associated with vector fields from the algebra $\mathcal{S}^\infty_\Omega$ and two discrete transformations, e.g.,*

$$(t, \lambda, \mu, \psi) \mapsto (-t, -\lambda, \mu, \psi) \quad \textit{and} \quad (t, \lambda, \mu, \psi) \mapsto (t, \lambda, -\mu, -\psi).$$

Corollary 5.3. *The factor group of the complete point symmetry group of the barotropic vorticity equation on the sphere* (5.2.1) *with respect to its connected component of the unity is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

5.5 Conclusion

In this paper we verified the claim raised in [18, 23] that the barotropic vorticity equation on the sphere possesses two independent (up to composition with each other and with continuous symmetry transformations) discrete symmetries. The computation involved two parts, an algebraic step and a step related to the direct method of finding point symmetries. In view of the structure of the maximal Lie invariance algebra $\mathcal{S}^\infty_0$, we were able to find a sufficiently large number of megaideals of $\mathcal{S}^\infty_0$ and then selected those of them which were essential for our consideration, i.e. the megaideals (5.4.1). This allowed us to derive important restrictions on the form of point symmetry transformations and therefore strongly economized the remaining computations which were necessary to be carried out using the direct method. We should in particular stress that by taking into account all the constraints that are derivable by the algebraic method, we already obtained a strongly restricted form of the admitted point symmetries. Only a single constraint, which relates the constants a_1, ε and c, could not be found from the transformation behavior of the megaideals and consequently had to be determined using the direct method. As the sBVE is a complicated third-order nonlinear partial differential equation in $(1+2)$ variables, not deriving the above restricted form would have rendered it quite problematic to compute the complete point symmetry group using only the direct method.

By Proposition 5.1 we also extended the number of possibilities to determine megaideals of Lie algebras. This will be crucial for the computation of the complete point symmetry group of other systems of differential equations as the method we proposed in [21] and applied in this paper heavily relies on the availability of a large number of megaideals of the associated maximal Lie invariance algebras.

Another novel feature of the present paper is the combining of a simplification of automorphisms via factoring out internal automorphisms as originally proposed in [58, 59] with the algebraic technique based on megaideals. This is advantageous for the case under consideration as the rotations from SO(3) in angular coordinates have a rather cumbersome representation, i.e. already the direct integration of the Lie equations associated with elements of $\mathfrak{so}(3)$ is a nontrivial problem. If the calculation of the complete point symmetry group G_0 would be done without factoring out internal automorphisms, the integration of the Lie equations would be implicitly repeated during the computation, which would considerably complicate the calculations within the algebraic method. As $\mathfrak{so}(3)$ is both a direct summand and a megaideal of $\mathcal{S}_0^\infty$, the extension of any automorphism of $\mathfrak{so}(3)$ to the complement of $\mathfrak{so}(3)$ in $\mathcal{S}_0^\infty$ by identity is an automorphism of $\mathcal{S}_0^\infty$. Moreover, any such automorphism of $\mathcal{S}_0^\infty$ is internal as the automorphism group of $\mathfrak{so}(3)$ coincides with the group of internal automorphisms. Hence we can easily factor out such automorphisms assuming in the course of the computation that the basis elements $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ are identically transformed. Factoring out other internal automorphisms does not essentially simplify the consideration.

To conclude, it often happens that some discrete symmetries of a system of differential equations are known but it is difficult to prove that there are no other discrete symmetries. It will therefore be instructive to test the refined algebraic method for the computation of discrete symmetries as presented in this paper with equations which are known to possess nontrivial discrete symmetries, such as the potential fast diffusion equation $v_t = v_{xx}/v_x$, cf. [121].

Chapter 6

Differential invariants for the Korteweg–de Vries equation

6.1 Introduction

Invariants and differential invariants are important objects associated with transformation groups. They play a role for finding invariant, partially invariant and differentially invariant solutions [52, 101, 112], in computer vision [104], for the construction of invariant discretization schemes [17, 22, 41, 71, 82, 102, 133] and in the study of invariant parameterization schemes [14, 16, 123].

There are two main ways to construct differential invariants for Lie group actions. The notation we use follows the book [101] and the papers [33, 45, 102, 104, 106, 107]. Let G be a (pseudo)group of transformations acting on the space of variables (x, u), where $x = (x^1, \dots, x^p)$ is the tuple of independent variables and $u = (u^1, \dots, u^q)$ is the tuple of dependent variables. Let $\mathfrak{g}$ be the Lie algebra of vector fields that is associated with G.

The first way for the computation of differential invariants uses the infinitesimal method [38, 52, 101, 112]. The criterion for a function I defined on a subset of the corresponding nth-order jet space to be a differential invariant of the maximal Lie invariance group G is that the condition

$$\mathrm{pr}^{(n)}\mathbf{v}(I) = 0, \tag{6.1.1}$$

holds for any vector field $\mathbf{v} \in \mathfrak{g}$. In equation (6.1.1), the vector field $\mathbf{v}$ is of the form $\mathbf{v} = \xi^i(x, u)\partial_{x^i} + \phi_\alpha(x, u)\partial_{u^\alpha}$ (the summation over double indices is applied), and $\mathrm{pr}^{(n)}\mathbf{v}$ denotes the standard nth prolongation of $\mathbf{v}$. In the framework of the infinitesimal method, the differential invariants I are computed by solving the system of quasilinear first-order partial differential equations of the form (6.1.1), where the vector field $\mathbf{v}$ runs through a generating set of $\mathfrak{g}$.

The second possibility for computing differential invariants uses moving frames [33, 45, 46]. The main advantage of the moving frame method is that it avoids the integration of differential equations, which is necessary in the infinitesimal approach. At the same time, using moving frames allows one to invoke the powerful recurrence relations, which can be helpful in studying the structure of the algebra of differential invariants.

In this paper, we study differential invariants for the maximal Lie invariance group of the Korteweg–de Vries (KdV) equation. This problem was already considered in [33, 106] and in [38] within the framework of the moving frame and infinitesimal approaches, respectively. Thus, on one hand it is instructive to compare and review the results available in the literature. On the other hand, we extend these results in the present paper. In particular, we explicitly present functional bases of differential invariants of arbitrary order for the aforementioned group.

The further organization of the paper is the following. In Section 6.2 we restate the maximal Lie invariance group of the KdV equation. Section 6.3 collects some results related to a moving frame for the maximal Lie invariance group of the KdV equation as presented in [33]. We also introduce an alternative moving frame in this section. Section 6.4 contains our main results, which are a complete list of functionally independent differential invariants for the maximal Lie invariance group of KdV equation of *any* order as well as the description of a basis of differential invariants for the new normalization introduced in Section 6.3. Section 6.5 contains some remarks related to the results of the paper.

6.2 Lie symmetries of the KdV equation

The KdV equation is undoubtedly one of the most important partial differential equations in mathematical physics. It describes the motion of long shallow-water waves in a channel. Here we will use it in the following dimensionless form:

$$u_t + uu_x + u_{xxx} = 0. \tag{6.2.1}$$

The KdV equation is completely integrable using inverse scattering [51]. The coefficients of each vector field $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ generating a one-parameter Lie symmetry group of the KdV equation satisfy the system of determining equations

$$\tau_x = \tau_u = \xi_u = \eta_t = \eta_x = 0, \quad \eta = \xi_t - \tfrac{2}{3}u\tau_t, \quad \eta_u = -\tfrac{2}{3}\tau_t = -2\xi_x \tag{6.2.2}$$

with the general solution

$$\tau = 3c_4t + c_1, \quad \xi = c_4x + c_3t + c_2, \quad \eta = -2c_4u + c_3,$$

where $c_1, \ldots, c_4$ are arbitrary constants. Hence the maximal Lie invariance algebra $\mathfrak{g}$ of (6.2.1) is spanned by the four vector fields

$$\partial_t, \quad \partial_x, \quad t\partial_x + \partial_u, \quad 3t\partial_t + x\partial_x - 2u\partial_u. \tag{6.2.3}$$

Associated with these basis elements are the one-parameter symmetry groups of (i) time translations, (ii) space translations, (iii) Galilean boosts and (iv) scalings. The most general Lie symmetry transformation of the KdV equation can be constructed using these elementary one-parameter groups:

$$T = e^{3\varepsilon_4}(t+\varepsilon_1), \quad X = e^{\varepsilon_4}(x+\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_3t), \quad U = e^{-2\varepsilon_4}(u+\varepsilon_3), \tag{6.2.4}$$

where $\varepsilon_1, \ldots, \varepsilon_4 \in \mathbb{R}$ are continuous group parameters. The KdV equation also admits a discrete point symmetry, given by simultaneous changes of the signs of the variables t and x.

The prolongation of the general element Q of the algebra $\mathfrak{g}$ has

$$\eta^\alpha = -(3\alpha_1+\alpha_2+2)c_4u_\alpha - \alpha_1c_3u_{\alpha_1-1,\alpha_2+1},$$

as the coefficient of ∂_{u_α}, where $\alpha = (\alpha_1, \alpha_2)$ is a multiindex, $\alpha_1, \alpha_2 \in \mathbb{N}\cup\{0\}$, and $u_\alpha = \partial^{\alpha_1+\alpha_2}u/\partial t^{\alpha_1}\partial x^{\alpha_2}$ as usual.

Using the chain rule, from the above transformation formula (6.2.4) one obtains the expressions for the transformed derivative operators,

$$\mathrm{D}_T = e^{-3\varepsilon_4}(\mathrm{D}_t - \varepsilon_3\mathrm{D}_x), \quad \mathrm{D}_X = e^{-\varepsilon_4}\mathrm{D}_x.$$

In [33] these operators were used for listing some of the lower order transformed partial derivatives of u. However, in order to obtain a closed formula for a functional basis of differential invariants of *arbitrary* order for the KdV equation, it is useful to attempt to derive a closed-form expression for the transformed derivatives of u. Such an expression is

$$\begin{aligned} U_\alpha &= e^{-(3\alpha_1+\alpha_2+2)\varepsilon_4}(\mathrm{D}_t - \varepsilon_3\mathrm{D}_x)^{\alpha_1}\mathrm{D}_x^{\alpha_2}u \\ &= e^{-(3\alpha_1+\alpha_2+2)\varepsilon_4}\sum_{k=0}^{\alpha_1}(-\varepsilon_3)^k\binom{\alpha_1}{k}u_{\alpha_1-k,\alpha_2+k}. \end{aligned} \tag{6.2.5}$$

In particular, the expressions for U_T and U_X are

$$U_T = e^{-5\varepsilon_4}(u_t - \varepsilon_3 u_x), \quad U_X = e^{-3\varepsilon_4}u_x.$$

6.3 A moving frame for the KdV equation

As the maximal Lie invariance group of the KdV equation is finite-dimensional, we only review the construction of moving frames for finite-dimensional group actions here. Details on the moving frame construction for Lie pseudogroups can be found, e.g., in [33, 107].

Definition 6.1. Let there be given a Lie group G acting on a manifold M. A *right moving frame* is a mapping $\rho\colon M \to G$ that satisfies the property $\rho(g\cdot z) = \rho(z)g^{-1}$ for any $g \in G$ and $z \in M$.

The theorem on moving frames, see e.g. [46, 104, 106], guarantees the existence of a moving frame in the neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z. Moving frames are constructed using a procedure called normalization, which is based on the selection of a submanifold (the cross-section) that intersects the group orbits only once and transversally.

There exist infinitely many possibilities to construct a moving frame. The single moving frames differ in the choice of the respective cross-sections. The moving frame constructed in [33] rests on the normalization conditions

$$T = 0, \quad X = 0, \quad U = 0, \quad U_T = 1, \tag{6.3.1}$$

i.e., it is defined on the first jet space J^1. It is necessary to construct the moving frame on the first jet space, as the maximal Lie invariance group of the KdV equation does not act freely on the space M, spanned by t, x and u. The action of G first becomes free when prolonged to J^1, which is then the proper space to construct the moving frame $\rho^{(1)}\colon J^1 \to G$ on. Solving the above algebraic system (6.3.1) for the group parameters $\varepsilon_1, \dots, \varepsilon_4$ yields the moving frame $\rho^{(1)}$

$$\varepsilon_1 = -t, \quad \varepsilon_2 = -x, \quad \varepsilon_3 = -u, \quad \varepsilon_4 = \frac{1}{5}\ln(u_t + uu_x), \tag{6.3.2}$$

which is well defined provided that $u_t + uu_x > 0$. This moving frame becomes singular when $u_t + uu_x = 0$. The latter condition is equivalent, on the manifold of the KdV equation, to the condition that $u_{xxx} = 0$ and implies, together with the KdV equation, that $u_{xx} = 0$.

Another possible normalization, leading to an alternative moving frame, is the following:

$$T = 0, \quad X = 0, \quad U = 0, \quad U_X = 1.$$

Solving the normalization conditions gives the associated moving frame

$$\varepsilon_1 = -t, \quad \varepsilon_2 = -x, \quad \varepsilon_3 = -u, \quad \varepsilon_4 = \frac{1}{3}\ln u_x, \tag{6.3.3}$$

which is well defined provided that $u_x > 0$.

Note that for $u_t + uu_x < 0$ (resp. $u_x < 0$) one can replace the condition $U_T = 1$ by $U_T = -1$ (resp. $U_X = 1$ by $U_X = -1$).

6.4 Differential invariants for the KdV equation

The above moving frames can now be used to construct differential invariants using the *method of invariantization* [33, 104, 106].

Definition 6.2. The *invariantization* of a function $f\colon M \to \mathbb{R}$ is the function defined by

$$\iota(f) = f(\rho(z)\cdot z).$$

We first construct the set of all functionally independent differential invariants for the maximal Lie invariance group of the KdV equation using the moving frame (6.3.2). An exhaustive list of differential invariants of any order was not given in [33]. Such a list is obtained by plugging the moving frame (6.3.2) into the transformed derivatives (6.2.5). This yields

$$I_\alpha = \iota(U_\alpha) = (u_t + uu_x)^{-(3\alpha_1+\alpha_2+2)/5}\sum_{k=0}^{\alpha_1}\binom{\alpha_1}{k}u^k u_{\alpha_1-k,\alpha_2+k}, \tag{6.4.1}$$

where $\alpha_1 > 1$ or $\alpha_2 > 0$. Invariantizing t, x, u and u_t, one recovers the normalization conditions (6.3.1) and the associated differential invariants are dubbed *phantom invariants.* The corresponding invariantized form of the KdV equation is $1 + I_{03} = 0$.

Using the alternative moving frame (6.3.3), invariantization of (6.2.5) leads to the following set of functionally independent differential invariants of the maximal Lie invariance group of the KdV equation,

$$I_\alpha = \iota(U_\alpha) = u_x^{-(3\alpha_1+\alpha_2+2)/3}\sum_{k=0}^{\alpha_1}\binom{\alpha_1}{k}u^k u_{\alpha_1-k,\alpha_2+k}, \tag{6.4.2}$$

where $\alpha_1 > 0$ or $\alpha_2 > 1$, and $H^1 = \iota(t) = 0$, $H^2 = \iota(x) = 0$, $I_{00} = \iota(u) = 0$ and $I_{01} = \iota(u_x) = 1$ exhaust the set phantom invariants for this moving frame. Then the invariantization of the KdV equation yields the invariant form $I_{10} + I_{03} = 0$. The advantage of the form (6.4.2) of differential invariants compared to the form (6.4.1), which follows from the normalization (6.3.1) chosen in [33], is that these invariants are singular only on the subset $u_x = 0$, which is contained in the subset $u_{xx} = 0$ on which the invariants (6.4.1) are singular (again, when restrict to the KdV equation).

In principle, by computing the form of differential invariants of any order we have already solved the problem to exhaustively describe all the differential invariants for the maximal Lie invariance group of the KdV equation. On the other hand, it is instructive to study the structure of the algebra of differential invariants in some more detail.

In particular, an interesting open problem in the theory of differential invariants is to find minimal generating set of differential invariants in an algorithmic way. This is the set of differential invariants that is sufficient to generate all differential invariants by means of acting on the generating invariants with the operators of invariant differentiation and taking combinations of the basis invariants with these invariant derivatives. Often the computation of the *syzygies* among the differential invariants is a crucial step to prove the minimality of a given generating set. The two operators of invariantization for the maximal Lie invariance group G of the KdV equation follow from the invariantization of the operators of total differentiation D_t and D_x and they are

$$\mathrm{D}^{\mathrm{i}}_t = \iota(\mathrm{D}_t) = (u_t + uu_x)^{-3/5}(\mathrm{D}_t + u\mathrm{D}_x),$$
$$\mathrm{D}^{\mathrm{i}}_x = \iota(\mathrm{D}_x) = (u_t + uu_x)^{-1/5}\mathrm{D}_x.$$

In [33] it was claimed that the invariants

$$I_{01} = \frac{u_x}{(u_t + uu_x)^{3/5}}, \quad I_{20} = \frac{u_{tt} + 2uu_{tx} + u^2u_{xx}}{(u_t + uu_x)^{8/5}}$$

form a generating set of the algebra of differential invariants for the KdV equation. While this is certainly true, this set is not minimal. In [106] it was shown that the differential invariant I_{01} is in fact sufficient to generate the entire algebra of differential invariants for the KdV equation. The crucial step missed in finding the minimal generating set in [33] was the use of the commutator formula for the operators of invariant differentiation $\mathrm{D}^{\mathrm{i}}_t$ and $\mathrm{D}^{\mathrm{i}}_x$, which is

$$[\mathrm{D}^{\mathrm{i}}_t, \mathrm{D}^{\mathrm{i}}_x] = \tfrac{3}{5}(I_{11} + I_{01}^2)\mathrm{D}^{\mathrm{i}}_t - \tfrac{1}{5}(I_{20} + 6I_{01})\mathrm{D}^{\mathrm{i}}_x. \tag{6.4.3}$$

From the recurrence relation

$$\mathrm{D}^{\mathrm{i}}_t I_{01} = -\tfrac{3}{5}I_{01}^2 + I_{11} - \tfrac{3}{5}I_{01}I_{20}$$

one can solve for I_{11} in terms of I_{01} and I_{20}. Applying the commutation relation (6.4.3) to the invariant I_{01} then allows solving for I_{20} solely in terms of I_{01}, which explicitly gives

$$I_{20} = \frac{[\mathrm{D}^{\mathrm{i}}_t, \mathrm{D}^{\mathrm{i}}_x]I_{01} - \frac{3}{5}(\mathrm{D}^{\mathrm{i}}_t I_{01} + \frac{8}{5}I_{01}^2)\mathrm{D}^{\mathrm{i}}_t I_{01} + \frac{6}{5}I_{01}\mathrm{D}^{\mathrm{i}}_x I_{01}}{\frac{9}{25}I_{01}\mathrm{D}^{\mathrm{i}}_t I_{01} - \frac{4}{5}\mathrm{D}^{\mathrm{i}}_x I_{01}},$$

which shows that I_{01} is indeed the minimal generating set of the algebra of differential invariants for the KdV equation.

We now repeat the computation of a basis of differential invariants for the moving frame (6.3.3). The associated operators of invariant differentiation for this moving frame are the same that were constructed in [38] within the framework of the infinitesimal approach,

$$\mathrm{D}^{\mathrm{i}}_t = u_x^{-1}(\mathrm{D}_t + u\mathrm{D}_x), \quad \mathrm{D}^{\mathrm{i}}_x = u_x^{-1/3}\mathrm{D}_x.$$

The computation of corresponding recurrence relations differs from that given in [33, 106] only in minor details. Identifying $c_3 = \xi_t$ and $c_4 = \frac{1}{3}\tau_t$, we obtain the invariantized forms

$$\hat\tau = \iota(\tau), \quad \hat\xi = \iota(\xi), \quad \hat\eta = \hat\eta^{00} = \iota(\eta),$$
$$\hat\eta^\alpha = \iota(\eta^\alpha) = -\frac{3\alpha_1 + \alpha_2 + 2}{3}I_\alpha\hat\tau^1 - \alpha_1 I_{\alpha_1-1,\alpha_2+1}\hat\eta, \quad \alpha_1 + \alpha_2 > 0,$$

and the first three forms $\hat\tau$, $\hat\xi$ and $\hat\eta$ jointly with $\hat\tau^1=\iota(\tau^1)$ make up, in view of the invariantized counterpart of the determining equations (6.2.2), a basis of the invariantized Maurer–Cartan forms of the algebra $\mathfrak g$. The recurrence formulas for the normalized differential invariants are

$$\mathrm{d_h}H^1=\omega^1+\hat\tau,\quad \mathrm{d_h}H^2=\omega^2+\hat\xi,\quad \mathrm{d_h}I_\alpha=I_{\alpha_1+1,\alpha_2}\omega^1+I_{\alpha_1,\alpha_2+1}\omega^2+\hat\eta^\alpha,$$

where the form $\omega^1=\iota(\mathrm dx)$ and $\omega^2=\iota(\mathrm dx)$ constitute the associated invariantized horizontal co-frame, $\mathrm{d_h}$ is the horizontal differential and so $\mathrm{d_h}F=(\mathrm D^{\mathrm i}_tF)\omega^1+(\mathrm D^{\mathrm i}_xF)\omega^2$. We take into account that $H^1=0$, $H^2=0$, $I_{00}=0$ and $I_{01}=1$ and solve the corresponding recurrence formulas with respect to the basis invariantized Maurer–Cartan forms,

$$\hat\tau=-\omega^1,\quad \hat\xi=-\omega^2,\quad \hat\eta=-I_{10}\omega^1-\omega^2,\quad \hat\tau^1=I_{11}\omega^1+I_{02}\omega^2.$$

Then splitting of the other recurrence formulas yields

$$\begin{aligned}\mathrm D^{\mathrm i}_tI_\alpha&=I_{\alpha_1+1,\alpha_2}-\frac{3\alpha_1+\alpha_2+2}{3}I_{11}I_\alpha+\alpha_1I_{10}I_{\alpha_1-1,\alpha_2+1},\\ \mathrm D^{\mathrm i}_xI_\alpha&=I_{\alpha_1,\alpha_2+1}-\frac{3\alpha_1+\alpha_2+2}{3}I_{02}I_\alpha+\alpha_1I_{\alpha_1-1,\alpha_2+1},\end{aligned}$$

where $\alpha_1>0$ or $\alpha_2>1$.

It is obvious from the above split recurrence formulas for that the whole set of differential invariants of the maximal Lie symmetry group of the KdV equation is generated by the two lowest-order normalized invariants

$$I_{10}=u_x^{-5/3}(u_t+uu_x),\quad I_{02}=u_x^{-4/3}u_{xx}.$$

At the same time, the differential invariant I_{02} is expressed in terms of invariant derivatives of I_{10} and hence a basis associated with the moving frame (6.3.3) consists of the single element I_{10}. Indeed, we have

$$[\mathrm D^{\mathrm i}_x,\mathrm D^{\mathrm i}_t]=-I_{02}\mathrm D^{\mathrm i}_t+(1+\tfrac13I_{11})\mathrm D^{\mathrm i}_x=-I_{02}\mathrm D^{\mathrm i}_t+(\tfrac13(\mathrm D^{\mathrm i}_xI_{10})+\tfrac59I_{10}I_{02}+\tfrac23)\mathrm D^{\mathrm i}_x$$

as $I_{11}=\mathrm D^{\mathrm i}_xI_{10}+\frac53I_{10}I_{02}-1$. Applying the commutation relation for $\mathrm D^{\mathrm i}_x$ and $\mathrm D^{\mathrm i}_t$ to I_{10} and solving the obtained equation with respect to I_{20}, we derive the requested expression,

$$I_{20}=\frac{[\mathrm D^{\mathrm i}_t,\mathrm D^{\mathrm i}_x]I_{10}-\frac13(\mathrm D^{\mathrm i}_xI_{10}+2)\mathrm D^{\mathrm i}_xI_{10}}{\frac59I_{10}\mathrm D^{\mathrm i}_xI_{10}-\mathrm D^{\mathrm i}_tI_{10}}.$$

6.5 Conclusion

The present paper is devoted to the construction of differential invariants for the maximal Lie invariance group of the KdV equation. We illustrate by examples that it is worthwhile to examine different possibilities for choosing the normalization conditions, which is a cornerstone for the moving frame computation. This is an important investigation as the form of differential invariants obtained depends strongly on the set of normalization equations chosen. In the present case of the maximal Lie invariance group of the KdV equation, using $U_X=1$ as a normalization condition instead of the condition $U_T=1$ chosen in [33] leads to the normalized differential invariants (6.4.2) which have a simpler form than the normalized differential invariants (6.4.1) associated with the latter condition. The same claim is true concerning the corresponding operators of invariant differentiation, recurrence formulas, etc. Moreover, the differential invariants (6.4.2) are singular only on a proper subset of the set of solutions of the KdV equation for which the differential invariants (6.4.1) are singular. The invariantized form of the KdV equation is more appropriate using the normalization condition $U_X = 1$. In contrast to the condition $U_T = 1$, this condition also naturally leads to the separation of differential invariants which involve only derivatives of u with respect to x that may be essential as the KdV equation is an evolution equation.

We also show that for Lie groups of rather simple structure, it is possible to construct functional bases of differential invariants of arbitrary order in an explicit and closed form like (6.4.1) and (6.4.2). This observation was first presented in [16] for an infinite-dimensional Lie pseudogroup. Such a closed-form expression is beneficial as it is generally simpler than the form of differential invariants obtained when acting with operators of invariant differentiation on basis differential invariants. It is difficult to conceive finding similar expressions for arbitrary order within the framework of the infinitesimal method in a reasonable way.

Chapter 7

On the ineffectiveness of constant rotation in the primitive equations

7.1 Introduction

A main motivation for the study of Lie point symmetries of differential equations is that they provide systematic tools which allow finding of ansatzes that reduce the number of independent variables in partial differential equations. Depending on the particular form of the reduction ansatz, the reduced differential equations can then be often integrated to yield exact solutions that are also particular solutions of the initial system of partial differential equations.

Another important application of symmetries of differential equations is that they can provide a necessary condition of whether two equations can be mapped to each other. This criterion is most effective in the case when the target equation is linear as then the initial equation is linearizable. For the equations of hydro-thermodynamics, Lie symmetries have proved to be extremely successful in finding transformations relating different equations. The most famous example for this finding is certainly the linearization of the Burgers equation by means of the Hopf–Cole transformation [101], which in fact is a non-invertible transformation, mapping the Burgers equation to the linear heat equation. Also other equations of hydrodynamics, such as the one-dimensional system of shallow-water equations, the Thomas equation, the potential Burgers equation, the cylindrical Korteweg–de Vries equation and the -2 diffusion equation are linearizable by point transformations [26, 25, 59, 77]. All these transformations can be found by invoking the structure of the maximal Lie invariance algebras of the equations involved. For invertible point transformations as will be considered in the present paper, the relevant necessary criterion for the existence of a mapping relating two system of differential equations to each other is that the maximal Lie invariance algebras of the initial and the target system are isomorphic [26, 25].

Quite recently, a number of point transformations were found that allow canceling terms related to the Coriolis force in the equations of fluid dynamics. Although somehow expectable from the physical point of view, these transformations are often nontrivial. Examples of particular models where such a transformation was already found are the vorticity equation in spherical coordinates [18, 23, 116], the barotropic potential vorticity equation [19] and the shallow-water equations on flat [35] and parabolic topography [36].

It is the purpose of the present paper to show that such a transformation eliminating the Coriolis force also exists for the more complex system of the primitive equations. The primitive equations are a system of nonlinear partial differential equations for the momentum, mass and energy conservation. They form the dynamical core of most of the modern large-scale weather and climate prediction models.

A further major result of this paper is the computation of the complete point symmetry group of the primitive equations using the algebraic method proposed in [21]. To the best of our knowledge, this is the first computation of the complete point symmetry group of a multidimensional system of nonlinear partial differential equations admitting an infinite dimensional maximal Lie invariance group of complicated structure. The core part of the method is the computation of wide sets of megaideals that place suitable restrictions on admitted point symmetries of the primitive equations. Without the use of megaideals, the computation of the complete point symmetry group would require the solution of a cumbersome nonlinear system of partial differential equations, which in general is a hopeless endeavor.

The further organization of this paper is as follows. In Section 7.2 the primitive equations are introduced. In Section 7.3 we compute the symmetries of the primitive equations and explicitly find a point transformation that allows canceling of the effects of a constant rotation. In Section 7.4 we determine the complete point symmetry group of the primitive equations using the algebraic method. Section 7.5 is devoted to the usage of the transformation found and the computation of selected exact solutions of the primitive equations. The final Section 7.6 briefly sums up the results of the paper.

7.2 The primitive equations

We consider the primitive equations on the plane using pressure coordinates, i.e. the pressure p is used as a vertical coordinate instead of the actual geometric height z. The advantage of using pressure coordinates is that the continuity equation reduces to a diagnostic equation in this case. The system of primitive equation then reads [67]

$$
\begin{aligned}
&\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \omega\mathbf{v}_p + f(-v,u)^{\mathrm{T}} + \nabla\phi = 0,\\
&\phi_p + \frac{R}{p}T = 0,\\
&u_x + v_y + \omega_p = 0,\\
&T_t + \mathbf{v}\cdot\nabla T + \omega T_p - \frac{R}{c_p}\frac{\omega}{p}T = \frac{J}{c_p},
\end{aligned}
\tag{7.2.1}
$$

where $\mathbf{v} = (u, v)$ is the horizontal component of the velocity vector, $\nabla = (\partial_x, \partial_y)$ is the two-dimensional nabla operator, ω is the vertical velocity in the pressure coordinate system, i.e. the material derivative of the pressure p, ϕ is the geopotential and T is the temperature. All the unknown functions, $\mathbf{v}$, ω, ϕ and T depend on (t, x, y, p). Subscriptions of functions denote differentiation with respect to the corresponding variables. The constants f, R, c_p in the above system are the Coriolis parameter, the gas constant for dry air and the specific heat of dry air at constant pressure. The function $J = J(t, x, y, p)$ is the external heating. The first equation is the momentum equation, the second equation is the hydrostatic equation, the third equation is the continuity equation and the last equation is the a version of the first law of hydrodynamics. From the physical point of view, the system (7.2.1) forms the dynamical core of most of the present day's atmospheric numerical models.

The physical constants R and c_p are always positive. Moreover, from the practical point of view, the thermodynamic relation $c_p = c_v + R$ applies for ideal gases. By definition, c_v is the specific heat at constant volume, which is the amount of energy needed to heat one kilogram of a compound by one Kelvin while holding the volume constant. As there is no compound which will be heated by one Kelvin without supplying energy (i.e. always $c_v > 0$), this implies that $c_p > R$. So as to simplify the subsequent expressions, we will put $\kappa = R/c_p$ subsequently where $0 < \kappa < 1$. For the Coriolis parameter f we distinguish between the cases of $f = 0$ (no rotation of the reference frame) and $f = \text{const}$ (constant rotation of the reference frame). There arises the question of whether the choice $f = \text{const}$ is a physically interesting one. By definition, $f = 2\Omega \sin\varphi$, where Ω is the angular velocity of the Earth and φ is the geographic latitude. Thus, $f = f(\varphi)$ and therefore changes along the meridians. On the other hand, this change is rather small and eventually can be neglected for domains extending only moderately in North–South direction. For example, for a domain extending approximately 300 kilometer in North–South direction, the relative change in the value of f from the South to the North is only about 5% around the mid-latitudes. Therefore, for processes that take place on relatively small domains, $f = \text{const}$ to a good approximation.

