On one class of Tribin functions related to s-symbol encodings of numbers with zero redundancy

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Theory of Approximation of Functions and Related Topics
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Outline

We construct a continuum class of continuous nowhere monotonic functions using abstract s-symbol representations that are topologically equivalent to the classical s-adic representation.

M. Pratsiovytyi, S. Ratushniak, O. Baranovskyi, and I. Lysenko, *A class of Tribin functions related to s-symbol encodings of numbers with zero redundancy*, Preprint, arXiv:2602.14103 [math.FA] (2026), 11 p. `doi:10.48550/arXiv.2602.14103`

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An encoding of real numbers and their g-representation

Let $s > 1$ be a fixed natural number,
$A_s = \{0, 1, 2, \ldots, s-1\}$ an s-adic alphabet (a set of digits),
and $L_s = A_s \times A_s \times \ldots$ a space of sequences of elements of the alphabet.

An encoding of numbers of the closed interval $[0, 1]$ with an alphabet A_s is a surjective mapping g of the space L_s into $[0, 1]$:
$L_s \xrightarrow{g} [0, 1]$.

$x = \Delta^{g}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$, $(\alpha_n) \in L_s$, is called a g-representation of the number x.

Topologically equivalent representations

Two encodings with zero redundancy (g_1-representation and g_2-representation of numbers) using a common alphabet A are called topologically equivalent if the projector

$$\Delta^{g_1}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \to \Delta^{g_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$$

is continuous and strictly monotonic.

The system of g-encoding is said to have zero redundancy if almost all numbers have a unique g-representation (perhaps, except for a countable set of numbers having two representations).

The main object of study i

Let $s > 2$ be a fixed natural number,
$\Delta^{s*}_{\alpha_1\alpha_2...\alpha_n...}$ an s-symbol representation of a number $x \in [0, 1]$ that is topologically equivalent to the classical s-adic representation:

$$x = \Delta^{s}_{\alpha_1\alpha_2...\alpha_n...} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \ldots + \frac{\alpha_n}{s^n} + \ldots, \quad (\alpha_n) \in L_s,$$

and $\Delta^{2*}_{\beta_1\beta_2...\beta_n...}$ a two-symbol representation of a number $y \in [0, 1]$ that is topologically equivalent to the classical binary representation:

$$y = \Delta^{2}_{\beta_1\beta_2...\beta_n...} = \frac{\beta_1}{2} + \frac{\beta_2}{2^2} + \ldots + \frac{\beta_n}{2^n} + \ldots, \quad (\beta_n) \in L_2.$$

Let $x \in [0,1]$. Let A_0 and $A_1 = A_s \setminus A_0$ be non-empty subsets of A_s. We define a function f by the following equality:

$$f(\Delta^{s^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{2^*}_{\beta_1\beta_2\ldots\beta_n\ldots}, \tag{1}$$

where

$$\beta_1 = \begin{cases} 0 & \text{if } \alpha_1 \in A_0, \\ 1 & \text{if } \alpha_1 \in A_1, \end{cases} \tag{2}$$

$$\beta_{n+1} = \begin{cases} \beta_n & \text{if } \alpha_{n+1} = \alpha_n, \\ 1 - \beta_n & \text{if } \alpha_{n+1} \neq \alpha_n. \end{cases} \tag{3}$$

Motivation

A function is called nowhere monotonic if it does not have any (arbitrarily small) interval of monotonicity.

Among nowhere monotonic functions,

- there exist functions of bounded and unbounded variation,
- non-differentiable functions and functions with "satisfactory" differential properties,
  - including singular functions (functions whose derivative is equal to zero almost everywhere with respect to the Lebesgue measure).

To provide an analytical definition for such functions and study their properties we use various systems of encoding for real numbers (systems of representation, or numeral systems).

Examples of functions defined by using various systems of encoding for real numbers.

- T. Takagi, *A simple example of the continuous function without derivative*, Tōkyō Sūgaku-Butsurigakkwai Hōkoku **1** (1901), 176–177.
- W. Sierpiński, *Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donnée*, C. R. Acad. Sci. Paris **162** (1916), 629–632.

A general construction of Cantor projectors.

M. V. Pratsiovytyi, *Continuous Cantor projectors*, Methods of the Study of Algebraic and Topological Structures, Kyiv State Pedagog. Inst., Kyiv, 1989, pp. 95–105 (in Russian).

A. F. Turbin and M. V. Pratsiovytyi, *Fractal sets, functions, and probability distributions*, Nauk. Dumka, Kyiv, 1992 (in Russian). MR 1353239 (96f:28010)

A simple example of a continuous nowhere differentiable function (the Tribin function):

$$\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \to \Delta^2_{\beta_1\beta_2\ldots\beta_n\ldots}$$

M. V. Pratsiovytyi, *Fractal properties of one continuous nowhere differentiable function*, Nauk. Zap. Nats. Pedagog. Univ. Mykhaila Drahomanova. Fiz.-Mat. Nauky (2002), no. 3, 351–362 (in Ukrainian).

