# Vector bundles and torsion free sheaves on degenerations of elliptic curves

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**Summary.** In this paper we give a survey about the classification of vector bundles and torsion free sheaves on degenerations of elliptic curves. Coherent sheaves on singular curves of arithmetic genus one can be studied using the technique of matrix problems or via Fourier-Mukai transforms, both methods are discussed here. Moreover, we include new proofs of classical results about vector bundles on elliptic curves.

## 1 Overview

The aim of this paper is to give a survey of results on the classification of vector bundles and torsion free sheaves on singular projective curves of arithmetic genus one. We include new proofs of some classical results on vector bundles on smooth elliptic curves, which use the technique of derived categories and Fourier-Mukai transforms and are simpler than the original ones. Some results about singular curves are new or at least presented in a new framework.

This research project had several sources of motivation and inspiration. Our study of vector bundles on degenerations of elliptic curves was originally motivated by the McKay correspondence for minimally elliptic surface singularities [Kah89]. Here we use as the main technical tool methods from the representation theory of associative algebras, see for example [BD05]. In particular, a key tool in our approach to classification problems is played by the technique of "representations of bunches of chains" or "Gelfand problems" [Bon92]. At last, but not least we want to mention that our research was strongly influenced by ideas and methods coming from the homological mirror symmetry [Kon95, PZ98, FMW99].

Many different questions concerning properties of the category of vector bundles and coherent sheaves on degenerations of elliptic curves is encoded in the following general set-up:

**Problem 1.** Let $\mathbb{E} \longrightarrow \mathbb{T}$ be a flat family of projective curves of arithmetic genus one such that the fiber $\mathbb{E}_t$ is smooth for $t \neq 0$ and singular for $t = 0$. What happens with the derived category $D^b(\mathsf{Coh}_{\mathbb{E}_t})$, when $t \to 0$?
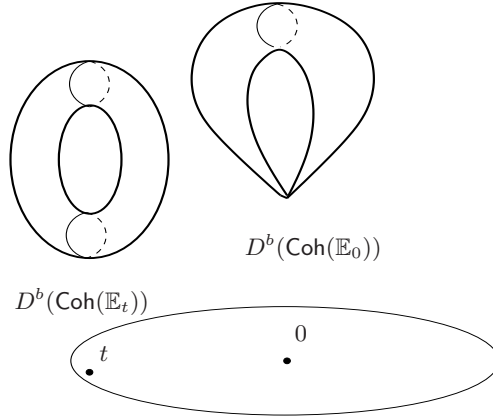


In order to start working on this question one has to consider the absolute case first, where the base $\mathbb{T}$ is $\mathsf{Spec}(\mathbf{k})$. In particular, one has to describe indecomposable vector bundles and indecomposable objects of the derived category of coherent sheaves on degenerations of elliptic curves and develop a technique to calculate homomorphism and extension spaces between indecomposable torsion free sheaves as well as various operations on them, like tensor products and dualizing.

For the first time we face this sort of problems when dealing with the McKay correspondence for *minimally elliptic singularities*. Namely, let $\mathbb{S} = \mathsf{Spec}(\mathbf{R})$ be the spectrum of a complete (or analytical) two-dimensional minimally elliptic singularity, $\pi : \widetilde{\mathbb{X}} \longrightarrow \mathbb{S}$ its minimal resolution, and $\mathbb{E}$ the exceptional divisor. Due to a construction of Kahn [Kah89], the functor[5] $M \mapsto res_{\mathbb{E}}(\pi^*(M)^{\vee\vee})$ establishes a bijection between the reflexive $\mathbf{R}$–modules (maximal Cohen-Macaulay modules) and the generically globally generated indecomposable vector bundles on $\mathbb{E}$ with vanishing first cohomology.

A typical example of a minimally elliptic singularity is a $T_{pqr}$–singularity, given by the equation $\mathbf{k}[[x, y, z]]/(x^p + y^q + z^r - \lambda xyz)$, where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$ and $\lambda \neq 0$. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then this singularity is *simple elliptic* and the exceptional divisor $\mathbb{E}$ is a smooth elliptic curve. Thus, in this case a description

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[5] $res_{\mathbb{E}}$ denotes the restriction to $\mathbb{E}$

of indecomposable maximal Cohen-Macaulay modules follows from Atiyah's classification of vector bundles on elliptic curves [Ati57]. The main result of Atiyah's paper essentially says:

**Theorem 1 (Atiyah).** *An indecomposable vector bundle $\mathcal{E}$ on an elliptic curve $\mathbb{E}$ is uniquely determined by its rank $r$, degree $d$ and determinant $det(\mathcal{E}) \in \mathrm{Pic}^d(\mathbb{E}) \cong \mathbb{E}$.*

In Section 2.3 we give a new proof of this result. However, if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then $\mathbb{S}$ is a so-called *cusp singularity* and in this case $\mathbb{E}$ is a cycle of $n$ projective lines $\mathbb{E}_n$, where $\mathbb{E}_1$ denotes a rational curve with one node.



$$\mathbb{E}_1 \qquad\qquad \mathbb{E}_6$$
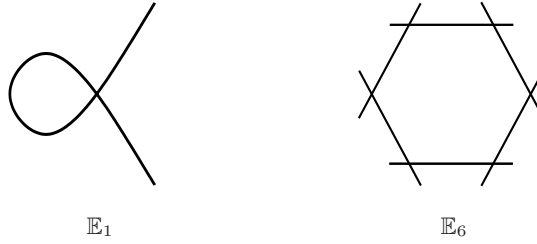
A complete classification of indecomposable vector bundles and torsion free sheaves on these curves in the case of an arbitrary base field $\mathbf{k}$ was obtained by Drozd and Greuel [DG01]. For algebraically closed fields there is the following description, which we prove in Section 3.2.

**Theorem 2.** *Let $\mathbb{E}_n$ denote a cycle of $n$ projective lines and $\mathbb{I}_k$ be a chain of $k$ projective lines, $\mathcal{E}$ an indecomposable torsion free sheaf on $\mathbb{E}_n$.*

1. *If $\mathcal{E}$ is locally free of rank $r$, then there is an étale covering $\pi_r : \mathbb{E}_{nr} \longrightarrow \mathbb{E}_n$, a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathbb{E}_{nr})$ and a natural number $m \in \mathbb{N}$ such that*

$$\mathcal{E} \cong \pi_{r*}(\mathcal{L}) \otimes \mathcal{F}_m,$$

   *where $\mathcal{F}_m$ is an indecomposable vector bundle, recursively defined by the sequences*

$$0 \longrightarrow \mathcal{F}_{m-1} \longrightarrow \mathcal{F}_m \longrightarrow \mathcal{O} \longrightarrow 0, \quad m \geq 2, \quad \mathcal{F}_1 = \mathcal{O}.$$

2. *If $\mathcal{E}$ is not locally free then there exists a finite map $p_k : \mathbb{I}_k \longrightarrow \mathbb{E}_n$ and a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathbb{I}_k)$ (where $k$, $p_k$ and $\mathcal{L}$ are determined by $\mathcal{E}$) such that $\mathcal{E} \cong p_{k*}(\mathcal{L})$.*

This classification is completely analogous to Oda's description of vector bundles on smooth elliptic curves [Oda71] and provides quite simple rules for the computation of the decomposition of the tensor product of any two vector bundles into a direct sum of indecomposable ones. It allows to describe the

dual sheaf of an indecomposable torsion free sheaf as well as the dimensions of homomorphism and extension spaces between indecomposable vector bundles (and in particular, their cohomology), see [Bur03, BDG01]. We carry out these computations in Section 3.2.

However, the way we prove this theorem, essentially uses ideas from representation theory and the technique of matrix problems [Bon92]. Using a similar approach, Theorem 2 was generalized by Burban and Drozd [BD04] to classify indecomposable complexes of the bounded (from the right) derived category of coherent sheaves $D^-(\mathsf{Coh}(\mathbb{E}))$ on a cycle of projective lines $\mathbb{E} = \mathbb{E}_n$.

The situation turns out to be quite different in the case of other singular projective curves of arithmetic genus one. For example, in the case of a cuspidal rational curve $zy^2 = x^3$ even a classification of indecomposable semi-stable vector bundles of a given slope is a *representation-wild* problem.[6] However, if we restrict our attention only on stable vector bundles, then this problem turns out to be *tame* again.[7] Moreover, the combinatorics of the answer is essentially the same as for smooth and nodal Weierstraß curves[8]:

**Theorem 3 (see [BD03, BK3]).** *Let $\mathbb{E}$ be a cuspidal cubic curve over an algebraically closed field $\mathbf{k}$ then*

- *The rank $r$ and the degree $d$ of a stable vector bundle $\mathcal{E}$ over $\mathbb{E}$ are coprime.*
- *A stable vector bundle $\mathcal{E}$ is determined completely by its rank $r$, its degree $d$, that should be coprime and its determinant $det(\mathcal{E}) \in \mathrm{Pic}^d(\mathbb{E}) \cong \mathbf{k}$.*

The technique of matrix problems is a very convenient tool for the study of vector bundles on a given singular projective rational curve of arithmetic genus one. However, to investigate the behavior of the category of coherent sheaves on genus one curves in families one needs other methods. One possible approach is provided by the technique of derived categories and Fourier-Mukai transforms [Muk81, ST01], see Section 3.4. The key idea of this method is that we can transform a sky-scraper sheaf into a torsion free sheaf by applying an auto-equivalence of the derived category. In a relative setting of elliptic fibrations with a section one can use *relative* Fourier-Mukai transforms to construct examples of relatively semi-stable torsion free sheaves, see for example [BK2].

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[6] An exact $\mathbf{k}$–linear category $\mathsf{A}$ over an algebraically closed field $\mathbf{k}$ is called wild if it contains as a full subcategory the category of finite-dimensional representations of *any* associative algebra.

[7] For a formal definition of tameness we refer to [DG01] where a wild-tame dichotomy for vector bundles and torsion free sheaves on reduced curves was proven.

[8] In this paper we call a plane cubic curve *Weierstraß curve*. If $\mathbf{k}$ is algebraically closed and $char(\mathbf{k}) \neq 2, 3$ then it can be written in the form $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$, where $g_2, g_3 \in \mathbf{k}$. It is singular if and only if $g_2^3 = 27g_3^2$ and unless $g_2 = g_3 = 0$ the singularity is a node, whereas in the case $g_2 = g_3 = 0$ the singularity is a cusp.

**Theorem 4 (see [BK1]).** *Let $\mathbb{E}$ be an* irreducible *projective curve of arithmetic genus one over an algebraically closed field* **k***. Then*

1. *The group of exact auto-equivalences of the derived category $D^b(\mathsf{Coh}(\mathbb{E}))$ transforms stable sheaves into stable ones and semi-stable sheaves into semi-stable ones.*
2. *For any rational number $\nu$ the abelian category $\mathsf{Coh}^\nu(\mathbb{E})$ of semi-stable coherent sheaves of slope $\nu$ is equivalent to the category $\mathsf{Coh}^\infty(\mathbb{E})$ of coherent torsion sheaves and this equivalence is induced by an exact auto-equivalence of $D^b(\mathsf{Coh}(\mathbb{E}))$.*
3. *For any coherent sheaf $\mathcal{F}$ on $\mathbb{E}$ such that $\mathsf{End}(\mathcal{F}) = \mathbf{k}$ there exists a point $x \in \mathbb{E}$ and $\Phi \in \mathsf{Aut}(D^b(\mathsf{Coh}(\mathbb{E})))$ such that $\mathcal{F} \cong \Phi(\mathbf{k}(x))$.*

This theorem shows a fundamental difference between a nodal and a cuspidal Weierstraß curve. Namely, let $\mathbb{E}$ be a singular Weierstraß curve and $s$ its singular point. Then the category of finite-dimensional modules over the complete local ring $\widehat{\mathcal{O}_{\mathbb{E},s}}$ has different representation types in the nodal and cuspidal cases. For a nodal curve, the category of finite dimensional representations of $\mathbf{k}[[x,y]]/xy$ is tame due to a result of Gelfand and Ponomarev [GP68]. In the second case, the category of finite length modules over the ring $\mathbf{k}[[x,y]]/(y^2 - x^3)$ is representation wild, see for example [Dro72].

The correspondence between sky-scraper sheaves and semi-stable vector bundles on irreducible Weierstraß curves was first discovered by Friedman, Morgan and Witten [FMW99] (see also [Teo00]) and afterwards widely used in the physical literature under the name "spectral cover construction".

**Theorem 5 (see [FMW99]).** *Let $\mathbb{E}$ be an irreducible Weierstraß curve, $p_0 \in \mathbb{E}$ a smooth point and $\mathcal{E}$ a semi-stable torsion free sheaf of degree zero. Then the sequence*

$$0 \longrightarrow H^0(\mathcal{E}(p_0)) \otimes \mathcal{O} \xrightarrow{ev} \mathcal{E}(p_0) \longrightarrow coker(ev) \longrightarrow 0$$

*is exact. Moreover, the functor $\Phi : \mathcal{E} \mapsto coker(ev)$ establish an equivalence between the category $\mathsf{Coh}^0(\mathbb{E})$ of semi-stable torsion-free sheaves of degree zero and the category of coherent torsion sheaves $\mathsf{Coh}^\infty(\mathbb{E})$.*

This correspondence between torsion sheaves and semi-stable coherent sheaves can be generalized to a relative setting of an elliptic fibration $\mathbb{E} \longrightarrow \mathbb{T}$. In [FMW99] it was used to construct vector bundles on $\mathbb{E}$ which are semi-stable of degree zero on each fiber, see also [BK2].

As was shown in [BK1], the functor $\Phi$ is the trace of a certain exact auto-equivalence of the derived category $D^b(\mathsf{Coh}(\mathbb{E}))$. Using this equivalence of categories and a concrete description of $\mathbf{k}[[x,y]]/xy$–modules in terms of their projective resolutions, one can get a description of semi-stable torsion free sheaves of degree zero on a nodal Weierstraß curve in terms of étale coverings [FM, BK1]. In Section 3.4 we give a short overview of some related results without going into details.

## 2 Vector bundles on smooth projective curves

In this section we review some classical results about vector bundles on smooth curves. However, we provide non-classical proofs which, as we think, are simpler and fit well in our approach to vector bundles and torsion free sheaves over singular curves. The behavior of the category of vector bundles on a smooth projective curve $\mathbb{X}$ is controlled by its genus $g(\mathbb{X})$.