One process that can be described with the model (7.2.1) for $f = \text{const}$ is the land–sea breeze. This is a circulation often induced by differential heating of a land-sea boundary, with winds directed landward during day and seaward during night. As the land–sea breeze can persist for several hours, the effect of the Coriolis force cannot be neglected. It is generally found that around six hours after the beginning of the sea breeze the circulation is weakened due to the effects of the Coriolis force [39]. This is why $f = \text{const}$ is essential in numerical models that aim to capture the land–sea circulation in an accurate way, see [39, 115] and references therein for a detailed review over numerical studies of this particular circulation pattern.

Another reason why it convenient to assume $f = \text{const}$ in the above system is the usage of Cartesian coordinates. For processes taking place on a large enough domain, the tangential plane approximation of the Earth is not reasonable any more. For such processes or for the general description of the global atmospheric circulation, it is more appropriate to study the primitive equations in spherical coordinates and to use $f = 2\Omega \sin\varphi$ without approximation.

Within the framework of group analysis of differential equations, the parameterized system (7.2.1) should be interpreted as a class of systems of differential equations with the arbitrary elements f, R, c_p and J. Two of the arbitrary elements are inessential, in that it is possible to scale $R = 1$ (by a scaling of T and c_p) and $f = 1$ if $f \neq 0$ (by a scaling of (t, x, y, p)). For physical reasons we will not make a use of these scaling. The possibility to set $f = 1$ is also not overly relevant as we will show in the following section that f can be set to zero by a point transformation.

7.3 Lie symmetries

In the following we will mostly be concerned with the system (7.2.1) in the case of $J = 0$, corresponding to the case of a non-heated atmosphere (adiabatic case). We now compute the Lie symmetries for this case using the infinitesimal invariance criterion [101]. The result gives the coefficients of the vector field

$$Q = \tau\partial_t + \xi^x\partial_x + \xi^y\partial_x + \xi^p\partial_p + \eta^u\partial_u + \eta^v\partial_v + \xi^\omega\partial_\omega + \eta^\phi\partial_\phi + \eta^T\partial_T,$$

which is the infinitesimal generator of the maximal Lie invariance algebra $\mathfrak{g}_f$ of the primitive equations (7.2.1). We have computed the maximal Lie invariance algebra using the Maple package DESOLV [31, 32, 152]. Splitting the general expression for Q gives the following basis elements of $\mathfrak{g}_f$:

$$
\begin{aligned}
&\mathcal{D}_1 = t\partial_t + \hat{f}ty\partial_x - \hat{f}tx\partial_y - (u - \hat{f}tv - \hat{f}y)\partial_u - (v + \hat{f}tu + \hat{f}x)\partial_v - \omega\partial_\omega \\
&\qquad - (2\phi + \hat{f}^2(x^2+y^2))\partial_\phi - 2T\partial_T, \\
&\mathcal{D}_2 = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2\phi\partial_\phi + 2T\partial_T, \quad \mathcal{D}_3 = p\partial_p + \omega\partial_\omega, \\
&\mathcal{P} = \partial_t, \quad \mathcal{J} = -y\partial_x + x\partial_y - v\partial_u + u\partial_v, \quad \mathcal{S} = p^\kappa(c_p\partial_\phi - \partial_T), \\
&\mathcal{X}(\boldsymbol{\gamma}) = \boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} + \boldsymbol{\gamma}_t\cdot\partial_{\mathbf{v}} - (\boldsymbol{\gamma}_{tt}\cdot\mathbf{x} + f(\gamma^1_t y - \gamma^2_t x))\partial_\phi, \quad \mathcal{Z}(\alpha) = \alpha\partial_\phi,
\end{aligned}
\tag{7.3.1}
$$

where $\hat{f} = f/2$, $\mathbf{x} = (x, y)$, $\partial_{\mathbf{x}} = (\partial_x, \partial_y)$, $\partial_{\mathbf{v}} = (\partial_u, \partial_v)$, $\boldsymbol{\gamma} = (\gamma^1, \gamma^2)$, and the parameters γ^1, γ^2 and α run through the set of smooth functions depending on t.

The associated one-parametric groups are (i) time-translations, (ii)–(iv) scalings, (v) planar rotations, (vi)–(vii) generalized Galilean transformations, (viii) gauging of the geopotential and (ix) generalized shifts.

The algebra $\mathfrak{g}_f$ is not singular in the Coriolis parameter f, which means that it is possible to set $f = 0$ in (7.3.1). The remaining question is whether there are additional infinitesimal generators extending the algebra $\mathfrak{g}_0$ when $f = 0$ in (7.2.1). Computing Lie symmetries of system (7.2.1) for $f = 0$ shows that this is not the case, i.e. the primitive equations in a nonrotating reference frame admit $\mathfrak{g}_0$ as the maximal Lie invariance algebra, whose basis elements are

$$
\begin{aligned}
&\mathcal{D}_1 = t\partial_t - u\partial_u - v\partial_v - \omega\partial_\omega - 2\phi\partial_\phi - 2T\partial_T, \\
&\mathcal{D}_2 = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2\phi\partial_\phi + 2T\partial_T, \quad \mathcal{D}_3 = p\partial_p + \omega\partial_\omega, \\
&\mathcal{P} = \partial_t \quad \mathcal{J} = -y\partial_x + x\partial_y - v\partial_u + u\partial_v, \quad \mathcal{S} = p^\kappa(c_p\partial_\phi - \partial_T), \\
&\mathcal{X}(\boldsymbol{\gamma}) = \boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} + \boldsymbol{\gamma}_t\cdot\partial_{\mathbf{v}} - \boldsymbol{\gamma}_{tt}\cdot\mathbf{x}\partial_\phi, \quad \mathcal{Z}(\alpha) = \alpha\partial_\phi,
\end{aligned}
\tag{7.3.2}
$$

The nonzero commutation relations among basis elements of $\mathfrak{g}_0$ are exhausted by

$$
\begin{aligned}
&[\mathcal{D}_1, \partial_t] = -\partial_t, \quad [\mathcal{D}_1, \mathcal{S}] = 2\mathcal{S}, \quad [\mathcal{D}_1, \mathcal{X}(\boldsymbol{\gamma})] = \mathcal{X}(t\boldsymbol{\gamma}_t), \quad [\mathcal{D}_1, \mathcal{Z}(\alpha)] = \mathcal{Z}(2\alpha + t\alpha_t), \\
&[\mathcal{D}_2, \mathcal{S}] = -2\mathcal{S}, \quad [\mathcal{D}_2, \mathcal{X}(\boldsymbol{\gamma})] = -\mathcal{X}(\boldsymbol{\gamma}), \quad [\mathcal{D}_2, \mathcal{Z}(\alpha)] = -2\mathcal{Z}(\alpha), \quad [\mathcal{D}_3, \mathcal{S}] = \kappa\mathcal{S}, \\
&[\partial_t, \mathcal{X}(\boldsymbol{\gamma})] = \mathcal{X}(\boldsymbol{\gamma}_t), \quad [\partial_t, \mathcal{Z}(\alpha)] = \mathcal{Z}(\alpha_t), \quad [\mathcal{J}, \mathcal{X}(\boldsymbol{\gamma})] = \mathcal{X}(\gamma^2, -\gamma^1), \\
&[\mathcal{X}(\boldsymbol{\gamma}), \mathcal{X}(\boldsymbol{\sigma})] = \mathcal{Z}(\boldsymbol{\sigma}\cdot\boldsymbol{\gamma}_{tt} - \boldsymbol{\gamma}\cdot\boldsymbol{\sigma}_{tt}).
\end{aligned}
$$

Based on the above commutation relations, one can see that the Lie algebra $\mathfrak{g}_0$ has the structure of $\mathfrak{g}_0 = (\mathfrak{g}_2 \oplus \mathfrak{g}_3) \Subset \mathfrak{i}$, where $\mathfrak{g}_2 = \langle \partial_t, \mathcal{D}_1 \rangle$ is a realization of the two-dimensional non-abelian algebra, $\mathfrak{g}_3 = \langle \mathcal{D}_2, \mathcal{D}_3, \mathcal{J} \rangle$ is a realization of the three-dimensional abelian algebra and $\mathfrak{i} = \langle \mathcal{X}(\boldsymbol{\gamma}), \mathcal{Z}(\alpha), \mathcal{S} \rangle$ is an infinite dimensional ideal in $\mathfrak{g}_0$, $\mathfrak{i} = (\langle \mathcal{X}(\boldsymbol{\gamma}) \rangle \Subset \langle \mathcal{Z}(\alpha) \rangle) \oplus \langle \mathcal{S} \rangle$, and $\langle \mathcal{Z}(\alpha) \rangle$ and $\langle \mathcal{S} \rangle$ are abelian ideals in the entire algebra $\mathfrak{g}_0$.

Upon redefining the basis elements in the algebra $\mathfrak{g}_f$ according to

$$
\partial_t \to \partial_t - \hat{f}\mathcal{J}, \quad \mathcal{X}(\boldsymbol{\gamma}) \to \mathcal{X}(\tilde{\boldsymbol{\gamma}})
$$

where $\tilde{\boldsymbol{\gamma}} = (\gamma^1\cos(\hat{f}t) - \gamma^2\sin(\hat{f}t), \gamma^1\sin(\hat{f}t) + \gamma^2\cos(\hat{f}t))$ and the remaining basis elements remain unchanged, they satisfy the same commutation relations as the basis elements of $\mathfrak{g}_0$. Therefore, the algebras $\mathfrak{g}_f$ and $\mathfrak{g}_0$ are isomorphic, which is a necessary condition for the existence of a point transformation mapping the primitive equations with $f \neq 0$ to the primitive equation in a resting reference frame ($f = 0$) [25]. This allows us to use the algebraic method for finding the transformation relating the two systems with $f = 0$ and $f \neq 0$ to each other.

Suppose that a point transformation

$$\mathcal{T}\colon\quad \tilde{z}^i = \mathcal{T}^i(t,x,y,p,u,v,\omega,\phi,T), \tag{7.3.3}$$

where $i \in \{t,x,y,p,u,v,\omega,\phi,T\}$, $(z^t,z^x,z^y,z^p,z^u,z^v,z^\omega,z^\phi,z^T) = (t,x,y,p,u,v,\omega,\phi,T)$ and $(\tilde{z}^t,\tilde{z}^x,\tilde{z}^y,\tilde{z}^p,\tilde{z}^u,\tilde{z}^v,\tilde{z}^\omega,\tilde{z}^\phi,\tilde{z}^T) = (\tilde{t},\tilde{x},\tilde{y},\tilde{p},\tilde{u},\tilde{v},\tilde{\omega},\tilde{\phi},\tilde{T})$, realize the above automorphism between the algebras $\mathfrak{g}_f$ and $\mathfrak{g}_0$. Then, the relation upon the corresponding basis elements Q and $\tilde{Q}$ of the algebras $\mathfrak{g}_f$ and $\mathfrak{g}_0$, respectively, reads

$$Q\mathcal{T}^i = \tilde{Q}\tilde{z}^i, \tag{7.3.4}$$

which is the usual rule for the transformation of vector fields.

Evaluating (7.3.4) for the transformation component of $\tilde{p}$, it follows from the transformation of $\mathcal{Z}$ that $\mathcal{T}^p_\phi = 0$ and from $\mathcal{S}$ that $\mathcal{T}^p_T = 0$. As $\mathcal{X}(\gamma)\mathcal{T}^p = 0$ must hold for arbitrary smooth functions γ, one obtains that $\mathcal{T}^p_x = \mathcal{T}^p_y = \mathcal{T}^p_u = \mathcal{T}^p_v = 0$. Moreover $\partial_t\mathcal{T}^p = 0$ and from the two different scaling operators $\mathcal{D}_1$ and $\mathcal{D}_3$ we conclude that $\tilde{p} = p$. Using similar arguments, one also finds that $\tilde{\omega} = \omega$, $\tilde{T} = T$ and $\tilde{t} = t$.

Now consider the transformation components of $\tilde{x}$ and $\tilde{y}$. In view of the operators $\mathcal{Z}$ and $\mathcal{S}$ we again conclude that $\mathcal{T}^x_\phi = \mathcal{T}^x_T = \mathcal{T}^y_\phi = \mathcal{T}^y_T = 0$. From the condition that $\mathcal{X}(\gamma)\mathcal{T}^x = \sigma^1$ and $\mathcal{X}(\gamma)\mathcal{T}^y = \sigma^2$ it follows that $\mathcal{T}^x = a^1(t)x + a^2(t)y + a^3(t)u + a^4(t)v + x^0(p,\omega)$ and $\mathcal{T}^y = b^1(t)x + b^2(t)y + b^3(t)u + b^4(t)v + y^0(p,\omega)$. The condition that $(\partial_t - \hat{f}\mathcal{J})\mathcal{T}^x = 0$ implies that $a^1_t - \hat{f}a^2 = 0$, $a^2_t + \hat{f}a^1 = 0$, $a^3_t - \hat{f}a^4 = 0$ and $a^4_t + \hat{f}a^3 = 0$. From $\mathcal{J}\mathcal{T}^x = -\mathcal{T}^y$ it follows that $b^1 = -a^2$, $b^2 = a^1$, $b^3 = -a^4$ and $b^4 = a^3$. The equation $\mathcal{D}_1\mathcal{T}^x = 0$ enforces that $a^3 = a^4 = 0$ and therefore also $b^3 = b^4 = 0$. Integration of the equations for a^1 and a^2 gives that $a^1 = c_1\cos(\hat{f}t) + c_2\sin(\hat{f}t)$ and $a^2 = c_2\cos(\hat{f}t) - c_1\sin(\hat{f}t)$. From the constraint $a^1\gamma^1 + a^2\gamma^2 = \sigma^1 = \cos(\hat{f}t)\gamma^1 - \sin(\hat{f}t)\gamma^2$ we conclude that $c_1 = 1$ and $c_2 = 0$. The action of the scaling operators $\mathcal{D}_1$ and $\mathcal{D}_3$ finally requires that $x^0 = y^0 = 0$. Therefore one obtains $\tilde{x} = \cos(\hat{f}t)x - \sin(\hat{f}t)y$ and $\tilde{y} = \sin(\hat{f}t)x + \cos(\hat{f}t)y$.

We next consider the transformation components of $\tilde{u}$ and $\tilde{v}$. Again the condition that $\mathcal{Z}(\alpha)\mathcal{T}^u = \mathcal{Z}(\alpha)\mathcal{T}^v = \mathcal{S}\mathcal{T}^u = \mathcal{S}\mathcal{T}^v = 0$ leads to $\mathcal{T}^u_\phi = \mathcal{T}^u_T = \mathcal{T}^v_\phi = \mathcal{T}^v_T = 0$. The condition that $\mathcal{X}(\gamma)\mathcal{T}^u = \sigma^1_t$ and $\mathcal{X}(\gamma)\mathcal{T}^v = \sigma^2_t$ implies that $\mathcal{T}^u = d^1(t)x + d^2(t)y + d^3(t)u + d^4(t) + u^0(p,\omega)$ and $\mathcal{T}^v = e^1(t)x + e^2(t)y + e^3(t)u + e^4(t) + v^0(p,\omega)$. The relations $d^1_t - \hat{f}d^2 = 0$, $d^2_t + \hat{f}d^1 = 0$, $d^3_t - \hat{f}d^4 = 0$ and $d^4_t + \hat{f}d^3 = 0$ and $e^1 = -d^2$, $e^2 = d^1$, $e^3 = -d^4$ and $e^4 = d^3$ follow from the transformations rules $(\partial_t - \hat{f}\mathcal{J})\mathcal{T}^u = 0$ and $\mathcal{J}\mathcal{T}^u = -\mathcal{T}^v$, respectively. The scaling transformation $\mathcal{D}_1\mathcal{T}^u = -\mathcal{T}^u$ enforces the relations the relation $d^4 = d^2/\hat{f}$ and $d^3 = -d^2/\hat{f}$. Integrating the equations for d_1 and d_2, including the constraint that $\mathcal{X}(\gamma)\mathcal{T}^u = \sigma^1_t$ and invoking the transformations of the scaling $\mathcal{D}_3$ leads to the transformation $\tilde{u} = \cos(\hat{f}t)u - \sin(\hat{f}t)v - \hat{f}(\sin(\hat{f}t)x + \cos(\hat{f}t)y)$ and $\tilde{v} = \sin(\hat{f}t)u + \cos(\hat{f}t)v + \hat{f}(\cos(\hat{f}t)x - \sin(\hat{f}t)y)$.

It thus remains to determine the transformation behavior of ϕ. From $\mathcal{Z}(\alpha)\mathcal{T}^\phi = \tilde{\alpha}$ and $\mathcal{S}\mathcal{T}^\phi = p^\kappa c_p$ it follows that $\mathcal{T}^\phi_\phi = 1$ and $\mathcal{T}^\phi_T = 0$. The actions $(\partial_t - \hat{f}\mathcal{J})\mathcal{T}^\phi = 0$ and $\mathcal{J}\mathcal{T}^\phi = 0$ imply that $\mathcal{T}^\phi_t = 0$. From $\mathcal{X}(\gamma)\mathcal{T}^\phi = -\tilde{\boldsymbol{\gamma}}_{tt}\cdot\tilde{\mathbf{x}}$ it follows that $\mathcal{T}^\phi_u = \mathcal{T}^\phi_v = 0$ and $\mathcal{T}^\phi_x = \hat{f}^2x$ and $\mathcal{T}^\phi_y = \hat{f}^2y$. The action of the scaling operators $\mathcal{D}_1$ and $\mathcal{D}_3$ on $\mathcal{T}^\phi$ implies that $\mathcal{T}^\phi_p = \mathcal{T}^\phi_\omega = 0$. Therefore, $\tilde{\phi} = \phi + \hat{f}^2(x^2+y^2)/2$.

It can be checked that all the equations from the condition $Q\mathcal{T}^i = \tilde{Q}\tilde{z}^i$ not used to derive the above form of the transformation $\tilde{z}^i = \mathcal{T}^i(t,x,y,p,u,v,\omega,\phi,T)$ reduce to identities. It can also be checked by direct substitution that the same transformation also relates the primitive equations with $f \neq 0$ to the equations in which $f = 0$. This proves the following theorem.

Theorem 7.1. *The primitive equations* (7.2.1) *in a reference frame with constant rotation can be transformed to the primitive equations in a reference frame at rest, i.e. $f = 0$ upon using the transformation*

$$\begin{aligned}
&\tilde t = t, \quad \tilde x = \cos(\hat f t)x - \sin(\hat f t)y, \quad \tilde y = \sin(\hat f t)x + \cos(\hat f t)y, \quad \tilde p = p,\\
&\tilde u = \cos(\hat f t)u - \sin(\hat f t)v - \hat f(\sin(\hat f t)x + \cos(\hat f t)y),\\
&\tilde v = \sin(\hat f t)u + \cos(\hat f t)v + \hat f(\cos(\hat f t)x - \sin(\hat f t)y),\\
&\tilde\omega = \omega, \quad \tilde\phi = \phi + \frac{f^2}{8}(x^2+y^2), \quad \tilde T = T,
\end{aligned} \tag{7.3.5}$$

where $\hat f = f/2$. The same transformation maps the maximal Lie invariance algebra $\mathfrak{g}_f$ to $\mathfrak{g}_0$.

Remark 7.2. In cylindrical coordinates, (r, θ, p), the transformation (7.3.5) takes the particular simple form

$$\begin{aligned}
&\tilde t = t, \quad \tilde r = r, \quad \tilde\theta = \theta + \frac{f}{2}t, \quad \tilde p = p,\\
&\tilde u^r = u^r, \quad \tilde u^\theta = u^\theta + \frac{f}{2}r, \quad \tilde\omega = \omega, \quad \tilde\phi = \phi + \frac{f^2}{8}r^2, \quad \tilde T = T,
\end{aligned}$$

where u^r and u^θ are the velocity components in radial and in azimuthal direction, respectively.

So as to derive the transformation (7.3.5) using the algebraic method it was necessary to assume $J = 0$, i.e. the system was required to be adiabatic. This assumption is crucial as for general functions $J = J(t, x, y, p)$ the primitive equations (7.2.1) only admit the gauging operators $\mathcal{Z}(\alpha)$ and $\mathcal{S}$. The span of these gauging operators is not enough to derive a sufficient number of equations for the transformation components (7.3.5). On the other hand, one can check the validity of this transformation for the case $J \neq 0$ by direct computation. As the differential operator $\mathbf{v} \cdot \nabla$ is invariant under the transformation (7.3.5) and this is the only term that is transformed in the temperature equation, the same transformation also maps the primitive equations for $J \neq 0$ in a rotating reference frame to the corresponding system in the resting reference frame with possibly another value of J.

Corollary 7.3. *Transformation* (7.3.5) *maps the non-adiabatic ($J \neq 0$) system of primitive equations in a reference frame with constant rotation to the non-adiabatic system of primitive equations in a reference frame at rest with possibly another value of J.*

Remark 7.4. Owing to the invariance of the advection operator $\mathbf{v} \cdot \nabla$ under the transformation (7.3.5), it is possible to extend the system of primitive equations (7.2.1) by equations of the form

$$S_t + \mathbf{v} \cdot \nabla S + \omega S_p = Q,$$

without introducting new nontrivial transformation components for the prognostic variable S and the source term Q, i.e. $\tilde S = S$ and $\tilde Q = Q$. Examples for physically relevant equations of the above form are, e.g., the moisture equation or any equation for a passively transported atmospheric tracer.

Remark 7.5. In the case $c_p = R$, the system of primitive equations (7.2.1), where we set $\Omega = 0$ without loss of generality, admits a wider maximal Lie invariance algebra $\hat{\mathfrak{g}}_0$ than $\mathfrak{g}_0$. Additional basis elements $\hat{\mathfrak{g}}_0$ in comparison with $\mathfrak{g}_0$ are

$$\mathcal{R}(\lambda) = 2\lambda\partial_{\tilde t} + \lambda_{\tilde t}\tilde x\partial_{\tilde x} + \lambda_{\tilde t}\tilde y\partial_{\tilde y} - 2\lambda_{\tilde t}p\partial_{\tilde p} - (\lambda_{\tilde t}\tilde u - \lambda_{\tilde t\tilde t}\tilde x)\partial_{\tilde u} - (\lambda_{\tilde t}\tilde v - \lambda_{\tilde t\tilde t}\tilde y)\partial_{\tilde v} -$$

$$(4\lambda_{\tilde t}\tilde\omega + 2\lambda_{\tilde t\tilde t}\tilde p)\partial_{\tilde\omega} - \left(2\lambda_{\tilde t}\tilde\phi + \frac{1}{2}\lambda_{\tilde t\tilde t\tilde t}(\tilde x^2+\tilde y^2)\right)\partial_{\tilde\phi} - 2\lambda_{\tilde t}\tilde T\partial_{\tilde T},$$
$$\mathcal{P}(\psi) = \psi\partial_{\tilde p} + \psi_{\tilde t}\partial_{\tilde\omega} + \frac{\psi\tilde T}{\tilde p}\partial_{\tilde T},$$

where λ and ψ run through the set of smooth functions of t. It is clear that the operator $\mathcal{R}(\lambda)$ is a generalization of the usual shifts in t ($\lambda = \mathrm{const}$) and the scaling operator $2\mathcal{D}_1+\mathcal{D}_2$ ($\lambda = t$). For arbitrary λ, $\mathcal{R}(\lambda)$ can be interpreted as re-parameterization of time. The operator $\mathcal{P}(\psi)$ in turn is a generalized Galilean boost in p-direction. At the same time, this example is unphysical as $c_p > R$.

7.4 Complete point symmetry group

The above consideration shows that without loss of generality it suffices to carry out group analysis of the primitive equations (7.2.1) only for the case $f = 0$ In this section we find the complete point symmetry group G^0 of Eqs. (7.2.1) with $f = 0$ by the algebraic method proposed in [21]. This method may be treated as an enhancement of the approach suggested in [58, 59] (see also [53]) by embedding the notion of megaideals [124]. See [44] for recent advances of the enhanced method. Its main benefit is that it can be applied even to systems of differential equations possessing infinite-dimensional Lie invariance algebras, which is the case for Eqs. (7.2.1), Although it can also simplify related computations for higher-dimensional Lie invariance algebras.

The algebra $\mathfrak{g} = \mathfrak{g}_0$ has the following obvious megaideals:

$$\mathfrak{g}' = \langle \mathcal{P}, \mathcal{S}, \mathcal{X}(\gamma), \mathcal{Z}(\alpha)\rangle, \quad \mathfrak{g}'' = \langle \mathcal{X}(\gamma), \mathcal{Z}(\alpha)\rangle, \quad \mathfrak{g}''' = \mathrm{Z}_{\mathfrak{g}''} = \langle \mathcal{Z}(\alpha)\rangle,$$
$$\mathrm{Z}_{\mathfrak{g}'} = \langle \mathcal{S}, \mathcal{Z}(1)\rangle, \quad \mathrm{Z}_{\mathfrak{g}'}\cap\mathfrak{g}''' = \langle \mathcal{Z}(1)\rangle,$$
$$\mathfrak{m}_1 = \mathrm{C}_{\mathfrak{g}}(\mathfrak{g}'') = \langle \mathcal{D}_3, \mathcal{S}, \mathcal{Z}(\alpha)\rangle, \quad \mathfrak{m}_1' = \langle\mathcal{S}\rangle, \quad \mathrm{C}_{\mathfrak{g}}(\mathfrak{m}_1) = \langle \mathcal{J}, \mathcal{X}(\gamma), \mathcal{Z}(\alpha)\rangle,$$

where $\mathfrak{a}'$, $\mathrm{Z}_{\mathfrak{a}}$ and $\mathrm{C}_{\mathfrak{a}}(\mathfrak{b})$ denote the derivative and the center of a Lie algebra $\mathfrak{a}$ and the centralizer of a subalgebra $\mathfrak{b}$ in $\mathfrak{a}$, respectively. Here and in what follows the parameters γ^1, γ^2 and α run through the set of smooth functions depending on t.

To find more megaideals of $\mathfrak{g}$, we apply Proposition 1 from [44] for various special choices of the megaideals $\mathfrak{i}_0$, $\mathfrak{i}_1$ and $\mathfrak{i}_2$ of $\mathfrak{g}$. This proposition states that the set $\mathfrak{s}$ of elements from $\mathfrak{i}_0$ whose commutators with arbitrary elements from $\mathfrak{i}_1$ belong to $\mathfrak{i}_2$ is also a megaideal of $\mathfrak{g}$. Thus, for $\mathfrak{i}_0 = \mathfrak{g}'''$, $\mathfrak{i}_1 = \mathfrak{g}'$ and $\mathfrak{i}_2 = \mathrm{Z}_{\mathfrak{g}'}\cap\mathfrak{g}''' = \langle\mathcal{Z}(1)\rangle$, we obtain $\mathfrak{s} = \langle \mathcal{Z}(1), \mathcal{Z}(t)\rangle$ and hence this is a megaideal. We reassign the last $\mathfrak{s}$ as $\mathfrak{i}_2$ and iterate the procedure with the same $\mathfrak{i}_0$ and $\mathfrak{i}_1$, which gives the series of megaideals $\langle \mathcal{Z}(1), \mathcal{Z}(t), \dots, \mathcal{Z}(t^n)\rangle$, $n\in\mathbb{N}_0$.

A convenient choice for $\mathfrak{i}_0$ and $\mathfrak{i}_1$ is $\mathfrak{i}_0 = \mathfrak{i}_1 = \mathfrak{g}$ when $\mathfrak{i}_2$ is varying. For $\mathfrak{i}_2 = \mathfrak{m}_1'$ and $\mathfrak{i}_2 = \mathfrak{g}''$ we respectively have the megaideals $\mathfrak{s} = \langle\mathcal{D}_3, \mathcal{S}\rangle =: \mathfrak{m}_2$ and $\mathfrak{s} = \langle \kappa\mathcal{D}_2 + 2\mathcal{D}_3, \mathcal{J}, \mathcal{X}(\gamma), \mathcal{Z}(\alpha)\rangle$. Then $\mathrm{C}_{\mathfrak{g}'}(\mathfrak{m}_2) = \langle \mathcal{P}, \mathcal{X}(\gamma), \mathcal{Z}(\alpha)\rangle$, and $\mathrm{C}_{\mathfrak{g}}(\mathfrak{m}_2) = \langle \mathcal{D}_1+\mathcal{D}_2, \mathcal{D}_2+2\mathcal{D}_3, \mathcal{J}, \mathcal{P}, \mathcal{X}(\gamma), \mathcal{Z}(\alpha)\rangle =: \mathfrak{m}_3$ are also megaideals, as well as $\mathrm{C}_{\mathfrak{m}_3}(\mathrm{Z}_{\mathfrak{g}'}\cap\mathfrak{g}''') = \langle \mathcal{D}_1+\mathcal{D}_2, \mathcal{J}, \mathcal{P}, \mathcal{X}(\gamma), \mathcal{Z}(\alpha)\rangle$.

Applying again Proposition 1 from [44] on the next step, we take $\mathfrak{i}_0 = \mathfrak{i}_1 = \mathrm{C}_{\mathfrak{g}'}(\mathfrak{m}_2)$ and $\mathfrak{i}_2 = \mathfrak{g}'''$ and derive the megaideal $\mathfrak{s} = \langle \mathcal{X}(1,0), \mathcal{X}(0,1), \mathcal{Z}(\alpha)\rangle =: \mathfrak{m}_4$. We reassign the last $\mathfrak{s}$ as $\mathfrak{i}_2$ and iterate the procedure with the same $\mathfrak{i}_0$ and $\mathfrak{i}_1$, which gives the series of megaideals

$$\langle \mathcal{X}(1,0), \mathcal{X}(0,1), \mathcal{X}(t,0), \mathcal{X}(0,t), \dots, \mathcal{X}(t^n,0), \mathcal{X}(0,t^n), \mathcal{Z}(\alpha)\rangle, \quad n\in\mathbb{N}_0.$$

Considering $\mathfrak{i}_0 = \mathfrak{g}$ and $\mathfrak{i}_1 = \mathfrak{m}_4\oplus\mathfrak{m}_1'$ with $\mathfrak{i}_2 = \mathfrak{g}'''$, we get $\mathfrak{s} = \langle \mathcal{D}_1, \mathcal{P}, \mathcal{S}, \mathcal{X}(\gamma), \mathcal{Z}(\alpha)\rangle$.