Independent examples.

- K. A. Bush, *Continuous functions without derivatives*, Amer. Math. Monthly **59** (1952), no. 4, 222–225. MR 0049278 (14,148b)
- W. Wunderlich, *Eine überall stetige und nirgends differenzierbare Funktion*, Elem. Math. **7** (1952), no. 4, 73–79. MR 0049279 (14,148c)

Using the Q_s^*-representation for real numbers,
a non-self-similar generalization of the classical s-adic representation.

M. V. Pratsiovytyi, O. M. Baranovskyi, and Yu. P. Maslova, *Generalization of the Tribin function*, J. Math. Sci. (N. Y.) **253** (2021), no. 2, 276–288. MR 4016749

References to the papers by V. Koval, O. Panasenko,
M. Pratsiovytyi, S. Ratushniak, and N. Vasylenko can be found in our preprint.

Well-definedness and continuity i

$$f(\Delta^{s^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{2^*}_{\beta_1\beta_2\ldots\beta_n\ldots}, \quad \alpha_n \in A_s = \{0, 1, 2, \ldots, s-1\},$$

$$\beta_1 = \begin{cases} 0 & \text{if } \alpha_1 \in A_0, \\ 1 & \text{if } \alpha_1 \in A_1, \end{cases} \quad \beta_{n+1} = \begin{cases} \beta_n & \text{if } \alpha_{n+1} = \alpha_n, \\ 1-\beta_n & \text{if } \alpha_{n+1} \neq \alpha_n, \end{cases}$$

Lemma
The function f is well defined.

Well-definedness and continuity ii

Theorem
Suppose given g_1- and g_2-representation for numbers in $[0,1]$ *are continuous representations with zero redundancy.*
A necessary and sufficient condition for the function ψ defined by equality

$$\psi(x = \Delta^{g_1}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{g_2}_{\beta_1\beta_2\ldots\beta_n\ldots}, \tag{4}$$

where $\beta_n = \varphi_n(\alpha_1, \ldots, \alpha_n)$, to be continuous at every point of $[0,1]$ *is that the function be well defined at every g_1-binary point.*

An encoding (g-representation) of numbers is called continuous if every g-cylinder is an interval, and cylinders of the same rank do not have common interior points (i.e., do not overlap).

Level sets

$$f(\Delta^{s^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{2^*}_{\beta_1\beta_2\ldots\beta_n\ldots}, \quad \alpha_n \in A_s = \{0, 1, 2, \ldots, s-1\},$$

$$\beta_1 = \begin{cases} 0 & \text{if } \alpha_1 \in A_0, \\ 1 & \text{if } \alpha_1 \in A_1, \end{cases} \quad \beta_{n+1} = \begin{cases} \beta_n & \text{if } \alpha_{n+1} = \alpha_n, \\ 1-\beta_n & \text{if } \alpha_{n+1} \neq \alpha_n, \end{cases}$$

Theorem
The range of the function f, $D_f = f([0,1])$, is the closed interval $[0,1]$. The image of a cylinder is a cylinder. The function f has finite and continuum level sets.

The level set y_0 of a function f is a set

$$f^{-1}(y_0) = \{x \colon f(x) = y_0\}.$$

Nowhere monotonicity

$$f(\Delta^{s*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{2*}_{\beta_1\beta_2\ldots\beta_n\ldots}, \quad \alpha_n \in A_s = \{0, 1, 2, \ldots, s-1\},$$

$$\beta_1 = \begin{cases} 0 & \text{if } \alpha_1 \in A_0, \\ 1 & \text{if } \alpha_1 \in A_1, \end{cases} \quad \beta_{n+1} = \begin{cases} \beta_n & \text{if } \alpha_{n+1} = \alpha_n, \\ 1 - \beta_n & \text{if } \alpha_{n+1} \neq \alpha_n, \end{cases}$$

Theorem
The function f is nowhere monotonic.

Variation

$$f(\Delta^{s^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^{2^*}_{\beta_1\beta_2\ldots\beta_n\ldots}, \quad \alpha_n \in A_s = \{0, 1, 2, \ldots, s-1\},$$

$$\beta_1 = \begin{cases} 0 & \text{if } \alpha_1 \in A_0, \\ 1 & \text{if } \alpha_1 \in A_1, \end{cases} \quad \beta_{n+1} = \begin{cases} \beta_n & \text{if } \alpha_{n+1} = \alpha_n, \\ 1 - \beta_n & \text{if } \alpha_{n+1} \neq \alpha_n, \end{cases}$$

Theorem
The function f is a continuous function of unbounded variation.

Summary

- We construct a continuum class of functions with complicated local structure, namely, a class of continuous nowhere monotonic functions.
- To this end, we use abstract s-symbol representations that are topologically equivalent to the classical s-adic representation.
- This class generalize certain known non-differentiable functions, including the Bush function, Wunderlich function, continuous Cantor projectors, Tribin function.

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