If $g(\mathbb{X}) = 0$ then $\mathbb{X}$ is a projective line $\mathbb{P}^1$ and any vector bundle on it splits into a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(n)$, $n \in \mathbb{Z}$. This result, usually attributed to Grothendieck [Gro57], was already known in an equivalent form to Birkhoff [Bir13]. We found it quite instructive to include Birkhoff's algorithmic proof in our survey.

A classification of vector bundles in the case of smooth elliptic curves i.e. $g(\mathbb{X}) = 1$ was obtained by Atiyah [Ati57]. He has shown that the category of vector bundles on an elliptic curve $\mathbb{X}$ is tame and an indecomposable vector bundle $\mathcal{E}$ is uniquely determined by its rank $r$, its degree $d$ and a point of the curve $x \in \mathbb{X}$. A modern, and in our opinion more conceptual way to prove Atiah's result uses the language of derived categories and is due to Lenzing and Meltzer [LM93]. In the case of an algebraically closed field of characteristics zero an alternative description of indecomposable vector bundles via étale coverings was found by Oda [Oda71]. This classification was a cornerstone in the proof of Polishchuk and Zaslow [PZ98] of Kontsevich's homological mirror symmetry conjecture for elliptic curves, see also [Kre01].

The case of curves of genus bigger than one is not considered in this survey. In this situation even the category of semi-stable vector bundles of slope one is representation wild and the main attention is drawn to the study of various moduli problems and properties of *stable* vector bundles, see for example [LeP97].

Throughout this section we do not require any assumptions about the base field **k**.

### 2.1 Vector bundles on the projective line

We are going to prove the following classical theorem.

**Theorem 6 (Birkhoff-Grothendieck).** *Any vector bundle $\mathcal{E}$ on the projective line $\mathbb{P}^1$ splits into a direct sum of line bundles:*

$$\mathcal{E} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(n)^{r_n}.$$

A proof of this result based on Serre duality and vanishing theorems can be found in a book of Le Potier [LeP97]. However, it is quite interesting to give another, completely elementary proof, based on a lemma proven by Birkhoff in 1913.

A projective line $\mathbb{P}^1$ is a union of two affine lines $\mathbb{A}^1_i$ $(i = 0, 1)$. If $(x_0 : x_1)$ are homogeneous coordinates in $\mathbb{P}^1$ then $\mathbb{A}^1_i = \{(x_0 : x_1)|x_i \neq 0\}$. The affine coordinate on $\mathbb{A}^1_0$ is $z = x_1/x_0$ and on $\mathbb{A}^1_1$ it is $z^{-1} = x_0/x_1$. Thus we can identify $\mathbb{A}^1_0$ with $\mathsf{Spec}(\mathbf{k}[z])$ and $\mathbb{A}^1_1$ with $\mathsf{Spec}(\mathbf{k}[z^{-1}])$, their intersection is then $\mathsf{Spec}(\mathbf{k}[z, z^{-1}])$. Certainly, any projective module over $\mathbf{k}[z]$ is free, i.e. all vector bundles over an affine line are trivial. Therefore to define a vector bundle over $\mathbb{P}^1$ one only has to prescribe its rank $r$ and a *gluing matrix* $M \in GL(r, \mathbf{k}[z, z^{-1}])$. Changing bases in free modules over $\mathbf{k}[z]$ and $\mathbf{k}[z^{-1}]$ corresponds to the transformations $M \mapsto T^{-1}MS$, where $S$ and $T$ are invertible matrices of the same size, over $\mathbf{k}[z]$ and $\mathbf{k}[z^{-1}]$ respectively.

**Proposition 1 (Birkhoff [Bir13]).** *For any matrix $M \in GL(r, \mathbf{k}[z, z^{-1}])$ there are matrices $S \in GL(r, \mathbf{k}[z])$ and $T \in GL(r, \mathbf{k}[z^{-1}])$ such that $T^{-1}MS$ is a diagonal matrix $\mathsf{diag}(z^{d_1}, \ldots, z^{d_r})$.*

*Proof.* One can diagonalize the matrix $M$ in three steps.

Step 1. Reduce the matrix $M = (a_{ij})$ to a lower triangular form with diagonal entries $a_{ii} = z^{m_i}$, where $m_i \in \mathbb{Z}$ and $m_1 \leq m_2 \leq \cdots \leq m_r$. Indeed, since $\mathbf{k}[z]$ is a discrete valuation ring, using invertible transformations of columns over $\mathbf{k}[z]$ we can reduce the first row $(a_{11}, a_{12}, \ldots, a_{1r})$ of $M$ to the form $(a_1, 0, \ldots, 0)$, where $a_1$ is the greatest common divisor of $a_{11}, a_{12}, \ldots, a_{1r}$.

Let $M'$ be the $(r - 1) \times (r - 1)$ matrix formed by the entries $a_{ij}$, $i, j \geq 2$. Since $\det(M) = a_1 \cdot \det(M')$ and $\det(M)$ is a unit in $\mathbf{k}[z, z^{-1}]$, it implies that $a_1 = z^{m_1}$ and $\det(M')$ is a unit in $\mathbf{k}[z, z^{-1}]$, too. Then we proceed with the matrix $M'$ inductively. Note, that the diagonal entries can be always reordered to satisfy $m_1 \leq m_2 \leq \cdots \leq m_r$.

Step 2. Consider the case of a lower-triangular $(2 \times 2)$-matrix

$$M = \begin{pmatrix} z^m & 0 \\ p(z, z^{-1}) & z^n \end{pmatrix}$$

with $m \leq n$. We show by induction on the difference $n - m$ that $M$ can be diagonalized performing invertible transformations of rows over $\mathbf{k}[z^{-1}]$ and invertible transformations of columns over $\mathbf{k}[z]$.

If $m = n$ then we can simply kill the entry $p = p(z, z^{-1})$. Assume now that $m < n$ and $p \neq 0$. Without loss of generality we may suppose that $p \in \langle z^{m+1}, \ldots, z^{n-1} \rangle$. Therefore there exist two mutually prime polynomials $a$ and $b$ in $\mathbf{k}[[z]]$ such that $ap + bz^n = z^d$ and $m < d < n$. Then $\begin{pmatrix} a & z^{n-d} \\ b & p/z^d \end{pmatrix}$ belongs to $GL(2, \mathbf{k}[z])$ and

$$\begin{pmatrix} z^m & 0 \\ p & z^n \end{pmatrix} \begin{pmatrix} a & z^{n-d} \\ b & p/z^d \end{pmatrix} = \begin{pmatrix} z^m a & z^{n+m-d} \\ z^d & 0 \end{pmatrix}.$$

In order to conclude the induction step it remains to note that $|n+m-2d| < |n-m|$.

Step 3. Let $M$ be a lower-diagonal matrix with the diagonal elements $z^{m_1}, \ldots, z^{m_r}$ with $m_1 \leq m_2 \leq \cdots \leq m_r$. We show by induction on $\sum_{i,j=1}^{r} |m_i - m_j|$ that $M$ can be diagonalized. This statement is obvious for $\sum_{i,j=1}^{r} |m_i - m_j| = 0$. Assume that $\sum_{i,j=1}^{r} |m_i - m_j| = N > 0$. Introduce an ordering on the set $\{(i,j)|1 \leq j \leq i \leq r\}$:

$$(2,1) < (2,3) < \cdots < (r-1,r) < (3,1) < \cdots < (r-2,r) < \cdots < (r,1).$$

Let $(i_0, j_0)$ be the smallest pair such that $a_{i_0 j_0} \neq 0$. Then we can apply the algorithm from the Step 2 to the $(2 \times 2)$ matrix formed by the entries $(j_0, j_0), (j_0, i_0), (i_0, j_0)$ and $(i_0, i_0)$ to diminish the sum $\sum_{i,j=1}^{r} |m_i - m_j|$. This completes the proof of Birkhoff's lemma.

Now it remains to note that $1 \times 1$ matrix $(t^d)$ defines the line bundle $\mathcal{O}_{\mathbb{P}^1}(-d)$. This implies the statement about the splitting of a vector bundle on a projective line into a direct sum of line bundles.

## 2.2 Projective curves of arithmetic genus bigger then one are vector bundle wild

In this subsection we are going to prove the following

**Theorem 7 (see [DG01]).** *Let $\mathbb{X}$ be an irreducible projective curve of arithmetic genus $g(\mathbb{X}) > 1$ over an algebraically closed field $\mathbf{k}$. Then the abelian category of semi-stable vector bundles of slope [9] one is representation wild.*

In order to show the wildness of a category $\mathsf{A}$ one frequently uses the following lemma:

**Lemma 1.** *Let $\mathsf{A}$ be an abelian category, $M, N \in \mathrm{Ob}(\mathsf{A})$ with $\mathsf{Hom}_{\mathsf{A}}(M, N) = 0$ and $\xi, \xi' \in \mathsf{Ext}_{\mathsf{A}}^1(N, M)$ two extensions*

$$\xi : 0 \longrightarrow M \overset{\alpha}{\longrightarrow} K \overset{\beta}{\longrightarrow} N \longrightarrow 0,$$

$$\xi' : 0 \longrightarrow M \overset{\alpha'}{\longrightarrow} K' \overset{\beta'}{\longrightarrow} N \longrightarrow 0.$$

*Then $K \cong K'$ if and only if there exist two isomorphisms $f : M \longrightarrow M$ and $g : N \longrightarrow N$ such that $f\xi = \xi' g$.*

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[9] The slope of a coherent sheaf $\mathcal{F}$ is $\mu(\mathcal{F}) = \frac{deg(\mathcal{F})}{rk(\mathcal{F})}$.

*Proof.* The statement is clear in one direction: if $f\xi = \xi'g$ in $\mathsf{Ext}^1_A(N, M)$, then $K \cong K'$ by the 5-Lemma.

Now suppose $K \cong K'$ and let $h : K \longrightarrow K'$ be an isomorphism. Then $im(h\alpha)$ is a subobject of $im(\alpha')$. Indeed, otherwise the map $\beta'h\alpha : M \longrightarrow N$ would be non-zero, a contradiction. Therefore we get the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\alpha} & K & \xrightarrow{\beta} & N & \longrightarrow & 0 \\
& & \downarrow{f} & & \downarrow{h} & & \downarrow{g} & & \\
0 & \longrightarrow & M & \xrightarrow{\alpha'} & K' & \xrightarrow{\beta'} & N & \longrightarrow & 0.
\end{array}
$$

In the same way we proceed with $h^{-1}$. Hence $f$ is an isomorphism, what proves the lemma.

Let us come back to the proof of the theorem. Suppose now that $\mathbb{X}$ is an irreducible projective curve of arithmetic genus $g > 1$, $\mathcal{O} = \mathcal{O}_{\mathbb{X}}$. Then for any two points $x \neq y$ from $\mathbb{X}$ we have $\mathsf{Hom}(\mathcal{O}(x), \mathcal{O}(y)) \cong H^0(\mathbb{X}, \mathcal{O}(y - x)) = 0$ and the Riemann-Roch theorem implies that $\mathsf{Ext}^1(\mathcal{O}(x), \mathcal{O}(y)) \cong H^1(\mathbb{X}, \mathcal{O}(y - x)) \cong \mathbf{k}^{g-1}$. Fix 5 different points $x_1, \ldots, x_5$ of the curve $\mathbb{X}$, choose non-zero elements $\xi_{ij} \in \mathsf{Ext}^1(\mathcal{O}(x_j), \mathcal{O}(x_i))$ for $i \neq j$ and consider vector bundles $\mathcal{F}(A, B)$, where $A, B \in Mat(n \times n, \mathbf{k})$ and $\mathcal{F}(A, B)$ is given as an extension

$$
0 \longrightarrow \underbrace{(\mathcal{O}(x_1) \oplus \mathcal{O}(x_2))^n}_{\mathcal{B}} \longrightarrow \mathcal{F}(A, B) \longrightarrow \underbrace{(\mathcal{O}(x_3) \oplus \mathcal{O}(x_4) \oplus \mathcal{O}(x_5))^n}_{\mathcal{A}} \longrightarrow 0
$$

corresponding to the element $\xi(A, B)$ of $\mathsf{Ext}^1(\mathcal{A}, \mathcal{B})$ presented by the matrix

$$
\begin{pmatrix} \xi_{13}I & \xi_{14}I & \xi_{15}I \\ \xi_{23}I & \xi_{24}A & \xi_{25}B \end{pmatrix},
$$

where $I$ denotes the unit $n \times n$ matrix. If $(A', B')$ is another pair of matrices, and $\mathcal{F}(A, B) \to \mathcal{F}(A', B')$ any morphism, then the previous lemma implies that there are morphisms $\phi : \mathcal{A} \to \mathcal{A}'$ and $\psi : \mathcal{B} \to \mathcal{B}'$ such that $\psi\xi(A, B) = \xi(A', B')\phi$ Now one can easily deduce that $\Phi = \mathsf{diag}(S, S, S)$ and $\Psi = \mathsf{diag}(S, S)$ for some matrix $S \in Mat(n \times n, k)$ such that $SA = A'S$ and $SB = B'S$.

If we consider a pair of matrices $(A, B)$ as a representation of the free algebra $\mathbf{k}\langle x, y \rangle$ in 2 generators, the correspondence $(A, B) \mapsto \mathcal{F}(A, B)$ becomes a *full, faithful and exact* functor $\mathbf{k}\langle x, y \rangle - \mathsf{mod} \longrightarrow \mathsf{VB}(\mathbb{X})$. In particular, it maps non-isomorphic modules to non-isomorphic vector bundles and indecomposable modules to indecomposable vector bundles. Using the terminology of representation theory of algebras, we say in this situation that the curve $\mathbb{X}$ is *vector bundle wild*. For a precise definition of wildness we refer to [DG01].

Recall that the algebra $\mathbf{k}\langle x, y \rangle$ here can be replaced by *any* finitely generated algebra $\Lambda = \mathbf{k}\langle a_1, \ldots, a_n \rangle$. Indeed, any $\Lambda$–module $M$ such that $dim_{\mathbf{k}}(M) = m$ is given by a set of matrices $A_1, \ldots, A_n$ of size $m \times m$. One

gets a full, faithful and exact functor $\Lambda - \mathrm{mod} \longrightarrow \mathbf{k}\langle x, y \rangle - \mathrm{mod}$ mapping the module $M$ to the $\mathbf{k}\langle x, y\rangle$–module of dimension $m \cdot n$ defined by the pair of matrices

$$
X = \begin{pmatrix} \lambda_1 I & 0 & \ldots & 0 \\ 0 & \lambda_2 I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n I \end{pmatrix} \quad Y = \begin{pmatrix} A_1 & I & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix},
$$

where $\lambda_1, \ldots, \lambda_n$ are different elements of the field $\mathbf{k}$. Thus a classification of vector bundles over $\mathbb{X}$ would imply a classifications of *all* representations of *all* finitely generated algebras, a goal that perhaps nobody considers as achievable (whence the name "wild").