Some of the above megaideals of $\mathfrak{g}_0$ can be neglected in the course of computing the complete point symmetry group G_0 of the primitive equations (7.2.1) with $f=0$ by the algebraic method. Indeed, the condition $G_*\mathfrak{i}\subseteq\mathfrak{i}$ for a megaideal $\mathfrak{i}$ may only result in constraints for components of point symmetry transformations that are consequences of those obtained in the course of the computation with other megaideals. In particular, this is the case if a megaideal $\mathfrak{i}$ is a sum of other megaideals. To optimize the computation, we select a minimal set of megaideals that allow us to easily derive a set of constraints for components of point symmetry transformations that is maximal within the algebraic framework. We choose the following megaideals from those we have computed:

$$
\begin{aligned}
&\langle\mathcal{Z}(1)\rangle,\quad \langle\mathcal{Z}(1),\mathcal{Z}(t)\rangle,\quad \langle\mathcal{S}\rangle,\quad \langle\mathcal{X}(1,0),\mathcal{X}(0,1),\mathcal{Z}(\alpha)\rangle,\\
&\langle\mathcal{X}(t,0),\mathcal{X}(0,t),\mathcal{X}(1,0),\mathcal{X}(0,1),\mathcal{Z}(\alpha)\rangle,\\
&\langle\mathcal{X}(t^2,0),\mathcal{X}(0,t^2),\mathcal{X}(t,0),\mathcal{X}(0,t),\mathcal{X}(1,0),\mathcal{X}(0,1),\mathcal{Z}(\alpha)\rangle,\\
&\langle\mathcal{J},\mathcal{X}(\gamma),\mathcal{Z}(\alpha)\rangle,\quad \langle\mathcal{P},\mathcal{X}(\gamma),\mathcal{Z}(\alpha)\rangle,\quad \langle\mathcal{D}_1+\mathcal{D}_2,\mathcal{J},\mathcal{P},\mathcal{X}(\gamma),\mathcal{Z}(\alpha)\rangle,\\
&\langle\mathcal{D}_3,\mathcal{S}\rangle,\quad \langle\mathcal{D}_1,\mathcal{P},\mathcal{S},\mathcal{X}(\gamma),\mathcal{Z}(\alpha)\rangle.
\end{aligned}
\tag{7.4.1}
$$

We additionally ordered the megaideal list in such a way that megaideals heading the list give more elementary equations of the form $\mathcal{T}^j_{z^i}=0$ with some $i,j\in\{t,x,y,p,u,v,\omega,\phi,T\}$ or allows us to specify the expressions for some $\mathcal{T}^j$.

The general form of point transformations that acts in the space of the independent and dependent variables of the primitive equations (7.2.1) is given by Eq. (7.3.3), where the corresponding Jacobian J does not vanish. For a point transformation $\mathcal{T}$ to be qualified as a point symmetry of the primitive equations (7.2.1) with $f=0$, its counterpart $\mathcal{T}_*$ push-forwarding vector fields should preserve each of the selected megaideals (7.4.1) of the algebra $\mathfrak{g}_0$. As a result, we obtain the conditions

$$\mathcal{T}_*\mathcal{Z}(1)=\mathcal{T}^i_\phi\partial_{\tilde z^i}=a_1\tilde{\mathcal{Z}}(1),\tag{7.4.2a}$$
$$\mathcal{T}_*\mathcal{Z}(t)=t\mathcal{T}^i_\phi\partial_{\tilde z^i}=a_2\tilde{\mathcal{Z}}(\tilde t)+a_3\tilde{\mathcal{Z}}(1),\tag{7.4.2b}$$
$$\mathcal{T}_*\mathcal{S}=p^\kappa(c_p\mathcal{T}^i_\phi-\mathcal{T}^i_T)\partial_{\tilde z^i}=a_4\tilde{\mathcal{S}},\tag{7.4.2c}$$
$$\mathcal{T}_*\mathcal{X}(1,0)=\mathcal{T}^i_x\partial_{\tilde z^i}=\tilde{\mathcal{X}}(b^{00}_{11},b^{00}_{21})+\tilde{\mathcal{Z}}(\tilde\alpha^{01}),\tag{7.4.2d}$$
$$\mathcal{T}_*\mathcal{X}(0,1)=\mathcal{T}^i_y\partial_{\tilde z^i}=\tilde{\mathcal{X}}(b^{00}_{12},b^{00}_{22})+\tilde{\mathcal{Z}}(\tilde\alpha^{02}),\tag{7.4.2e}$$
$$\mathcal{T}_*\mathcal{X}(t,0)=(t\mathcal{T}^i_x+\mathcal{T}^i_u)\partial_{\tilde z^i}=\tilde{\mathcal{X}}(b^{11}_{11}\tilde t+b^{10}_{11},b^{11}_{21}\tilde t+b^{10}_{21})+\tilde{\mathcal{Z}}(\tilde\alpha^{11}),\tag{7.4.2f}$$
$$\mathcal{T}_*\mathcal{X}(0,t)=(t\mathcal{T}^i_y+\mathcal{T}^i_v)\partial_{\tilde z^i}=\tilde{\mathcal{X}}(b^{11}_{12}\tilde t+b^{10}_{12},b^{11}_{22}\tilde t+b^{10}_{22})+\tilde{\mathcal{Z}}(\tilde\alpha^{12}),\tag{7.4.2g}$$
$$\begin{aligned}\mathcal{T}_*\mathcal{X}(t^2,0)&=(t^2\mathcal{T}^i_x+2t\mathcal{T}^i_u-2x\mathcal{T}^i_\phi)\partial_{\tilde z^i}\\&=\tilde{\mathcal{X}}(b^{22}_{11}\tilde t^2+b^{21}_{11}\tilde t+b^{20}_{11},b^{22}_{21}\tilde t^2+b^{21}_{21}\tilde t+b^{20}_{21})+\tilde{\mathcal{Z}}(\tilde\alpha^{21}),\end{aligned}\tag{7.4.2h}$$
$$\begin{aligned}\mathcal{T}_*\mathcal{X}(0,t^2)&=(t^2\mathcal{T}^i_y+2t\mathcal{T}^i_v-2y\mathcal{T}^i_\phi)\partial_{\tilde z^i}\\&=\tilde{\mathcal{X}}(b^{22}_{12}\tilde t^2+b^{21}_{12}\tilde t+b^{20}_{12},b^{22}_{22}\tilde t^2+b^{21}_{22}\tilde t+b^{20}_{22})+\tilde{\mathcal{Z}}(\tilde\alpha^{22}),\end{aligned}\tag{7.4.2i}$$
$$\mathcal{T}_*\mathcal{J}=(x\mathcal{T}^i_y-y\mathcal{T}^i_x+u\mathcal{T}^i_v-v\mathcal{T}^i_u)\partial_{\tilde z^i}=a_5\tilde{\mathcal{J}}+\mathcal{X}(\tilde\gamma^3)+\tilde{\mathcal{Z}}(\tilde\alpha^3),\tag{7.4.2j}$$
$$\mathcal{T}_*\mathcal{P}=\mathcal{T}^i_t\partial_{\tilde z^i}=a_6\tilde{\mathcal{P}}+\mathcal{X}(\tilde\gamma^4)+\tilde{\mathcal{Z}}(\tilde\alpha^4),\tag{7.4.2k}$$
$$\mathcal{T}_*(\mathcal{D}_1+\mathcal{D}_2)=(t\mathcal{T}^i_t+x\mathcal{T}^i_x+y\mathcal{T}^i_y+\omega\mathcal{T}^i_\omega)\partial_{\tilde z^i}=a_7(\tilde{\mathcal{D}}_1+\tilde{\mathcal{D}}_2)+a_8\tilde{\mathcal{P}}+\mathcal{X}(\tilde\gamma^5)+\tilde{\mathcal{Z}}(\tilde\alpha^5),\tag{7.4.2l}$$
$$\mathcal{T}_*\mathcal{D}_3=(p\mathcal{T}^i_p+\omega\mathcal{T}^i_\omega)\partial_{\tilde z^i}=a_9\tilde{\mathcal{D}}_3+a_{10}\tilde{\mathcal{S}},\tag{7.4.2m}$$
$$\begin{aligned}\mathcal{T}_*\mathcal{D}_1&=(t\mathcal{T}^i_t-u\mathcal{T}^i_u-v\mathcal{T}^i_v-\omega\mathcal{T}^i_\omega-2\phi\mathcal{T}^i_\phi-2T\mathcal{T}^i_T)\partial_{\tilde z^i}\\&=a_{11}\tilde{\mathcal{D}}_1+a_{12}\tilde{\mathcal{P}}+a_{13}\tilde{\mathcal{S}}+\mathcal{X}(\tilde\gamma^6)+\tilde{\mathcal{Z}}(\tilde\alpha^6),\end{aligned}\tag{7.4.2n}$$

where $i \in \{t, x, y, p, u, v, \omega, \phi, T\}$, and we assume summation with respect to repeated indices; a_s, $s = 1, \dots, 13$, b^{00}_{kl}, b^{10}_{kl}, b^{11}_{kl}, b^{20}_{kl}, b^{21}_{kl} and b^{22}_{kl}, $k, l = 1, 2$, are constants; $\tilde{\boldsymbol{\gamma}}^m = (\tilde\gamma^{m1}, \tilde\gamma^{m2})$, $m = 3, \dots, 6$, and the parameters $\tilde\alpha^{0l}$, $\tilde\alpha^{1l}$, $\tilde\alpha^{2l}$, $\tilde\gamma^{ml}$, $\tilde\gamma^{ml}$ and $\tilde\alpha^m$ are smooth functions depending on $\tilde t$.

We will derive constraints on $\mathcal{T}$ by sequentially equating the coefficients of vector fields in the conditions (7.4.2) and by taking into account the constraints obtained in previous steps.

Thus, the condition (7.4.2a) directly implies that $\mathcal{T}^\phi_\phi = a_1$ and $\mathcal{T}^i_\phi = 0$ if $i \neq \phi$. Then the constant a_1 is nonzero since the Jacobian J does not vanish. The equation $a_1 t = a_2 \tilde t + a_3$ derived from the condition (7.4.2b) gives that $a_2 \neq 0$ and hence the component $\mathcal{T}^t$ depends only on t and the dependence is affine,

$$\mathcal{T}^t = a_1 a_2^{-1} t - a_3 a_2^{-1}.$$

This completely specifies expression for $\mathcal{T}^t$ and also implies that $\partial_{\tilde t} = a_1^{-1} a_2 \partial_t$.

The condition (7.4.2c) is split into the equations $p^\kappa(c_p \mathcal{T}^T_\phi - \mathcal{T}^T_T) = -a_4(\mathcal{T}^p)^\kappa$, $p^\kappa(c_p \mathcal{T}^\phi_\phi - \mathcal{T}^\phi_T) = a_4 c_p (\mathcal{T}^p)^\kappa$ and $c_p \mathcal{T}^i_\phi - \mathcal{T}^i_T = 0$ for $i \neq \phi, T$. Therefore, $\mathcal{T}^T_T = a_4(\mathcal{T}^p/p)^\kappa$, $\mathcal{T}^\phi_T = c_p a_1 - c_p a_4 (\mathcal{T}^p/p)^\kappa$, $\mathcal{T}^i_T = 0$ for $i \neq \phi, T$, and $a_4 \neq 0$.

Considering simultaneously the pairs of the conditions (7.4.2d) and (7.4.2e), (7.4.2f) and (7.4.2g), as well as (7.4.2h) and (7.4.2i), we derive that

$$\begin{aligned}
&\mathcal{T}^x_x = b^{00}_{11},\ \mathcal{T}^y_x = b^{00}_{21},\quad \mathcal{T}^\phi_x = \tilde\alpha^{01},\\
&\mathcal{T}^x_y = b^{00}_{12},\ \mathcal{T}^y_y = b^{00}_{22},\quad \mathcal{T}^\phi_y = \tilde\alpha^{02},
\end{aligned}\qquad \mathcal{T}^i_x = \mathcal{T}^i_y = 0,\ i \neq x, y, \phi;$$

$$\begin{aligned}
&\mathcal{T}^x_u = b^{11}_{11}\tilde t + b^{10}_{11} - b^{00}_{11} t,\ \mathcal{T}^y_u = b^{11}_{21}\tilde t + b^{10}_{21} - b^{00}_{21} t,\quad \mathcal{T}^u_u = b^{11}_{11},\ \mathcal{T}^v_u = b^{11}_{21},\quad \mathcal{T}^\phi_u = \tilde\alpha^{11} - t\tilde\alpha^{01},\\
&\mathcal{T}^x_v = b^{11}_{12}\tilde t + b^{10}_{12} - b^{00}_{12} t,\ \mathcal{T}^y_v = b^{11}_{22}\tilde t + b^{10}_{22} - b^{00}_{22} t,\quad \mathcal{T}^u_v = b^{11}_{12},\ \mathcal{T}^v_v = b^{11}_{22},\quad \mathcal{T}^\phi_v = \tilde\alpha^{12} - t\tilde\alpha^{02},\\
&\mathcal{T}^i_u = \mathcal{T}^i_v = 0,\ i = t, p, \omega, T;
\end{aligned}$$

$$\begin{aligned}
&b^{00}_{kl} t^2 + 2t(b^{11}_{kl}\tilde t + b^{10}_{kl} - b^{00}_{kl} t) = b^{22}_{kl}\tilde t^2 + b^{21}_{kl}\tilde t + b^{20}_{kl},\quad 2b^{11}_{kl} t = 2b^{22}_{kl}\tilde t + b^{21}_{kl},\quad k, l = 1, 2\\
&2t\tilde\alpha^{11} - t^2\tilde\alpha^{01} - 2a_1 x = -2b^{22}_{11}\tilde x - 2b^{22}_{21}\tilde y + \tilde\alpha^{21},\\
&2t\tilde\alpha^{12} - t^2\tilde\alpha^{02} - 2a_1 y = -2b^{22}_{12}\tilde x - 2b^{22}_{22}\tilde y + \tilde\alpha^{22},
\end{aligned}$$

The last two equations imply that $|b^{22}_{kl}| \neq 0$ (otherwise, the Jacobian J equals zero) and thus the transformation components $\tilde x = \mathcal{T}^x$ and $\tilde y = \mathcal{T}^y$ depend only on (t, x, y). More precisely, in terms of the constants b^{00}_{kl} we have the representation

$$\mathcal{T}^x = b^{00}_{11} x + b^{00}_{12} y + \beta^1(t), \quad \mathcal{T}^y = b^{00}_{21} x + b^{00}_{22} y + \beta^2(t),$$

where β^k are smooth functions of t. As $\mathcal{T}^x_u = \mathcal{T}^y_u = \mathcal{T}^x_v = \mathcal{T}^y_v = 0$, we obtain $b^{11}_{kl}\tilde t + b^{10}_{kl} - b^{00}_{kl} t = 0$. Then $b^{00}_{kl} = a_1 a_2^{-1} b^{11}_{kl}$ and $b^{00}_{kl} = a_1^2 a_2^{-2} b^{22}_{kl}$, i.e., $B^{00} = a_1 a_2^{-1} B^{11}$ and $B^{00} = a_1^2 a_2^{-2} B^{22}$, where we use the matrix notation $B^{00} = (b^{00}_{kl})$, $B^{11} = (b^{11}_{kl})$ and $B^{22} = (b^{22}_{kl})$. On the other hand, $-2(B^{22})^{\mathrm T} B^{00} = -2a_1 E$, where E is the 2×2 unit matrix, i.e., $(B^{00})^{\mathrm T} B^{00} = a_1^3 a_2^{-2} E$, which implies, e.g., for the $(1,1)$-entry that $(b^{00}_{11})^2 + (b^{00}_{12})^2 = a_1^3 a_2^{-2}$. Therefore, $a_1 > 0$ and thus we can represent the matrix B^{00} in the form

$$B^{00} = a_1^{3/2} a_2^{-1} O,$$

where O is a 2×2 orthogonal matrix. This completes specifying the expressions for $\mathcal{T}^x$ and $\mathcal{T}^y$.

The representation for B^{00} implies $b^{00}_{11} = b^{00}_{22}$ and $b^{00}_{12} = -b^{00}_{21}$. Using this, we derive from the condition (7.4.2j) that $B^{00}\mathbf{x} = a_5\tilde{\mathbf{x}} + (\tilde\gamma^{32}, -\tilde\gamma^{31})^{\mathsf T}$, which gives $a_5 = 1$, $\beta^1(t) = \tilde\gamma^{32}(\tilde t)$ and

$\beta^2(t) = -\tilde\gamma^{31}(\tilde t)$. In view of the above equations for derivatives $\mathcal{T}^j_{z^i}$ with $i,j = u,v$, briefly representable as $(\mathcal{T}^j_{z^i})_{i,j=u,v} = B^{11}$, we also get from the condition (7.4.2j) that $a_1^{-1}a_2B^{00}\mathbf{v} = a_5\tilde{\mathbf{v}} + (\tilde\gamma^{32}_{\tilde t}, -\tilde\gamma^{31}_{\tilde t})^\mathsf{T}$. Arranging the last equation results to finally specifying the expressions for $\mathcal{T}^u$ and $\mathcal{T}^v$,

$$\mathcal{T}^u = \frac{a_2}{a_1}(b^{00}_{11}u + b^{00}_{12}v + \beta^1_t(t)), \quad \mathcal{T}^v = \frac{a_2}{a_1}(b^{00}_{21}u + b^{00}_{22}v + \beta^2_t(t)),$$

As the derivatives $\mathcal{T}^\phi_u$ and $\mathcal{T}^\phi_v$ may depend only on t, the condition (7.4.2j) gives that $\mathcal{T}^\phi_u = \mathcal{T}^\phi_v = 0$. The variables x and y are involved in the expression of $\mathcal{T}^\phi$ only within the summand $-a_2^2a_1^{-2}\boldsymbol{\beta}_{tt}\cdot B^{11}\mathbf{x}$.

The condition (7.4.2k) obviously implies the elementary equations $\mathcal{T}^p_t = \mathcal{T}^\omega_t = \mathcal{T}^T_t = 0$ and the constraint that the transformation component $\mathcal{T}^\phi$ may involve the variable t only via the above summand and one more summand that depends only on t.

Two more elementary equations, $\mathcal{T}^p_\omega = \mathcal{T}^T_\omega = 0$, follows from the condition (7.4.2l). The equation implied by (7.4.2l) for $\mathcal{T}^t$ is $t\mathcal{T}^t_t = a_7\mathcal{T}^t + a_8$, which gives $a_7 = 1$. Then the equation implied for $\mathcal{T}^\omega$ takes he form $\omega\mathcal{T}^\omega_\omega = \mathcal{T}^\omega$. The main feature of $\mathcal{T}^\phi$ obtained from (7.4.2l) is that the term $\omega\mathcal{T}^\phi_\omega$ depends only on t, x and y.

Consider equations yielded by the condition (7.4.2m). Thus, the equation for $\mathcal{T}^T$ is $p\mathcal{T}^T_p = -a_{10}p^\kappa$, and hence $\mathcal{T}^T_{pT} = 0$. As $\mathcal{T}^T_T = a_4(\mathcal{T}^p/p)^\kappa$, we get that $(\mathcal{T}^p/p)_p = 0$, i.e. $p\mathcal{T}^p_p = \mathcal{T}^p$. The equation for $\mathcal{T}^p$ is $p\mathcal{T}^p_p = a_9\mathcal{T}^p$. Therefore, $a_9 = 1$. Then the equation for $\mathcal{T}^\omega$ takes the form $p\mathcal{T}^\omega_p + \omega\mathcal{T}^\omega_\omega = \mathcal{T}^\omega$ and, after combining with the analogous equation that is obtained from the condition (7.4.2l), reduces to the equation $\mathcal{T}^\omega_p = 0$.

Collecting coefficients of $\partial_{\tilde t}$, $\partial_{\tilde T}$ and $\partial_{\tilde\phi}$ in (7.4.2n) results in the equations $t\mathcal{T}^t_t = a_{11}\mathcal{T}^t + a_{12}$, $-2T\mathcal{T}^T_T = -2a_{11}\mathcal{T}^T - a_{13}(\mathcal{T}^p)^\kappa$ and

$$t\mathcal{T}^\phi_t - \omega\mathcal{T}^\phi_\omega - \phi\mathcal{T}^\phi_\phi - 2T\mathcal{T}^\phi_T = -2a_{11}\mathcal{T}^\phi + a_{13}c_p(\mathcal{T}^p)^\kappa - \tilde\gamma^6_{tt}\cdot\tilde{\mathbf{x}} + \tilde\alpha^6.$$

The essential consequence of the first equation is $a_{11} = 1$. Then the third equation implies that $\mathcal{T}^\phi$ does not depend on and ω since we have already proved that all summands in the left hand side of the equation as well as $\mathcal{T}^p$ have this property. From the second and third equations it is obvious he variable p is involved in the expressions of $\mathcal{T}^T$ and $\mathcal{T}^\phi$ only within the summands $-\frac12 a_{13}(\mathcal{T}^p)^\kappa$ and $\frac12 a_{13}c_p(\mathcal{T}^p)^\kappa$, respectively.

We introduce notation of the following constants:

$$\varepsilon_0 = -\frac{a_3}{a_2}, \quad \varepsilon_1 = \frac{a_1}{a_2} \neq 0, \quad \varepsilon_2 = \frac{a_1^{3/2}}{a_2} > 0, \quad \varepsilon_3 = \frac{\mathcal{T}^p}{p} > 0, \quad \varepsilon_4 = \frac12 a_{12}\varepsilon_3^\kappa,$$
$$\varepsilon_5 = \frac{\mathcal{T}^\omega}{\omega} \neq 0, \quad \varepsilon_6 = a_4\varepsilon_3^\kappa.$$

The constant ε_3 should be greater than zero for both physical and mathematical reasons since the exponent ε_3^κ should be well defined for all κ: $0 < \kappa < 1$ and, in view of the physical interpretation of the variable p, both its initial and transformed values should simultaneously be positive. The constant ε_2 can be assumed positive since the parameters ε_2 and O are defined up to simultaneously alternating their signs. Collecting all the restrictions we have derived for the components of the transformation $\mathcal{T}$ within the algebraic approach and using the above notation, we obtain the preliminary representation of this transformation,

$$\begin{aligned}
&\tilde t = \varepsilon_1 t + \varepsilon_0, \quad \tilde{\mathbf{x}} = \varepsilon_2 O\mathbf{x} + \boldsymbol{\beta}(t), \quad \tilde p = \varepsilon_3 p,\\
&\tilde{\mathbf{v}} = \frac{\varepsilon_2}{\varepsilon_1}O\mathbf{v} + \frac{1}{\varepsilon_1}\boldsymbol{\beta}_t(t), \quad \tilde\omega = \varepsilon_5\omega,\\
&\tilde\phi = \frac{\varepsilon_2^2}{\varepsilon_1^2}\phi + c_p\left(\frac{\varepsilon_2^2}{\varepsilon_1^2} - \varepsilon_6\right)T + \varepsilon_4 c_p p^\kappa - \frac{\varepsilon_2}{\varepsilon_1^2}\boldsymbol{\beta}_{tt}(t)\cdot O\mathbf{x} + \alpha(t), \quad \tilde T = \varepsilon_6 T - \varepsilon_4 p^\kappa.
\end{aligned} \tag{7.4.3}$$

Not all parameters in the representation (7.4.3) are independent. For the transformation $\mathcal{T}$ to really be a point symmetry of the primitive equations (7.2.1), some parameters have to satisfy additional constraints that cannot be derived within the framework of the algebraic approach. This is why the computation should be completed by the direct method. The application of the direct method can be simplified by factoring out *a priori* known continuous transformations. Thus, we can set $\varepsilon_0 = \varepsilon_4 = 0$, $\boldsymbol{\beta} = \mathbf{0}$, $\alpha = 0$ and O to be equal to the diagonal matrix with the diagonal entries -1 and 1.

We calculate expressions for transformed derivatives and substitute them to the primitive equations (7.2.1) written in terms of the transformed variables, which are with tildes. Then we choose u_t, v_t, ϕ_p, ω_p and T_t as principal derivatives, express them in terms of other (parametric) derivatives from (7.2.1), substitute the obtained expressions into the system derived on the previous step. Splitting the resulting system with respect to parametric derivatives gives the missing equations,

$$\varepsilon_5 = \varepsilon_3, \quad \varepsilon_6 = \frac{\varepsilon_2^2}{\varepsilon_1^2}.$$

This equations jointly with the representation (7.4.3) leads to the following assertion:

Theorem 7.6. *The complete point symmetry group of the primitive equations* (7.2.1) *consists of the transformations*

$$\begin{aligned}
&\tilde{t} = \varepsilon_1 t + \varepsilon_0, \quad \tilde{\mathbf{x}} = \varepsilon_2 O\mathbf{x} + \boldsymbol{\beta}(t), \quad \tilde{p} = \varepsilon_3 p, \\
&\tilde{\mathbf{v}} = \frac{\varepsilon_2}{\varepsilon_1} O\mathbf{v} + \frac{1}{\varepsilon_1}\boldsymbol{\beta}_t(t), \quad \tilde{\omega} = \varepsilon_3\omega, \\
&\tilde{\phi} = \frac{\varepsilon_2^2}{\varepsilon_1^2}\phi + \varepsilon_4 c_p p^\kappa - \frac{\varepsilon_2}{\varepsilon_1^2}\boldsymbol{\beta}_{tt}(t)\cdot O\mathbf{x} + \alpha(t), \quad \tilde{T} = \frac{\varepsilon_2^2}{\varepsilon_1^2}T - \varepsilon_4 p^\kappa,
\end{aligned}$$

where ε_0, …, ε_4 are arbitrary constants with $\varepsilon_1 \neq 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$; $\boldsymbol{\beta} = (\beta^1, \beta^2)$; the parameters β^1, β^2 and α run through the set of smooth functions of t; O is an arbitrary 2×2 orthogonal matrix.

Corollary 7.7. *The discrete symmetries of the primitive equations* (7.2.1) *are exhausted, up to combining with continuous symmetries and with each other, by two involutions, which are the inversion of time, $t \to -t$, and simultaneous mirror mappings in the (x, y)- and (u, v)-planes, $(x, y, u, v) \to (-x, y, -u, v)$.*

7.5 Exact solutions

Finding the transformation (7.3.5) has two more immediate benefits. It allows one to take arbitrary exact solutions of the primitive equations in the resting reference frame to exact solutions of the primitive equations in a constantly rotating reference frame and vice versa. This transformation is also important because it enables one to carry out Lie reductions using the simplified Lie invariance algebra $\mathfrak{g}_0$, spanned by the operators (7.3.2) and then to extend the solutions obtained to the rotating case. Examples for both of the above usages are presented in this section.

Physically, the simple solution of the nonrotating primitive equations,

$$u = u_0(p), \quad v = v_0(p), \quad \omega = 0, \quad \phi = \phi(p), \quad T = T(p),$$

where the relation between T and p is given via the hydrostatic equation, describes a stably stratified atmosphere with a horizontally homogeneous horizontal wind field, a vanishing vertical velocity and horizontally homogeneous fields of geopotential and temperature. The inverse of transformation (7.3.5) takes this solution to

$$\tilde{u} = \cos\left(\frac{f}{2}t\right)u_0(p) + \sin\left(\frac{f}{2}t\right)v_0(p) + \frac{f}{2}y,$$
$$\tilde{v} = -\sin\left(\frac{f}{2}t\right)u_0(p) + \cos\left(\frac{f}{2}t\right)v_0(p) - \frac{f}{2}x,$$
$$\tilde{\omega} = 0, \quad \tilde{\phi} = \phi(p) - \frac{f^2}{8}(x^2+y^2), \quad \tilde{T} = T(p),$$

which is a solution of the primitive equations in a constantly rotating reference frame. This solution now is horizontally isotropic in the geopotential, while there is still no vertical velocity. Physically, this means that the effects of a constant rotation cannot lead to vertical motion if the initial vertical velocity is vanishing. The above solution then describes the inertia motion of fluid particles under the action of the Coriolis force, cf. [35] for the corresponding solution of the rotating shallow-water equations. This type of motion can be frequently observed for buoys in the ocean.

To systematically carry out Lie reductions of the primitive equations (7.2.1) with $f = 0$, it is necessary to compute an optimal list of inequivalent subalgebras, which forms the cornerstone of the reduction procedure. We do not aim to establish a complete list of inequivalent subalgebras of dimensions one, two and three here, which for the proper cases would allow reduction of the number of independent variables by one, two or three. In other words, the corresponding reduced systems would be systems of partial differential equations in two independent variables, systems of ordinary differential equations and systems of algebraic equations, respectively.

Instead, we consider the Lie reduction with respect to the subalgebra

$$\mathfrak{s} = \langle \mathcal{X}(\boldsymbol{\gamma}) + a^1\mathcal{S}, \mathcal{X}(\boldsymbol{\sigma}) + a^2\mathcal{S}\rangle,$$

where a^1 and a^2 are arbitrary constants, the pairs $\boldsymbol{\gamma} = (\gamma^1, \gamma^2)$ and $\boldsymbol{\sigma} = (\sigma^1, \sigma^2)$ of smooth functions of t are linearly independent and $\boldsymbol{\gamma}_{tt}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}_{tt}\cdot\boldsymbol{\gamma} = 0$. Note that in this case the operators $\mathcal{X}(\boldsymbol{\gamma})$ and $\mathcal{X}(\boldsymbol{\sigma})$ commute and form a proper subalgebra that is suitable for Lie reduction. Previous experience shows, that this algebra is indeed an element of the optimal list of two-dimensional subalgebras of the primitive equations, see the corresponding results for the vorticity equation, the Euler equations, the Navier–Stokes equations and the magneto-hydrodynamic equations [49, 119, 120, 130]. An appropriate reduction ansatz corresponding to this subalgebra is

$$\mathbf{v} = \hat{\mathbf{v}} + \frac{\boldsymbol{\sigma}^\perp\cdot\mathbf{x}}{\delta}\boldsymbol{\gamma}_t - \frac{\boldsymbol{\gamma}^\perp\cdot\mathbf{x}}{\delta}\boldsymbol{\sigma}_t,$$
$$\omega = \hat{\omega},$$
$$\phi = \hat{\phi} + c_p\frac{p^\kappa}{\delta}(a^1\boldsymbol{\sigma}^\perp - a^2\boldsymbol{\gamma}^\perp)\cdot\mathbf{x} - \frac{\boldsymbol{\sigma}^\perp\cdot\mathbf{x}}{2\delta}\boldsymbol{\gamma}_{tt}\cdot\mathbf{x} - \frac{\boldsymbol{\gamma}^\perp\cdot\mathbf{x}}{2\delta}\boldsymbol{\sigma}_{tt}\cdot\mathbf{x},$$
$$T = p^\kappa\hat{T} + \frac{p^\kappa}{\delta}(a^1\boldsymbol{\sigma}^\perp - a^2\boldsymbol{\gamma}^\perp)\cdot\mathbf{x},$$

where $\boldsymbol{\gamma}^\perp = (\gamma^2, -\gamma^1)$, $\boldsymbol{\sigma}^\perp = (\sigma^2, -\sigma^1)$, $\delta = \gamma^1\sigma^2 - \gamma^2\sigma^1 = \boldsymbol{\gamma}\cdot\boldsymbol{\sigma}^\perp = -\boldsymbol{\gamma}^\perp\cdot\boldsymbol{\sigma} \neq 0$ (δ can be assumed to be positive up to simultaneously alternating signs, e.g., of $\boldsymbol{\gamma}^1$ and $\boldsymbol{\gamma}^2$), and quantities with hat depend on the invariant independent variables t and p. By the way, for each pair $\boldsymbol{\beta} = (\beta^1, \beta^2)$ one has the representation

$$\boldsymbol{\beta} = \frac{\boldsymbol{\sigma}^\perp \cdot \boldsymbol{\beta}}{\delta}\boldsymbol{\gamma} - \frac{\boldsymbol{\gamma}^\perp \cdot \boldsymbol{\beta}}{\delta}\boldsymbol{\sigma} = -\frac{\boldsymbol{\sigma} \cdot \boldsymbol{\beta}}{\delta}\boldsymbol{\gamma}^\perp + \frac{\boldsymbol{\gamma} \cdot \boldsymbol{\beta}}{\delta}\boldsymbol{\sigma}^\perp.$$

The above ansatz reduces the primitive equations (7.2.1) with $f = 0$ to

$$\hat{\mathbf{v}}_t + \hat{\omega}\hat{\mathbf{v}}_p + \frac{\boldsymbol{\sigma}^\perp \cdot \hat{\mathbf{v}}}{\delta}\boldsymbol{\gamma}_t - \frac{\boldsymbol{\gamma}^\perp \cdot \hat{\mathbf{v}}}{\delta}\boldsymbol{\sigma}_t + c_p\frac{p^\kappa}{\delta}(a^1\boldsymbol{\sigma}^\perp - a^2\boldsymbol{\gamma}^\perp) = 0, \tag{7.5.1a}$$

$$\hat{\phi}_p + Rp^{\kappa-1}\hat{T} = 0, \tag{7.5.1b}$$

$$\frac{\delta_t}{\delta} + \hat{\omega}_p = 0, \tag{7.5.1c}$$

$$\hat{T}_t + \hat{\omega}\hat{T}_p + \frac{1}{\delta}(a^1\boldsymbol{\sigma}^\perp - a^2\boldsymbol{\gamma}^\perp) \cdot \hat{\mathbf{v}} = 0, \tag{7.5.1d}$$

Upon integrating the reduced continuity equation (7.5.1c) to yield

$$\hat{\omega} = \delta_t\delta^{-1}p + \chi(t),$$

it is clear that the above system is reduced to a linear system of four (1+1)-dimensional first-order partial differential equations, which can be solved in the following way. We make the change of dependent variables $\hat{\mathbf{v}} = G\check{\mathbf{v}}$, where the 2×2 nondegenerate matrix-function $G = G(t)$ is chosen as a solution of the equation $G_t - HG = 0$ with the 2×2 matrix-function $H = H(t)$ defined by $H\mathbf{x} = \delta^{-1}(\boldsymbol{\sigma}^\perp \cdot \mathbf{x})\boldsymbol{\gamma}_t - \delta^{-1}(\boldsymbol{\gamma}^\perp \cdot \mathbf{x})\boldsymbol{\sigma}_t$. Let G^{-1} denote the inverse of the matrix G. Then the first two equations (7.5.1a) reduce to

$$\check{\mathbf{v}}_t + \hat{\omega}\check{\mathbf{v}}_p + c_p\frac{p^\kappa}{\delta}G^{-1}(a^1\boldsymbol{\sigma}^\perp - a^2\boldsymbol{\gamma}^\perp) = 0, \tag{7.5.2}$$

which is an inhomogeneous system of two decoupled linear partial differential equations. The change of the independent variables

$$\tau = t, \quad \xi = \frac{p}{\delta(t)} - \theta(t) \quad \text{with} \quad \theta(t) = \int_{t_0}^{t} \frac{\chi(t')}{\delta(t')}\mathrm{d}t',$$

maps the system (7.5.2) to the system of trivial ordinary differential equations with the independent variable τ, where ξ plays the role of parameter. The general solution of the latter system can be found by quadratures. This gives the following expression for $\hat{\mathbf{v}}$:

$$\hat{\mathbf{v}}(t,p) = G(t)\mathbf{v}^0(\xi) - c_pG(t)\int_{t_0}^{t}(\delta(\tau))^{\kappa-1}\big(\xi + \theta(\tau)\big)^\kappa G^{-1}(\tau)\big(a^1\boldsymbol{\sigma}^\perp(\tau) - a^2\boldsymbol{\gamma}^\perp(\tau)\big)\,\mathrm{d}\tau,$$

where $\mathbf{v}^0$ is a pair of arbitrary smooth functions of ξ.

In order to solve the equation (7.5.1d), we substitute the obtained expression for $\hat{\mathbf{v}}$ into it, switch again to the variables (τ, ξ) and integrate with respect to τ. As a result, we have

$$\hat{T}(t,p) = T^0(\xi) - \int_{t_0}^{t} \frac{a^1\boldsymbol{\sigma}^\perp(\tau) - a^2\boldsymbol{\gamma}^\perp(\tau)}{\delta(\tau)} \cdot \hat{\mathbf{v}}\Big(\tau, \delta(\tau)\big(\xi + \theta(\tau)\big)\Big)\,\mathrm{d}\tau,$$

where T^0 is an arbitrary smooth function of ξ.

The last step of solving the system (7.5.1) is the integration of the equation (7.5.1b) with respect to p, which gives

$$\hat{\phi} = \phi^0(t) - R\int_{p_0}^{p} \tilde{p}^{\kappa-1}\hat{T}(t,\tilde{p})\,\mathrm{d}\tilde{p},$$

where ϕ^0 is an arbitrary smooth function of t, which can be neglected.

Substituting the expressions derived for the values with hats into the ansatz, we obtain the entire family of $\mathfrak{s}$-invariant solutions of the primitive equations (7.2.1).

7.6 Conclusion

The present paper is devoted to an investigation of the system of primitive equations from its symmetry point of view. We found a point transformation that allows to cancel the effects of a constant Coriolis force in this system. This transformation might be relevant for application of the primitive equations on the f-plane, such as in studies of land-sea breezes.

In practice, the primitive equations (7.2.1) as presented in Section 7.2 are forced and damped by external mechanisms, such as external heating (including J), phase transitions of water (including an equation for moisture) and bottom friction (including friction in the momentum equations). The presence of these additional mechanisms substantially narrows the number of admitted Lie symmetries. It was discussed in Section 7.2 that for arbitrary J only $\mathcal{Z}(\alpha)$ and $\mathcal{S}$ are admitted as Lie symmetries. Symmetry breaking due to external forcing terms thus substantially hinders the applicability of symmetry methods. On the other hand by omitting these external influences it is possible to arrive at a system that has a wide maximal Lie invariance (pseudo)group and is therefore accessible to the machinery of group analysis. Eventually, results derived for the simplified equations can be extended to the usual system. In this manner we have shown in the present paper that the same transformation (7.3.5) that maps the rotating primitive equations to the non-rotating ones in the adiabatic case extends trivially also to the non-adiabatic case, without the necessity to modify J in the transformed equation. In other words (7.3.5) in not an equivalence transformation.

In a similar manner, it can be checked that the transformation (7.3.5) also maps the *dissipative* primitive equations in a constantly rotating reference frame to the dissipative primitive equations in a resting reference frame. This means that attaching classical friction, $\nu\Delta\mathbf{v}$, to the right-hand side of the momentum equations in (7.2.1) does not require to modify transformation (7.3.5) in order to set f to zero.

In Section 7.2 we discussed an application of the model of primitive equations for the choice $f=\text{const}$, i.e. on the f-plane. For domains extending farther in North–South direction, the latitudinal variation of the Coriolis parameter becomes relevant. The next order of accuracy approximation for f in a Cartesian plane is $f=f_0+\beta y$, $\beta=\text{const}$, i.e. a linear variation of the rotation. It can be checked that in this case, the primitive equations (7.2.1) only admit a six-parametric maximal Lie invariance group which therefore cannot be isomorphic to the maximal Lie invariance algebra $\mathfrak{g}_0$ (7.3.2) computed in Section 7.3. This at once implies that there cannot exist a point transformation which maps the case of $f=f_0+\beta y$ to the primitive equations in a resting reference frame.

In Section 7.4 we established another main result of this paper by computing the complete point symmetry group of the primitive equations using the algebraic method. This computation was rather elaborate due to the wide spaces of both independent and dependent variables of the primitive equations. It was necessary to establish a suitable set of megaideals, finding of which crucially relied on the iterative use of Proposition 1 from [42]. Without this set of megaideals it would have been overly difficult to simplify the determining equations of point symmetry transformations for the primitive equations enough to enable their direct integration. This example thus shows the power of the algebraic method for finding complete point symmetry groups for large systems of nonlinear partial differential equations admitting infinite dimensional Lie pseudogroups, which would be challenging with the conventional direct method.

In Section 7.5 we shortly discussed the construction of exact solutions of the primitive equations with rotation. The mapping (7.3.5) allows one to carry over solutions of the nonrotating equations to the equations in a rotating reference frame. This is important from the point of view of exact solutions that can be obtained from Lie reduction, as the operators from the algebra $\mathfrak{g}_0$ are considerable simpler than those from the algebra $\mathfrak{g}_f$. In the paper, we derived an exact solution of the primitive equations that arises from a completely integrable case of Lie reduction. The case considered is certainly the most important example of reduction with respect to two-dimensional subalgebras. We should also like to stress that upon constructing the optimal lists of inequivalent subalgebras, a considerable fraction of these lists will not be suitable for Lie reduction. In particular, all algebras including $\langle \mathcal{Z}(\alpha)\rangle$, $\langle \mathcal{S}\rangle$ or some combination of these two basis elements as subalgebra will not allow one to find a reduction ansatz. Moreover, a number of cases arising from three-dimensional subalgebras will also not be needed for reduction. More precisely, reductions using algebras including $\langle \mathcal{X}(\boldsymbol{\gamma}) + a^1\mathcal{S}, \mathcal{X}(\boldsymbol{\sigma}) + a^2\mathcal{S}\rangle$, $a^1, a^2 \in \mathbb{R}$, $\gamma^i_{tt}\sigma^i - \sigma^i_{tt}\gamma^i = 0$ for $i = 1, 2$ as a subalgebra are not required the system of differential equations resulting from reduction using this subalgebra can be completely integrated by quadratures.

In view of the remarks of the previous paragraph, despite we have not systematically followed the steps of group-invariant reduction, the results obtained are in a certain sense a substantial part of the exact solutions of the primitive equations that can be found by Lie reduction. A more detailed exposure of the group analysis of the system of primitive equations will be presented elsewhere.

Chapter 8

Invariant parameterization and turbulence modeling on the beta-plane

8.1 Introduction

As atmospheric and oceanic numerical models get increasingly complex, it becomes more and more challenging to propose valuable conceptual paradigms for those processes that the model is still not able to capture owing to its limited spatial and temporal resolution. This problem is common to all numerical models irrespectively of their eventual degree of sophistication [143, 144]. In the beginning of numerical modeling in geophysical fluid dynamics, it was often the lack of computer power that dictated which processes had to be parameterized, even with a concise understanding of these processes. As computers became more capable, the problem of parameterization shifted to processes occurring on rather fine scales where it can be difficult to retrieve accurate experimental data. Accordingly, for various processes that should be taken into account in order to improve the forecast range of a numerical model, there is still no satisfactory general understanding. This naturally makes it difficult to set up valuable parameterization schemes, which for this reason is usually an elaborate task.

On the other hand, processes that occur in geophysical fluid dynamics and that can be described using differential equations also might have certain structural or geometrical properties. Such properties can be conservation of mass or energy or other fundamental conservation laws. Real-world processes are generally also invariant under specific transformation groups, as e.g. the Galilean group. This is why one can ask the question whether it is reasonable to construct parameterization schemes for processes possessing certain structural features in a manner such that these features are preserved in the closed model. In this way, even if a model is not able to explicitly resolve processes, loosely speaking, it takes into account some of their significant properties. This study was initiated in [98] for the problem of finding invariant turbulence closure schemes for the filtered Navier–Stokes equations. In the present paper we aim to give a further instance for invariant parameterization schemes by constructing closure ansatzes that retain certain Lie point symmetries of the barotropic vorticity equation on the beta-plane.

This possible stream of constructing geometrically motivated parameterization schemes in some sense parallels the present general trend in numerical modeling to design specially adapted discretizations of differential equations that capture a range of their qualitative or global features, such as conservation laws, a Hamiltonian structure or symmetry properties. Especially the possibility of constructing discretization schemes that have the same symmetries as the original differential equations they are a model of, as proposed and discussed e.g. in [41, 70, 71, 133, 146], is of immediate relevance to the present work. This is because, strictly speaking, a discretization of a system of differential equation is in practice not enough to set up a valuable numerical model. There always has to be a model for the unresolved parts of the dynamics. (Neglecting them is in general not a good idea as for nonlinear differential equations these parts will, sooner or later, spoil the numerical integration.) Then, if one aims to construct an invariant discrete counterpart of some relevant physical model, care should also be taken about the invariance characteristics of the processes that are not explicitly resolved. This is where the method of finding invariant parameterization schemes comes into play. The combination of invariant discretization schemes for the resolved part of the model with invariant parameterization schemes for the unresolved parts yields a completely invariant numerical description of a given system of differential equations. Such a fully invariant model might be closer to a true geometric numerical integration scheme than solely a symmetry preserving discretization without any closure for the subgrid-scale terms or with some non-invariant closure.

Perhaps the most relevant usage of the barotropic vorticity equation is related to two-dimensional turbulence. Turbulence on the beta-plane (or, more general, on the rotating sphere) is peculiar in that it allows for the combination of turbulent and wave-like effects. It is believed to explain the emergence of strong jets and band-like structure on giant planets in our solar system and is therefore the subject of intensive investigation, see e.g. [57, 89, 134, 137, 147] and references therein. In the present paper we focus on freely decaying turbulence on the beta-plane by using invariant hyperdiffusion terms to initiate the energy–enstrophy cascades. These cascades are likely responsible for the emergence of coherent, stable structures (vortices) that are ubiquitous in large-scale geophysical fluid dynamics.

The outline of the paper is as follows. In the subsequent Section 8.2, we discuss and slightly extend the concept of invariant parameterization schemes as introduced in [98] and [123]. Special attention will be paid to methods related to invariant parameterization schemes and inverse group classification. In Section 8.3 we present the maximal Lie invariance algebra $\mathfrak{g}_1$ and the maximal Lie invariance pseudogroup of the barotropic vorticity equation on the beta-plane. The computation of the algebra $\mathfrak{g}_1$ is briefly described in Appendix 8.10. A concise description of the general method for computing differential invariants of Lie (pseudo)groups using the method of moving frames is given in Section 8.4. In Section 8.5 the algebra of differential invariants is determined for the maximal Lie invariance pseudogroup of the vorticity equation. The related computation can be found in Appendix 8.11. Two examples for invariant parameterization schemes constructed out of existing schemes using the invariantization process are presented in Section 8.6. Section 8.7 is devoted to the application of differential invariants in turbulence on the beta-plane. In particular, invariant hyperdiffusion schemes are introduced. The vorticity equation on the beta-plane is integrated numerically using both invariant and non-invariant hyperdiffusion and the corresponding enstrophy spectra are obtained. In Section 8.8 we discuss the possibility of deriving invariant parameterization schemes that also respect conservation laws. As an example, an invariant diffusion term is constructed that preserves the entire maximal Lie invariance pseudogroup of the vorticity equation and also preserves conservation of energy, circulation and momentum. The results are summarized and further discussed in the final Section 8.9, in which we also indicate possible future research directions in the field of invariant parameterization.

8.2 Invariant parameterization schemes

The problem of finding parameterization or closure schemes for subgrid-scale terms in averaged differential equations that admit Lie symmetries of the original (unaveraged) differential equation was first raised in [98], see also [99, 131]. Recently, we put this idea into the framework of group classification [123], by showing that any problem of constructing invariant parameterization schemes amounts in solving a (possibly complicated) group classification problem.

As for the classical group classification, there are two principal ways to construct parameterization schemes, the direct and the inverse one [123]. In the direct approach, one replaces the terms to be parameterized with arbitrary functions depending on the mean variables and derivatives thereof. This is in the line with the general definition of all physical parameterization schemes, which are concerned to express the unknown subgrid-scale terms using the information included in the grid-scale (mean) quantities. The form of dependency of the arbitrary functions on the mean variables is guided by physical intuition. It determines the properties of all the families of invariant parameterization schemes that can be derived (e.g. the highest order of derivatives that can arise). Once the general form of the arbitrary function is chosen, one is left with a possibly rather general class of differential equations, which is amenable with tools from usual group classification, see e.g. [15, 43, 127]. This in particular will lead to a list of families of mutually inequivalent parameterization schemes that admit different Lie invariance algebras. One can then select those families that preserve the most essential symmetry features of the process to be parameterized. The final step is to suitably narrow the selected families by including other desirable physical properties into the invariant parameterization scheme.

In the present paper, however, we will be solely concerned with the inverse approach, which is why we will discuss it in a more extended manner. The inverse approach rests on the fact that any system of differential equations can be rewritten in terms of differential invariants of its maximal Lie invariance group, provided that the prolongation of the group to the corresponding jet space acts semi-regularly [101]. This property can be used in the course of the parameterization problem in the following way: Suppose that we are given a Lie group G regarded as important to be preserved for valuable parameterization schemes as a Lie symmetry group. Computing a basis of differential invariants of G along with a complete set of its independent operators of invariant differentiation, see e.g. [45, 46, 105, 112], serves to exhaustively describe the entire algebra of differential invariants of G. As any combination of these differential invariants will necessarily be invariant with respect to G, assembling them together to a parameterization will immediately lead to a closure scheme admitting G as a Lie symmetry group.

The key question hence lies in the correct selection of a suitable symmetry group. The initial point for the selection is given by symmetry properties of the model to be parameterized. In the course of the parameterization one can intend to preserve the whole Lie symmetry group of the initial model or its proper subgroups. The choice for an invariance group for parameterization obviously should not solely be justified using mathematical arguments. Sometimes, it can be motivated from obvious physical reasons. If the process to be parameterized can be described within the framework of classical mechanics then any reasonable parameterization for that process should be invariant under the Galilean group. Moreover, for turbulence closure schemes, scale invariance might be of particular importance. For processes that can be described in the framework of a variational principle and respect certain conservation laws, it might be reasonable that the parameterization scheme to be developed respects the associated Noether symmetries.

There are several processes in fluid mechanics that are intimately linked to the presence of certain boundary conditions (e.g., turbulence near walls, boundary layer convection, etc.). For such processes the inclusion of the particular boundaries is an integral part in the formulation of a parameterization scheme. At first glance, to find invariant parameterization schemes for such processes it is inevitable to single out those subgroups of the maximal Lie invariance group G of the system $\mathcal{L}$ of differential equations describing the process of interest that are compatible with a particular boundary value problem. The main complication with this approach is that most of boundary value problems admit no symmetries, see e.g. [25]. At the same time, it is more natural to assume that symmetries of $\mathcal{L}$ act as equivalence transformations on a joint class of physically relevant boundary value problems for $\mathcal{L}$, i.e., these transformations send a particular boundary value problem to another problem from the same class [22]. Even the basic physical symmetries including shifts in space and time, rotations, scalings, Galilean boosts or Lorentz transformations, which are related to fundamental properties of the space and the time (homogeneity, isotropy, similarity, Galilean or special relativity principle, respectively), usually act on boundary value problems in much the same way as equivalence transformations. This is why it is the generation of a group of well-defined equivalence transformations on a properly chosen class of boundary value problems that can serve as a criterion for selecting a subgroup of G to be taken into account in the course of invariant parameterization of $\mathcal{L}$.

Employing techniques of inverse group classification does not automatically lead to ready-to-use parameterizations, but it gives a frame in which parameterizations can be defined without the violation of basic invariance properties. Examples of the violation have been reported in the recent literature. See, e.g., [98] for a discussion about the Smagorinsky model in the filtered Navier–Stokes equations violating scale invariance and [123] for a note on the Kuo convection schemes that describes a Galilean invariant process in a non-invariant fashion. The construction of parameterization schemes that fail to preserve essential symmetries can be easily avoided by applying the above methods of inverse group classification. This may help to restrict the large number of possible closure schemes using geometrical reasoning and thereby may assist in finding a proper description of unresolved subgrid-scale processes.

There is yet a second possibility to construct invariant parameterization schemes using the inverse group classification approach, which has not been reported in [123]. It rests on the construction of *moving frames* for the Lie group G with respect to which parameterizations under study should be invariant. It is a general feature of a moving frame that it allows constructing of invariant counterparts of differential functions. This property enables the construction of an invariant parameterization scheme out of a non-invariant one. It is simply necessary to apply the moving frame corresponding to the selected Lie group G to the specific closed differential equation. More precisely, consider a system $\mathcal{L}$ of differential equations

$$L^l(x, u_{(n)}) = 0, \quad l = 1, \dots, m.$$

The dependent variables u can be represented according to $u = \bar{u} + u'$, where $\bar{u}$ is the averaged or filtered part of the dynamics (i.e. the resolved or grid-scale part) and u' denotes the departure of u from the mean or filtered part $\bar{u}$ (i.e. the subgrid-scale part). Numerical models in geophysical fluid dynamics are formulated as equations for the resolved part, which are obtained from $L^l = 0$ by averaging or filtering, leading to

$$\tilde{L}^l(x, \bar{u}_{(n)}, w) = 0, \quad l = 1, \dots, m,$$

where $\tilde{L}^l$ are smooth differential functions of their arguments. The particular form of $\tilde{L}^l$ depends on the actual averaging rule chosen and the form of the initial system $\mathcal{L}$. The unknown subgrid-scale terms that arise in the course of averaging (e.g. by using the Reynolds averaging rule for products, $\overline{ab} = \bar{a}\bar{b} + \overline{a'b'}$) are collected in the tuple w. These terms have to be parameterized in order to close the above system of averaged differential equations. A local parameterization scheme establishes a particular functional relation

$$w = \theta(x, \bar{u}_{(r)})$$

between the subgrid-scale and grid-scale quantities. Let there be given a moving frame $\rho^{(j)}$ of order $j = \max(r, n)$ for the selected Lie group G, see Section 8.4. Any particular parameterization scheme can then be invariantized via replacing $\tilde{L}^l$ and θ by their invariantized counterparts,

$$\iota(\tilde{L}^l(x, \bar{u}_{(n)}, w)) = \tilde{L}^l(\rho^{(j)} \cdot x, \rho^{(j)} \cdot \bar{u}_{(n)}, w) \quad \text{and} \quad \iota(\theta(x, \bar{u}_{(r)})) = \theta(\rho^{(j)} \cdot x, \rho^{(j)} \cdot \bar{u}_{(r)}).$$

Example 8.1. It is instructive to illustrate this idea with a simple example. Let us consider the famous Korteweg–de Vries (KdV) equation,

$$u_t + uu_x + u_{xxx} = 0.$$

Its maximal Lie invariance group $G_{\rm KdV}$ is four-dimensional and the most general transformation leaving the KdV equation invariant is

$$(T, X, U) = (e^{3\varepsilon_4}(t + \varepsilon_1), e^{\varepsilon_4}(x + \varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_3 t), e^{-2\varepsilon_4}(u + \varepsilon_3)), \tag{8.2.1}$$

where ε_1, $\dots \varepsilon_4$ are arbitrary constants. Let us now apply the classical Reynolds averaging to the KdV equation. This yields

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{u}_{xxx} = -\frac{1}{2}(\overline{u'^2})_x,$$

where the right-hand side is the term we seek closure for. A simple closure ansatz is the down-gradient parameterization, i.e. we close the above equation by setting $\overline{u'^2}/2 = -\kappa\bar{u}_x$, where for the sake of simplicity we use $\kappa = \text{const}$. This yields the closed KdV equation

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{u}_{xxx} = \kappa\bar{u}_{xx}. \tag{8.2.2}$$

However, it is easily verified that this equation is not invariant under the transformation (8.2.1). Namely, the scale invariance is lost, i.e. the closed KdV equation is invariant only under the three-parameter group of transformations associated with the group parameters ε_1, ε_2 and ε_3. To restore scale invariance, we can invariantize the closed KdV equation (8.2.2) using the moving frame associated with the group $G_{\rm KdV}$.

Moving frames for the group $G_{\rm KdV}$ were constructed in [42, 33]. It is convenient to invariantize Eq. (8.2.2) using the moving frame with

$$\varepsilon_1 = -t, \quad \varepsilon_2 = -x, \quad \varepsilon_3 = -\bar{u}, \quad \varepsilon_4 = \frac{1}{3}\ln\bar{u}_x.$$

This is done by firstly applying the transformations (8.2.1) to (8.2.2) which yields

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{u}_{xxx} = \kappa e^{\varepsilon_4}\bar{u}_{xx},$$

showing explicitly that this equation fails to be scale invariant. The invariantization is completed upon substituting the moving frame for ε_4 giving

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{u}_{xxx} = \kappa\sqrt[3]{\bar{u}_x}\bar{u}_{xx}.$$

It is readily checked that this closed equation is invariant under the same symmetry group $G_{\rm KdV}$ as is the original KdV equation. The price for restoring scale invariance of the closed KdV equation invoking the simple down-gradient parameterization is that the closure scheme becomes nonlinear. We will observe the same effect when invariantizing linear hyperdiffusion models for the vorticity equation on the beta-plane, which will be shown in detail below.

8.3 Lie symmetries of the vorticity equation

The barotropic vorticity equation on the beta-plane is a simple but still genuine meteorological model. It has the form

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x = 0, \quad \text{or} \quad \zeta^{\rm a}_t + J(\psi,\zeta^{\rm a}) = 0. \tag{8.3.1}$$

Here $J(a,b) := a_xb_y - a_yb_x$, $\psi = \psi(t,x,y)$ is the stream function, $\zeta = \psi_{xx}+\psi_{yy}$ is the vorticity and $\zeta^{\rm a} = \zeta + \mathsf{f} = \zeta + \mathsf{f}_0 + \beta y$ is the absolute vorticity under the β-plane approximation $\mathsf{f} = \mathsf{f}_0 + \beta y$ of the Coriolis parameter f, β is a nonzero constant parameter (the differential rotation). The constant f_0 is dynamically inessential and can be neglected.

The maximal Lie invariance algebra $\mathfrak{g}_1$ of Eq. (8.3.1) is spanned by the vector fields

$$\mathcal{D} = t\partial_t - x\partial_x - y\partial_y - 3\psi\partial_\psi, \quad \partial_t, \quad \partial_y, \quad \mathcal{X}(\tilde f) = \tilde f(t)\partial_x - \tilde f_t(t)y\partial_\psi \quad \mathcal{Z}(\tilde g) = \tilde g(t)\partial_\psi,$$

where the parameters $\tilde f$ and $\tilde g$ run through the set of smooth functions of t [18, 60]. More details on how the above vector fields are obtained can be found in Appendix 8.10. The vorticity equation (8.3.1) also admits two independent discrete symmetries, which alternate signs of the pairs (t,x) and (y,ψ), see [21] for more details. Such discrete symmetries will not be taken into account in the course of construction of differential invariants. Any nonzero value of β can be gauged to one by a scaling transformation.

The one-parameter Lie (pseudo)groups generated by the above vector fields read

$$\begin{aligned}
\Gamma_{\varepsilon_1}&: \quad (t,x,y,\psi) \mapsto (e^{\varepsilon_1}t, e^{-\varepsilon_1}x, e^{-\varepsilon_1}y, e^{-3\varepsilon_1}\psi)\\
\Gamma_{\varepsilon_2}&: \quad (t,x,y,\psi) \mapsto (t+\varepsilon_2, x, y, \psi)\\
\Gamma_{\varepsilon_3}&: \quad (t,x,y,\psi) \mapsto (t, x, y+\varepsilon_3, \psi)\\
\Gamma_{f}&: \quad (t,x,y,\psi) \mapsto (t, x+f(t), y, \psi - f_t(t)y)\\
\Gamma_{g}&: \quad (t,x,y,\psi) \mapsto (t, x, y, \psi + g(t)),
\end{aligned}$$

where $\varepsilon_i \in \mathbb{R}$ and $f(t) := \varepsilon_4\tilde f(t)$ and $g(t) := \varepsilon_5\tilde g(t)$. Accordingly, the admitted Lie symmetries of the barotropic vorticity equation on the beta-plane are scalings, time translations, translations in y-direction, generalized Galilean boosts in the x-direction and gaugings of the stream function with smooth time-dependent summands.

We will compose transformations from these one-parameter Lie (pseudo)groups in the following way $\Gamma = \Gamma_{\varepsilon_1}\circ\Gamma_{\varepsilon_2}\circ\Gamma_{\varepsilon_3}\circ\Gamma_f\circ\Gamma_g$ to a transformation Γ from the maximal Lie symmetry pseudogroup G_1 of the vorticity equation (8.3.1). Any transformation of G_1 then has the form

$$(T,X,Y,\Psi) = \big(e^{\varepsilon_1}(t+\varepsilon_2),\, e^{-\varepsilon_1}(x+f(t)),\, e^{-\varepsilon_1}(y+\varepsilon_3),\, e^{-3\varepsilon_1}(\psi + g(t) - f_t(t)y)\big). \tag{8.3.2}$$

In what follows, we set $h(t,y) = g(t) - f_t(t)y$ for convenience and use the substitution $h_y = -f_t$, whenever h_y occurs.

Note that the maximal Lie invariance algebra $\mathfrak{g}_0$ of the usual vorticity equation, which is also called the barotropic vorticity equation on the f-plane and corresponds to the value $\beta = 0$, is much wider than the algebra $\mathfrak{g}_1$ and contains $\mathfrak{g}_1$ as a proper subalgebra [6, 7]. The algebra $\mathfrak{g}_0$ is spanned by the vector fields from $\mathfrak{g}_1$ jointly with the vector fields

$$t\partial_t - \psi\partial_\psi, \quad -y\partial_x + x\partial_y, \quad -ty\partial_x + tx\partial_y + \tfrac{1}{2}(x^2+y^2)\partial_\psi, \quad \tilde h(t)\partial_y + \tilde h_t(t)x\partial_\psi,$$

where the parameter $\tilde h$ runs through the set of smooth functions of t. This means that in addition to the transformations from G_1 the maximal Lie symmetry pseudogroup G_0 of the usual vorticity equation also contains one more family of scalings, usual rotations in the (x,y)-plane, rotations depending on t with constant angle velocities and generalized Galilean boosts in the y-direction.