### 2.3 Vector bundles on elliptic curves

In this subsection we shall discuss a classification of indecomposable coherent sheaves over smooth elliptic curves. Modulo some facts about derived categories we give a self-contained proof of Atiyah's classification of indecomposable vector bundles which is probably simpler than the original proof.

**Definition 1.** *An elliptic curve $\mathbb{E}$ over a field $\mathbf{k}$ is a smooth projective curve of genus one having a $\mathbf{k}$–rational point $p_0$.*

The category $\mathsf{Coh}(\mathbb{E})$ of coherent sheaves on an elliptic curve $\mathbb{E}$ has the following properties, sometimes called *"dimension one Calabi-Yau property"*:

- It is abelian, $\mathbf{k}$–linear, finite-dimensional, noetherian and of global dimension one.
- Serre Duality: for any two coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on $\mathbb{E}$ there is an isomorphism
$$
\mathsf{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathsf{Ext}^1(\mathcal{F}, \mathcal{G})^*,
$$
functorial in both arguments.

It is interesting to note that these properties almost characterize the category of coherent sheaves on an elliptic curve:

**Theorem 8 (Reiten $-$ van den Bergh [RV02]).** *Let $\mathbf{k}$ be an algebraically closed field and $\mathsf{A}$ an indecomposable abelian Calabi-Yau category of dimension one. Then $\mathsf{A}$ is equivalent either to the category of finite-dimensional $\mathbf{k}[[t]]$–modules or to the category of coherent sheaves on an elliptic curve $\mathbb{E}$.*

This theorem characterizes Calabi-Yau abelian categories of global dimension one. We shall need one more formula to proceed with a classification of indecomposable coherent sheaves.

**Theorem 9 (Riemann–Roch formula).** *For any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on an elliptic curve $\mathbb{E}$ there is an integral bilinear Euler form*

$$\langle \mathcal{F}, \mathcal{G} \rangle := dim_{\mathbf{k}} \, \mathsf{Hom}(\mathcal{F}, \mathcal{G}) - dim_{\mathbf{k}} \, \mathsf{Ext}^1(\mathcal{F}, \mathcal{G}) = \begin{vmatrix} deg(\mathcal{G}) & deg(\mathcal{F}) \\ rk(\mathcal{G}) & rk(\mathcal{F}) \end{vmatrix}.$$

*In particular, $\langle \, , \, \rangle$ is anti-symmetric: $\langle \mathcal{F}, \mathcal{G} \rangle = -\langle \mathcal{G}, \mathcal{F} \rangle$.*

Now we are ready to start with the classification of indecomposable coherent sheaves.

**Theorem 10 (Atiyah).** *Let $\mathbb{E}$ be an elliptic curve over a field $\mathbf{k}$. Then*

1. *Any indecomposable coherent sheaf $\mathcal{F}$ is semi-stable.*
2. *If $\mathcal{F}$ is semi-stable and indecomposable then all its Jordan-Hölder factors are isomorphic.*
3. *A coherent sheaf $\mathcal{F}$ is stable if and only if $\mathsf{End}(\mathcal{F}) = \boldsymbol{K}$, where $\mathbf{k} \subset \boldsymbol{K}$ is some finite field extension.*

*Proof.* It is well-known that any coherent sheaf $\mathcal{F} \in \mathsf{Coh}(E)$ has a Harder-Narasimhan filtration

$$0 \subset \mathcal{F}_n \subset \ldots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{F}$$

whose factors $\mathcal{A}_\nu := \mathcal{F}_\nu / \mathcal{F}_{\nu+1}$ are semi-stable with decreasing slopes $\mu(\mathcal{A}_n) > \mu(\mathcal{A}_{n-1}) > \ldots > \mu(\mathcal{A}_0)$. Using the definition of semi-stability, this implies $\mathsf{Hom}(\mathcal{A}_{\nu+i}, \mathcal{A}_\nu) = 0$ for all $\nu \geq 0$ and $i > 0$. Therefore, $\mathsf{Ext}^1(\mathcal{A}_0, \mathcal{F}_1) \cong \mathsf{Hom}(\mathcal{F}_1, \mathcal{A}_0)^* = 0$, and the exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{A}_0 \to 0$ must split. In particular, if $\mathcal{F}$ is indecomposable, we have $\mathcal{F}_1 = 0$ and $\mathcal{F} \cong \mathcal{A}_0$ and $\mathcal{F}$ is semi-stable.

The full sub-category of $\mathsf{Coh}(\mathbb{E})$ whose objects are the semi-stable sheaves of a fixed slope is an abelian category in which any object has a Jordan-Hölder filtration with stable factors. If $\mathcal{F}$ and $\mathcal{G}$ are non-isomorphic stable sheaves which have the same slope then $\mathsf{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$. Based on this fact we deduce that an indecomposable semi-stable sheaf has all its Jordan-Hölder factors isomorphic to each other.

It is well-known that any non-zero automorphism of a stable coherent sheaf $\mathcal{F}$ is invertible, i.e. $\mathsf{End}(\mathcal{F})$ is a field $\boldsymbol{K}$. Since $\mathbb{E}$ is projective, the field extension $\mathbf{k} \subset \boldsymbol{K}$ is finite. On a smooth elliptic curve, the converse is true as well, which equips us with a useful homological characterization of stability.

To see this, suppose that all endomorphism of $\mathcal{F}$ are invertible but $\mathcal{F}$ is not stable. This implies the existence of an epimorphism $\mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ stable and $\mu(\mathcal{F}) \geq \mu(\mathcal{G})$. Serre duality implies $dim_{\mathbf{k}} \, \mathsf{Ext}^1(\mathcal{G}, \mathcal{F}) = dim_{\mathbf{k}} \, \mathsf{Hom}(\mathcal{F}, \mathcal{G}) > 0$, hence, $\langle \mathcal{G}, \mathcal{F} \rangle = dim_{\mathbf{k}} \, \mathsf{Hom}(\mathcal{G}, \mathcal{F}) - dim_{\mathbf{k}} \, \mathsf{Ext}^1(\mathcal{G}, \mathcal{F}) < dim_{\mathbf{k}} \, \mathsf{Hom}(\mathcal{G}, \mathcal{F})$. By Riemann-Roch formula $\langle \mathcal{G}, \mathcal{F} \rangle = (\mu(\mathcal{F}) - \mu(\mathcal{G})) / \mathrm{rk}(\mathcal{F}) \, \mathrm{rk}(\mathcal{G}) > 0$, hence $\mathsf{Hom}(\mathcal{G}, \mathcal{F}) \neq 0$. But this produces a non-zero composition $\mathcal{F} \to \mathcal{G} \to \mathcal{F}$ which is not an isomorphism, in contradiction to the assumption that $\mathsf{End}(\mathcal{F})$ is a field.

*Remark 1.* Usually one speaks about the stability of vector bundles on projective varieties in the case of an algebraically closed field of characteristics zero. However, due to a result of Rudakov [Rud97] one can use the stability notion for fairly general abelian categories.

The following classical fact was, probably first, proven by Dold [Dol60]:

**Proposition 2.** *Let* A *be an abelian category of global dimension one and* $\mathcal{F}$ *an object of the derived category* $D^b(\mathsf{A})$. *Then there is an isomorphism* $\mathcal{F} \cong \bigoplus\limits_{i \in \mathbb{Z}} H^i(\mathcal{F})[-i]$, *i.e. any object of* $D^b(\mathsf{A})$ *splits into a direct sum of its homologies.*

This proposition in particular means that the derived category $D^b(\mathsf{A})$ of a hereditary abelian category A and the abelian category A itself have the same representation type. However, it turns out that the derived category has a richer structure and more symmetry then the corresponding abelian category.

First of all note that the group $\mathsf{Aut}(D^b(\mathsf{Coh}(\mathbb{E})))$ acts on the $K$–group $K(\mathbb{E})$ of $\mathsf{Coh}(\mathbb{E})$ preserving the Euler form $\langle\ ,\ \rangle$. Hence, it leaves invariant the radical of the Euler form $\mathsf{rad}\langle\ ,\ \rangle = \{\mathcal{F} \in K(\mathbb{E})|\langle\mathcal{F},\ \rangle = 0\}$ and induces an action on $K(\mathbb{E})/\mathsf{rad}\langle\ ,\ \rangle$. Since by Riemann-Roch theorem $Z : K(\mathbb{E})/\mathsf{rad}\langle\ ,\ \rangle \xrightarrow{Z} \mathbb{Z}^2$ is an isomorphism, where $Z(\mathcal{F}) := (\mathrm{rk}(\mathcal{F}), \deg(\mathcal{F})) \in \mathbb{Z}^2$, we get a group homomorphism $\mathsf{Aut}(D^b(\mathsf{Coh}(\mathbb{E}))) \longrightarrow SL(2, \mathbb{Z})$. We call the pair $Z(\mathcal{F}) \in \mathbb{Z}^2$ the *charge* of $\mathcal{F}$.

**Theorem 11 (Mukai, [Muk81]).** *Let* $\mathbb{E}$ *be an elliptic curve. Then the group homomorphism* $\mathsf{Aut}(D^b(\mathsf{Coh}(\mathbb{E}))) \longrightarrow SL(2, \mathbb{Z})$ *is surjective.*

*Proof.* By the definition of an elliptic curve there is a **k**–rational point $p_0$ on $\mathbb{E}$ inducing an exact equivalence $\mathcal{O}(p_0) \otimes -$. Let $\mathcal{P} = \mathcal{O}_{\mathbb{E} \times \mathbb{E}}(\Delta - (p_0 \times \mathbb{E}) - (\mathbb{E} \times p_0))$ then the Fourier-Mukai transform

$$\Phi_{\mathcal{P}} : D^b(\mathsf{Coh}(\mathbb{E})) \to D^b(\mathsf{Coh}(\mathbb{E})), \quad \Phi_{\mathcal{P}}(\mathcal{F}) = \boldsymbol{R}\pi_{2*}(\mathcal{P} \otimes \pi_1^* \mathcal{F})$$

is an exact auto-equivalence of the derived category, see [Muk81]. The actions of $\mathcal{O}(p_0) \otimes -$ and $\Phi_{\mathcal{P}}$ on $K(\mathbb{E})/\mathsf{rad}\langle\ ,\ \rangle$ in the basis $\{[\mathcal{O}], [\mathbf{k}(p_0)]\}$ are given by the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ which are known to generate $SL(2, \mathbb{Z})$. This shows the claim.

The technique of derived categories makes it easy to give a classification of indecomposable coherent sheaves on an elliptic curve.

**Theorem 12.** *Let* $\mathcal{F}$ *be an indecomposable coherent sheaf on an elliptic curve* $\mathbb{E}$. *Then there exits a torsion sheaf* $\mathcal{T}$ *and an exact auto-equivalence* $\Phi \in \mathsf{Aut}(D^b(\mathsf{Coh}(\mathbb{E})))$ *such that* $\mathcal{F} \cong \Phi(\mathcal{T})$.

*Proof.* Let $\mathcal{F}$ be an indecomposable coherent sheaf on $\mathbb{E}$ with the charge $Z(\mathcal{F}) = (r, d)$, $r > 0$. Let $h = g.c.d.(r, d)$ be the greatest common divisor, then

there exists a matrix $F \in SL(2, \mathbb{Z})$ such that $F \left( \begin{smallmatrix} r \\ d \end{smallmatrix} \right) = \left( \begin{smallmatrix} 0 \\ h \end{smallmatrix} \right)$. We can lift the matrix $F$ to an auto-equivalence $\Phi \in \mathsf{Aut}(D^b(\mathsf{Coh}(\mathbb{E})))$, then $Z(\Phi(\mathcal{F})) = \left( \begin{smallmatrix} 0 \\ h \end{smallmatrix} \right)$. Since $\mathsf{Aut}(D^b(\mathsf{Coh}(\mathbb{E})))$ maps indecomposable objects of the derived category into indecomposable ones and since the only indecomposable objects in the derived category are shifts of indecomposable coherent sheaves, we can conclude that $\Phi(\mathcal{F})$ is isomorphic to a shift of some indecomposable sheaf of rank zero, what proves the theorem.

Let $M_{\mathbb{E}}(r, d)$ denote the set of indecomposable vector bundles on $\mathbb{E}$ of rank $r$ and degree $d$.

**Theorem 13 (Atiyah).** *Let $\mathbb{E}$ be an elliptic curve. Then for any integer $h > 0$ there exists a unique indecomposable vector bundle $\mathcal{F}_h \in M_{\mathbb{E}}(h, 0)$ such that $H^0(\mathcal{F}_h) \neq 0$. The vector bundles $\mathcal{F}_h$ are called* unipotent. *Moreover, the following properties hold:*

1. *$H^0(\mathcal{F}_h) = H^1(\mathcal{F}_h) = \mathbf{k}$ for all $h \geq 1$.*
2. *If $char(\mathbf{k}) = 0$ then $\mathcal{F}_h \cong Sym^{h-1}(\mathcal{F}_2)$. Moreover*

$$\mathcal{F}_e \otimes \mathcal{F}_f \cong \bigoplus_{i=0}^{f-1} \mathcal{F}_{e+f-2i-1}$$

*Sketch of the proof.* Since $\mathcal{F}_h$ is indecomposable of degree zero, it has a unique Jordan-Hölder factor $\mathcal{L} \in \mathrm{Pic}^0(\mathbb{E})$. From the assumption $\mathsf{Hom}(\mathcal{O}, \mathcal{F}_h) \neq 0$ we conclude that $\mathcal{L} \cong \mathcal{O}$, so each bundle $\mathcal{F}_h$ can be obtained by recursive self-extensions of the structure sheaf. Since by Theorem 12 the category of semi-stable vector bundles of degree zero is equivalent to the category of torsion sheaves, we conclude that the category of semi-stable sheaves with the Jordan-Hölder factor $\mathcal{O}$ is equivalent to the category of finite-dimensional $\mathbf{k}[[t]]$-modules. The exact sequence

$$0 \longrightarrow \mathcal{F}_{h-1} \longrightarrow \mathcal{F}_h \longrightarrow \mathcal{O} \longrightarrow 0.$$

corresponds via Fourier-Mukai transform $\Phi_{\mathcal{P}}$ to

$$0 \longrightarrow \mathbf{k}[[t]]/t^{h-1} \longrightarrow \mathbf{k}[[t]]/t^h \longrightarrow \mathbf{k} \longrightarrow 0.$$

In the same way we conclude that $\mathsf{Hom}(\mathcal{O}, \mathcal{F}_h) = \mathsf{Hom}_{\mathbf{k}[[t]]}(\mathbf{k}, \mathbf{k}[[t]]/t^h) = \mathbf{k}$. Moreover, one can show that $\Phi_{\mathcal{P}}(\mathbf{k}[[t]]/t^e \otimes_{\mathbf{k}} \mathbf{k}[[t]]/t^f) \cong \mathcal{F}_e \otimes \mathcal{F}_f$, hence we have the same rules for the decomposition of the tensor product of unipotent vector bundles and of nilpotent Jordan cells, see [Ati57, Oda71, Muk81, PH05].