Remark 8.2. In order to set up a numerical model, a decision has to be taken about which boundary conditions should be implemented. It is very rare that the numbers of symmetries admitted by a differential equation is not reduced for an associated boundary value problem. The most immediate boundary conditions in the atmospheric sciences are periodic ones. However, a periodic domain implies a fixed domain size and therefore breaks the scale invariance of Eq. (8.3.1). On the other hand, scale invariance is an equivalence transformation of the class of *all* periodic boundary value problems of the vorticity equation, see also the discussion in [22]. A more serious problem is that the periodicity in y-direction is not natural for the beta-plane. On the other hand using a channel model (rigid walls in the North and in the South of the domain) breaks also the translational invariance in y-direction thereby reducing the admitted Lie symmetry group even stronger than in the presence of doubly periodic boundary conditions (though, in contrast to usual hyperdiffusion, it would not be necessary to define an additional boundary condition for the invariant hyperdiffusion as by definition $\psi_x = 0$ at the walls of the channel and the diffusion term therefore vanishes there). This is why we will use doubly periodic boundary conditions although $\beta \neq 0$ here. Despite this slight inconsistency, doubly periodic boundary conditions are used quite extensively in studying turbulence properties on the beta-plane [89, 134, 147].

The above form (8.3.1) of the vorticity equation is not particularly useful for a numerical evaluation. The reason is, of course, that any numerical model can be run only at a finite resolution, which requires a suitably chosen averaging or filtering of Eq. (8.3.1). As from the point of view of invariant parameterization schemes the precise type of averaging is only of secondary importance, we will employ a classical Reynolds averaging to Eq. (8.3.1) in the paper. This leads to the averaged vorticity equation

$$\bar{\zeta}_t + \bar{\psi}_x\bar{\zeta}_y - \bar{\psi}_y\bar{\zeta}_x + \beta\bar{\psi}_x = \overline{(\zeta'\psi'_y)_x} - \overline{(\zeta'\psi'_x)_y}, \tag{8.3.3}$$

where the right-hand side of this equation denotes the eddy-vorticity flux, which we aim to parameterize subsequently. For the sake of notational simplicity, we will omit bars over the mean quantities from now on.

Slightly more generally, the vorticity equation (8.3.1) can be augmented with external forcings F and dissipative terms D yielding a general expression of the form

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x = F + D. \tag{8.3.4}$$

A further question we aim to address is whether symmetries might be helpful in deriving invariant expressions for F and D. As by definition F denotes *external* forcing terms, it is not immediately clear why symmetries of the vorticity equation should place restrictions on the form of F. However, as we shall show, symmetries are valuable in finding invariant diffusion terms D that can be used in the course of turbulence modeling. For the sake of simplicity we therefore will use Eq. (8.3.4) for the case of $F = 0$ and $D \neq 0$, i.e. we assume that no external forcing acts on the system to which a damping is attached. Physically, the presence of F and D can be interpreted as symmetry breaking in the vorticity equation (8.3.1). Which symmetries are to be broken and which are to be preserved can be controlled upon expressing the term D via differential invariants derived in Section 8.5. This is a comprehensive problem and not all of the cases arising might be interesting from the physical point of view. We therefore restrict ourselves on the case where D or the eddy vorticity flux in Eq. (8.3.3) can be represented in such a manner that the resulting equation admits all the transformations from the maximal Lie invariance pseudogroup (8.3.2). This is the approach proposed in [98] and it appears to be suitable for the beta-plane equation.

8.4 Algorithm for the construction of differential invariants

Given a transformation pseudogroup G in the space of p independent variables $x = (x^1, \dots, x^p)$ and q dependent variables $u = (u^1, \dots, u^q)$, in order to exhaustively describe its differential invariants one should find either a functional basis of differential invariants of any fixed order or a complete set of independent operators of invariant differentiation and a minimal set of differential invariants generating all differential invariants via invariant differentiation and functional combination. Within the framework of the method of moving frames the solution of this problem is split into two parts [33]. It is convenient to compute normalized differential invariants and operators of invariant differentiation using the explicit expressions for transformations from G. The corresponding computation consists of two procedures, *normalization* and *invariantization*. At the same time, the derivation of syzygies (i.e., relations involving operators of invariant differentiation) between normalized differential invariants is mostly based on the determining equations of G, and an important tool for this is given by *recurrence formulas*. In this section we briefly describe related basic notions and results, paying the main attention to the computational realization of algorithms in fixed local coordinates. See [33, 45, 46, 105, 107, 108] for detail and rigorous presentations.

In what follows the index j runs from 1 to p, the index a runs from 1 to q. We use two kinds of integer tuples for the indexing of objects. One of these kinds is given by the usual multi-indices of the form $\alpha = (\alpha_1, \dots, \alpha_p)$, where $\alpha_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $|\alpha| = \alpha_1 + \dots + \alpha_p$. By δ_j we denote the p-index whose jth entry equals 1 and whose other entries are zero. Thus, both the derivative $\partial^{|\alpha|}u^a/(\partial x^1)^{\alpha_1} \cdots (\partial x^p)^{\alpha_p}$ and the associated variable of the jet space $\mathrm{J}^\infty(x|u)$ are denoted by u^a_α, $\mathrm{D}^\alpha = \mathrm{D}_1^{\alpha_1} \cdots \mathrm{D}_p^{\alpha_p}$, etc. Here $\mathrm{D}_j = \partial_{x^j} + \sum_{\alpha,a} u^a_{\alpha+\delta_j}\partial_{u^a_\alpha}$ is the operator of total differentiation with respect to the variable x^j. The other kind of index tuples is presented by $J = (j_1, \dots, j_\kappa)$, where $1 \leqslant j_k \leqslant p$, $k = 1, \dots, \kappa$, $\kappa \in \mathbb{N}_0$. Such index tuples are used for the indexing of compositions of operators of invariant differentiation, which do not commute. Namely, we write $\mathrm{D}^{\mathrm{i}}_J$ for $\mathrm{D}^{\mathrm{i}}_{j_1} \cdots \mathrm{D}^{\mathrm{i}}_{j_\kappa}$. The symbol $\mathrm{d_h}$ denotes the horizontal differential, $\mathrm{d_h}F = \sum_{j=1}^p (\mathrm{D}_jF)\mathrm{d}x^j$ for a differential function $F = F[u]$, i.e. a function of x^j and u^a_α.

The normalization procedure for the pseudogroup G consists of three steps:

1. Choose a parameterization (local coordinates) of G and find explicit formulas for the prolonged action of G in terms of the jet variables.

2. Choose a subset of the transformed jet variables and equate the expressions for them to chosen constants.

3. Solve the obtained system of normalization equations as a system of algebraic equations with respect to the parameters of the pseudogroup G including the derivatives of the functional parameters.

The second step is nothing but a choice of an appropriate (coordinate) cross-section of the G-orbits. This should be implemented in a way ensuring that the system from the third step will be well defined and solvable.

The normalization procedure results in the construction of a moving frame ρ for the pseudogroup G, which is, roughly speaking, an equivariant map from the jet space to G. Once the moving frame is constructed it can be used to map any object $\chi(x, u_{(n)})$ defined on an open subset of the jet space (a differential function, a differential operator or a differential form) to its invariant counterpart, $\iota(\chi(x, u_{(n)})) = \chi(\rho^{(n)}(x, u_{(n)}) \cdot (x, u_{(n)}))$. To carry out this in practice, one should replace all occurrences of the pseudogroup parameters in the transformed version of the object by their expressions obtained with the normalization procedure.

Thus, the invariantization of the coordinate functions x^j and u^a_α of the jet space yields the so-called *normalized differential invariants* $H^j = \iota(x^j)$ and $I^a_\alpha = \iota(u^a_\alpha)$. In fact, the invariantized coordinate functions whose transformed counterparts were used to set up the normalization equations are equal to the respective constants chosen in the course of normalization and hence these objects are called *phantom differential invariants*. Non-phantom normalized differential invariants are functionally independent and any differential invariant can be represented as a function of normalized differential invariants. Invariantization of the *operators of total differentiation*, D_j, gives the operators of invariant differentiation, $\mathrm{D}^{\mathrm{i}}_j$, which upon acting on differential invariants produce other differential invariants. Note that the domain of the jet space, where invariantized objects are well defined, depends on what cross-section is chosen.

In order to determine the algebra of differential invariants the normalized differential invariants and the operators of invariant differentiation play a key role. It has been proved [112] that for any Lie (pseudo)group the algebra of differential invariants can be completely described upon finding a *finite* generating set of differential invariants. As stated above, all the other differential invariants are then a suitable combination of the basis differential invariants or their invariant derivatives. The hardest part in describing the algebra of differential invariants is usually to find a *minimal* generating set of these invariants. Proving the minimality of a given basis usually involves the computation of the *syzygies* among the differential invariants, meaning functional relations among the differentiated differential invariants $\mathrm{D}^{\mathrm{i}}_J I^a_\alpha$, $S(\dots,\mathrm{D}^{\mathrm{i}}_J I^a_\alpha,\dots)=0$.

In general, the normalized differential invariants are derived from invariantization of the derivatives of the dependent variables, whereas the differentiated differential invariants are obtained by acting on normalized differential invariants of lower order with the operators of invariant differentiation. The central point is that the operations of invariant differentiation and invariantization of a differential function in general do not commute. Roughly speaking, the failure of commutation of these two operations is quantified by the so-called recurrence relations

$$\mathrm{d_h} H^j = \omega^j + \hat\xi^j, \quad \mathrm{d_h} I^a_\alpha = \sum_{j=1}^p I^a_{\alpha+\delta_j}\omega^j + \hat\varphi^{a,\alpha}, \tag{8.4.1}$$

where $\omega^j = \iota(\mathrm{d}x^j)$ [33, 107]. The forms $\hat\xi^j = \iota(\xi^j)$ and $\hat\varphi^\alpha_a = \iota(\varphi^\alpha_a)$ are the invariantizations of the coefficients of the general prolonged infinitesimal generator

$$Q_\infty = \sum_{j=1}^p \xi^j\partial_{x^j} + \sum_{\alpha\geqslant 0}\sum_{a=1}^q \varphi^{a,\alpha}\partial_{u^a_\alpha}, \quad \varphi^{a,\alpha} = \mathrm{D}^\alpha\left(\varphi^a - \sum_{j=1}^p \xi^j u^a_{\delta_j}\right) + \sum_{j=1}^p \xi^j u^a_{\alpha+\delta_j},$$

of G. More rigorously, here ξ^j and u^a_α are interpreted as coordinate functions on the space of prolonged infinitesimal generators of G, i.e., first-order differential forms in the jet space. Hence their invariantizations should also be forms, which are called *invariantized Maurer–Cartan forms*.

The left-hand sides of the relations (8.4.1) are zero for phantom differential invariants. If the cross-section is chosen in a proper way, the recurrence relations for the phantom invariants can be solved for the independent invariantized Maurer–Cartan forms, which in turn can be plugged into the relations for the non-phantom differential invariants. Collecting coefficients of ω^j then yields a closed description of the relation between normalized and differentiated differential invariants, which in turn might enable the determination of a basis of differential invariants. For this latter task, specialized methods from computational algebra can be applied [108], which is, however, not necessary in the present case due to the relatively simple structure of the maximal Lie invariance pseudogroup G_1 of Eq. (8.3.1).

8.5 Differential invariants for the beta-plane vorticity equation

In order to derive the moving frame for the maximal Lie invariance pseudogroup G_1 of the barotropic vorticity equation on the beta-plane, it is necessary to prolong the group actions to the derivatives of ψ. For this aim, we have to derive expressions for the implicit differentiation operators, D_T, D_X and D_Y. They can be determined as the dual of the lifted horizontal coframe for G_1, which reads

$$\begin{aligned}
\mathrm{d_h}T &= (T_t+\psi_tT_\psi)\mathrm{d}t+(T_x+\psi_xT_\psi)\mathrm{d}x+(T_y+\psi_yT_\psi)\mathrm{d}y = e^{\varepsilon_1}\mathrm{d}t\\
\mathrm{d_h}X &= (X_t+\psi_tX_\psi)\mathrm{d}t+(X_x+\psi_xX_\psi)\mathrm{d}x+(X_y+\psi_yX_\psi)\mathrm{d}y = e^{-\varepsilon_1}f_t\mathrm{d}t-e^{-\varepsilon_1}\mathrm{d}x\\
\mathrm{d_h}Y &= (Y_t+\psi_tY_\psi)\mathrm{d}t+(Y_x+\psi_xY_\psi)\mathrm{d}x+(Y_y+\psi_yY_\psi)\mathrm{d}y = e^{-\varepsilon_1}\mathrm{d}y.
\end{aligned}$$

Therefore, the required implicit differentiation operators are

$$\mathrm{D}_T=e^{-\varepsilon_1}(\mathrm{D}_t-f_t\mathrm{D}_x),\quad \mathrm{D}_X=e^{\varepsilon_1}\mathrm{D}_x,\quad \mathrm{D}_Y=e^{\varepsilon_1}\mathrm{D}_y, \tag{8.5.1}$$

where D_t, D_x and D_y denote the usual operators of total differentiation with respect to t, x and y, respectively, $\mathrm{D}_t=\partial_t+\sum_\alpha\psi_{\alpha+\delta_1}\partial_{\psi_\alpha}$, $\mathrm{D}_x=\partial_x+\sum_\alpha\psi_{\alpha+\delta_2}\partial_{\psi_\alpha}$ and $\mathrm{D}_y=\partial_y+\sum_\alpha\psi_{\alpha+\delta_3}\partial_{\psi_\alpha}$. Here and in what follows $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index running through $\mathbb{N}_0^3$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$, $\delta_1=(1,0,0)$, $\delta_2=(0,1,0)$, $\delta_3=(0,0,1)$ and the variable $\psi_\alpha=\psi_{\alpha_1\alpha_2\alpha_3}$ of the jet space corresponds to the derivative $\partial^{|\alpha|}\psi/\partial t^{\alpha_1}\partial x^{\alpha_2}\partial y^{\alpha_3}$. We also use the notation $f_{(k)}=\mathrm{d}^kf/\mathrm{d}t^k$ and $h_{(k)}=\partial^kh/\partial t^k$, $k\in\mathbb{N}_0$. The transformed derivatives $\Psi_\alpha=\partial^{|\alpha|}\Psi/\partial T^{\alpha_1}\partial X^{\alpha_2}\partial Y^{\alpha_3}$, $|\alpha|>0$, are then

$$\begin{aligned}
\Psi_\alpha &= \mathrm{D}_T^{\alpha_1}\mathrm{D}_X^{\alpha_2}\mathrm{D}_Y^{\alpha_3}\Psi = e^{(\alpha_2+\alpha_3-\alpha_1-3)\varepsilon_1}(\mathrm{D}_t-f_t\mathrm{D}_x)^{\alpha_1}\mathrm{D}_x^{\alpha_2}\mathrm{D}_y^{\alpha_3}(\psi+h)\\
&= e^{(\alpha_2+\alpha_3-\alpha_1-3)\varepsilon_1}\left((\mathrm{D}_t-f_t\mathrm{D}_x)^{\alpha_1}\psi_{0\alpha_2\alpha_3}+\left\{\begin{array}{ll}-f_{(\alpha_1+1)}, & \alpha_2=0,\ \alpha_3=1\\ h_{(\alpha_1)}, & \alpha_2=\alpha_3=0\end{array}\right\}\right).
\end{aligned}$$

We carry out the normalization procedure in the domain of the jet space which is defined by the condition $\psi_x\neq0$. We choose the conditions for normalization, which allow us to express all the pseudogroup parameters (including the derivatives of functional pseudogroup parameters) in terms of variables of the jet space:

$$T=X=Y=0,\quad \Psi_{k00}=\Psi_{k01}=0,\quad k=0,1,\dots,\quad \Psi_{010}=\varepsilon, \tag{8.5.2}$$

where $\varepsilon=\operatorname{sgn}\psi_x$. These conditions yield a complete moving frame for the maximal Lie invariance pseudogroup of the vorticity equation, which explicitly reads

$$\begin{aligned}
&\varepsilon_1=\ln\sqrt{|\psi_x|},\quad \varepsilon_2=-t,\quad \varepsilon_3=-y,\quad f=-x,\\
&f_{(k+1)}=(\mathrm{D}_t-\psi_y\mathrm{D}_x)^k\psi_y,\quad h_{(k)}=-(\mathrm{D}_t-\psi_y\mathrm{D}_x)^k\psi,\quad k=0,1,\dots.
\end{aligned} \tag{8.5.3}$$

The series of equalities for $f_{(k+1)}$ an $h_{(k)}$ is proved by induction with respect to k using the equations

$$f_{(k+1)}=(\mathrm{D}_t-f_t\mathrm{D}_x)^k\psi_y,\quad h_{(k)}=-(\mathrm{D}_t-f_t\mathrm{D}_x)^k\psi.$$

The nontrivial normalized differential invariants are found via invariantizing the derivatives ψ_α for the values of α for which Ψ_α are not involved in the construction of the above moving frame, i.e., for

$$\alpha\in A=\mathbb{N}_0^3\setminus\{(k,0,0),\,(k,0,1),\,(0,1,0),\,k\in\mathbb{N}_0\}.$$

In other words, for each $\alpha \in A$ we should substitute the expressions (8.5.3) for the pseudogroup parameters into the expressions for Ψ_α. (The invariantization of the coordinate functions chosen for the normalization conditions (8.5.2) are equal to the corresponding normalization constants and are the phantom normalized differential invariants for the moving frame (8.5.3).) As a result, we obtain the differential invariants

$$I_\alpha = \iota(\psi_\alpha) = |\psi_x|^{(\alpha_2+\alpha_3-\alpha_1-3)/2}(\mathrm{D}_t - \psi_y \mathrm{D}_x)^{\alpha_1}\psi_{0\alpha_2\alpha_3}, \quad \alpha \in A.$$

The order of I_α as a differential function of ψ equals $|\alpha|$. It is also obvious that any finite number of the invariants I_α are functionally independent. This agrees with the general theory of moving frames [33, 45, 107], which also implies a stronger assertion.

Theorem 8.3. *For each $r \geqslant 2$ the functions $I_\alpha = |\psi_x|^{(\alpha_2+\alpha_3-\alpha_1-3)/2}(\mathrm{D}_t - \psi_y \mathrm{D}_x)^{\alpha_1}\psi_{0\alpha_2\alpha_3}$, where $\alpha \in A$ and $|\alpha| \leqslant r$, form a local functional basis of differential invariants of order not greater than r for the maximal Lie invariance pseudogroup G_1 of the barotropic vorticity equation on the beta-plane.*

The description of differential invariants of G_1 given in Theorem 8.3 is sufficient for applications within the framework of invariant parameterization. At the same time, it is interesting and useful to have more information on the structure of the algebra of differential invariants of the pseudogroup G_1 including the operators of invariant differentiation.

Theorem 8.4. *The algebra of differential invariants of the maximal Lie invariance pseudogroup of the barotropic vorticity equation on the beta-plane* (8.3.1) *is generated, in the domain Ω_1 of the jet space where $\mathrm{D}_x^2(\sqrt{|\psi_x|}\,) \neq 0$, by the single differential invariant $I_{020} = \psi_{xx}/\sqrt{|\psi_x|}$ along with the three operators of invariant differentiation*

$$\mathrm{D}^{\mathrm{i}}_t = \frac{1}{\sqrt{|\psi_x|}}(\mathrm{D}_t - \psi_y \mathrm{D}_x), \quad \mathrm{D}^{\mathrm{i}}_x = \sqrt{|\psi_x|}\mathrm{D}_x, \quad \mathrm{D}^{\mathrm{i}}_y = \sqrt{|\psi_x|}\mathrm{D}_y.$$

All other differential invariants are functions of I_{020} and invariant derivatives thereof. The proof of this theorem is presented in detail in Appendix 8.11.

8.6 Invariantization of parameterization schemes

The Replacement Theorem states that any differential invariant $I(x, u_{(n)})$ of order n can be expressed in terms of the normalized differential invariants via replacing any argument of $I(x, u_{(n)})$ by its respective invariantization, see [46]. In particular, any system of differential equations can be represented using the normalized differential invariants of its associated maximal Lie invariance group. The invariantization of the vorticity equation (8.3.1) in view of the moving frame (8.5.3) reads $(I_{120} + I_{102}) + (I_{021} + I_{003}) + \beta = 0$, or, explicitly

$$\frac{\zeta_t - \psi_y \zeta_x}{\psi_x} + \zeta_y + \beta = 0. \tag{8.6.1}$$

This is the fully invariant representation of the barotropic vorticity equation on the beta-plane.

Differential invariants computed in the previous section can be assembled together to invariant parameterizations of the eddy-vorticity flux in the averaged vorticity equation (8.3.3). Alternatively, we can invariantize any existing parameterization scheme under the moving frame action (8.5.3). The following two examples implement this idea.

Example 8.5. A classical albeit simple parameterization for the eddy-vorticity flux is

$$\mathrm{evf} := \overline{(\zeta'\psi'_y)}_x - \overline{(\zeta'\psi'_x)}_y = \mathrm{D}_x(K\zeta_x) + \mathrm{D}_y(K\zeta_y),$$

where $K = K(x,y)$ might be considered as a spatially dependent function. The most straightforward way to cast this parameterization into the related invariant one is by applying the moving frame (8.5.3) to the terms on the right-hand side. This yields

$$\begin{aligned}\mathrm{evf}^{\mathrm{i}} &= \mathrm{D}^{\mathrm{i}}_x(K(I_{030}+I_{012})) + \mathrm{D}^{\mathrm{i}}_y(K(I_{021}+I_{003})) = K(I_{040}+2I_{022}+I_{004})\\ &= K\sqrt{|\psi_x|}(\zeta_{xx}+\zeta_{yy}),\end{aligned}$$

where $\mathrm{evf}^{\mathrm{i}} = \iota(\mathrm{evf})$ and $K = \mathrm{const}$ now as $\iota(x) = \iota(y) = 0$. The invariant representation of the closed barotropic vorticity equation then reads

$$\frac{\zeta_t - \psi_y\zeta_x}{\psi_x} + \zeta_y + \beta = K\sqrt{|\psi_x|}(\zeta_{xx}+\zeta_{yy}).$$

Example 8.6. The anticipated (potential) vorticity method was originally proposed by Sadourny and Basdevant [136]. The idea of this method is to approximate the diffusion effect in the vorticity equation by a weighted upwind estimate of the vorticity itself, i.e. by employing

$$\zeta^{\mathrm{a}}_t + J(\psi,\zeta^{\mathrm{a}}) = \nu J(\psi, \Delta^n J(\psi,\zeta^{\mathrm{a}})),$$

where ν is a constant, $n \in \mathbb{N}_0$ and ζ^{a} is the absolute vorticity. Here and in what follows $\Delta = \nabla^2$ is the two-dimensional Laplacian. The purpose of adding the specific forcing term on the right-hand side of the vorticity equation is to suppress the high frequency noise in the vorticity field and at the same time to ensure that energy is conserved during the integration while enstrophy is dissipated. The latter properties can be easily verified upon multiplying Eq. (8.3.1) with the stream function ψ and any function of the absolute vorticity ζ^{a}, respectively, and integrating over the domain Ω, see also [148].

There is a problem with this parameterization scheme in that it is not Galilean invariant. Galilean invariance (as well as the proper scale invariance), however, can be easily included by the method of invariantization. For the sake of demonstration, we consider the case of $n = 0$ here, which is the original version of the anticipated vorticity closure. Upon using the moving frame (8.5.3), we obtain

$$\iota(J(\psi, J(\psi,\zeta^{\mathrm{a}}))) = \frac{1}{\sqrt{|\psi_x|}}J(\psi_y,\zeta^{\mathrm{a}}) + \sqrt{|\psi_x|}\zeta^{\mathrm{a}}_{yy}.$$

Attaching this to the invariant representation of the vorticity equation (8.6.1), the vorticity equation with fully invariant closure reads ($\varepsilon = \operatorname{sgn}\psi_x$)

$$\zeta^{\mathrm{a}}_t + J(\psi,\zeta^{\mathrm{a}}) = \nu\sqrt{|\psi_x|}(\varepsilon J(\psi_y,\zeta^{\mathrm{a}}) + \psi_x\zeta^{\mathrm{a}}_{yy}). \qquad (8.6.2)$$

It is obvious that this parameterization is quite different from that proposed in [136]. It cannot be brought in the form of nested Jacobian operators and it does not conserve energy any more (for the derivation of conservative invariant closure schemes, see Section 8.8). On the other hand, the inherent invariance of the closed vorticity equation (8.6.2) under Galilean and scale symmetry is an appealing property for itself and might be relevant e.g. when vorticity dynamics is studied in a moving coordinate frame.

Quite recently, an approximate scale invariant formulation of the anticipated potential vorticity method was proposed in [34] using scale analysis techniques and physical reasoning. The motivation for this study is that modern weather and climate models might be required to operate on grids with variable resolution. Unfortunately, varying resolution in an atmospheric numerical model is not a simple task as most of the parameterization schemes employed are definitely not scale invariant, but rather tuned to yield best results on some fixed grid. This means that painful efforts might be necessary in order to adjust all the parameterization schemes of a numerical model to different spatial-temporal resolutions. Having a general method at hand that allows deriving of scale insensitive closure schemes is therefore of potential practical interest in numerical geophysical fluid dynamics. Albeit simple, the method of invariantization of existing parameterization schemes may give appropriate closure schemes that are both physically meaningful and respect those symmetries that might be essential for a specific process to be represented numerically.

These are only two examples for fully invariant closure schemes. See one more example in the next section. In principle, each term of the form $S(I^1,\dots,I^N)$, where S is a smooth function of its arguments and I^1, ..., I^N are differential invariants of G_1, satisfies the same requirement when added to the right hand side of Eq. (8.6.1). In other words, the general form of closure ansatzes for Eq. (8.6.1), which are invariant with respect to the entire group G_1, is

$$\zeta_t+\psi_x\zeta_y-\psi_y\zeta_x+\beta\psi_x=\psi_x S(I^1,\dots,I^N).$$

8.7 Application of invariant parameterizations to turbulence modeling

In this section, we give an application in which we aim to demonstrate in practice the ideas outlined above and in [123]. This example deals with turbulence properties of the two-dimensional incompressible Euler equations. Strictly speaking, turbulence is a three-dimensional problem as a two-dimensional turbulent flow is not stable with respect to fully three-dimensional perturbations to that flow [137]. Nevertheless, there are countless studies concerning the turbulent properties of two-dimensional flow simply because it is a relevant problem in large-scale geophysical fluid dynamics, which behaves as approximately two-dimensional.

In short, the first theoretical results concerning two-dimensional turbulence were derived in [10, 76], following the pioneering work on three-dimensional turbulence done by Kolmogorov [75]. Extensive numerical studies have been carried out since then attempting to verify distinct aspects of the theory proposed [11, 12, 30, 47, 86]. The two-dimensional case is especially peculiar, as it admits infinitely many conservation laws including the conservation of energy. The energy in the barotropic vorticity is purely kinetic and can be represented in different ways using doubly periodic boundary conditions as

$$\mathcal{E}=\frac{1}{2}\int_\Omega \mathbf{v}^2\mathrm{d}A=\frac{1}{2}\int_\Omega(\nabla\psi)^2\mathrm{d}A=-\frac{1}{2}\int_\Omega\psi\zeta\mathrm{d}A, \qquad (8.7.1)$$

where $\Omega=[0,L_x[\times[0,L_y[$ and $\mathrm{d}A=\mathrm{d}x\,\mathrm{d}y$. The special form of Eq. (8.3.1) leads to the following class of conservation laws

$$\mathcal{C}_g=\int_\Omega g(\zeta^{\mathrm{a}})\mathrm{d}A,$$

for any smooth function g of the absolute vorticity $\zeta^{\mathrm{a}}=\zeta+\mathsf{f}_0+\beta y$. The most relevant realization of the above conservation laws in the present context is the enstrophy, given for the particular value $g=(\zeta^{\mathrm{a}})^2/2$.

First of all, consider the case of no differential rotation ($\beta = 0$), i.e. the Coriolis parameter f is approximated by the constant f_0, which is referred to as the f-plane approximation. It is the simultaneous conservation of energy and enstrophy in this case that leads to the remarkable behavior of two-dimensional turbulence [137, 147]. Starting with a random initial velocity (or stream function field), energy is transported to the large scale, while enstrophy is transported to the smaller scales. This cascade is associated with an organization of the vortices, with vortices of the same sign merging into bigger ones (though the precise mechanisms of the cascade including the role of the vortices are not yet fully understood). In order to initiate these fluxes of energy to the larger scale and enstrophy to the smaller scale and thus the process of organization, it is necessary to place a sink of enstrophy at the very small scales. This sink acts as a remover of enstrophy while ideally conserving energy, as the latter is transported away from the small scales on which the dissipation acts (which in practice is hard to realize in a numerical simulation using a finite number of grid points). It is believed that the form of the energy spectrum in a range above which dissipation is acting (inertial range) can be derived using scaling theory in a similar manner as it was shown by Kolmogorov for the three-dimensional case.

The energy and enstrophy spectra $E(k)$ and $C(k)$ are defined by

$$\bar{\mathcal{E}} = \frac{1}{2L_xL_y}\int_\Omega \mathbf{v}^2 \mathrm{d}A = \frac{1}{2L_xL_y}\int_\Omega (\nabla\psi)^2 \mathrm{d}A = \int E(k)\mathrm{d}k,$$
$$\bar{\mathcal{C}} = \frac{1}{2L_xL_y}\int_\Omega \zeta^2 \mathrm{d}A = \frac{1}{2L_xL_y}\int_\Omega (\Delta\psi)^2 \mathrm{d}A = \int C(k)\mathrm{d}k,$$

where $\bar{\mathcal{E}}$ and $\bar{\mathcal{C}}$ are the average energy and average enstrophy, $k = \sqrt{(k^x)^2 + (k^y)^2}$ is the scalar wave number, k^x and k^y are the wave numbers in x- and y-direction, respectively. The possibility of using a single wave number is due to the assumption of isotropy that is generally made in turbulence theory and which is reasonable in the case of vanishing differential rotation. According to the theory, the form of the energy spectrum in the inertial range should follow

$$E(k) \propto k^{-3}.$$

This is referred to as the *enstrophy cascade* in two-dimensional turbulence. Analogously, the *enstrophy spectrum* in the inertial range should follow

$$C_{\mathrm{ens}}(k) \propto k^{-1} = k^2E(k).$$

The impact of the beta-term in the vorticity equation on the turbulent cascades was first studied in [134]. In this seminal paper, it was remarked that the Rossby wave solutions admitted by the beta-plane equation can act as a source of anisotropization of turbulence at the larger scale. Qualitatively, at some stage the size of the vortices is big enough that they are exposed to the effect of differential rotation, which essentially hinders the tendency of vortex growth due to the inverse energy cascade. Rather, the vortices evolve into Rossby wave and eventually to the formation of zonal jets as observed e.g. on giant planets. Depending on the precise setting used (e.g. strength of the differential rotation, additional energy injection to the system), the results of turbulence simulations can vary, but often energy spectra steeper than those predicted theoretically can be found [57, 89, 134].