*Remark 2.* Atiyah's original proof from 1957 was written at the time when the formalism of derived and triangulated categories was not developed yet. However, his construction of a bijection between $M_{\mathbb{E}}(r, d)$ and $M_{\mathbb{E}}(h, 0)$ corresponds exactly to the action of the group of exact equivalences of the derived category of coherent sheaves on the set of indecomposable objects. This was probably for the first time observed by Lenzing and Meltzer in [LM93]. For further elaborations, see [Pol03, PH05, BK3].

Actually, Atiyah's description of indecomposable vector bundles on an elliptic curve $\mathbb{E}$ is more precise.

**Theorem 14 (Atiyah).** *Let $\mathbb{E}$ be an elliptic curve over an algebraically closed field $\mathbf{k}$. For any pair of coprime integers $(r,d)$ with $r > 0$ pick up some $\mathcal{E}(r,d) \in M_{\mathbb{E}}(r,d)$. Then*

1. *$M_{\mathbb{E}}(r,d) = \{\mathcal{E}(r,d) \otimes \mathcal{L} |\, \mathcal{L} \in \mathrm{Pic}^0(\mathbb{E})\}$.*
2. *$\mathcal{E}(r,d) \otimes \mathcal{L} \cong \mathcal{E}(r,d)$ if and only if $\mathcal{L}^r \cong \mathcal{O}$.*
3. *The map $\det : M_{\mathbb{E}}(r,d) \longrightarrow M_{\mathbb{E}}(1,d)$ is a bijection.*
4. *If $char(\mathbf{k}) = 0$, then $\mathcal{F}_h \otimes - : M_{\mathbb{E}}(r,d) \longrightarrow M_{\mathbb{E}}(rh,dh)$ is a bijection.*

*Remark 3.* If $\mathbf{k}$ is not algebraically closed and $\mathbb{E}$ is a smooth projective curve of genus one over $\mathbf{k}$, without $\mathbf{k}$–rational points, then we miss the generator $\mathcal{O}(p_0)$ in the group of exact auto-equivalences $D^b(\mathsf{Coh}(X))$ and the method used for elliptic curves can not be immediately applied. This problem was solved in a paper of Pumplün [Pum04].

We may sum up the discussed properties of indecomposable coherent sheaves on elliptic curves:

**Proposition 3.** *Let $\mathbb{E}$ be an elliptic curve over a field $\mathbf{k}$. Then*

1. *Any indecomposable coherent sheaf $\mathcal{F}$ on $\mathbb{E}$ is semi-stable with a unique stable Jordan-Hölder factor.*
2. *An indecomposable vector bundle is determined by its charge $(r,d) \in \mathbb{Z}^2$ and a closed point $x$ of the curve $\mathbb{E}$.*
3. *Let $\mathsf{Coh}^\nu(\mathbb{E})$ be the category of semi-stable sheaves of slope $\nu$. Then for any $\mu, \nu \in \mathbb{Q} \cup \{\infty\}$ the abelian categories $\mathsf{Coh}^\nu(\mathbb{E})$ and $\mathsf{Coh}^\mu(\mathbb{E})$ are equivalent and this equivalence is induced by an auto-equivalence of $D^b(\mathsf{Coh}(\mathbb{E}))$.*
4. *In particular, each category $\mathsf{Coh}^\mu(\mathbb{E})$ is equivalent to the category of coherent torsion sheaves.*
5. *If $\mathcal{F} \in \mathsf{Coh}^\nu(\mathbb{E}), \mathcal{G} \in \mathsf{Coh}^\mu(\mathbb{E})$ and $\nu < \mu$ then $\mathsf{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$ and*

$$dim_{\mathbf{k}} \mathsf{Hom}(\mathcal{F}, \mathcal{G}) = deg(\mathcal{F}) rk(\mathcal{G}) - deg(\mathcal{G}) rk(\mathcal{F}).$$

*The case $\nu > \mu$ is dual by Serre duality.*

This gives a pretty complete description of the category of coherent sheaves on an elliptic curve and of its derived category. However, in applications one needs another description of indecomposable vector bundles, see [Pol02, PZ98].

The following form of Atiyah's classification is due to Oda [Oda71]. It was used by Polishchuk and Zaslow in their proof of the homological mirror symmetry conjecture for elliptic curves, see [PZ98, Kre01].

**Theorem 15 (Oda).** *Let $\mathbf{k} = \mathbb{C}$ and $\mathbb{E} = \mathbb{E}_\tau = \mathbb{C}/\langle 1, \tau \rangle$ be an elliptic curve, $\mathcal{E} \in M_{\mathbb{E}}(rh, dh)$ an indecomposable vector bundle, where $gcd(r,d) = 1$. Then there exists a unique line bundle $\mathcal{L}$ of degree $d$ on $\mathbb{E}_{r\tau}$ such that*

$$\mathcal{E} \cong \pi_*(\mathcal{L}) \otimes \mathcal{F}_h \cong \pi_*(\mathcal{L} \otimes \mathcal{F}_h),$$

*where* $\pi : \mathbb{E}_{r\tau} \longrightarrow \mathbb{E}_\tau$ *is an étale covering of degree* $r$.

*Proof.* Let $\mathcal{L}$ be a line bundle on $\mathbb{E}_{r\tau}$ of degree $d$. Since the morphism $\pi$ is étale, $\pi_*(\mathcal{L})$ is a vector bundle on $\mathbb{E}_\tau$ of rank $r$. The Todd class of an elliptic curve is trivial, hence by Grothendieck-Riemann-Roch theorem we obtain $deg(\pi_*(\mathcal{L})) = deg(\mathcal{L}) = d$.

Now let us show that $\pi_*(\mathcal{L})$ is indecomposable. To do this it suffices to prove that $\mathsf{End}(\pi_*\mathcal{L}) = \mathbb{C}$. Consider the fiber product diagram

$$
\begin{array}{ccc}
\widetilde{\mathbb{E}} & \xrightarrow{\ p_1\ } & \mathbb{E}_{r\tau} \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
\mathbb{E}_{r\tau} & \xrightarrow{\ \pi_2\ } & \mathbb{E}_\tau .
\end{array}
$$

One can easily check that $\widetilde{\mathbb{E}}$ is a union of $r$ copies of the elliptic curve $\mathbb{E}_{r\tau}$: $\widetilde{\mathbb{E}} = \coprod_{i=1}^{r} \mathbb{E}_{r\tau}^i$, where each $p_1^{(i)} : \mathbb{E}_{r\tau}^i \longrightarrow \mathbb{E}_{r\tau}$, $i = 1, \ldots, r$ can be chosen to be the identity map and $p_2^{(i)}(z) = z + \frac{i}{r}\tau$.

Since all morphisms $\pi_i, p_i, i = 1, 2$ are affine and flat, the functors $\pi_{i*}, \pi_i^*, p_{i*}, p_i^*$ are exact. Moreover, $p_i^! = p_i^*$ since the canonical sheaf of an elliptic curve is trivial. Using the base change isomorphism and Grothendieck duality we have

$$\mathsf{Hom}_{\mathbb{E}_\tau}\big(\pi_{1*}\mathcal{L}, \pi_{2*}\mathcal{L}\big) \cong \mathsf{Hom}_{\mathbb{E}_{r\tau}}\big(\pi_2^*\pi_{1*}\mathcal{L}, \mathcal{L}\big)$$
$$\cong \mathsf{Hom}_{\mathbb{E}_{r\tau}}\big(p_{2*}p_1^*\mathcal{L}, \mathcal{L}\big) \cong \mathsf{Hom}_{\widetilde{\mathbb{E}}}\big(p_1^*\mathcal{L}, p_2^*\mathcal{L}\big).$$

It remains to note that $\mathsf{Hom}_{\widetilde{\mathbb{E}}}\big(p_1^{(i)*}\mathcal{L}, p_2^{(i)*}\mathcal{L}\big) = 0$ for $i \neq 0$.

If $\mathcal{E}$ is an indecomposable vector bundle on $\mathbb{E}_\tau$ of rank $rh$ and degree $dh$, then by Theorem 14 there exists $\mathcal{M} \in \mathrm{Pic}^0(\mathbb{E}_\tau)$ such that $\mathcal{E} \cong \pi_*(\mathcal{L}) \otimes \mathcal{M} \otimes \mathcal{F}_h$. By the projection formula $\mathcal{E} \cong \pi_*(\mathcal{L} \otimes \pi^*\mathcal{M}) \otimes \mathcal{F}_h$. Moreover, passing to an étale covering kills the ambiguity in the choice of $\mathcal{M}$.

It remains to show that $\pi^*(\mathcal{F}_h) \cong \mathcal{F}_h$. To do this it suffices see that $\pi^*(\mathcal{F}_2) \cong \mathcal{F}_2$, since $\mathcal{F}_h \cong Sym^{h-1}(\mathcal{F}_2)$ and the inverse image commutes with all tensor operations. The only property we have to check is that $\pi^*(\mathcal{F}_2)$ does not split. It is equivalent to say that the map $\pi^* : H^1(\mathcal{O}_{\mathbb{E}_\tau}) \longrightarrow H^1(\mathcal{O}_{\mathbb{E}_{r\tau}})$ is non-zero.

This follows from the commutativity of the diagram:

$$
\begin{array}{ccccccc}
\mathbb{Z}^2 & =\!\!= & H_1(\mathbb{E}_\tau, \mathbb{Z}) & \xrightarrow{\ \cong\ } & H^1(\mathbb{E}_\tau, \mathbb{Z}) & \hookrightarrow & H^1(\mathbb{E}_\tau, 0) \\
{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & r \end{smallmatrix}\right)}\big\uparrow & & {\scriptstyle \pi_*}\big\uparrow & & {\scriptstyle \pi^*}\big\downarrow & & {\scriptstyle \pi^*}\big\downarrow \\
\mathbb{Z}^2 & =\!\!= & H_1(\mathbb{E}_{r\tau}, \mathbb{Z}) & \longrightarrow & H^1(\mathbb{E}_{r\tau}, \mathbb{Z}) & \hookrightarrow & H^1(\mathbb{E}_{r\tau}, 0).
\end{array}
$$

# 3 Vector bundles and torsion free sheaves on singular curves of arithmetic genus one

In this paper we discuss two approaches for the study of the category of coherent sheaves on a singular projective curve of arithmetic genus one. The first uses the technique of derived categories and Fourier-Mukai transforms. Its key point is that any semi-stable torsion free sheaf on an irreducible Weierstraß curve can be obtained from a torsion sheaf by applying an auto-equivalence of the derived category. This technique can be generalized to the case of elliptic fibrations: we can transform a family of torsion sheaves to a family of sheaves, which are semi-stable on each fiber.

However, the approach via Fourier-Mukai transforms allows to describe only semi-stable sheaves. In order to get a description of all indecomposable torsion free sheaves, another technique turns out to be useful. Namely, we relate vector bundles on a singular rational curve $\mathbb{X}$ and on its normalization $\widetilde{\mathbb{X}} \xrightarrow{p} \mathbb{X}$. The inverse image functor $p^* : \mathsf{VB}(\mathbb{X}) \longrightarrow \mathsf{VB}(\widetilde{\mathbb{X}})$ can map non-isomorphic bundles into isomorphic ones. The full information about the fibers of this map is encoded in a certain *matrix problem*. In the case of an algebraically closed field this approach leads to a very concrete description of indecomposable vector bundles on cycles of projective lines via étale coverings (no assumption on $char(\mathbf{k})$ is needed).

Combining both methods, we get a quite complete description of the category of torsion free sheaves on a nodal Weierstraß curve.

## 3.1 Vector bundles on singular curves via matrix problems

Let $\mathbb{X}$ be a reduced projective curve over a field $\mathbf{k}$. Introduce the following notation:

- $p : \widetilde{\mathbb{X}} \longrightarrow \mathbb{X}$ the normalization of $\mathbb{X}$;
- $\mathcal{O} = \mathcal{O}_{\mathbb{X}}$ and $\widetilde{\mathcal{O}} = p_*\mathcal{O}_{\widetilde{\mathbb{X}}}$;
- $\mathcal{J} = Ann_{\mathcal{O}}(\widetilde{\mathcal{O}}/\mathcal{O})$ the conductor of $\mathcal{O}$ in $\widetilde{\mathcal{O}}$;
- $\mathcal{A} = \mathcal{O}/\mathcal{J}$ and $\widetilde{\mathcal{A}} = \widetilde{\mathcal{O}}/\mathcal{J}$.

Note that $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are skyscraper sheaves supported at the singular locus of $\mathbb{X}$. Since the morphism $p$ is affine, $p_*$ identifies the category of coherent sheaves $\mathsf{Coh}(\widetilde{\mathbb{X}})$ and the category $\mathsf{Coh}_{\widetilde{\mathcal{O}}}$ of coherent modules on the ringed space $(\mathbb{X}, \widetilde{\mathcal{O}})$. Let $\mathbb{S}$ be the subscheme of $\mathbb{X}$ defined by the conductor $\mathcal{J}$, $\widetilde{\mathbb{S}}$ its scheme-theoretic pull-back on $\widetilde{\mathbb{X}}$ and $\mathcal{I} = \mathcal{I}_{\widetilde{\mathbb{S}}}$ its ideal sheaf on $\widetilde{\mathbb{X}}$. Then $p_*$ also induces an equivalence between the category of $\mathcal{O}_{\widetilde{\mathbb{X}}}/\mathcal{I}$–modules and the category of $\widetilde{\mathcal{A}}$–modules.

For a sheaf of algebras $\Lambda \in \{\mathcal{O}, \widetilde{\mathcal{O}}, \mathcal{A}, \widetilde{\mathcal{A}}\}$ on the topological space $\mathbb{X}$, denote by $\mathsf{TF}_\Lambda$ the category of torsion free coherent $\Lambda$–modules and by $\mathsf{VB}_\Lambda$ its full subcategory of locally free sheaves. The usual way to deal with vector bundles

on a singular curve is to lift them to the normalization, and then work on a smooth curve, see for example [Ses209]. Passing to the normalization we loose information about the isomorphism classes of objects of $\mathsf{VB}_\mathcal{O}$ since non-isomorphic vector bundles can have isomorphic inverse images. In order to describe the fibers of the map $\mathsf{VB}_\mathcal{O} \longrightarrow \mathsf{VB}_{\widetilde{\mathcal{O}}}$ and to be able to deal with arbitrary torsion free sheaves we introduce the following definition:

**Definition 2.** The category of triples $\mathsf{T}_\mathbb{X}$ *is defined as follows:*

1. *Its objects are triples $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$, where $\widetilde{\mathcal{F}}$ is a locally free $\widetilde{\mathcal{O}}$–module, $\mathcal{M}$ is a coherent $\mathcal{A}$–module and $\tilde{i} : \mathcal{M} \otimes_\mathcal{O} \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}}$ is an epimorphism of $\widetilde{\mathcal{A}}$–modules, which induces a monomorphism of $\mathcal{A}$–modules $i : \mathcal{M} \longrightarrow \mathcal{M} \otimes_\mathcal{O} \widetilde{\mathcal{A}} \xrightarrow{\tilde{i}} \widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}}$.*

2. *A morphism $(\widetilde{\mathcal{F}}_1, \mathcal{M}_1, \tilde{i}_1) \xrightarrow{(F,f)} (\widetilde{\mathcal{F}}_2, \mathcal{M}_1, \tilde{i}_2)$ is given by a pair $(F, f)$, where $\widetilde{\mathcal{F}}_1 \xrightarrow{F} \widetilde{\mathcal{F}}_2$ is a morphism of $\widetilde{\mathcal{O}}$-modules and $\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2$ is a morphism of $\mathcal{A}$-modules, such that the following diagram*

$$
\begin{array}{ccc}
\mathcal{M}_1 \otimes_\mathcal{O} \widetilde{\mathcal{A}} & \xrightarrow{\ \tilde{i}_1\ } & \widetilde{\mathcal{F}}_1 \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}} \\
{\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle \bar{F}} \\
\mathcal{M}_2 \otimes_\mathcal{O} \widetilde{\mathcal{A}} & \xrightarrow{\ \tilde{i}_2\ } & \widetilde{\mathcal{F}}_2 \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}}
\end{array}
$$

*is commutative in $\mathsf{Coh}_{\widetilde{\mathcal{A}}}$, where $\bar{F} = F \otimes id$ and $\bar{f} = \varphi \otimes id$.*

The main reason to introduce the formalism of triples is the following theorem:

**Theorem 16.** *The functor $\mathsf{TF}_\mathcal{O} \xrightarrow{\Psi} \mathsf{T}_\mathbb{X}$ mapping a torsion free sheaf $\mathcal{F}$ to the triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$, where $\widetilde{\mathcal{F}} = \mathcal{F} \otimes_\mathcal{O} \widetilde{\mathcal{O}} / \mathsf{tor}(\mathcal{F} \otimes_\mathcal{O} \widetilde{\mathcal{O}})$, $\mathcal{M} = \widetilde{\mathcal{F}} \otimes_\mathcal{O} \mathcal{A}$ and $\tilde{i} : \mathcal{F} \otimes_\mathcal{O} \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}}$, is an equivalence of categories. Moreover, the category of vector bundles $\mathsf{VB}_\mathcal{O}$ is equivalent to the full subcategory of $\mathsf{T}_\mathbb{X}$ consisting of those triples $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$, for which $\mathcal{M}$ is a free $\mathcal{A}$–module and $\tilde{i}$ is an isomorphism.*

*Sketch of the proof.* We construct the quasi-inverse functor $\mathsf{T}_\mathbb{X} \xrightarrow{\Psi'} \mathsf{TF}_\mathbb{X}$ as follows. Let $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$ be some triple. Consider the pull-back diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J}\widetilde{\mathcal{F}} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & \mathcal{J}\widetilde{\mathcal{F}} & \longrightarrow & \widetilde{\mathcal{F}} & \longrightarrow & \widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}} & \longrightarrow & 0
\end{array}
$$

in the category of $\mathcal{O}$–modules. Since the pull-back is functorial, we get a functor $\mathsf{T}_\mathbb{X} \xrightarrow{\Psi'} \mathsf{Coh}(\mathbb{X})$. Since the map $i$ is injective, $\mathcal{F} \longrightarrow \widetilde{\mathcal{F}}$ is injective as well, so $\mathcal{F}$ is torsion free. It remains to show that the functors $\Psi$ and $\Psi'$ are quasi-inverse to each other. We refer to [DG01] for the details of the proof.

*Remark 4.* There is a geometric way to interpret the above construction of the category of triples. Let $\mathbb{X}$ be a singular curve, $\widetilde{\mathbb{X}} \xrightarrow{p} \mathbb{X}$ its normalization, $s : \mathbb{S} \longrightarrow \mathbb{X}$ the inclusion of the closed subscheme defined by the conductor ideal and $\tilde{s} : \widetilde{\mathbb{S}} \longrightarrow \widetilde{\mathbb{X}}$ its pull-back on the normalization. Consider the Cartesian diagram

$$
\begin{array}{ccc}
\widetilde{\mathbb{S}} & \xrightarrow{\tilde{s}} & \widetilde{\mathbb{X}} \\
{\scriptstyle \tilde{p}} \downarrow & & \downarrow {\scriptstyle p} \\
\mathbb{S} & \xrightarrow{s} & \mathbb{X}.
\end{array}
$$

Theorem 16 says that a torsion free sheaf $\mathcal{F}$ on a singular curve $\mathbb{X}$ can be reconstructed from its "normalization" $p^*(\mathcal{F})/\mathsf{tor}(p^*\mathcal{F})$, its pull-back $s^*\mathcal{F}$ on $\mathbb{S}$, and the "gluing map" $\tilde{p}^*s^*\mathcal{F} \longrightarrow \tilde{s}^*p^*\mathcal{F} \longrightarrow \tilde{s}^*\big(p^*\mathcal{F}/\mathsf{tor}(p^*\mathcal{F})\big)$.

Now let us see how this construction can be used to classify torsion free sheaves on degenerations of elliptic curves. Let $char(\mathbf{k}) \neq 2$ and $\mathbb{E}$ be a nodal Weierstraß curve, given by the equation $zy^2 - x^3 - zx^2 = 0$, $s = (0:0:1)$ its singular point, $\mathbb{P}^1 = \widetilde{\mathbb{E}} \xrightarrow{p} \mathbb{E}$ the normalization map. Choose coordinates on $\mathbb{P}^1$ in such a way that the preimages of $s$ are $0 = (0:1)$ and $\infty = (1:0)$.

The previous theorem says that a torsion free sheaf $\mathcal{F}$ on the curve $\mathbb{E}$ is uniquely determined by the corresponding triple $\Psi(\mathcal{F}) = (\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$. Here $\widetilde{\mathcal{F}}$ is a locally free $\widetilde{\mathcal{O}}$–module, or as we have seen, a locally free $\mathcal{O}_{\mathbb{P}^1}$-module. Using the notation $\widetilde{\mathcal{O}}(n) = p_*(\mathcal{O}_{\mathbb{P}^1}(n))$, due to the theorem of Birkhoff-Grothendieck, $\widetilde{\mathcal{F}} \cong \bigoplus_{n \in \mathbb{Z}} \widetilde{\mathcal{O}}(n)^{r_n}$.

Since $\mathcal{A} = \mathcal{O}/\mathcal{J} = \mathbf{k}(s)$ and $\widetilde{\mathcal{A}} = \widetilde{\mathcal{O}}/\mathcal{J} = (\mathbf{k} \times \mathbf{k})(s)$, the sheaf $\mathcal{M}$ can be identified with its stalk at $s$ and the map $\tilde{i} : \mathcal{M} \otimes_\mathcal{A} \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}}$ can be viewed as a pair $(i(0), i(\infty))$ of linear maps of $\mathbf{k}$–vector spaces. In order to write $\tilde{i}$ in terms of matrices we identify $\widetilde{\mathcal{O}}(n) \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}}$ with $p_*(\mathcal{O}_{\mathbb{P}^1}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}/\mathcal{I})$. The choice of coordinates on $\mathbb{P}^1$ fixes two canonical sections $z_0$ and $z_1$ of $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ and we use the trivializations

$$
\mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{I} \longrightarrow \mathbf{k}(0) \times \mathbf{k}(\infty)
$$

given by $\zeta \otimes 1 \mapsto (\zeta/z_1^n(0), \zeta/z_0^n(\infty))$. Note, that this isomorphism only depends on the choice of coordinates of $\mathbb{P}^1$. In such a way we supply the $\widetilde{\mathcal{A}}$–module $\widetilde{\mathcal{F}} \otimes \widetilde{\mathcal{A}} = \widetilde{\mathcal{F}}(0) \oplus \widetilde{\mathcal{F}}(\infty)$ with a basis and get isomorphisms $\widetilde{\mathcal{F}}(0) \cong \bigoplus_{n \in \mathbb{Z}} \mathbf{k}(0)^{r_n}$ and $\widetilde{\mathcal{F}}(\infty) \cong \bigoplus_{n \in \mathbb{Z}} \mathbf{k}(\infty)^{r_n}$. With respect to all choices the morphism $\tilde{i}$ is given by two matrices $i(0)$ and $i(\infty)$, divided into horizontal blocks:

From the definition of the category of triples it follows that the matrices $i(0)$ and $i(\infty)$ have to be of full row rank and the transposed matrix $(i(0)|i(\infty))^t$ has to be monomorphic. Vector bundles on $\mathbb{E}$ correspond to invertible square matrices $i(0)$ and $i(\infty)$.

$$i(0) \qquad\qquad i(\infty)$$

Of course, for a fixed $\widetilde{\mathcal{F}} = \bigoplus_{n \in \mathbb{Z}} \widetilde{\mathcal{O}}(n)^{r_n}$ and $\mathcal{M} = \mathbf{k}^N(s)$, two different pairs of matrices $(i(0)|i(\infty))$ and $(i'(0)|i'(\infty))$ can define isomorphic torsion free sheaves on $\mathbb{E}$. However, since the functor $\Psi : \mathsf{TF}_{\mathcal{O}} \longrightarrow \mathsf{T}_{\mathbb{X}}$ preserves isomorphism classes of indecomposable objects, two triples $(\widetilde{\mathcal{F}}, \mathcal{M}, \widetilde{i})$ and $(\widetilde{\mathcal{F}}, \mathcal{M}, \widetilde{i'})$ define isomorphic torsion free sheaves if and only if there are automorphisms $F : \widetilde{\mathcal{F}} \longrightarrow \widetilde{\mathcal{F}}$ and $f : \mathcal{M} \longrightarrow \mathcal{M}$ such that $\bar{F}\widetilde{i} = \widetilde{i'}\bar{f}$.

An endomorphism $F$ of $\widetilde{\mathcal{F}} = \bigoplus_{n \in \mathbb{Z}} \widetilde{\mathcal{O}}(n)^{r_n}$ can be written in a matrix form: $F = (F_{kl})$, where $F_{kl}$ is a $r_l \times r_k$–matrix with coefficients in the vector space $\mathsf{Hom}(\widetilde{\mathcal{O}}(k), \widetilde{\mathcal{O}}(l)) \cong \mathbf{k}[z_0, z_1]_{k-l}$. In particular, the matrix $F$ is lower triangular and the diagonal $r_n \times r_n$ blocks $F_{nn}$ are just matrices over $\mathbf{k}$. The morphism $F$ is an isomorphism if and only if all $F_{nn}$ are invertible. Let $r = rank(\widetilde{\mathcal{F}})$. With respect to the chosen trivialization of $\mathcal{O}_{\mathbb{P}^1}(n)$ at $0$ and $\infty$ the map $\bar{F} : \mathbf{k}^r(0) \oplus \mathbf{k}^r(\infty) \longrightarrow \mathbf{k}^r(0) \oplus \mathbf{k}^r(\infty)$ is given by the pair of matrices $(F(0), F(\infty))$ and we have the following transformation rules for the pair $(i(0), i(\infty))$:

$$\big(i(0), i(\infty)\big) \mapsto \big(F(0)^{-1}i(0)S, F(\infty)^{-1}i(\infty)S\big)$$

where $F$ is an automorphism of $\bigoplus_{n \in \mathbb{Z}} \widetilde{\mathcal{O}}(n)^{r_n}$ and $S$ an automorphism of $\mathbf{k}^N$.

Note, that the matrices $F_{kl}(0)$ and $F_{kl}(\infty)$, $k, l \in \mathbb{Z}, k > l$ can be *arbitrary* and $F_{nn}(0) = F_{nn}(\infty)$ can be *arbitrary invertible* for $n \in \mathbb{Z}$. As a result we get the following *matrix problem*.

**Matrix problem for a nodal Weierstraß curve**. We have two matrices $i(0)$ and $i(\infty)$ of the same size and both of full row rank. Each of them is divided into horizontal blocks labeled by integers (they are called sometimes weights). Blocks of $i(0)$ and $i(\infty)$, labeled by the same integer, have the same size. We are allowed to perform only the following transformations:

1. We can simultaneously do any elementary transformations of columns of $i(0)$ and $i(\infty)$.
2. We can simultaneously do any invertible elementary transformations of rows inside of any two conjugated horizontal blocks.
3. We can in each of the matrices $i(0)$ and $i(\infty)$, independently add a scalar multiple of any row with lower weight to any row with higher weight.

The main idea is that we can transform the matrix $i$ into a canonical form which is quite analogous to the Jordan normal form.

These types of matrix problems are well-known in representation theory. First they appeared in a work of Nazarova and Roiter [NR69] about the classification of $\mathbf{k}[[x,y]]/(xy)$–modules. They are called, sometimes, "Gelfand problems" or "representations of bunches of chains".

*Example 1.* Let $\mathbb{E}$ be a nodal Weierstraß curve.