In practice, the sink of enstrophy at the small scales is usually implemented by adding a hyperviscosity of the form

$$D = (-1)^{n-1}\nu\Delta^n\zeta \tag{8.7.2}$$

for $n \in \mathbb{N}^+$ to the right-hand side of Eq. (8.3.1), cf. Eq. (8.3.4). However, it can easily be checked that this form of hyperviscosity is not invariant under the Lie symmetry pseudogroups of the beta-plane and f-plane equations. More specifically, it violates the scale invariance of Eq. (8.3.1). *From the theoretical point of view, this violation appears to be especially odd, as it is precisely the scale invariance of the Euler equations that is used to derive the form of the energy spectrum in the inertial range.*

Theorem 8.3 directly implies that the invariantization $\iota(D) = (-1)^{n-1}\nu\sqrt{|\psi_x|^{2n-1}}\Delta^n\zeta$ is a differential invariant of the maximal Lie invariance pseudogroup of the vorticity equation. In view of the results of Section 8.6, we conclude that the form of the diffusion term obtained in the course of the invariantization is

$$\tilde{D} = |\psi_x|\iota(D) = (-1)^{n-1}\nu\sqrt{|\psi_x|^{2n+1}}\Delta^n\zeta.$$

The completely invariant formulation of the vorticity equation on the beta-plane with hyperdiffusion therefore reads

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x = (-1)^{n-1}\nu\sqrt{|\psi_x|^{2n+1}}\Delta^n\zeta. \qquad (8.7.3)$$

Note, however, that the price for introducing an invariant enstrophy sink is the *nonlinearity* of the (hyper)diffusion term. More generally, the situation is alike to the problem of finding a relation between the Reynolds stresses and the mean strain rate in the Reynolds averaged Navier–Stokes equations or in large–eddy simulations thereof. It was pointed out that establishing a relationship between the *nonlinear* Reynolds stresses and the *linear* strain rate (i.e. invoking the Boussinesq hypothesis) may lead to unrealistic results for certain turbulent flows such as in rotating or stratified fluids or those exposed to abrupt changes of the mean strain rate, see the discussions in [118, 154]. It is therefore worthwhile pointing out that the requirement of preserving the entire maximal Lie invariance pseudogroup of the barotropic vorticity equation on the beta-plane automatically yields nonlinear hyperdiffusion terms. For $n = 1$, the right-hand side of Eq. (8.7.3) can be considered as a generalized down-gradient parameterization for the eddy-vorticity flux, which is also a nonlinear quantity. That is, requiring a (hyper)diffusion scheme to be scale invariant, it is indispensable to use *nonlinear* (hyper)diffusion.

It is important to note that the anisotropic coefficient $\sqrt{|\psi_x|^{2n+1}}$ arises due to the special form of normalization conditions (8.5.2) we have chosen in Section 8.5 for the construction of the moving frame. This form is by no means unique but rather a consequence of the moving frame we have invoked. The situation is comparable to the discretization of differential equations, which can also be done in multiple ways. Some schemes have better properties than others and ultimately it is necessary to both analyze and test the various schemes for different sets of problems. Having more than one possibility to construct invariant subgrid-scale schemes out of a given non-invariant scheme should therefore be considered as an advantage rather than as a drawback of the proposed method. The knowledge of the complete set of differential invariants, which is obtained as a byproduct when determining the invariantization map for a given group action, allows one to derive series of invariant closure schemes starting from that obtained as a direct result of the invariantization of the given initial scheme. This is facilitated by recombining a given invariant scheme using the differential invariants, as any functional combination of differential invariants is again a differential invariant.

A number of alternative (isotropic) forms of a completely invariant nonlinear hyperviscosity term for the vorticity equation on the beta-plane can therefore be suggested, e.g.

$$\tilde{D} = (-1)^{n-1}\nu\zeta^{2n+1}\Delta^n\zeta, \quad \tilde{D} = (-1)^{n-1}\nu\nabla(\zeta^{2n+1}\nabla\Delta^{n-1}\zeta), \quad \text{etc.},$$

which are derived upon recombining the differential invariants derived in Theorem 8.3. Due to the wide possibility for varying ansatzes for invariant parameterizations we can control different desirable conditions which proper invariant closure schemes should additionally satisfy, cf. Section 8.8.

Subsequently we will exclusively work with Eq. (8.7.3). Our motivation for choosing the anisotropic hyperdiffusion (8.7.3) rather than any of the above isotropic ones stems from recent experiments on turbulence which suggest that contrary to the Kolmogorov hypothesis the small scales might indeed feel the effects from the large scale being anisotropic, i.e. that anisotropy can propagate through to the very small scales, see e.g. [139]. However, future tests will be conducted so as to compare the different forms of invariant hyperdiffusion.

We give some numerical experiments using Eq. (8.7.3) and compare it with the respective non-invariant model that employs classical hyperdiffusion (8.7.2). Both models are integrated using a finite difference scheme and biharmonic dissipation is used in all the experiments, i.e. $n = 2$. The nonlinear terms on the left-hand side are discretized using the Arakawa Jacobian operator [8], which guarantees energy and enstrophy conservation of the spatial discretization in the case of vanishing dissipation, $\nu = 0$. A leapfrog scheme is used for the time stepping in conjunction with a Robert–Asselin–Williams filter [155], in order to suppress the computational mode. The size of the domain is $L_x = L_y = 2\pi$, with a default of $N = 1024$ grid points in each direction, $\beta = 1$. The initial condition is a Gaussian random stream function field, with the initial energy spectrum given by the function $E(k) \propto k^3 \exp(-3k^2/k_p^2)$, where $k_p = 64$. No normalization of the initial energy was used. The value of ν was chosen to be $\nu_{\rm inv} = 1 \cdot 10^{-10}$ in the invariant case and $\nu_{\rm ninv} = 2 \cdot 10^{-9}$ for the non-invariant simulations. Note that the value of $\nu_{\rm ninv}$ has been selected to lie in between the values given in [30] for the two integrations using 512^2 and 4096^2 grid points. The value of $\nu_{\rm inv}$ has been chosen so that $\nu_{\rm inv} \approx \max(\nu_{\rm ninv}\sqrt{|\psi_x|^5})$ initially for the sake of comparison.

Both models have been integrated for approximately 10 000 time steps using $\Delta t = 1 \cdot 10^{-3}$. Hence, all the results presented below were evaluated at approximately $t = 10$, which should be long enough so that inertial ranges can form in the energy and enstrophy spectra. Below, we shall like to present the enstrophy spectra for fully developed freely decaying turbulence using both the invariant and the non-invariant hyperdiffusion terms. As was said above, according to the Batchelor–Kraichnan theory the enstrophy spectrum should be of the form k^{-1} in the intertial range. However, finding experimental evidence for a spectrum of this form proved rather hard and most numerical simulations carried out so far yield steeper spectra.

In Fig. 8.1*a* we show the enstrophy spectrum found from the simulation using invariant hyperdiffusion. In the region between approximately $k = 100$ up to $k = 300$ the spectrum follows k^{-1} almost perfectly. That is, the invariant hyperdiffusion of the form used in (8.7.3) leads to an experimental verification of the Batchelor–Kraichnan theory.

In Fig. 8.1*b* we show the corresponding enstrophy spectrum obtained using conventional (non-invariant) hyperdiffusion. As in the majority of turbulence simulations, also we obtain a spectrum in the inertial range that is *steeper* than k^{-1}, lying between k^{-1} and k^{-2}, in this case. Moreover, it is instructive to note that the lower parts of the spectra (up to the respective inertial ranges) are rather similar for both schemes, while differences occur within the inertial and in the diffusion ranges. This observation underpins that the proposed nonlinear invariant hyperdiffusion is physically acting as a viscousity term in Eq. (8.7.3).

Fig. 8.2 shows the associated vorticity fields obtained using the invariant and non-invariant hyperdiffusion schemes at the end of the integration. Note that the value of β chosen is rather

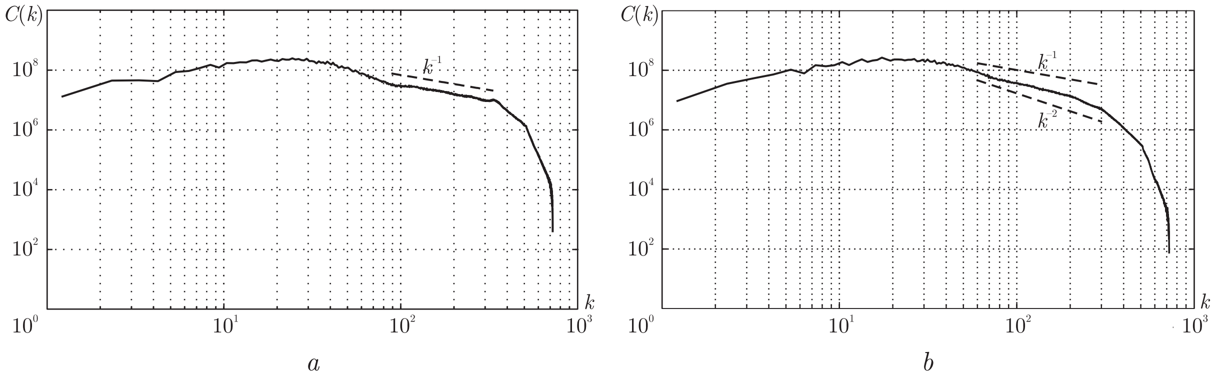

Figure 8.1: Enstrophy spectrum at approximately $t = 10$ using (a) invariant hyperdiffusion and (b) non-invariant hyperdiffusion.

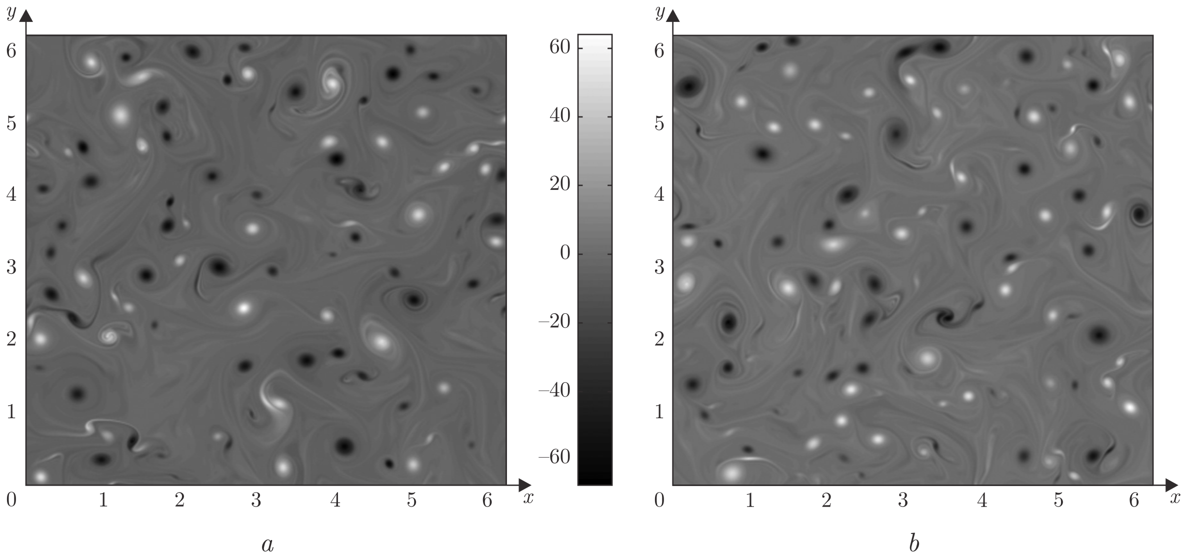

Figure 8.2: Vorticity field at approximately $t = 10$ using (a) invariant hyperdiffusion and (b) non-invariant hyperdiffusion.

small (and much smaller as compared to the value of $\beta = 3$ used in [89]) so the effects of differential rotation on the vorticity fields are rather minimal. Both fields look qualitatively similar verifying that invariant hyperdiffusion is capable of producing a physically meaningful vorticity field.

Remark 8.7. Decaying turbulence simulations are an important class of tests for numerical integration schemes. On the other hand, from the point of view of both the theory and application, it is generally more instructive when Eq. (8.3.1) is augmented with some forcing which supplies energy to the system and thereby prevents turbulence from dying out. As it is then usually necessary to damp out the energy which is otherwise piling up at small wave numbers (large scales) due to the inverse energy cascade, an additional drag term is introduced in Eq. (8.3.1). This drag term can be either physical (e.g. linear Ekman drag due to bottom friction) or, similar as hyperviscosity, scale selective. In the latter case, one uses a *hypoviscosity* [40], which is given by adding a term proportional to $\Delta^{-n}\zeta$, which acts scale selective by emphasizing the large scale and thus is effectively energy removing. Again, one could raise the question whether such a hypofriction should possess some invariance properties, but this is beyond the scope of the present paper and should be considered in a forthcoming study.

8.8 Conservative invariant parameterizations

A parameterization is called *conservative* if the corresponding closed system of differential equations possesses nonzero conservation laws. Special attention should be paid to parameterizations possessing conservation laws that have a clear physical interpretation (such as the conservation of energy, mass, momentum, etc.) and that originate from the conservation laws of the initial system of equations. If a parameterization is both conservative and invariant with respect to a pseudogroup of transformations, it is called a *conservative invariant parameterization.*

The general method for singling out conservative parameterizations among invariant closure ansatzes is based on the usage of the Euler operators, i.e. variational derivatives with respect to the dependent variables [101]. Suppose that $\tilde{\mathcal{L}}_\theta$: $\tilde{L}^l(x,\bar{u}_{(n)},\theta)=0$, $l=1,\dots,m$, $\theta=\theta(I^1,\dots,I^N)$ represent a family of local parameterizations for a system $\mathcal{L}$: $L^l(x,u_{(n)})=0$, $l=1,\dots,m$, which are invariant with respect to a pseudogroup G. Here $\tilde{L}^l$ are fixed smooth functions of their arguments. The tuple θ of arbitrary elements consists of smooth functions of certain differential invariants I^1, ..., I^N of G. It runs through a set of such tuples constrained by a system of differential equations, where I^1, ..., I^N play the role of independent variables. We require the tuples $(\lambda^{m1},\dots,\lambda^{ml})$, $m=1,\dots,M$, of differential functions of u to be characteristics of M linearly independent local conservation laws of the system $\tilde{\mathcal{L}}_\theta$ for some values of θ, i.e. for each m the combination $\lambda^{m1}\tilde{L}^1+\dots+\lambda^{ml}\tilde{L}^l$ is a total divergence. The theorem on characterization of total divergences [101, Theorem 4.7] then implies that

$$\mathsf{E}^a(\lambda^{m1}\tilde{L}^1+\dots+\lambda^{ml}\tilde{L}^l)=0 \qquad (8.8.1)$$

for each $m=1,\dots,M$ and $a=1,\dots,q$, where E^a is the Euler operator associated with the dependent variable u^a, $\mathsf{E}^a f=\sum_\alpha(-\mathrm{D})^\alpha f_{u^a_\alpha}$. Splitting Eqs. (8.8.1) with respect to derivatives of u wherever this is possible, one constructs the system of determining equations with respect to θ, which should be solved in order to derive the corresponding conservative invariant parameterizations.

As the direct computation is too cumbersome, we use some heuristic arguments and look for a diffusion ansatz for the barotropic vorticity equation on the beta-plane which satisfies the following relevant and valuable conditions:

- It is invariant with respect to the entire maximal Lie invariance pseudogroup G_1 of Eq. (8.3.1).
- The subgrid-scale term or, more generally, the sink term to be represented is a differential function of the vorticity.
- This expression is as similar as possible to the hyperviscosity term (8.7.2).
- And, last but not least, the parameterization is conservative. More precisely, it possesses all the conservation laws of Eq. (8.3.1) with zero-order characteristics.

An example of such a parameterization is given by

$$\zeta_t+\psi_x\zeta_y-\psi_y\zeta_x+\beta\psi_x=D,\quad D=\nu\Delta\frac{\Delta\zeta^7}{\zeta}=7\nu\Delta(\zeta^5\Delta\zeta+6\zeta^4(\nabla\zeta)^2). \qquad (8.8.2)$$

All the properties listed above can be checked for the sink term (8.8.2). Thus, the expression for D from (8.8.2) involves only the vorticity and its derivatives and is quite similar to (8.7.2). Moreover, the diffusion D is a globally defined differential function which is a polynomial of its arguments. The invariance of Eq. (8.8.2) with respect to G_1 can be simply checked using the infinitesimal invariance criterion. A more sophisticated way to check this invariance is to rewrite Eq. (8.8.2) in terms of normalized invariants of the pseudogroup G_1, which will not be done explicitly here. As an unexpected but valuable bonus we have that the maximal Lie symmetry pseudogroup of Eq. (8.8.2) with the same term D in the case of the f-plane ($\beta = 0$) is even wider than G_1. It also includes the usual rotations of the variables (x, y) and the generalized Galilean boosts in y-direction, which belong to the Lie symmetry pseudogroup G_0 of the barotropic vorticity equation on the f-plane. This in particular means that the parameterization (8.8.2) is isotropic.

The space of zero-order characteristics of Eq. (8.3.1) is generated by the characteristics $\lambda = f(t)$, $\lambda = g(t)y$ and $\lambda = \psi$, where f and g run through the set of smooth functions of t. The physically most important of these characteristics are $\lambda = 1$, $\lambda = y$ and $\lambda = \psi$, which are associated with the conservation of circulation, x-momentum and energy. Any zero-order characteristic of Eq. (8.3.1) is a characteristic of Eq. (8.8.2). Indeed, denoting

$$L := \zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x - D$$

we derive that

$$\begin{aligned}
fL &= \mathrm{D}_x\left(f\psi_{xt} + f\psi\zeta_y + f\beta\psi - \nu f\mathrm{D}_x\frac{\Delta\zeta^7}{\zeta}\right) + \mathrm{D}_y\left(f\psi_{yt} - f\psi\zeta_x - \nu f\mathrm{D}_y\frac{\Delta\zeta^7}{\zeta}\right),\\
gyL &= \mathrm{D}_x\left(gy\psi_{xt} + gy\psi\zeta_y - \frac{g}{2}(\psi_y)^2 + gy\beta\psi - \nu gy\mathrm{D}_x\frac{\Delta\zeta^7}{\zeta}\right)\\
&\quad + \mathrm{D}_y\left(gy\psi_{yt} - g\psi_y - gy\psi\zeta_x + g\psi\psi_{xy} - \nu gy\mathrm{D}_y\frac{\Delta\zeta^7}{\zeta} + \nu g\frac{\Delta\zeta^7}{\zeta}\right),\\
\psi L &= \mathrm{D}_t\left(-\frac{1}{2}(\nabla\psi)^2\right) + \mathrm{D}_x\left(\psi\psi_{xt} + \frac{1}{2}\psi^2\zeta_y + \frac{\beta}{2}\psi^2 - \nu\psi\mathrm{D}_x\frac{\Delta\zeta^7}{\zeta} + \nu\psi_x\frac{\Delta\zeta^7}{\zeta} - \nu\mathrm{D}_x\zeta^7\right)\\
&\quad + \mathrm{D}_y\left(\psi\psi_{yt} - \frac{1}{2}\psi^2\zeta_x - \nu\psi\mathrm{D}_y\frac{\Delta\zeta^7}{\zeta} + \nu\psi_y\frac{\Delta\zeta^7}{\zeta} - \nu\mathrm{D}_y\zeta^7\right).
\end{aligned}$$

If we grant that the vorticity equation coupled with some diffusive term possesses a smaller number of conservation laws (e.g. owing to the special physical properties of this diffusion), we can use a simpler form for the expression D. For example, the differential function $D = \nu\Delta\zeta^4$ leads to a parameterization which is invariant with respect to the entire pseudogroup G_1 and possesses conservation laws with the characteristics $\lambda = f(t)$, $\lambda = g(t)y$ for arbitrary values of the smooth parameter-functions f and g.

The parameterization (8.8.2) demonstrates the feasibility of combining invariant and conservative properties of closure schemes. This possibility is important for two obvious reasons. Firstly, conservation laws incorporate relevant physical information that is worth being preserved by a parameterization scheme. Secondly, from the point of view of constructing parameterization schemes, the requirement of preserving both symmetries and conservation laws leads to a more specific class of schemes than considering either only symmetries or only conservation laws. The additional narrowing of the class of admitted schemes using geometric constraints can then help to reduce the number of schemes that must be tested numerically so as to find the optimal parameterization for a given process.

8.9 Conclusion and discussion

The differential invariants of the Lie symmetry pseudogroup G_1 of the barotropic vorticity equation on the beta-plane are computed using the technique of moving frames for Lie pseudogroups.

A basis of these differential invariants along with the associated operators of invariant differentiation is established. Together, they serve to completely describe the algebra of differential invariants of G_1. Although differential invariants have many applications (such as the integration of ordinary differential equations [101], computation of so-called differentially invariant solutions [52, 112], the construction of invariant numerical discretization schemes [41], etc.), in the paper we focus on their usage in the construction of invariant closure schemes or, perhaps more generally, invariant diffusion terms for the averaged vorticity equation. This is one of the two general methods proposed in [123] to derive parameterization schemes with symmetry properties. As an alternative to the direct usage of elementary differential invariants that can be build together to yield invariant closure schemes, we propose the method of invariantization of existing parameterization schemes. This method is along the line of invariantization of existing discretization schemes as introduced in [70, 71]. Although this method is straightforward to apply, a potential complication is that the result depends on the particular choice of the moving frame and therefore does not lead to a unique invariant counterpart of an existing non-invariant scheme. As a consequence, it might be necessary to modify invariantized closure schemes and to test different invariantizations in order to devise physically valuable closures.

The differential invariants derived are used to construct invariant hyperdiffusion terms in order to model the behavior of two-dimensional freely decaying turbulence. The resulting enstrophy spectrum exhibits an arc of approximate k^{-1} slope which is the theoretically derived shape for the postulated enstrophy inertial range. It should be stressed, though, that the obtained enstrophy spectrum should be taken with a pinch of salt. Since the derivation of the theoretical form of the spectra in [10, 76] it has been tried in numerous studies to obtain these spectra in numerical simulations. Although results often vary, spectra are found with a steeper slope than the predicted k^{-1} curve as described in [11, 12, 30, 81, 89, 138]. It seems to be generally agreed today that the presence of the stable coherent vortices, which is the main feature of two-dimensional turbulence, has a strong impact on the derived enstrophy spectra. This holds in the case of turbulence both on the f-plane and on the beta-plane. The introduction of an invariant hyperdiffusion-like term certainly complicates the situation as diffusion then is coupled *nonlinearly* to the vorticity equation. On the other hand, it was indicated that the presence of the beta-term in the vorticity equation allows for a nonlocal transfer of anisotropy from the larger to the smaller scales [89]. A nonlinear diffusion term has the potential to support such a nonlocal scale interaction and thereby serves as a potential parameterization scheme for numerical models. It should be stressed in this context that in all the simulations we have carried out, the rate of energy dissipation was lower than using classical hyperdiffusion even in quite low-resolution numerical experiments.

Apart from the discussion above, the possibility of constructing hyperdiffusion-like enstrophy sink terms that lead to scale invariant enstrophy spectra seems to be a valuable property for itself. It is precisely the scale invariance of the Euler equations that is used to predict the behavior of two-dimensional turbulence in the inertial range and therefore the availability of dissipative versions of the vorticity equation having the same invariance properties as the inviscid vorticity equation might be a general advantage. Heuristically, one can expect that an invariant closure scheme should be better adapted for the problem of reproducing features that have been derived using symmetries (as the isotropic enstrophy spectrum), similarly as an invariant discretization scheme often reproduces better invariant exact solutions of a differential equation than non-invariant discretization schemes [133]. This assumption is supported by the proved relevance of Lie symmetries in turbulence theory [100]. The results obtained in the present paper do not contradict this assumption, keeping in mind especially that the premises invoked to obtain the theoretical form of the spectra are at present under revision. In this context, it should again be stressed that there is a multitude of invariant parameterization schemes or invariant diffusion terms that can be coupled to the vorticity equation on the beta-plane. The fact that already the simplest invariantized version (8.7.3) of the hyperdiffusion term (which has obvious weaknesses) shows quite good properties in the course of our numerical tests is a motivating result which is worth pointing out. Nevertheless, in order to verify and better assess the ability of invariant hyperdiffusion schemes to model turbulence on the beta-plane, further theoretical and numerical studies must be carried out.

The method we propose in this paper is fully generalizable. It is the number of variables of a model and its symmetry group that determine whether the method is computationally more complicated to realize. Thus, the relative simplicity of constructing diffusion schemes that are invariant under the entire maximal Lie invariance group is a particular feature of the beta-plane vorticity equation, which is computationally more involved for vorticity dynamics on the f-plane. The complication with the latter model is that the corresponding maximal Lie invariance pseudogroup G_0 is even wider than G_1. This makes it much harder to derive reasonably simple closure schemes that are invariant under the entire pseudogroup G_0, see the discussion in [123], where a generating set of differential invariants of G_0 and a complete set of its independent operators of invariant differentiation are determined. A possible remedy for this complication is to consider closure schemes that are invariant only under certain subgroups of the maximal Lie invariance pseudogroup of the f-plane equation. As highlighted in the present paper, the selection of such subgroups can be justified for physical reasons when boundaries come into play.

Another novel feature of the present paper is the explicit inclusion of conservation laws in invariant closure schemes. The chance of constructing such conservative invariant parameterization schemes is of obvious physical relevance. For physical processes that do not violate particular conservation laws, it is natural to require the associated parameterization to be also conservative. It was demonstrated in the paper for the vorticity equation on the beta-plane that the concepts of invariant and conservative parameterization schemes can be united to yield closure ansatzes that preserve both all the symmetries and certain conservation laws of this equation. The construction of further invariant conservative closure schemes as well as their exhaustive testing will be a next major challenge in the application of ideas of group analysis to the parameterization problem.

8.10 Appendix: Symmetries of the vorticity equation on the beta-plane

We aim to detail the computation of the maximal Lie invariance algebra $\mathfrak{g}_1$ of the vorticity equation (8.3.1) here. Full expositions on finding Lie symmetries of differential equations can be found in the standard textbooks [6, 25, 101, 112]. More details on the symmetries (and exact solutions) of the vorticity equation are presented in [18].

Given a generator

$$Q = \tau(t,x,y,\psi)\partial_t + \xi(t,x,y,\psi)\partial_x + \eta(t,x,y,\psi)\partial_y + \varphi(t,x,y,\psi)\partial_\psi. \tag{8.10.1}$$

of a one-parameter point symmetry group of the vorticity equation

$$\Delta = \zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x = 0, \quad \zeta = \psi_{xx} + \psi_{yy},$$

the infinitesimal invariance criterion [101, 112] implies $Q_3(\Delta) = 0$, which has to hold on the manifold $\Delta = 0$, where Q_3 denotes the third prolongation of the vector field Q. Explicitly, the prolonged vector field Q_3 is defined by $Q_3 = Q + \sum_{0<|\alpha|\leqslant 3}\varphi^\alpha\partial_{\psi_\alpha}$ and the coefficients of Q_3 are derived from the general prolongation formula,

$$\varphi^\alpha = \mathrm{D}_t^{\alpha_1}\mathrm{D}_x^{\alpha_2}\mathrm{D}_y^{\alpha_3}(\varphi - \tau\psi_{\delta_1} - \xi\psi_{\delta_2} - \eta\psi_{\delta_3}) + \tau\psi_{\alpha+\delta_1} + \xi\psi_{\alpha+\delta_2} + \eta\psi_{\alpha+\delta_3}. \qquad (8.10.2)$$

Here we use the notation introduced in the beginning of Section 8.5. Then the condition $Q_3(\Delta) = 0$ expands to

$$\varphi^{120} + \varphi^{102} + \varphi^{010}\zeta_y + \psi_x(\varphi^{021} + \varphi^{003}) - \varphi^{001}\zeta_x - \psi_y(\varphi^{030} + \varphi^{012}) + \beta\varphi^{010} = 0,$$

and the constraint that $Q_3(\Delta) = 0$ has to hold only on the manifold of $\Delta = 0$ is taken into account by substituting $\psi_{txx} = -\psi_{tyy} - \psi_x\zeta_y + \psi_y\zeta_x - \beta\psi_x$ wherever ψ_{txx} occurs. As the coefficients of Q are only functions of t, x, y and ψ, the expanded condition can be split with respect to the various derivatives of ψ. This splitting yields the determining equations for the coefficients of the vector field Q,

$$\begin{aligned}&\tau_x = \tau_y = \tau_\psi = \xi_y = \xi_\psi = \eta_t = \eta_x = \eta_\psi = \varphi_x = 0,\\ &\xi_x = \eta_y = -\tau_t, \quad \varphi_y = -\xi_t, \quad \varphi_\psi = -3\tau_t.\end{aligned} \qquad (8.10.3)$$

The general solution of this system of determining equations reads

$$\tau = c_1 t + c_2, \quad \xi = -c_1 x + \tilde f(t), \quad \eta = -c_1 y + c_3, \quad \varphi = -3c_1\psi - \tilde f_t y + \tilde g(t),$$

where $\tilde f$ and $\tilde g$ run through the set of smooth functions of t. Thus, the maximal Lie invariance algebra of infinitesimal symmetries of the barotropic vorticity equation on the beta-plane is spanned by the vector fields

$$\mathcal{D} = t\partial_t - x\partial_x - y\partial_y - 3\psi\partial_\psi, \quad \partial_t, \quad \partial_y, \quad \mathcal{X}(\tilde f) = \tilde f(t)\partial_x - \tilde f_t(t) y\partial_\psi \quad \mathcal{Z}(\tilde g) = \tilde g(t)\partial_\psi.$$

8.11 Appendix: Algebra of differential invariants for the vorticity equation

In this appendix we present the details for the proof of Theorem 8.4 which exhaustively describes the algebra of differential invariants for the maximal Lie invariance pseudogroup of the barotropic vorticity equation on the beta-plane.