- The following triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$ defines an indecomposable vector bundle of rank 2 on $\mathbb{E}$: the normalization $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}} \oplus \widetilde{\mathcal{O}}(n), n \neq 0$, $\mathcal{M} = \mathbf{k}^2(s)$ and matrices:

$$i(0) = \begin{array}{c} \\ \boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}} \\ \end{array} \begin{array}{c} 0 \\ \\ n \end{array} \quad \text{and} \quad i(\infty) = \begin{array}{c} \\ \boxed{\begin{array}{cc} 0 & 1 \\ \lambda & 0 \end{array}} \\ \end{array} \begin{array}{c} 0 \\ \\ n \end{array} \quad \lambda \in \mathbf{k}^*.$$

- The triple $(\widetilde{\mathcal{O}}(-1), \mathbf{k}^2, \tilde{i} = \boxed{1 \ \ 0} \ \boxed{0 \ \ 1})$ describes the unique torsion free sheaf that is not locally free of degree zero, which compactifies the Jacobian $\mathrm{Pic}^0(\mathbb{E})$.

A Gelfand matrix problem is determined by a certain partially ordered set together with an equivalence relation on it. Such a poset with an equivalence relation is called a *bunch of chains.* Before giving a general definition, we give an example describing the matrix problem which corresponds to a nodal Weierstraß curve.

There are two infinite sets $E_0 = \{E_0(k)|k \in \mathbb{Z}\}$ and $E_\infty = \{E_\infty(k)|k \in \mathbb{Z}\}$ with the ordering $\cdots < E_*(-1) < E_*(0) < E_*(1) < \ldots, * \in \{0,\infty\}$ and two one-point sets $\{F_0\}$ and $\{F_\infty\}$. On the union

$$\mathbf{E} \bigcup \mathbf{F} = (E_0 \cup E_\infty) \bigcup (F_0 \cup F_\infty)$$

we introduce an equivalence relation: $E_0(k) \sim E_\infty(k)$, where $k \in \mathbb{Z}$ and $F_0 \sim F_\infty$.
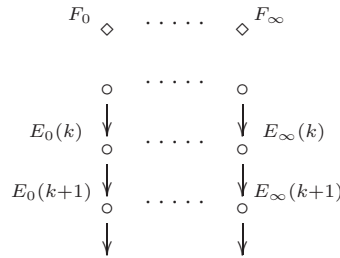


This picture contains complete information about the corresponding matrix problem. The circles denote elements of $\mathbf{E}$, the diamonds denote elements $F_0$ and $F_\infty$, dotted lines connect equivalent elements and vertical arrows describe the partial order in $E_0$ and $E_\infty$. The sets $E_0 \cup F_0$ and $E_\infty \cup F_\infty$ correspond to matrices $i(0)$ and $i(\infty)$ respectively, elements $E_0(k)$ and $E_\infty(k)$

label their horizontal stripes, $k \in \mathbb{Z}$. We also say that a row from the horizontal block $E_*(k)$ has weight $k$, $* \in \{0, \infty\}$. The equivalence relation $E_0(k) \sim E_\infty(k)$ means that horizontal blocks of weight $k$ have the same number of rows and $F_0 \sim F_\infty$ tells that $i(0)$ and $i(\infty)$ have the same number of columns. Moreover, elementary transformations inside of two conjugated blocks have to be done simultaneously. The total order on $E_0$ and $E_\infty$ means that we can add any scalar multiple of any row of smaller weight to any row with a bigger weight. Such transformations can be performed in the matrices $E_0$ and $E_\infty$ independently.

**Definition 3.** *Let* **k** *be an arbitrary field. A cycle of $n$ projective lines is a projective curve $\mathbb{E}_n$ over* **k** *with $n$ irreducible components, each of them is isomorphic to $\mathbb{P}^1$. We additionally assume that all components intersect transversally with intersection matrix of type $\widetilde{A}_n$ and the completion of the local rings of any singular point of $\mathbb{E}_n$ is isomorphic to* **k**$[[u, v]]/uv$. *In a similar way, a chain of $k$ projective lines $\mathbb{I}_k$ is a configuration of projective $k$ lines with intersection matrix of type $A_{k-1}$.*

*Remark 5.* Let **k** $= \mathbb{R}$ and $\mathbb{E}$ be the cubic curve $zy^2 = x^3 - zx^2$. Then $s = (0 : 0 : 1)$ is the singular point of $\mathbb{E}$ and $\widehat{\mathcal{O}_{\mathbb{E},s}} = \mathbb{R}[[u, v]]/(u^2 + v^2)$. Then $\mathbb{E}$ is not a cycle of projective lines in the sense of Definition 3 and the combinatorics of the indecomposable vector bundles on $\mathbb{E}$ will be considered elsewhere.

Let $\mathbb{E}$ be either a chain or a cycle of projective lines and $\widetilde{\mathbb{E}}$ its normalization. The matrix problem we get is given by the following partially ordered set.

Consider the set of pairs $\{(\mathbb{L}, a)\}$, where $\mathbb{L}$ is an irreducible component of $\widetilde{\mathbb{E}}$ and $a \in \mathbb{L}$ a preimage of a singular point. To each such pair we attach a totally ordered set $E_{(\mathbb{L},a)} = \{E_{(\mathbb{L},a)}(i)|i \in \mathbb{Z}\}$, where $\cdots < E_{(\mathbb{L},a)}(-1) < E_{(\mathbb{L},a)}(0) < E_{(\mathbb{L},a)}(1) < \ldots$ and a one-point set $F_{(\mathbb{L},a)}$. On the union

$$\mathbf{E} \bigcup \mathbf{F} = \bigcup_{(\mathbb{L},a)} (E_{(\mathbb{L},a)} \cup F_{(\mathbb{L},a)}),$$

we introduce an equivalence relation:

1. $F_{(\mathbb{L}',a')} \sim F_{(\mathbb{L}'',a'')}$, where $a'$ and $a''$ are preimages of the same singular point $a \in \mathbb{E}$
2. $E_{(\mathbb{L},a')}(k) \sim F_{(\mathbb{L},a'')}(k)$ for $k \in \mathbb{Z}$ and $a', a'' \in \mathbb{L}$.

Such a partially ordered set with an equivalence relation is called a bunch of chains [Bon92]. A representation of such a bunch of chains is given by a set of matrices $M(\mathbb{L}, a)$, for each element $(\mathbb{L}, a)$. Every matrix $M(\mathbb{L}, a)$ is divided into horizontal blocks labelled by the elements of $E_{(\mathbb{L},a)}$. Of course, all but finitely many labels corresponds to empty blocks. The principle of conjugation of blocks is the same as for a rational curve with one node.

The category of representations of a bunch of chains is additive and has two types of indecomposable representations: *bands* and *strings* [Bon92]. Hence,

the technique of representations of bunches of chains allows to describe inde-composable torsion free sheaves on chains and cycles of projective lines.

Let $\mathbb{E} = \mathbb{E}_n$ be a cycle of $n$ projective lines, $\{a_1, a_2, \ldots, a_n\}$ the set of singular points of $\mathbb{E}$, $\widetilde{\mathbb{E}} \xrightarrow{p} \mathbb{E}$ the normalization of $\mathbb{E}$, $\widetilde{\mathbb{E}} = \coprod_{i=1}^{n} \mathbb{L}_i$, where each $\mathbb{L}_i$ is isomorphic to a projective line and $\{a_i', a_i''\} = p^{-1}(a_i)$. Assume that $a_i', a_{i+1}'' \in \mathbb{L}_i$, where $a_{n+1}'' = a_1''$. Fix coordinates on each projective line $\mathbb{L}_i$ in such a way, that $a_i' = (0 : 1)$ and $a_{i+1}'' = (1 : 0)$.

**Definition 4.** *A band $\mathcal{B}(\boldsymbol{d}, m, p(t))$ is an indecomposable vector bundle of rank rmd. It is determined by the following parameters:*

1. *$\boldsymbol{d} = (d_1, d_2, \ldots, d_n, d_{n+1}, d_{n+2}, \ldots, d_{2n}, \ldots, d_{rn-n+1}, d_{rn-n+2}, \ldots, d_{rn}) \in \mathbb{Z}^{rn}$ is a sequence of degrees on the normalized curve $\widetilde{\mathbb{E}}$. This sequence should be non-periodic, i.e. not of the form $\mathbf{e}^s = \underbrace{ee\ldots e}_{s\,times}$, where $\boldsymbol{e} = e_1, e_2, \ldots, e_{qn}$ is another sequence and $q = \frac{r}{s}$.*
2. *$p(t) = t^k + a_1 t^{d-1} + \cdots + a_k \in \mathbf{k}[t]$ is an irreducible polynomial of degree $k$, $p(t) \neq 0$.*
3. *$m \in \mathbb{Z}^+$ is a positive integer.*

*In particular, one can recover from the sequence $\boldsymbol{d}$ the pull-back of $\mathcal{B}(\boldsymbol{d}, m, p(t))$ on the l-th irreducible component of $\widetilde{\mathbb{E}}$: it is*

$$p_l^*(\mathcal{B}(\boldsymbol{d}, m, p(t))) \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{L}_l}(d_{l+in})^{mk}.$$

*A string $\mathcal{S}(\boldsymbol{d}, f)$ depends only on two discrete parameters $f \in \{1, 2, \ldots, n\}$ and $\boldsymbol{d} = (d_1, d_2, \ldots, d_t)$, $t > 1$.*

Now we are going to explain the way of construction of gluing matrices of triples corresponding to bands $\mathcal{B}(\boldsymbol{d}, m, p(t))$ and strings $\mathcal{S}(\boldsymbol{e}, f)$.

**Algorithm 1. Bands**. Let $\boldsymbol{d} = (d_1, d_2, \ldots, d_{rn}) \in \mathbb{Z}^{rn}$ be a non-periodic sequence, $m \in \mathbb{Z}^+$ and $p(t) \in \mathbf{k}[t]$ an irreducible polynomial of degree $k$. We have $2n$ matrices $M(\mathbb{L}_i, a_i')$ and $M(\mathbb{L}_i, a_{i+1}''), i = 1, \ldots, n$ occurring in the triple, corresponding to $\mathcal{B}(\boldsymbol{d}, m, p(t))$. Each of them has size $mrk \times mrk$. Divide these matrices into $mk \times mk$ square blocks. Consider the sequences $\boldsymbol{d}(i) = d_i d_{i+n} \ldots d_{i+(r-1)n}$ and label the horizontal strips of $M(\mathbb{L}_i, a_i')$ and $M(\mathbb{L}_i, a_{i+1}'')$ by integers occurring in each $\boldsymbol{d}(i)$. If some integer $d$ appears $l$ times in $\boldsymbol{d}(i)$ then the horizontal strip corresponding to the label $d$ consists of $l$ substrips having $mk$ rows each. Recall now an algorithm of writing the components of the matrix $\tilde{i}$ in a normal form [Bon92]:

1. Start with the sequence $(\mathbb{L}_1, a'_1) \xrightarrow{1} (\mathbb{L}_1, a''_2) \xrightarrow{1} (\mathbb{L}_2, a'_2) \xrightarrow{1} (\mathbb{L}_2, a''_3) \xrightarrow{1}$ $\cdots \xrightarrow{1} (\mathbb{L}_n, a'_1) \xrightarrow{2} (L_1, a''_1) \xrightarrow{2} \cdots \xrightarrow{r} (\mathbb{L}_n, a'_n) \xrightarrow{r} (\mathbb{L}_n, a''_1) \xrightarrow{1}$ . It is convenient to imagine this sequence as a cyclic word broken at the place $(\mathbb{L}_1, a'_1)$.

2. Unroll the sequence $\boldsymbol{d}$. This means that we write over each $(\mathbb{L}_i, a)$ the corresponding term of the subsequence $\boldsymbol{d}(i)$ together with the number of its previous occurrences in $\boldsymbol{d}(i)$ including the current one:

$$(\mathbb{L}_1, a'_1)^{(d_1,1)} \xrightarrow{1} (\mathbb{L}_2, a'_2)^{(d_2,1)} \xrightarrow{1} (\mathbb{L}_2, a''_3)^{(d_2,1)} \xrightarrow{1}$$

$$\cdots \xrightarrow{r} (\mathbb{L}_n, a''_1)^{d_{rn},*} \xrightarrow{r} (\mathbb{L}_n, a'_1)^{(d_{rn},1)} \xrightarrow{1} .$$

3. Now we can fill the entries of the matrices $M(\mathbb{L}, a)$. Consider each arrow $(\mathbb{L}, a)^{(d,i)} \xrightarrow{l}$ . Then insert the matrix $I_{mk}$ in the block $((d,i),l)$ of the matrix $M(\mathbb{L}, a)$, which is defined as the intersection of the $i$-th substrip of the horizontal strip labeled by $d$ and the $l$-th vertical strip.