A complete set of independent operators of invariant differentiation is derived by invariantization of the usual operators of total differentiation, yielding

$$\mathrm{D}_t^{\mathrm{i}} = \frac{1}{\sqrt{|\psi_x|}}(\mathrm{D}_t - \psi_y\mathrm{D}_x), \quad \mathrm{D}_x^{\mathrm{i}} = \sqrt{|\psi_x|}\mathrm{D}_x, \quad \mathrm{D}_y^{\mathrm{i}} = \sqrt{|\psi_x|}\mathrm{D}_y. \qquad (8.11.1)$$

This is practically realized via substituting the expressions (8.5.3) for the pseudogroup parameters into the implicit differentiation operators (8.5.1). Any operator of invariant differentiation related to the pseudogroup G_1 is locally a combination of the operators (8.11.1) with functional coefficients depending only on differential invariants of G_1. The commutation relations between the operators $\mathrm{D}_t^{\mathrm{i}}$, $\mathrm{D}_x^{\mathrm{i}}$ and $\mathrm{D}_y^{\mathrm{i}}$ are

$$\begin{aligned}&[\mathrm{D}_t^{\mathrm{i}}, \mathrm{D}_x^{\mathrm{i}}] = \frac{\varepsilon}{2}I_{020}\mathrm{D}_t^{\mathrm{i}} + \left(I_{011} + \frac{\varepsilon}{2}I_{110}\right)\mathrm{D}_x^{\mathrm{i}},\\ &[\mathrm{D}_t^{\mathrm{i}}, \mathrm{D}_y^{\mathrm{i}}] = \frac{\varepsilon}{2}I_{011}\mathrm{D}_t^{\mathrm{i}} + I_{002}\mathrm{D}_x^{\mathrm{i}} + \frac{\varepsilon}{2}I_{110}\mathrm{D}_y^{\mathrm{i}},\\ &[\mathrm{D}_x^{\mathrm{i}}, \mathrm{D}_y^{\mathrm{i}}] = \frac{\varepsilon}{2}I_{020}\mathrm{D}_y^{\mathrm{i}} - \frac{\varepsilon}{2}I_{011}\mathrm{D}_x^{\mathrm{i}}.\end{aligned} \qquad (8.11.2)$$

In order to completely describe the algebra of differential invariants of G_1, it remains to establish a basis of differential invariants such that any differential invariant of G_1 can be represented as a function of basis elements and their invariant derivatives. It is also necessary to compute a complete system of syzygies between basis invariants. For this aim, we will evaluate the recurrence relations between the normalized differential invariants and the differentiated differential invariants as detailed in [33, 107]. The starting point for the application of the general algorithm to the maximal Lie invariance pseudogroup G_1 of the vorticity equation on the beta-plane is the system of determining equations for the coefficients of a vector field (8.10.1) from the maximal Lie invariance algebra of Eq. (8.3.1), which is given through system (8.10.3). Consider the prolonged operator $Q_\infty = Q + \sum_{|\alpha|>0} \varphi^\alpha \partial_{\psi_\alpha}$. The coefficients of Q_∞ are calculated by the standard prolongation formula (8.10.2). In view of the determining equations, the coefficients φ^α take the form

$$\varphi^\alpha = (\alpha_2+\alpha_3-\alpha_1-3)\tau_t\psi_\alpha - \sum_{k=1}^{\alpha_1}\binom{\alpha_1}{k}\xi_{(k)}\psi_{\alpha-k\delta_1+\delta_2} + \left\{\begin{array}{ll} -\xi_{(\alpha_1+1)}, & \alpha_2=0,\ \alpha_3=1 \\ \varphi_{(\alpha_1)}, & \alpha_2=\alpha_3=0 \end{array}\right\},$$

where $\xi_{(k)} = \partial^k\xi/\partial t^k$ and $\varphi_{(k)} = \partial^k\varphi/\partial t^k$, $k=0,1,2,\dots$. We collect the coefficients of Q and their derivatives appearing in the expressions for the prolonged coefficients of Q and denote the associated invariantized objects, which are differential forms, as $\hat\tau^0 = \iota(\tau)$, $\hat\tau^1 = \iota(\tau_t)$, $\hat\xi^k = \iota(\xi_{(k)})$, $\hat\eta = \iota(\eta)$ and $\hat\varphi^k = \iota(\varphi_{(k)})$. In the course of the normalization (8.5.2) the invariantized counterparts $\hat\varphi^\alpha = \iota(\varphi^\alpha)$ of the prolonged coefficients of Q are

$$\hat\varphi^{j00} = \hat\varphi^j - \varepsilon\hat\xi^j - \sum_{k=1}^{j-1}\binom{j}{k} I_{j-k,10}\hat\xi^k \quad \text{if} \quad j>0, \quad \hat\varphi^{j01} = -\hat\xi^{j+1} - \sum_{k=1}^{j}\binom{j}{k} I_{j-k,11}\hat\xi^k,$$

$$\hat\varphi^\alpha = (\alpha_2+\alpha_3-\alpha_1-3)I_\alpha\hat\tau^1 - \sum_{k=1}^{\alpha_1}\binom{\alpha_1}{k} I_{\alpha-k\delta_1+\delta_2}\hat\xi^k \quad \text{if} \quad \alpha_2>0 \quad \text{or} \quad \alpha_3>1.$$

For lower values of $|\alpha|$, $0<|\alpha|\leqslant 3$, we calculate

$$\begin{aligned}
&\hat\varphi^{100} = \hat\varphi^1 - \varepsilon\hat\xi^1, \quad \hat\varphi^{010} = -2\hat\tau^1, \quad \hat\varphi^{001} = -\hat\xi^1,\\
&\hat\varphi^{200} = \hat\varphi^2 - \varepsilon\hat\xi^2 - 2I_{110}\hat\xi^1, \quad \hat\varphi^{110} = -3I_{110}\hat\tau^1 - I_{020}\hat\xi^1, \quad \hat\varphi^{101} = -\hat\xi^2 - I_{011}\hat\xi^1,\\
&\hat\varphi^{020} = -I_{020}\hat\tau^1, \quad \hat\varphi^{011} = -I_{011}\hat\tau^1, \quad \hat\varphi^{002} = -I_{002}\hat\tau^1,\\
&\hat\varphi^{300} = \hat\varphi^3 - \varepsilon\hat\xi^3 - 3I_{110}\hat\xi^2 - 3I_{210}\hat\xi^1,\\
&\hat\varphi^{210} = -4I_{210}\hat\tau^1 - I_{020}\hat\xi^2 - 2I_{120}\hat\xi^1, \quad \hat\varphi^{201} = -\hat\xi^3 - I_{011}\hat\xi^2 - 2I_{111}\hat\xi^1,\\
&\hat\varphi^{120} = -2I_{120}\hat\tau^1 - I_{030}\hat\xi^1, \quad \hat\varphi^{111} = -2I_{111}\hat\tau^1 - I_{021}\hat\xi^1, \quad \hat\varphi^{102} = -2I_{102}\hat\tau^1 - I_{012}\hat\xi^1,\\
&\hat\varphi^{030} = \hat\varphi^{021} = \hat\varphi^{012} = \hat\varphi^{003} = 0.
\end{aligned}$$

From the recurrence relations for the phantom invariants $H^0 = \iota(t)$, $H^1 = \iota(x)$, $H^2 = \iota(y)$, $I_{i00} = \iota(\psi_{i00})$, $I_{i01} = \iota(\psi_{i01})$, $i = 0,1,\dots$, and $I_{010} = \iota(\psi_{010})$, which are

$$\begin{aligned}
&\mathrm{d_h}H^0 = \omega^1 + \hat\tau^0 = 0, \quad \mathrm{d_h}H^1 = \omega^2 + \hat\xi^0 = 0, \quad \mathrm{d_h}H^2 = \omega^3 + \hat\eta = 0, \quad \mathrm{d_h}I_{000} = \omega^2 + \hat\varphi^0 = 0,\\
&\mathrm{d_h}I_{j00} = I_{j10}\omega^2 + \hat\varphi^j - \varepsilon\hat\xi^j - \sum_{k=1}^{j-1}\binom{j}{k} I_{j-k,10}\hat\xi^k = 0, \quad j = 1,2,\dots,\\
&\mathrm{d_h}I_{j01} = I_{j11}\omega^2 + I_{j02}\omega^3 - \hat\xi^{j+1} - \sum_{k=1}^{j}\binom{j}{k} I_{j-k,11}\hat\xi^k = 0, \quad j = 0,1,\dots,\\
&\mathrm{d_h}I_{010} = I_{110}\omega^1 + I_{020}\omega^2 + I_{011}\omega^3 - 2\hat\tau^1 = 0,
\end{aligned}$$

where $\omega^1 = \iota(\mathrm{d}t)$, $\omega^2 = \iota(\mathrm{d}x)$ and $\omega^3 = \iota(\mathrm{d}y)$, we derive expressions for the invariantized Maurer–Cartan forms

$$\hat\tau^0 = -\omega^1, \quad \hat\xi^0 = -\omega^2, \quad \hat\eta = -\omega^3, \quad \hat\varphi^0 = -\omega^2, \quad \hat\tau^1 = \tfrac{1}{2}(I_{110}\omega^1 + I_{020}\omega^2 + I_{011}\omega^3),$$
$$\hat\xi^j = I_{j-1,11}\omega^2 + I_{j-1,02}\omega^3 - \sum_{k=1}^{j-1}\binom{j-1}{k} I_{j-k-1,11}\hat\xi^k,$$
$$\hat\varphi^j = -I_{j10}\omega^2 + \varepsilon\hat\xi^j + \sum_{k=1}^{j-1}\binom{j}{k} I_{j-k,10}\hat\xi^k,$$

$j = 1, 2, \dots$. The forms $\hat\xi^j$ should be calculated recursively starting from $j = 1$. Thus,

$$\hat\xi^1 = I_{011}\omega^2 + I_{002}\omega^3,$$
$$\hat\xi^2 = (I_{111} - I_{011}^2)\omega^2 + (I_{102} - I_{011}I_{002})\omega^3,$$
$$\hat\xi^3 = (I_{211} - 3I_{011}I_{111} + I_{111}^3)\omega^2 + (I_{202} - 3I_{011}I_{102} + I_{011}^2 I_{002})\omega^3, \quad \dots \quad .$$

In general, $\hat\xi^j = \hat\xi^{j,2}\omega^2 + \hat\xi^{j,3}\omega^3$, where the coefficients $\hat\xi^{j,2}$ and $\hat\xi^{j,3}$ are expressed in terms of normalized invariants I_α with $|\alpha| \leqslant j + 1$.

The recurrence relations for non-phantom normalized invariants correspondingly read

$$\mathrm{d_h} I_{\alpha_1\alpha_2\alpha_3} = I_{\alpha+\delta_1}\omega^1 + I_{\alpha+\delta_2}\omega^2 + I_{\alpha+\delta_3}\omega^3 + (\alpha_2 + \alpha_3 - \alpha_1 - 3)I_\alpha\hat\tau^1$$
$$- \sum_{k=1}^{\alpha_1}\binom{\alpha_1}{k} I_{\alpha-k\delta_1+\delta_2}\hat\xi^k \quad \text{if} \quad \alpha_2 > 0 \quad \text{or} \quad \alpha_3 > 1.$$

As by definition $\mathrm{d_h}F = (\mathrm{D}^{\mathrm{i}}_t F)\omega^1 + (\mathrm{D}^{\mathrm{i}}_x F)\omega^2 + (\mathrm{D}^{\mathrm{i}}_y F)\omega^3$, the above recurrence relations can be split into a list of equations for first-order invariant derivatives of normalized differential invariants I_α with $\alpha_2 > 0$ or $\alpha_3 > 1$ by taking into account the expressions for the invariantized Maurer–Cartan forms:

$$\begin{aligned}
\mathrm{D}^{\mathrm{i}}_t I_\alpha &= I_{\alpha+\delta_1} + \frac{\alpha_2 + \alpha_3 - \alpha_1 - 3}{2} I_{110} I_\alpha,\\
\mathrm{D}^{\mathrm{i}}_x I_\alpha &= I_{\alpha+\delta_2} + \frac{\alpha_2 + \alpha_3 - \alpha_1 - 3}{2} I_{020} I_\alpha - \sum_{k=1}^{\alpha_1}\binom{\alpha_1}{k} I_{\alpha-k\delta_1+\delta_2}\hat\xi^{k,2},\\
\mathrm{D}^{\mathrm{i}}_y I_\alpha &= I_{\alpha+\delta_3} + \frac{\alpha_2 + \alpha_3 - \alpha_1 - 3}{2} I_{011} I_\alpha - \sum_{k=1}^{\alpha_1}\binom{\alpha_1}{k} I_{\alpha-k\delta_1+\delta_2}\hat\xi^{k,3}.
\end{aligned} \tag{8.11.3}$$

We only present the closed expressions for the first-order invariant derivatives of I_α with $|\alpha| \leqslant 3$:

$$\mathrm{D}^{\mathrm{i}}_t I_{110} = I_{210} - \tfrac{3}{2}I_{110}^2, \quad \mathrm{D}^{\mathrm{i}}_x I_{110} = I_{120} - \tfrac{3}{2}I_{110}I_{020} - I_{011}I_{020},$$
$$\mathrm{D}^{\mathrm{i}}_y I_{110} = I_{111} - \tfrac{3}{2}I_{110}I_{011} - I_{020}I_{002},$$
$$\mathrm{D}^{\mathrm{i}}_t I_{020} = I_{120} - \tfrac{1}{2}I_{110}I_{020}, \quad \mathrm{D}^{\mathrm{i}}_x I_{020} = I_{030} - \tfrac{1}{2}I_{020}^2, \quad \mathrm{D}^{\mathrm{i}}_y I_{020} = I_{021} - \tfrac{1}{2}I_{011}I_{020},$$
$$\mathrm{D}^{\mathrm{i}}_t I_{011} = I_{111} - \tfrac{1}{2}I_{110}I_{011}, \quad \mathrm{D}^{\mathrm{i}}_x I_{011} = I_{021} - \tfrac{1}{2}I_{011}I_{020}, \quad \mathrm{D}^{\mathrm{i}}_y I_{011} = I_{012} - \tfrac{1}{2}I_{011}^2,$$
$$\mathrm{D}^{\mathrm{i}}_t I_{002} = I_{102} - \tfrac{1}{2}I_{110}I_{002}, \quad \mathrm{D}^{\mathrm{i}}_x I_{002} = I_{012} - \tfrac{1}{2}I_{020}I_{002}, \quad \mathrm{D}^{\mathrm{i}}_y I_{002} = I_{003} - \tfrac{1}{2}I_{011}I_{002},$$
$$\mathrm{D}^{\mathrm{i}}_t I_{210} = I_{310} - 2I_{110}I_{210}, \quad \mathrm{D}^{\mathrm{i}}_x I_{210} = I_{220} - 2I_{020}I_{210} - 2I_{011}I_{120} + (I_{011}^2 - I_{111})I_{020},$$
$$\mathrm{D}^{\mathrm{i}}_y I_{210} = I_{211} - 2I_{011}I_{210} - 2I_{002}I_{120} + (I_{002}I_{011} - I_{102})I_{020},$$
$$\mathrm{D}^{\mathrm{i}}_t I_{201} = I_{301} - 2I_{110}I_{201}, \quad \mathrm{D}^{\mathrm{i}}_x I_{201} = I_{211} - 2I_{020}I_{201} - 3I_{001}I_{111} + I_{011}^3,$$
$$\mathrm{D}^{\mathrm{i}}_y I_{201} = I_{202} - 2I_{011}I_{201} - 2I_{002}I_{111} - I_{011}I_{102} + I_{002}I_{011}^2,$$

$$\begin{aligned}
&\mathrm{D}^{\mathrm{i}}_t I_{120} = I_{220} - I_{110}I_{120}, \quad \mathrm{D}^{\mathrm{i}}_x I_{120} = I_{130} - I_{020}I_{120} - I_{011}I_{030},\\
&\mathrm{D}^{\mathrm{i}}_y I_{120} = I_{121} - I_{011}I_{120} - I_{002}I_{030},\\
&\mathrm{D}^{\mathrm{i}}_t I_{111} = I_{211} - I_{110}I_{111}, \quad \mathrm{D}^{\mathrm{i}}_x I_{111} = I_{121} - I_{020}I_{111} - I_{011}I_{021},\\
&\mathrm{D}^{\mathrm{i}}_y I_{111} = I_{112} - I_{011}I_{111} - I_{002}I_{021},\\
&\mathrm{D}^{\mathrm{i}}_t I_{102} = I_{202} - I_{110}I_{102}, \quad \mathrm{D}^{\mathrm{i}}_x I_{102} = I_{112} - I_{020}I_{102} - I_{011}I_{012},\\
&\mathrm{D}^{\mathrm{i}}_y I_{102} = I_{103} - I_{011}I_{102} - I_{002}I_{012},\\
&\mathrm{D}^{\mathrm{i}}_t I_{030} = I_{130}, \quad \mathrm{D}^{\mathrm{i}}_x I_{030} = I_{040}, \quad \mathrm{D}^{\mathrm{i}}_y I_{030} = I_{031},\\
&\mathrm{D}^{\mathrm{i}}_t I_{021} = I_{121}, \quad \mathrm{D}^{\mathrm{i}}_x I_{021} = I_{031}, \quad \mathrm{D}^{\mathrm{i}}_y I_{021} = I_{022},\\
&\mathrm{D}^{\mathrm{i}}_t I_{012} = I_{112}, \quad \mathrm{D}^{\mathrm{i}}_x I_{012} = I_{022}, \quad \mathrm{D}^{\mathrm{i}}_y I_{012} = I_{013},\\
&\mathrm{D}^{\mathrm{i}}_t I_{003} = I_{103}, \quad \mathrm{D}^{\mathrm{i}}_x I_{003} = I_{013}, \quad \mathrm{D}^{\mathrm{i}}_y I_{003} = I_{004}.
\end{aligned}$$

In principle, it is possible to read off the generating differential invariants from the above split recurrence relations. The expressions for $I_{\alpha+\delta_1}$, $I_{\alpha+\delta_2}$ and $I_{\alpha+\delta_3}$ derived from (8.11.3) only involve first-order invariant derivatives of I_α and normalized invariants of orders not greater than $|\alpha|$. This implies that a generating set of differential invariants consists of invariantized derivatives which are minimal with respect to the usual partial ordering of derivatives and are not phantom invariants. We have four such minimal elements,

$$I_{110} = \frac{\psi_{tx} - \psi_y\psi_{xx}}{\sqrt{|\psi_x|^3}}, \quad I_{020} = \frac{\psi_{xx}}{\sqrt{|\psi_x|}}, \quad I_{011} = \frac{\psi_{xy}}{\sqrt{|\psi_x|}}, \quad I_{002} = \frac{\psi_{yy}}{\sqrt{|\psi_x|}}.$$

All the other invariantized derivatives are expressed via invariant derivatives of I_{110}, I_{020}, I_{011} and I_{002}. As was indicated above, not all differentiated differential invariants are necessarily functionally independent, which is encoded in syzygies of the algebra of differential invariants. Taking into account these syzygies can further reduce the number of generating differential invariants thereby allowing one a more concise description of the basis of differential invariants. In the present case, we find the following lower-order syzygies:

$$\begin{aligned}
&\mathrm{D}^{\mathrm{i}}_t I_{011} - \mathrm{D}^{\mathrm{i}}_y I_{110} = I_{110}I_{011} + I_{020}I_{002},\\
&\mathrm{D}^{\mathrm{i}}_t I_{020} - \mathrm{D}^{\mathrm{i}}_x I_{110} = I_{020}(I_{110} + I_{011}),\\
&\mathrm{D}^{\mathrm{i}}_y I_{011} - \mathrm{D}^{\mathrm{i}}_x I_{002} = \tfrac12 I_{020}I_{002} - \tfrac12 I_{011}^2,\\
&\mathrm{D}^{\mathrm{i}}_x I_{011} - \mathrm{D}^{\mathrm{i}}_y I_{020} = 0,\\
&(\mathrm{D}^{\mathrm{i}}_y)^2 I_{110} - \mathrm{D}^{\mathrm{i}}_t\mathrm{D}^{\mathrm{i}}_x I_{002} = \tfrac12(\mathrm{D}^{\mathrm{i}}_t - I_{011})(I_{020}I_{002}) - (\mathrm{D}^{\mathrm{i}}_y + I_{011})(\tfrac32 I_{110}I_{011} + I_{020}I_{002})\\
&\qquad -I_{011}\mathrm{D}^{\mathrm{i}}_y I_{110} - I_{002}\mathrm{D}^{\mathrm{i}}_y I_{020},\\
&(\mathrm{D}^{\mathrm{i}}_y)^2 I_{020} - (\mathrm{D}^{\mathrm{i}}_x)^2 I_{002} = \tfrac12 \mathrm{D}^{\mathrm{i}}_x(I_{020}I_{002}) - \tfrac12 \mathrm{D}^{\mathrm{i}}_y(I_{011}I_{020}).
\end{aligned}$$

From the two first syzygies we can express the invariants I_{011} and I_{002} via invariant derivatives of I_{110} and I_{020},

$$\begin{aligned}
&I_{011} = \frac{\mathrm{D}^{\mathrm{i}}_t I_{020} - \mathrm{D}^{\mathrm{i}}_x I_{110}}{I_{020}} - I_{110},\\
&I_{002} = \frac{1}{I_{020}}(\mathrm{D}^{\mathrm{i}}_t - I_{110})\left(\frac{\mathrm{D}^{\mathrm{i}}_t I_{020} - \mathrm{D}^{\mathrm{i}}_x I_{110}}{I_{020}} - I_{110}\right) - \frac{\mathrm{D}^{\mathrm{i}}_y I_{110}}{I_{020}}.
\end{aligned}$$

Another way of finding relations between generating invariants is to use the commutation relations between the operators of invariant differentiation. Evaluating each equality from (8.11.2) on an element I from the above generating set, we obtain a system of linear algebraic equations with respect to the other elements of these sets, which can be solved on the domain of the jet space where the determinant of the matrix associated with the system does not vanish. It is convenient to choose, e.g., $I = I_{020}$. Then, we derive the representations

$$
\begin{aligned}
I_{011} &= \frac{I_{020}\mathrm{D}^{\mathrm{i}}_y I_{020} - 2\varepsilon[\mathrm{D}^{\mathrm{i}}_x, \mathrm{D}^{\mathrm{i}}_y]I_{020}}{\mathrm{D}^{\mathrm{i}}_x I_{020}},\\
I_{110} &= \frac{2\varepsilon[\mathrm{D}^{\mathrm{i}}_t, \mathrm{D}^{\mathrm{i}}_x]I_{020} - I_{020}\mathrm{D}^{\mathrm{i}}_t I_{020}}{\mathrm{D}^{\mathrm{i}}_x I_{020}} - 2\varepsilon I_{011},\\
I_{002} &= \frac{[\mathrm{D}^{\mathrm{i}}_t, \mathrm{D}^{\mathrm{i}}_y]I_{020}}{\mathrm{D}^{\mathrm{i}}_x I_{020}} - \frac{\varepsilon}{2}\frac{\mathrm{D}^{\mathrm{i}}_t I_{020}}{\mathrm{D}^{\mathrm{i}}_x I_{020}} I_{011} - \frac{\varepsilon}{2}\frac{\mathrm{D}^{\mathrm{i}}_y I_{020}}{\mathrm{D}^{\mathrm{i}}_x I_{020}} I_{110},
\end{aligned}
$$

which are defined on the domain Ω_1 of the jet space where $\mathrm{D}^{\mathrm{i}}_x I_{020} \neq 0$, i.e., $\mathrm{D}_x^2(\sqrt{|\psi_x|}\,) \neq 0$.

As a result, it is straightforward to establish Theorem 8.4.

Chapter 9

Summary and conclusion

This thesis was devoted to the study of parameterization schemes with symmetry properties. We developed and refined several techniques related to the group classification of differential equations. As was shown in [123] these methods lie at the heart of any invariant parameterization scheme.

We demonstrated the effectiveness of the newly developed group classification tools by reconsidering classification problems related to a class of diffusion–convection and a class of nonlinear wave equations. Both classes had been considered previously but could not be classified exhaustively. In particular, the problem of complete group classification of the class of nonlinear wave equations has been a standing problem in this field for more than 20 years.

Having effective classification strategies at ones disposal is of crucial importance for the invariant parameterization problem. The reason for this is that the classes of differential equations that arise in the construction of symmetry-preserving parameterization schemes are usually much more involved as the classes of equations normally considered in the field of group analysis. This is the reason why the bulk of this thesis was devoted to a careful extension of the existing group classification methods.

To make the proposed methods for finding invariant parameterization schemes practical also for non-specialists, it will be important to *automate* several of the steps required by using computer algebra systems. The sophistication of packages for the computation of Lie symmetries and exact solutions of systems of differential equations as nowadays available for environments like `Maple` or `Mathematica` is unfortunately not paralleled by similar packages for classes of differential equations. That is, group classification to date is still a problem that has to be solved largely by hand, which can become quite cumbersome for the complicated classes of equations arising in parameterization problems. Here we believe that greater progress will be made for inverse classification problems rather than for the direct ones in the nearer future. This is due to the equivariant moving frame method, which can be used in the inverse classification problem and which has the potential of a full automatization using computer algebra packages. Essentially all the steps required in the construction of a moving frame are entirely algorithmic and thus do not call for a case-by-case consideration as typically required in direct classification problems. In fact, `Maple` already features a package for the computation of moving frames. This makes the invariantization approach for parameterization schemes introduced in this thesis a viable way for finding symmetry-preserving closure models.

On the practical side, the methods conceived for constructing invariant parameterization schemes have been used for problems related to turbulence modeling on the beta-plane, i.e. the so-called geostrophic turbulence. Geostrophic turbulence is of central importance in the atmospheric and planetary sciences as it is believed to explain the emergence of jets and coherent structures on the planetary scale. Two-dimensional turbulence is perhaps even harder to grasp than three-dimensional turbulence as it is always inherently an idealization. True two-dimensional turbulence is difficult to realize experimentally and thus one usually relies on numerical simulations to study its properties. Unfortunately, the Batchelor–Kraichnan theory of two-dimensional turbulence is quite hard to verify, especially when it comes to the description of the energy spectrum in the enstrophy inertial range. Here several of the numerical simulations carried out previously report energy spectra that are considerably steeper than the theoretically predicted k^{-3} slope. In turn, using invariantized hyperdiffusion we showed that producing such a k^{-3} spectrum numerically is also practically possible. The extension of this study to the model of turbulence in a barotropic ocean as introduced in Chapter 2 once more proved the practicality of invariant parameterization schemes. Jointly these results give the hope that invariant parameterizations will play an important role also for other physical processes that have to be parameterized in Earth simulation models.

There are several directions in which it is possible to further extend the work reported in this thesis. First and foremost, it will be necessary to generalize the proposed methods to allow constructing invariant *nonlocal* parameterization schemes. Nonlocal parameterizations are important because for several processes in geophysical fluid dynamics it is insufficient to parameterize them using only the information given in a neighborhood around each point. An important example for such a process is convection, which usually affects a significant part of an atmospheric column. Nonlocal parameterization schemes are often in the form of integro-differential equations. From the mathematical side, the study of nonlocal symmetry-preserving parameterization schemes therefore boils down to the group analysis of integro-differential equations, which is not at all a well-investigated subject. Here, once again, considerable research on methods of group analysis will be required before practical examples for invariant nonlocal parameterization schemes can be constructed.

A further direction that seems promising from both a conceptual and a practical point of view is the combination of invariant and *conservative* parameterization schemes. Conservative parameterization schemes are parameterizations that lead to closed systems of differential equations preserving certain conservation laws as admitted by the original system of differential equations. They were briefly introduced in Chapter 2, see also [14, 16], but due to the complexity of classification problems for conservation laws new efficient methods will be required to tackle the complex models of hydro-thermodynamics. Extending the range of applicability of such methods and applying them to real parameterization problems thus appears to be a logical next step in the framework of geometry-preserving parameterization.

In conclusion, structure-preserving parameterization appears to be a natural approach to the problem of finding consistent subgrid-scale models for the averaged or filtered governing equations of hydrodynamics and geophysical fluid dynamics. As it becomes increasingly difficult to devise a unified general parameterization methodology for the multitude of dynamically active processes in the Earth system, resorting to the paradigm of preserving the geometry of the fundamental equations describing fluid flow is attractive for good reasons. Most importantly, it reduces the parameterization problem to first principles. It is known that finding a good parameterization is as much an art as it is science. The concept of expressing unresolved processes using only the information contained in the resolved part of a model is and will always be incomplete. By bringing the preservation of geometry in the center of the parameterization problem, a consistent framework is made available that uses symmetries of differential equations in the very way they are implied on physical grounds: to describe the essential properties of the laws of nature.

Bibliography

[1] Akhatov I.S., Gazizov R.K. and Ibragimov N.K., Nonlocal symmetries. Heuristic approach, *J. Math. Sci.* **55** (1991), 1401–1450.

[2] Ames W.F., Anderson R.L., Dorodnitsyn V.A., Ferapontov E.V., Gazizov R.K., Ibragimov N.H. and Svirshchevskii S.R., *CRC handbook of Lie group analysis of differential equations. Vol. 1. Symmetries, exact solutions and conservation laws. Edited by N. H. Ibragimov*, CRC Press, Boca Raton, 1994.

[3] Anco S. and Bluman G., Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classifications, *Eur. J. App. Math.* **13** (2002), 545–566.

[4] Anco S. and Bluman G., Direct construction method for conservation laws of partial differential equations. Part II: General treatment, *Eur. J. App. Math.* **13** (2002), 567–585.

[5] Anderson R.L., Baikov V.A., Gazizov R.K., Hereman W., Ibragimov N.H., Mahomed F.M., Meleshko S.V., Nucci M.C., Olver P.J., Sheftel' M.B., Turbiner A.V. and Vorob'ev E.M., *CRC handbook of Lie group analysis of differential equations. Vol. 3. New trends in theoretical developments and computational methods*, CRC Press, Boca Raton, 1996.

[6] Andreev V.K., Kaptsov O.V., Pukhnachov V.V. and Rodionov A.A., *Applications of group-theoretical methods in hydrodynamics*, Kluwer, Dordrecht, 1998.

[7] Andreev V.K. and Rodionov A.A., Group analysis of equations of planar flows of an ideal fluid in Lagrangian coordinates, *Dokl. Akad. Nauk SSSR* **298** (1988), 1358–1361.

[8] Arakawa A., Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I, *J. Comp. Phys.* **1** (1966), 119–143.

[9] Basarab-Horwath P., Lahno V. and Zhdanov R., The structure of Lie algebras and the classification problem for partial differential equations, *Acta Appl. Math.* **69** (2001), 43–94.

[10] Batchelor G.K., Computation of the energy spectrum in homogeneous two-dimensional turbulence, *Phys. Fluids* **12**, Suppl. II (1969), 233–239.

[11] Benzi R., Paladin G., Patarnello S., Santangelo P. and Vulpiani A., Intermittency and coherent structures in two-dimensional turbulence, *J. Phys. A* **19** (1986), 3771–3784.

[12] Benzi R., Patarnello S. and Santangelo P., Self-similar coherent structures in two-dimensional decaying turbulence, *J. Phys. A* **21** (1988), 1221–1237.

[13] Bihlo A., *Symmetry methods in the atmospheric sciences*, Ph.D. thesis, University of Vienna, 2010.

[14] Bihlo A. and Bluman G., Conservative parameterization schemes, *J. Math. Phys.* **54** (2013), 083101 (24 pp), arXiv:1209.4279.

[15] Bihlo A., Dos Santos Cardoso-Bihlo E.M. and Popovych R.O., Complete group classification of a class of nonlinear wave equations, *J. Math. Phys.* **53** (2012), 123515 (32 pages), arXiv:1106.4801.

[16] Bihlo A., Dos Santos Cardoso-Bihlo E.M. and Popovych R.O., Invariant parameterization and turbulence modeling on the beta-plane, *Phys. D* **269** (2014), 48–62, arXiv:1112.1917.