4. Put at the $((d_{rn},1),r)$-th place of $M(\mathbb{L}_n, a''_1)$ the Frobenius block $J_m(p(t))$.

**Strings**. Let $\boldsymbol{e} = (e_1, e_2, \ldots, e_s) \in \mathbb{Z}^s$ and $f \in \{1, 2, \ldots, n\}$. The algorithm to write the matrices for the torsion free sheaf $\mathcal{S}(\boldsymbol{d}, f)$ is essentially the same as for bands. The parameter $f$ denotes the number of the component $\mathbb{L}_f$ of $\widetilde{\mathbb{E}}$ which the sequence $(\mathbb{L}_f, a'_f) \longrightarrow (\mathbb{L}_f, a''_{f+1}) \longrightarrow \cdots \longrightarrow (\mathbb{L}_{f+s}, a'_{f+s}) \longrightarrow$ $(\mathbb{L}_{f+s}, a''_{f+s+1})$ starts with. Then we unroll $\boldsymbol{e}$ using the same algorithm as for bands. The only difference is that we insert instead of $I_{mk}$ the unit $(1 \times 1)$–matrix. Note, that some of the matrices $M(\mathbb{L}_i, a'_i)$ and $M(\mathbb{L}_i, a''_i)$ can be non-square, but they are automatically of full row rank

*Example 2.* Let $\mathbb{E} = \mathbb{E}_2$ be a cycle of two projective lines, $\boldsymbol{d} = (0, 1, 1, 3, 1, -2)$ and $p(t) \in \mathbf{k}[[t]]$ an irreducible polynomial of degree $k$. Then $\mathcal{B}(\boldsymbol{d}, m, p(t))$ is a vector bundle of degree $3m$ with the normalization

$$\left(\mathcal{O}_{\mathbb{L}_1}^{mk} \oplus \mathcal{O}_{\mathbb{L}_1}(1)^{2mk}\right) \oplus \left(\mathcal{O}_{\mathbb{L}_2}(-2)^{mk} \oplus \mathcal{O}_{\mathbb{L}_2}(1)^{mk} \oplus \mathcal{O}_{\mathbb{L}_2}(3)^{mk}\right)$$

and gluing matrices

The corresponding unrolled sequence looks as follows:

$(\mathbb{L}_1, a_1')^{(0,1)} \xrightarrow{1} (\mathbb{L}_1, a_2'')^{(0,1)} \xrightarrow{1} (\mathbb{L}_2, a_2')^{(1,1)} \xrightarrow{1} (\mathbb{L}_2, a_1'')^{(1,1)} \xrightarrow{2}$
$(\mathbb{L}_1, a_1')^{(1,1)} \xrightarrow{2} (\mathbb{L}_1, a_2'')^{(1,1)} \xrightarrow{2} (\mathbb{L}_2, a_2')^{(3,1)} \xrightarrow{2} (\mathbb{L}_2, a_1'')^{(3,1)} \xrightarrow{3}$
$(\mathbb{L}_1, a_1')^{(1,2)} \xrightarrow{3} (\mathbb{L}_1, a_2'')^{(1,2)} \xrightarrow{3} (\mathbb{L}_2, a_2')^{(-2,1)} \xrightarrow{3} (\mathbb{L}_2, a_1'')^{(-2,1)} \xrightarrow{1} .$

Let $f = 2$ and $\boldsymbol{d} = (-1, 0, 1, -1, 1)$. Then the corresponding torsion free sheaf $\mathcal{S}(\boldsymbol{d}, f)$ has normalization

$$\widetilde{\mathcal{F}} = \left(\mathcal{O}_{\mathbb{L}_1}(-1) \oplus \mathcal{O}_{\mathbb{L}_1}\right) \oplus \left(\mathcal{O}_{\mathbb{L}_2}(-1) \oplus \mathcal{O}_{\mathbb{L}_2}(1)^2\right)$$

and gluing matrices

$$M(\mathbb{L}_1, a_1') = \begin{array}{|c|c|c|} \hline & 1 & \\ \hline 1 & & \\ \hline \end{array} \begin{array}{c} {}^{-1} \\ {}_{0} \end{array} \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \hline \end{array} = M(\mathbb{L}_1, a_2'')$$

$$M(\mathbb{L}_2, a_1'') = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array} \begin{array}{c} {}^{-1} \\ {}_{1} \end{array} \begin{array}{|c|c|c|} \hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array} = M(\mathbb{L}_2, a_2')$$

where $J_m$ is the Frobenius block corresponding to the $\mathbf{k}[t]$-module $\mathbf{k}[t]/p(t)^m$. The corresponding unrolled sequence is

$(\mathbb{L}_2, a_2')^{(-1,1)} \xrightarrow{1} (\mathbb{L}_2, a_1'')^{(-1,1)} \xrightarrow{1} (\mathbb{L}_1, a_1')^{(0,1)} \xrightarrow{1} (\mathbb{L}_1, a_2'')^{(0,1)} \xrightarrow{1}$
$(\mathbb{L}_2, a_2')^{(1,1)} \xrightarrow{2} (\mathbb{L}_2, a_1'')^{(1,1)} \xrightarrow{2} (\mathbb{L}_1, a_1')^{(-1,1)} \xrightarrow{2} (\mathbb{L}_1, a_2'')^{(-1,1)} \xrightarrow{2}$
$(\mathbb{L}_2, a_2')^{(1,2)} \xrightarrow{3} (\mathbb{L}_2, a_1'')^{(1,2)} \xrightarrow{3} .$

Summing everything up, we get the following theorem.

**Theorem 17 ([DG01]).** *Let $\mathbb{E} = \mathbb{E}_n$ be a cycle of $n$ projective lines over a filed $\mathbf{k}$. Then*

- *any indecomposable vector bundle on $\mathbb{E}$ is isomorphic to some $\mathcal{B}(\boldsymbol{d}, m, p(t))$, where $\boldsymbol{d} = (d_1, d_2, \ldots, d_{rn}) \in \mathbb{Z}^{rn}$ is a non-periodic sequence, $m \in \mathbb{Z}^+$ and $p(t) = t^k + a_1 t^{k-1} + \cdots + a_k \in \mathbf{k}[t]$ is an irreducible polynomial, $p(t) \neq 0$.*
- *Any torsion free and not locally free coherent sheaf is isomorphic to some $\mathcal{S}(\boldsymbol{d}, t)$, where $t \in \{1, 2, \ldots, n\}$ and $\boldsymbol{d} \in \mathbb{Z}^{rn}$.*

*The only isomorphisms between indecomposable vector bundles are generated by*

- $\mathcal{B}(\boldsymbol{d}, m, p(t)) \cong \mathcal{B}(\boldsymbol{d}^\circ, m, q(t))$, $\boldsymbol{d}^\circ = d_{rn}, d_{rn-1}, \ldots, d_1$ *and* $q(t) = \frac{t^k}{a_k} p(1/t)$
- $\mathcal{B}(\boldsymbol{d}, m, p(t)) \cong \mathcal{B}(\boldsymbol{d}', m, p(t))$, *with*

$$\boldsymbol{d}' = d_{n+1}, d_{n+2}, \ldots, d_{2n}, d_{2n+1}, \ldots, d_1, d_2, \ldots, d_n.$$

*The only isomorphisms between strings are $\mathcal{S}(\boldsymbol{e}, f) \cong \mathcal{S}(\boldsymbol{e}^\circ, f^\circ)$, where $\boldsymbol{e}^\circ$ is the opposite sequence $\boldsymbol{e}^\circ = e_s, e_{s-1}, \ldots, e_1$. If $s = nk + s'$ with $0 \leq s' < n$, then $f^\circ = s' + f$ taken modulo $n$.*

*Remark 6.* If $\mathbb{E} = \mathbb{E}_1$ is a Weierstraß nodal curve, then there is no choice for the parameter $f$ in the definition of a string and one simply uses the notation $\mathcal{S}(\boldsymbol{d})$. If the field $\mathbf{k}$ is algebraically closed, we write $\mathcal{B}(\boldsymbol{d}, m, \lambda)$ instead of $\mathcal{B}(\boldsymbol{d}, m, t - \lambda)$.

As a direct corollary of the combinatorics of strings and bands we obtain the following theorem

**Theorem 18 ([DG01]).** *Let $\mathbb{X} = \mathbb{I}_n$ be a chain of $n$ projective lines, then any vector bundle $\mathcal{E}$ on $\mathbb{X}$ splits into a direct sum of lines bundles. Moreover, $\mathrm{Pic}(\mathbb{I}_n) \cong \mathbb{Z}^n$ and a line bundle is determined by its restrictions on each irreducible component. A description of torsion free sheaves is similar: any indecomposable torsion free sheaf is isomorphic to the direct image of a line bundle on a subchain of projective lines.*

### 3.2 Properties of torsion free sheaves on cycles of projective lines

Throughout this subsection, let $\mathbf{k}$ be an algebraically closed field and $\mathbb{E} = \mathbb{E}_n$ a cycle of $n$ projective lines over $\mathbf{k}$. As we have seen in the previous subsection, indecomposable vector bundles on $\mathbb{E}$ are bands $\mathcal{B}(\boldsymbol{d}, m, \lambda)$ and indecomposable torsion free but not locally free sheaves are strings $\mathcal{S}(\boldsymbol{d}, f)$. They were described in terms of a certain problem of linear algebra. However, in the case of an algebraically closed field there is a geometric way to present the classification of indecomposable torsion free sheaves on $\mathbb{E}$ without appealing to the formalism of bunches of chains. This description, which we give here for the first time, is completely parallel to Oda's description of vector bundles on elliptic curves [Oda71].

We start with a lemma describing unipotent vector bundles on $\mathbb{E}$.

**Lemma 2.** *For any integer $m \geq 1$ there exists a unique indecomposable vector bundle $\mathcal{F}_m$ on $\mathbb{E}_n$ appearing in the exact sequence*

$$0 \longrightarrow \mathcal{F}_{m-1} \longrightarrow \mathcal{F}_m \longrightarrow \mathcal{O} \longrightarrow 0, \qquad \mathcal{F}_1 = \mathcal{O}.$$

*In our notation we have $\mathcal{F}_m \cong \mathcal{B}(\boldsymbol{0}, m, 1)$, where $\boldsymbol{0} = \underbrace{(0, 0, \ldots, 0)}_{n \, times}$.*

*Sketch of the proof.* Since the dualizing sheaf $\omega_{\mathbb{E}} \cong \mathcal{O}$ is trivial, we have $\mathsf{Ext}^1(\mathcal{O}, \mathcal{O}) = \mathbf{k}$ and there is a unique non-split extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{O} \longrightarrow 0.$$

Then using the same arguments as in [Ati57], we can inductively construct indecomposable vector bundles $\mathcal{F}_m$, $m \geq 1$ such that $H^0(\mathcal{F}_m) \cong H^1(\mathcal{F}_m) = \mathbf{k}$ together with exact sequences

$$0 \longrightarrow \mathcal{F}_{m-1} \longrightarrow \mathcal{F}_m \longrightarrow \mathcal{O} \longrightarrow 0, \qquad \mathcal{F}_1 = \mathcal{O}.$$

On the other hand, $\mathcal{B}(\mathbf{0}, m, 1)$ is the unique indecomposable vector bundle on $\mathbb{E}$ of rank $m$ and normalization $\mathcal{O}_{\widetilde{\mathbb{E}}}^m$ with a non-zero section. Hence $\mathcal{F}_m \cong \mathcal{B}(\mathbf{0}, m, 1)$.

The proof of the following proposition is straightforward:

**Proposition 4.** *Let* $\Psi : \mathsf{VB}(\mathbb{E}) \longrightarrow \mathsf{T}_{\mathbb{E}}$ *be the functor establishing an equivalence between the category of vector bundles on* $\mathbb{E}$ *and the category of triples. Then* $\Psi$ *preserves tensor products:* $\Psi(\mathcal{E} \otimes \mathcal{F}) \cong \Psi(\mathcal{E}) \otimes \Psi(\mathcal{F})$, *where* $(\widetilde{\mathcal{E}}, \mathcal{M}, \tilde{i}) \otimes (\widetilde{\mathcal{F}}, \mathcal{N}, \tilde{j}) = (\widetilde{\mathcal{E}} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{F}}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \tilde{i} \otimes \tilde{j})$. *In particular,*

- *We have an isomorphism* $\mathcal{B}((d), m, \lambda) \cong \mathcal{B}((d), 1, \lambda) \otimes \mathcal{F}_m$.
- *There is the following rule for a decomposition of the tensor product of two unipotent vector bundles:*

$$\mathcal{F}_f \otimes \mathcal{F}_g \cong \bigoplus_i \mathcal{F}_{h_i},$$

  *where integers* $h_i$ *are the same as in the decomposition* $\mathbf{k}[t]/t^f \otimes_{\mathbf{k}} \mathbf{k}[t]/t^g \cong \bigoplus_{i \in \mathbb{Z}} \mathbf{k}[t]/t^{h_i}$ *in the category of* $\mathbf{k}[t]$*–modules.*
- *In particular, if* $\mathbf{k}$ *is of characteristics zero, we have*

$$\mathcal{F}_f \otimes \mathcal{F}_g \cong \bigoplus_{j=1}^{g} \mathcal{F}_{f-g-1+2j}.$$

Now we formulate a geometric description of indecomposable torsion free sheaves on a cycle of projective lines in the case of an algebraically closed field.

**Theorem 19.** *Let* $\mathbb{E} = \mathbb{E}_n$ *be a cycle of* $n$ *projective lines and* $\mathbb{I}_k$ *be a chain of* $k$ *projective lines,* $\mathcal{E}$ *an indecomposable torsion free sheaf on* $\mathbb{E}_n$.

1. *If* $\mathcal{E}$ *is locally free of rank* $r$, *then there is an étale covering* $\pi_r : \mathbb{E}_{nr} \longrightarrow \mathbb{E}_n$, *a line bundle* $\mathcal{L} \in \mathrm{Pic}(\mathbb{E}_{nr})$ *and a natural number* $m \in \mathbb{N}$ *such that*

$$\mathcal{E} \cong \pi_{r*}(\mathcal{L} \otimes \mathcal{F}_m).$$
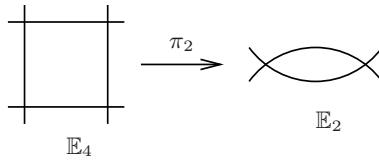
*2. If $\mathcal{E}$ is not locally free then there exists a map $p_k : \mathbb{I}_k \longrightarrow \mathbb{E}_n$ and a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathbb{I}_k)$ such that $\mathcal{E} \cong p_{k*}(\mathcal{L})$.*
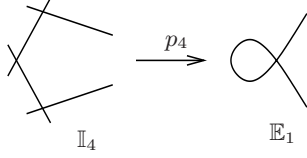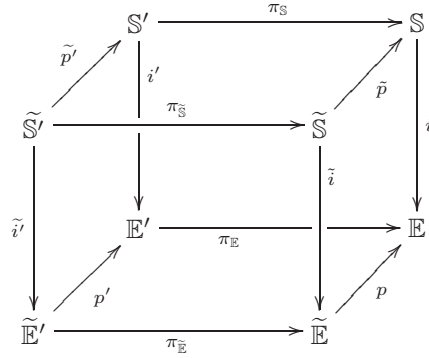


*Proof.* Let $\mathbb{E}$ be a cycle of projective lines and $\pi_{\mathbb{E}} : \mathbb{E}' \longrightarrow \mathbb{E}$ an étale covering of degree $r$, $\mathcal{F}$ a torsion free sheaf on $\mathbb{E}'$. In the notation of Remark 4 we have the commutative diagram



in which all squares are pull-back diagrams. In order to prove the theorem we have to compute the triple describing the torsion free sheaf $\pi_*(\mathcal{F})$. Note that each map $\mathbb{I} \longrightarrow \mathbb{E}$ from a chain of projective lines to a cycle of projective lines factors through an étale covering $\mathbb{E}' \longrightarrow \mathbb{E}$. So, in order to prove the second part of the theorem about the characterization of strings we may consider an étale covering of $\mathbb{E}$ as well.