[17] Bihlo A. and Nave J.C., Invariant discretization schemes using evolution–projection techniques, *SIGMA* **9** (2013), 23 pages, arXiv:1209.5028.

[18] Bihlo A. and Popovych R.O., Lie symmetries and exact solutions of the barotropic vorticity equation, *J. Math. Phys.* **50** (2009), 123102 (12 pages), arXiv:0902.4099.

[19] Bihlo A. and Popovych R.O., Symmetry analysis of barotropic potential vorticity equation, *Comm. Theor. Phys.* **52** (2009), 697–700, arXiv:0811.3008.

[20] Bihlo A. and Popovych R.O., Lie symmetry analysis and exact solutions of the quasi-geostrophic two-layer problem, *J. Math. Phys.* **52** (2011), 033103 (24 pages), arXiv:1010.1542.

[21] Bihlo A. and Popovych R.O., Point symmetry group of the barotropic vorticity equation, in *Proceedings of 5th Workshop "Group Analysis of Differential Equations & Integrable Systems" (June 6–10, 2010, Protaras, Cyprus)*, 2011 pp. 15–27.

[22] Bihlo A. and Popovych R.O., Invariant discretization schemes for the shallow-water equations, *SIAM J. Sci. Comput.* **34** (2012), B810–B839, arXiv:1201.0498.

[23] Bihlo A. and Popovych R.O., Lie reduction and exact solutions of vorticity equation on rotating sphere, *Phys. Lett. A* **376** (2012), 1179–1184, arXiv:1112.3019.

[24] Bihlo A. and Popovych R.O., Conservative parameterization schemes for the two-dimensional Euler equations, (in preparation), 2014.

[25] Bluman G. and Kumei S., *Symmetries and differential equations*, Springer, New York, 1989.

[26] Bluman G.W., Cheviakov A.F. and Anco S.C., *Application of symmetry methods to partial differential equations*, Springer, New York, 2010.

[27] Bochev P.B. and Hyman J.M., Principles of mimetic discretizations of differential operators, in *Compatible spatial discretizations*, Springer, pp. 89–119, 2006.

[28] Borovskikh A.V., Group classification of the eikonal equations for a three-dimensional nonhomogeneous medium, *Mat. Sb.* **195** (2004), 23–64, in Russian; translation in *Sb. Math.*, 195, (2004), no. 3–4, 479–520.

[29] Borovskikh A.V., The two-dimensional eikonal equation, *Siberian Math. J.* **47** (2006), 813–834.

[30] Bracco A., McWilliams J.C., Murante G., Provenzale A. and Weiss J.B., Revisiting freely decaying two-dimensional turbulence at millennial resolution, *Phys. Fluids* **12** (2000), 2931–2941.

[31] Butcher J., Carminati J. and Vu K.T., A comparative study of some computer algebra packages which determine the Lie point symmetries of differential equations, *Comput. Phys. Comm.* **155** (2003), 92–114.

[32] Carminati J. and Vu K., Symbolic computation and differential equations: Lie symmetries, *J. Symb. Comput.* **29** (2000), 95–116.

[33] Cheh J., Olver P.J. and Pohjanpelto J., Algorithms for differential invariants of symmetry groups of differential equations, *Found. Comput. Math.* **8** (2008), 501–532.

[34] Chen Q., Gunzburger M. and Ringler T., A scale-invariant formulation of the anticipated potential vorticity method, *Mon. Wea. Rev.* **139** (2011), 2614–2629.

[35] Chesnokov A.A., Symmetries and exact solutions of the rotating shallow-water equations, *Eur. J. Appl. Math.* **20** (2009), 461–477.

[36] Chesnokov A.A., Properties and exact solutions of the equations of motion of shallow water in a spinning paraboloid, *J. Appl. Math. Mech.* **75** (2011), 350–356.

[37] Cheviakov A.F., GeM software package for computation of symmetries and conservation laws of differential equations, *Comput. Phys. Comm.* **176** (2007), 48–61.

[38] Chupakhin A.P., Differential invariants: theorem of commutativity, *Commun. Nonlinear Sci. Numer. Simul.* **9** (2004), 25–33.

[39] Crosman E.T. and Horel J.D., Sea and lake breezes: A review of numerical studies, *Bound.-Lay. Meteorol.* **137** (2010), 1–29.

[40] Danilov S. and Gurarie D., Forced two-dimensional turbulence in spectral and physical space, *Phys. Rev. E* **63** (2001), 061208 (12 pages).

[41] Dorodnitsyn V., *Applications of Lie Groups to Difference Equations*, vol. 8 of *Differential and integral equations and their applications*, Chapman & Hall/CRC, Boca Raton, FL, 2011.

[42] Dos Santos Cardoso-Bihlo E.M., Differential invariants for the Korteweg–de Vries equations, in *Proceedings of Sixth International Workshop "Group Analysis of Differential Equations & Integrable Systems" (June 17–21, 2012, Protaras, Cyprus)*, 2012 pp. 71–79.

[43] Dos Santos Cardoso-Bihlo E.M., Bihlo A. and Popovych R.O., Enhanced preliminary group classification of a class of generalized diffusion equations, *Commun. Nonlinear Sci. Numer. Simulat.* **16** (2011), 3622–3638, arXiv:1012.0297.

[44] Dos Santos Cardoso-Bihlo E.M. and Popovych R.O., Complete point symmetry group of the vorticity equation on a rotating sphere, *J. Engrg. Math.* **82** (2013), 31–38, arXiv:1206.6919.

[45] Fels M. and Olver P.J., Moving coframes: I. A practical algorithm, *Acta Appl. Math.* **51** (1998), 161–213.

[46] Fels M. and Olver P.J., Moving coframes. II. Regularization and theoretical foundations, *Acta Appl. Math.* **55** (1999), 127–208.

[47] Fox S. and Davidson P., Freely decaying two-dimensional turbulence, *J. Fluid Mech.* **659** (2010), 351–364.

[48] Frisch U., *Turbulence: The Legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge, 1995.

[49] Fushchych W.I. and Popovych R.O., Symmetry reduction and exact solutions of the Navier–Stokes equations, *J. Nonlinear Math. Phys.* **1** (1994), 75–113,156–188, arXiv:math-ph/0207016.

[50] Gandarias M.L., Torrisi M. and Valenti A., Symmetry classification and optimal systems of a non-linear wave equation, *Internat. J. Non-Linear Mech.* **39** (2004), 389–398.

[51] Gardner C.S., Greene J.M., Kruskal M.D. and Miura R.M., Method for solving the Korteweg–de Vries equation, *Phys. Rev. Lett.* **19** (1967), 1095–1097.

[52] Golovin S.V., Applications of the differential invariants of infinite dimensional groups in hydrodynamics, *Commun. Nonlinear Sci. Numer. Simul.* **9** (2004), 35–51.

[53] Gray R.J., *Automorphisms of Lie Algebras*, Ph.D. Thesis, University of Surrey, Surrey, 2013.

[54] Harin A.O., On a countable-dimensional subalgebra of the equivalence algebra for equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *J. Math. Phys.* **34** (1993), 3676–3682.

[55] Head A.K., LIE, a PC program for Lie analysis of differential equations, *Comput. Phys. Comm.* **77** (1993), 241–248, (See also http://www.cmst.csiro.au/LIE/LIE.htm).

[56] Huang D. and Ivanova N.M., Group analysis and exact solutions of a class of variable coefficient nonlinear telegraph equations, *J. Math. Phys.* **48** (2007), 073507, 23 pp.

[57] Huang H.P., Galperin B. and Sukoriansky S., Anisotropic spectra in two-dimensional turbulence on the surface of a rotating sphere, *Phys. Fluids* **13** (2001), 225–240.

[58] Hydon P.E., How to construct the discrete symmetries of partial differential equations, *Eur. J. Appl. Math.* **11** (2000), 515–527.

[59] Hydon P.E., *Symmetry methods for differential equations*, Cambridge University Press, Cambridge, 2000.

[60] Ibragimov N.H., Aksenov A.V., Baikov V.A., Chugunov V.A., Gazizov R.K. and Meshkov A.G., *CRC handbook of Lie group analysis of differential equations. Vol. 2. Applications in engineering and physical sciences. Edited by N. H. Ibragimov*, CRC Press, Boca Raton, 1995.

[61] Ibragimov N.H. and Khabirov S.V., Contact transformation group classification of nonlinear wave equations, *Nonlin. Dyn.* **22** (2000), 61–71.

[62] Ibragimov N.H., Kovalev V.F. and Pustovalov V.V., Symmetries of integro-differential equations: a survey of methods illustrated by the Benny equations, *Nonlin. Dyn.* **28** (2002), 135–153.

[63] Ibragimov N.H., Torrisi M. and Valenti A., Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *J. Math. Phys.* **32** (1991), 2988–2995.

[64] Ibragimov N.H., Torrisi M. and Valenti A., Differential invariants of nonlinear equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *Commun. Nonlinear Sci. Numer. Simul.* **9** (2004), 69–80.

[65] Ivanova N.M., Popovych R.O. and Sophocleous C., Group analysis of variable coefficient diffusion-convection equations. I. Enhanced group classification, *Lobachevskii J. Math.* **31** (2010), 100–122.

[66] Jeffrey A., Acceleration wave propagation in hyperelastic rods of variable cross-section, *Wave Motion* **4** (1982), 173–180.

[67] Kasahara A., Various vertical coordinate systems used for numerical weather prediction, *Mon. Wea. Rev.* **102** (1974), 509–522.

[68] Katkov V.L., A class of exact solutions of the equation for the forecast of the geopotential, *Izv. Akad. Nauk. SSSR Ser. Fiz. Atmosfer. i Oceana* **1** (1965), 630–631.

[69] Khabirov S.V., A property of the determining equations for an algebra in the group classification problem for wave equations, *Sibirsk. Mat. Zh.* **50** (2009), 647–668, in Russian; translation in *Sib. Math. J.*, 50:515–532, 2009.

[70] Kim P., Invariantization of numerical schemes using moving frames, *BIT Numerical Mathematics* **47** (2007), 525–546.

[71] Kim P., Invariantization of the Crank–Nicolson method for Burgers' equation, *Phys. D* **237** (2008), 243–254.

[72] Kingston J.G. and Sophocleous C., On point transformations of a generalised Burgers equation, *Phys. Lett. A* **155** (1991), 15–19.

[73] Kingston J.G. and Sophocleous C., On form-preserving point transformations of partial differential equations, *J. Phys. A* **31** (1998), 1597–1619.

[74] Kingston J.G. and Sophocleous C., Symmetries and form-preserving transformations of one-dimensional wave equations with dissipation, *Int. J. Non-Lin. Mech.* **36** (2001), 987–997.

[75] Kolmogoroff A.N., The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers **30** (1941), 299–303.

[76] Kraichnan R.H., Inertial ranges in two-dimensional turbulence, *Phys. Fluids* **10** (1967), 1417–1423.

[77] Kumei S. and Bluman G.W., When nonlinear differential equations are equivalent to linear differential equations, *SIAM J. Appl. Math.* **42** (1982), 1157–1173.

[78] Lagno V.I. and Samoilenko A.M., Group classification of nonlinear evolution equations: I. Invariance under semisimple local transformation groups, *Differ. Equ.* **38** (2002), 384–391.

[79] Lahno V., Zhdanov R. and Magda O., Group classification and exact solutions of nonlinear wave equations, *Acta Appl. Math.* **91** (2006), 253–313.

[80] Lahno V.I. and Spichak S.V., Group classification of quasilinear elliptic-type equations. I. Invariance with respect to Lie algebras with nontrivial Levi decomposition, *Ukrainian Math. J.* **59** (2007), 1719–1736.

[81] Legras B., Santangelo P. and Benzi R., High-resolution numerical experiments for forced two-dimensional turbulence, *Europhys. Lett.* **5** (1988), 37–42.

[82] Levi D. and Winternitz P., Continuous symmetries of difference equations, *J. Phys. A* **39** (2006), R1–R63.

[83] Lie S., Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen, *Arch. for Math.* **6** (1881), 328–368, (Translation by N.H. Ibragimov: S. Lie, On Integration of a Class of Linear Partial Differential Equations by Means of Definite Integrals, *CRC Handbook of Lie Group Analysis of Differential Equations*, 2:473–508, 1994).

[84] Lie S., *Theorie der Transformationsgruppen*, vol. 1–3, B. G. Teubner, Leipzig, 1888, 1890, 1893.

[85] Lie S., *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, B.G. Teubner, Leipzig, 1891.

[86] Lilly D.K., Numerical simulation of developing and decaying two-dimensional turbulence, *J. Fluid Mech.* **45** (1971), 395–415.

[87] Lisle I.G., *Equivalence transformations for classes of differential equations*, Ph.D. thesis, University of British Columbia, 1992.

[88] Magadeev B.A., Group classification of nonlinear evolution equations, *Algebra i Analiz* **5** (1993), 141–156, (in Russian); English translation in *St. Petersburg Math. J.* 5:345–359, 1994.

[89] Maltrud M.E. and Vallis G.K., Energy spectra and coherent structures in forced two-dimensional and beta-plane turbulence., *J. Fluid Mech.* **228** (1991), 321–342.

[90] Marshall D.P. and Adcroft A.J., Parameterization of ocean eddies: Potential vorticity mixing, energetics and Arnold's first stability theorem, *Ocean Modell.* **32** (2010), 188–204.

[91] Meleshko S.V., Group classification of equations of two-dimensional gas motions, *Prikl. Mat. Mekh.* **58** (1994), 56–62, in Russian; translation in *J. Appl. Math. Mech.*, 58:629–635.

[92] Meleshko S.V., Generalization of the equivalence transformations, *J. Nonlin. Math. Phys.* **3** (1996), 170–174.

[93] Nadjafikhah M. and Bakhshandeh-Chamazkoti R., Symmetry group classification for general Burgers' equation, *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010), 2303–2310.

[94] Nikitin A.G., Group Classification of Systems of Nonlinear Reaction-Diffusion Equations, *Ukr. Math. Bull.* **2** (2005), 153–204.

[95] Nikitin A.G., Group classification of systems of non-linear reaction-diffusion equations with general diffusion matrix. I. Generalized Ginzburg-Landau equations, *J. Math. Anal. Appl.* **324** (2006), 615–628.

[96] Nikitin A.G., Group classification of systems of non-linear reaction-diffusion equations with general diffusion matrix. II. Generalized Turing systems, *J. Math. Anal. Appl.* **332** (2007), 666–690.

[97] Nikitin A.G. and Popovych R.O., Group classification of nonlinear Schrödinger equations, *Ukrainian Math. J.* **53** (2001), 1255–1265.

[98] Oberlack M., *Invariant modeling in large-eddy simulation of turbulence*, *in: Annual research briefs*, Stanford University, 1997.

[99] Oberlack M., *Symmetrie, Invarianz und Selbstähnlichkeit in der Turbulenz*, Habilitationsschrift, Rheinisch–Westfählische Technische Hochschule Aachen, 2000.

[100] Oberlack M., A unified approach for symmetries in plane parallel turbulent shear flows, *J. Fluid Mech.* **427** (2001), 299–328.

[101] Olver P.J., *Application of Lie groups to differential equations*, Springer, New York, 2000.

[102] Olver P.J., Geometric foundations of numerical algorithms and symmetry, *Appl. Algebra Engrg. Comm. Comput.* **11** (2001), 417–436.

[103] Olver P.J., Moving frames, *J. Symbolic Comput.* **36** (2003), 501–512.

[104] Olver P.J., A survey of moving frames, in *Proceedings of the 6th international conference on Computer Algebra and Geometric Algebra with Applications*, Springer-Verlag, 2004 pp. 105–138.

[105] Olver P.J., Generating differential invariants, *J. Math. Anal. Appl.* **333** (2007), 450–471.

[106] Olver P.J., Recent advances in the theory and application of Lie pseudo-groups, in *AIP Conference Proceedings*, vol. 1260, vol. 1260, 2010 pp. 35–63.

[107] Olver P.J. and Pohjanpelto J., Moving frames for Lie pseudo-groups, *Canadian J. Math.* **60** (2008), 1336–1386.

[108] Olver P.J. and Pohjanpelto J., Differential invariant algebras of Lie pseudo-groups, *Adv. Math.* **222** (2009), 1746–1792.

[109] Olver P.J., Pohjanpelto J. and Valiquette F., On the structure of Lie pseudo-groups, *SIGMA* **5** (2009), paper 077, 14 pp.

[110] Oron A. and Rosenau P., Some symmetries of the nonlinear heat and wave equations, *Phys. Lett. A* **118** (1986), 172–176.

[111] Ovsiannikov L.V., Group properties of nonlinear heat equation, *Dokl. AN SSSR* **125** (1959), 492–495, in Russian.

[112] Ovsiannikov L.V., *Group analysis of differential equations*, Acad. Press, New York, 1982.

[113] Ovsjannikov L.V., *Gruppovye svoistva differentsialnykh uravnenii.*, Izdat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1962.

[114] Ovsjannikov L.V. and Ibragimov N.H., Group analysis of the differential equations of mechanics, in *General mechanics*, vol. 2, Moscow, pp. 5–52, 1975. In Russian.

[115] Physick W., A numerical model of the sea-breeze phenomenon over a lake or gulf, *J. Atmos. Sci.* **33** (1976), 2107–2135.

[116] Platzman G.W., The spectral form of the vorticity equation, *J. Meteor.* **17** (1960), 635–644.

[117] Pocheketa O.A. and Popovych R.O., Extended symmetry analysis of generalized Burgers equations, (in preparation), 2014.

[118] Pope S.B., *Turbulent flows*, Cambridge University Press, Cambridge, 2000.

[119] Popovych G., On reduction of the Euler equations by means of two-dimensional algebras, *J. Nonlinear Math. Phys.* **3** (1967), 441–446.

[120] Popovych H.V., Lie, partially invariant and nonclassical submodels of the Euler equations, in *Proceedings of Institute of Mathematics of NAS of Ukraine*, vol. 43/1, Kyiv, vol. 43/1, 2002 pp. 178–183.

[121] Popovych R., Vaneeva O. and Ivanova N., Potential nonclassical symmetries and solutions of fast diffusion equation, *Phys. Lett. A* **362** (2007), 166–173.

[122] Popovych R.O., Classification of admissible transformations of differential equations, in *Collection of Works of Institute of Mathematics*, vol. 3, Kyiv, pp. 239–254, 2006.

[123] Popovych R.O. and Bihlo A., Symmetry preserving parameterization schemes, *J. Math. Phys.* **53** (2012), 073102 (36 pages), arXiv:1010.3010.

[124] Popovych R.O., Boyko V.M., Nesterenko M.O. and Lutfullin M.W., Realizations of real low-dimensional Lie algebras, *J. Phys. A* **36** (2003), 7337–7360, see arXiv:math-ph/0301029v7 for an extended and revised version.

[125] Popovych R.O. and Ivanova N.M., New results on group classification of nonlinear diffusion–convection equations, *J. Phys. A* **37** (2004), 7547–7565.

[126] Popovych R.O., Ivanova N.M. and Eshraghi H., Group classification of (1+1)-dimensional Schrödinger equations with potentials and power nonlinearities, *J. Math. Phys.* **45** (2004), 3049–3057.

[127] Popovych R.O., Kunzinger M. and Eshraghi H., Admissible transformations and normalized classes of nonlinear Schrödinger equations, *Acta Appl. Math.* **109** (2010), 315–359.

[128] Popovych R.O., Kunzinger M. and Ivanova N.M., Conservation laws and potential symmetries of linear parabolic equations, *Acta Appl. Math.* **100** (2008), 113–185.

[129] Popovych R.O. and Vaneeva O.O., More common errors in finding exact solutions of nonlinear differential equations: Part I, *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010), 3887–3899, arXiv:0911.1848v2.

[130] Popovych V., On classes of Lie solutions of MHD equations, expressed via the general solution of the heat equation, *J. Nonlinear Math. Phys.* **4** (1997), 149–151.

[131] Razafindralandy D., Hamdouni A. and Béghein C., A class of subgrid-scale models preserving the symmetry group of Navier–Stokes equations, *Commun. Nonlinear Sci. Numer. Simul.* **12** (2007), 243–253.

[132] Razafindralandy D., Hamdouni A. and Oberlack M., Analysis and development of subgrid turbulence models preserving the symmetry properties of the Navier–Stokes equations, *Eur. J. Mech. B/Fluids* **26** (2007), 531–550.

[133] Rebelo R. and Valiquette F., Symmetry preserving numerical schemes for partial differential equations and their numerical tests, *J. Difference Equ. Appl.* **19** (2013), 738–757.

[134] Rhines P.B., Waves and turbulence on a beta-plane, *J. Fluid Mech.* **69** (1975), 417–443.

[135] Rocha Filho T.M. and Figueiredo A., [SADE] a Maple package for the symmetry analysis of differential equations, *Comput. Phys. Comm.* **182** (2011), 467–476.

[136] Sadourny R. and Basdevant C., Parameterization of subgrid scale barotropic and baroclinic eddies in quasi-geostrophic models: Anticipated potential vorticity method, *J. Atmos. Sci.* **42** (1985), 1353–1363.

[137] Salmon R., *Lectures on geophysical fluid dynamics*, Oxford University Press, New York, 1998.

[138] Santangelo P., Benzi R. and Legras B., The generation of vortices in high-resolution, two-dimensional decaying turbulence and the influence of initial conditions on the breaking of self-similarity, *Phys. Fluids* **1** (1989), 1027–1034.

[139] Shen X. and Warhaft Z., The anisotropy of the small scale structure in high Reynolds number ($R_\lambda \sim 1000$) turbulent shear flow, *Phys. Fluids* **12** (2000), 2976–2989.

[140] Song L. and Zhang H., Preliminary group classification for the nonlinear wave equation $u_{tt} = f(x,u)u_{xx} + g(x,u)$, *Nonlinear Anal.* **70** (2009), 3512–3521.

[141] Speziale C.G., Galilean invariance of subgrid-scale stress models in the large-eddy simulation of turbulence, *J. Fluid Mech.* **156** (1985), 55–62.

[142] Staniforth A. and Thuburn J., Horizontal grids for global weather and climate prediction models: a review, *Q. J. R. Meteorol. Soc.* **138** (2012), 1–26.

[143] Stensrud D.J., *Parameterization schemes: Keys to understanding numerical weather prediction models*, Cambridge University Press, Cambridge, 2007.

[144] Stull R.B., *An introduction to boundary layer meteorology*, vol. 13 of *Atmospheric Sciences Library*, Kluwer Academic Publishers, Dortrecht, 1988.

[145] Thuburn J. and Cotter C.J., A framework for mimetic discretization of the rotating shallow-water equations on arbitrary polygonal grids, *SIAM J. Sci. Comput.* **34** (2012), B203–B225.

[146] Valiquette F. and Winternitz P., Discretization of partial differential equations preserving their physical symmetries, *J. Phys. A* **38** (2005), 9765–9783.

[147] Vallis G.K., *Atmospheric and Oceanic Fluid Dynamics*, Cambridge University Press, Cambridge, 2006.

[148] Vallis G.K. and Hua B.L., Eddy viscosity of the anticipated potential vorticity method, *J. Atmos. Sci.* **45** (1988), 617–627.

[149] Vaneeva O.O., Johnpillai A.G., Popovych R.O. and Sophocleous C., Enhanced group analysis and conservation laws of variable coefficient reaction-diffusion equations with power nonlinearities, *J. Math. Anal. Appl.* **330** (2007), 1363–1386.

[150] Vaneeva O.O., Popovych R.O. and Sophocleous C., Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, *Acta Appl. Math.* **106** (2009), 1–46.

[151] Vaneeva O.O., Popovych R.O. and Sophocleous C., Extended group analysis of variable coefficient reaction-diffusion equations with exponential nonlinearities, *J. Math. Anal. Appl.* **396** (2012), 225–242.

[152] Vu K.T., Butcher J. and Carminati J., Similarity solutions of partial differential equations using DESOLV, *Comput. Phys. Comm.* **176** (2007), 682–693.

[153] Vu K.T., Jefferson G.F. and Carminati J., Finding higher symmetries of differential equations using the MAPLE package DESOLVII, *Comput. Phys. Comm.* **183** (2012), 1044–1054.

[154] Wilcox D.C., *Turbulence modeling for CFD*, DCW Industries, Inc., La Cañada, CA, 1993.

[155] Williams P.D., A proposed modification to the Robert-Asselin time filter, *Mon. Wea. Rev.* **137** (2009), 2538–2546.

[156] Winternitz P. and Gazeau J.P., Allowed transformations and symmetry classes of variable coefficient Korteweg-de Vries equations, *Phys. Lett. A* **167** (1992), 246–250.

[157] Wittkopf A., *Algorithms and implementations for differential elimination*, Ph.D. thesis, Simon Fraser University Burnaby, BC, Canada, 2004.

[158] Wolf T., A comparison of four approaches to the calculation of conservation laws, *Eur. J. App. Math.* **13** (2002), 129–152.

[159] Zhdanov R.Z. and Lahno V.I., Group classification of heat conductivity equations with a nonlinear source, *J. Phys. A* **32** (1999), 7405–7418.

Curriculum Vitae

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Academic education

Nov 2008	Degree in Meteorology (M.Sc.) at the University of Vienna
Aug 2010–	Ph.D. candidate in meteorology at the University of Vienna

Professional experience

May 2010–	Ph.D. student at the Wolfgang Pauli Institute and the Faculty of Mathematics, University of Vienna
Jan 2009–Feb 2010	Operational meteorologist at MeteoServe Company, Vienna
Feb 2003–Feb 2006	Climatologist at the Central Institute for Meteorology and Geodynamics (ZAMG), Vienna

Research projects

April 2014–	Project member in FWF project P23714 *Geometry of algebras of generalized functions* (P.I. Prof. Dr. Michael Grosser)
April 2013–March 2014	Project member in FWF project P25064 *Extended group analysis of differential equations* (P.I. D.Sc. Roman O. Popovych)
March 2010–March 2014	Member of COST-project ES0905 *Basic Concepts for Convection Parameterization in Weather Forecast and Climate Models* (P.I. Dr. Jun-Ichi Yano)
April 2009–Mar 2012	Project member in FWF project P20632 *Classification problems of group analysis* (P.I. D.Sc. Roman O. Popovych)
Jan 2009–Feb 2010	Project leader of DIBMETSAT (Digital image processing - supported meteorology - services for air traffic management) for MeteoServe Company
Feb 2003–Feb 2006	Project member in Reclip:more project *Research for Climate Protection* (P.I. Dr. Wolfgang Loibl)

Thesis

- Dos Santos Cardoso, E.M., 2008. *Distance measures for statistical downscaling.* M.Sc. thesis, University of Vienna, 160 pp. (Supervisor: Prof. Dr. Georg Skoda)

Publications

Published and accepted

- Bihlo, A., E.M. Dos Santos Cardoso-Bihlo and R.O. Popovych, 2014. Invariant parameterization and turbulence modeling on the beta-plane. *Physica D* **269**, 48–62, arXiv:1112.1917.
- Dos Santos Cardoso-Bihlo E.M., Differential invariants for the Korteweg–de Vries equation, *Proceedings of the Sixth International Workshop "Group Analysis of Differential Equations and Integrable Systems" (Protaras, Cyprus, June 17–21, 2012)*, University of Cyprus, Nicosia, 2013, 17–29, arXiv:1307.4452.
- Dos Santos Cardoso-Bihlo E. and Popovych R.O., Complete point symmetry group of vorticity equation on rotating sphere, *J. Engrg. Math.* **82** (2013), 31-38, arXiv:1206.6919.
- Bihlo, A., E.M. Dos Santos Cardoso-Bihlo and R.O. Popovych, 2012. Complete group classification of a class of nonlinear wave equations. *J. Math. Phys.* **53** (2012), 123515, 32 pp., arXiv:1106.4801.
- Dos Santos Cardoso-Bihlo, E.M., A. Bihlo and R.O. Popovych, 2011. Enhanced preliminary group classification of a class of generalized diffusion equations. *Commun. Nonlinear Sci. Numer. Simul.* **16** (9), 3622-3638, arXiv:1012.0297v1.

Preprints

- Bihlo, A., E.M. Dos Santos Cardoso-Bihlo and R.O. Popovych, 2014. Invariant and conservative parameterization schemes, in volume 2 of *Parameterization of Atmospheric Convection*, (R. S. Plant and J. I. Yano, Eds.), Imperial College Press, in press.
- Dos Santos Cardoso-Bihlo, E.M. and R.O. Popovych, 2014. On the ineffectiveness of constant rotation in the primitive equations.

Selected presentations

- Dec 2012: "Complete group classification of a class of nonlinear wave equations", poster presentation at the 2012 CMS Winter Meeting, Montréal
- May 2012: "Invariant parameterization schemes", poster presentation at the conference *Symmetries of Differential Equations: Frames, Invariants and Applications*, University of Minnesota
- May 2012: "Invariant turbulence modeling", poster presentation at the conference *Symmetries of Differential Equations: Frames, Invariants and Applications*, University of Minnesota
- Feb 2012: "Invariant parameterization schemes", talk at the Department of Meteorology at the University of Reading, UK
- Jun 2011: "Group analysis and invariant parameterization in dynamic meteorology", talk at the Faculty of Geosciences, Geography and Astronomy at the University of Vienna
- Jun 2006: "Distance measures for statistical downscaling", talk at the Faculty of Geosciences, Geography and Astronomy at the University of Vienna
- March 2006 "Empirical Downscaling of dynamically downscaled ERA40 fields", poster presentation at University of Natural Resources and Life Sciences, Vienna
- Oct 2004 "An introduction to R", course at the Central Institute for Meteorology and Geodynamics
- Oct 2003: "Geographical refinement of dynamically and statistically downscaled air temperature and precipitation for Austria", talk at GIS meeting, Budapest, Hungary

Conference attendances

7 Dec–10 Dec 2012	2012 Canadian Mathematical Society winter meeting, Montréal, QC, Canada
17–20 May 2012	Conference *Symmetries of Differential Equations: Frames, Invariants and Applications*, University of Minnesota, MN, USA
1–2 Feb 2012	COST working group meeting, Reading, UK
2 May 2011	COST working group meeting, Munich, Germany
22–25 March 2011	COST workshop, Cambridge, UK
6–10 June 2010	Fifth International Workshop *Group Analysis of Differential Equations and Integrable Systems*, Protaras, Cyprus
16–17 March	9th Austrian Climate Day *Climate, Climate Change and Implications*
25 Oct 2003	GIS meeting, Budapest, Hungary

Short-term scientific visits

May–June 2012	Research stay at the Department of Mathematics at the University of British Columbia, Vancouver, BC, Canada

Reviews

- British Journal of Mathematics & Computer Science
- Zeitschrift für Naturforschung A

Skills

Language skills	Portuguese, German, English, Spanish, French
Programming skills	Fortran, IDL, MatLab, R, SQL
Software skills	Excel, LaTeX, Linux, Maple, MS-Office, Windows, SPSS, GIS