Note the following simple fact about pull-back diagrams:

**Lemma 3.** *Let*



*be a pull-back diagram where all maps $f, g, f', g'$ are affine. Then for any coherent sheaf $\mathcal{F}$ on $\mathbb{Y}$ it holds $g^* f_* \mathcal{F} \cong f'_* g'^* \mathcal{F}$.*

The morphism $p^* \pi_{\mathbb{E}*}(\mathcal{F})/\mathrm{tor}(p^* \pi_{\widetilde{E}*}(\mathcal{F})) \longrightarrow \pi_{\widetilde{\mathbb{E}}*}\big(p'^*(\mathcal{F})/\mathrm{tor}(p'^*(\mathcal{F}))\big)$ is an isomorphism. Indeed, we have a surjection

$$p^* \pi_{\mathbb{E}*}(\mathcal{F}) \xrightarrow{\cong} \pi_{\widetilde{\mathbb{E}}*} p'^* \mathcal{F} \longrightarrow \pi_{\widetilde{\mathbb{E}}*}(p'^*(\mathcal{F})/\mathsf{tor}(p'^*(\mathcal{F})))$$

which induces a surjective map

$$p^* \pi_{\mathbb{E}*}(\mathcal{F})/\mathsf{tor}(p^* \pi_{\widetilde{\mathbb{E}}*}(\mathcal{F})) \longrightarrow \pi_{\widetilde{\mathbb{E}}*}\big(p'^*(\mathcal{F})/\mathsf{tor}(p'^*(\mathcal{F}))\big)$$

of torsion free sheaves. Since both sheaves have the same rank on each irreducible component of $\widetilde{\mathbb{E}}$, we conclude that this map is also injective and therefore an isomorphism.

We need one more simple statement about étale coverings.

**Lemma 4.** *Let $\pi : \mathbb{Y} \longrightarrow \mathbb{X}$ be an étale map of reduced schemes and $\mathcal{F}$ a coherent sheaf on $\mathbb{Y}$. Then there is a canonical isomorphism $\pi_*(\mathcal{F}/\mathsf{tor}(\mathcal{F})) \longrightarrow \pi_*(\mathcal{F})/\mathsf{tor}(\pi_*(\mathcal{F}))$.*

*Proof.* The canonical map $\mathcal{F} \longrightarrow \mathcal{F}/\mathsf{tor}(\mathcal{F})$ induces the morphism $\pi_*(\mathcal{F}) \longrightarrow \pi_*(\mathcal{F}/\mathsf{tor}(\mathcal{F}))$. Since $\pi$ is étale, the sheaf $\pi_*(\mathcal{F}/\mathsf{tor}(\mathcal{F}))$ is torsion free and the induced morphism $\pi_*(\mathcal{F}/\mathsf{tor}(\mathcal{F})) \longrightarrow \pi_*(\mathcal{F})/\mathsf{tor}(\pi_*(\mathcal{F}))$ is an isomorphism since it is an isomorphism on the stalks.

Let $\widetilde{\varepsilon} : \widetilde{p}'^* i'^*(\mathcal{F}) \longrightarrow \widetilde{i'}^*(p'^*(\mathcal{F})/\mathsf{tor}(p'^*(\mathcal{F})))$ be the gluing map describing the torsion free sheaf $\mathcal{F}$ in the corresponding triple. From the commutativity of the diagram

$$\begin{array}{ccc}
\widetilde{p}^* i^* \pi_{\mathbb{E}*}(\mathcal{F}) & \longrightarrow & \widetilde{i}^*\big(p^* \pi_{\mathbb{E}*}(\mathcal{F})/\mathsf{tor}(p^* \pi_{\mathbb{E}*}(\mathcal{F}))\big) \\
\downarrow & & \downarrow \\
 & & \widetilde{i}^*\big(\pi_{\widetilde{\mathbb{E}}*} p'^*(\mathcal{F})/\mathsf{tor}(\pi_{\widetilde{\mathbb{E}}*} p'^*(\mathcal{F}))\big) \\
 & & \downarrow \\
\widetilde{p}^* \pi_{\mathbb{S}*} i_n^*(\mathcal{F}) & & \widetilde{i}^*\big(\pi_{\widetilde{\mathbb{E}}*}(p'^*(\mathcal{F})/\mathsf{tor}(p'^*(\mathcal{F})))\big) \\
\downarrow & & \downarrow \\
\pi_{\widetilde{\mathbb{S}}*} \widetilde{p}'^* i'^*(\mathcal{F}) & \xrightarrow{\pi_{\widetilde{\mathbb{S}}*}(\widetilde{\varepsilon})} & \pi_{\widetilde{\mathbb{S}}*} \widetilde{i'}^*\big(p'^*(\mathcal{F})/\mathsf{tor}(p'^*(\mathcal{F}))\big)
\end{array}$$

we conclude that the direct image sheaf $\pi_{\mathbb{E}*} \mathcal{F}$ is described by the gluing matrices $\pi_{\widetilde{\mathbb{S}}*}(\widetilde{\varepsilon})$, which are exactly the matrices constructed in the Algorithm 1. This completes the proof.

*Remark 7.* For a given cycle of projective lines $\mathbb{E}$ over an arbitrary field $\mathbf{k}$ there is always an étale covering $\pi : \mathbb{E}' \longrightarrow \mathbb{E}$ of a given degree $r$. For example, let $\mathbb{E} = \mathbb{E}_1$ be a rational curve with one node, $\mathbb{X}_i = \mathbb{P}^1$ $(i = 1, 2)$, $f_i : \mathsf{Spec}(\mathbf{k} \times \mathbf{k}) \longrightarrow \mathbb{X}_i$ be two closed embeddings with the image $0$ and $\infty$ and $g_i : \mathbb{X}_i \longrightarrow \mathbb{E}$ two normalization maps mapping the points $0$ and $\infty$ on $\mathbb{X}_i$ to

the singular point of $\mathbb{E}$. Then the push-out of $\mathbb{X}_1$ and $\mathbb{X}_2$ over $\mathsf{Spec}(\mathbf{k} \times \mathbf{k})$ (in the category of all schemes and affine maps) is a cycle of two projective lines $\mathbb{E}_2$ and the induced map $g : \mathbb{E}_2 \longrightarrow \mathbb{E}_1$ is an étale covering of degree two. The general case can be considered in a similar way. Note that this is quite different to the case of elliptic curves, where the existence of an étale covering of a given degree strongly depends on the arithmetics of the curve.

Similarly to the proof of Theorem 19 we have the following proposition.

**Proposition 5.** *Let* $\pi_r : \mathbb{E}_{nr} \longrightarrow \mathbb{E}_n$ *be an étale covering of degree* $r$, $\widetilde{\mathbb{E}}_n = \coprod_{i=1}^{n} \mathbb{L}_i$ *and* $\widetilde{\mathbb{E}}_{nr} = \coprod_{j=1}^{nr} \mathbb{L}'_j$ *be the normalizations of* $\mathbb{E}_n$ *and* $\mathbb{E}_{nr}$. *Let* $\{a_1, a_2, \ldots, a_n\}$ *be the set of singular points of* $\mathbb{E}_n$ *and* $\{b_1, b_2, \ldots, b_n, b_{n+1}, \ldots, b_{nr}\}$ *the singular points of* $\mathbb{E}_{nr}$ *and* $\pi_r^{-1}(a_i) = \{b_i, b_{i+n}, \ldots, b_{i+(r-1)}\}$. *Assume* $\mathcal{E}$ *is a vector bundle of rank* $l$, *given by the triple* $(\widetilde{\mathcal{E}}, \mathcal{A}^l, \tilde{i})$, *where* $\widetilde{\mathcal{E}} \cong \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2 \oplus \cdots \oplus \widetilde{\mathcal{E}}_n$ *and* $\tilde{i}$ *is given by matrices* $M(\mathbb{L}_1, a'_1)$, $M(\mathbb{L}_1, a''_2)$, $\ldots$, $M(\mathbb{L}_n, a'_1)$. *Then the pull-back* $\pi_r^*(\mathcal{E})$ *corresponds to the triple* $(\widetilde{\mathcal{E}}', \mathcal{A}'^l, \tilde{i}')$ *where* $\widetilde{\mathcal{E}}'|_{\mathbb{L}'_{i+nj}} \cong \widetilde{\mathcal{E}}_i$, $0 \le j \le r-1$ *and* $\tilde{i}'$ *is given by matrices* $M(\mathbb{L}', b') = M(\mathbb{L}, a')$ *and* $M(\mathbb{L}', b'') = M(\mathbb{L}, a'')$ *if* $\pi_r(\mathbb{L}') = \mathbb{L}$ *and* $\pi_r(b) = a$.

From this proposition follows the following corollary:

**Corollary 1.** *Let Let* $\mathbb{E} = \mathbb{E}_n$ *be a cycle of* $n$ *projective lines and* $\pi_r : \mathbb{E}_{rd} \longrightarrow \mathbb{E}_n$ *an étale covering of degree* $r$. *Then*

1. $\pi_r^*\mathcal{B}(\boldsymbol{d}, 1, \lambda) \cong \mathcal{B}(\boldsymbol{d}^r, 1, \lambda^r)$.
2. *If* $char(\mathbf{k}) = 0$, *then* $\pi_r^*(\mathcal{F}_m) \cong \mathcal{F}_m$. *In particular, we have an isomorphism*
$$\mathcal{B}(\boldsymbol{d}, m, \lambda) \cong \mathcal{B}(\boldsymbol{d}, 1, \lambda) \otimes \mathcal{F}_m.$$

*Proof.* The proof of the first part is straightforward. To prove the second, note that $\pi_r^*(\mathcal{F}_m)$ is given by a triple, isomorphic to $(\widetilde{\mathcal{O}}^m, \mathcal{A}^m, \tilde{i})$, where $\tilde{i}$ is given by matrices $M(\mathbb{L}_1, a'_1) = I_m$, $M(\mathbb{L}_1, a''_2) = I_m$, $\ldots$, $M(\mathbb{L}_n, a'_n) = I_m$, $M(\mathbb{L}_n, a'_1) = J_m(1)^r$, where $J_m(1)$ is the Jordan $(m \times m)$–block with the eigenvalue 1. If $char(\mathbf{k}) = 0$ then $J_m(1)^r \sim J_m(1)$ and we get the claim. Note, that in the case $char(\mathbf{k}) = p$ we have $J_p(1)^p = I_p$ that implies $\pi_p^*(\mathcal{F}_p) \cong \mathcal{O}^p$. To complete the proof of the second claim note, that

$$\mathcal{B}(\boldsymbol{d}, m, \lambda) \cong \pi_{r*}(\mathcal{L}(\boldsymbol{d}, \lambda) \otimes \mathcal{F}_m) \cong \pi_{r*}(\mathcal{L}(\boldsymbol{d}, \lambda) \otimes \pi_r^*\mathcal{F}_m) \cong$$

$$\pi_{r*}(\mathcal{L}(\boldsymbol{d}, \lambda)) \otimes \mathcal{F}_m \cong \mathcal{B}(\boldsymbol{d}, 1, \lambda) \otimes \mathcal{F}_m.$$

*Remark 8.* As we have already seen, the technique of étale coverings requires special care in the case of positive characteristics. For example, let $\mathbb{E} = \mathbb{E}_1$ be a Weierstraß nodal curve and $\pi_2 : \mathbb{E}_2 \longrightarrow \mathbb{E}_1$ an étale covering of degree 2.

Then the vector bundle $\pi_{2*}(\mathcal{O})$ corresponds to the triple $(\widetilde{\mathcal{O}}^2, \mathbf{k}^2(s), \tilde{i})$, where $\tilde{i}$ is given by matrices

$$i(0) = \begin{array}{|cc|} \hline 1 & 0 \\ 0 & 1 \\ \hline \end{array} \quad \text{and} \quad i(\infty) = \begin{array}{|cc|} \hline 0 & 1 \\ 1 & 0 \\ \hline \end{array}$$

Then for $char(\mathbf{k}) \neq 2$ we have

$$\begin{array}{|cc|} \hline 0 & 1 \\ 1 & 0 \\ \hline \end{array} \sim \begin{array}{|cc|} \hline 1 & 0 \\ 0 & -1 \\ \hline \end{array}$$

and $\pi_{2*}\mathcal{O} \cong \mathcal{O} \oplus \mathcal{B}(0,1,-1)$. However, for $char(\mathbf{k}) = 2$

$$\begin{array}{|cc|} \hline 0 & 1 \\ 1 & 0 \\ \hline \end{array} \sim \begin{array}{|cc|} \hline 1 & 1 \\ 0 & 1 \\ \hline \end{array}$$

and $\pi_{2*}\mathcal{O} \cong \mathcal{F}_2$.

From Theorem 19 one can derive formulas for the cohomology groups of indecomposable torsion free sheaves, a formula for the dual of an indecomposable torsion free sheaf and rules for the computation of the direct sum decomposition of two indecomposable vector bundles. This is what we are going to describe now.

**Lemma 5 ([BDG01, BK1]).** *If $\boldsymbol{d} = (d_1, \ldots, d_{rn}) \in \mathbb{Z}^{rn}, \boldsymbol{e} = (e_1, e_2, \ldots, e_k)$, $\lambda \in \boldsymbol{k}^*, m \geq 1, 1 \leq f \leq n$, we have:*

*(i)* $\mathcal{B}(\boldsymbol{d}, m, \lambda)^\vee \cong \mathcal{B}(-\boldsymbol{d}, m, \lambda^{-1})$

*(ii)* $\mathcal{S}(\boldsymbol{e}, f)^\vee \cong \mathcal{S}(\boldsymbol{\kappa} - \boldsymbol{e}, f)$ *with* $\boldsymbol{\kappa} = \begin{cases} (-1, 0, \ldots, 0, -1) & \text{if } k \geq 2 \\ -2 & \text{if } k = 1. \end{cases}$

*Proof.* If $f : \mathbb{X} \to \mathbb{E}$ is a finite morphism, $\mathcal{F}$ a coherent sheaf on $\mathbb{X}$ and $\mathcal{G}$ a coherent sheaf on $\mathbb{E}$, there is a natural isomorphism of $f_*\mathcal{O}_\mathbb{X}$–modules

$$f_*\mathcal{H}om_\mathbb{X}(\mathcal{F}, f^!\mathcal{G}) \cong \mathcal{H}om_\mathbb{E}(f_*\mathcal{F}, \mathcal{G}).$$

Recall that $f^!\omega_\mathbb{E}$ is a dualizing sheaf on $\mathbb{X}$ if $\omega_\mathbb{E}$ is one on $\mathbb{E}$. In our situation $\omega_\mathbb{E} \cong \mathcal{O}_\mathbb{E}$ and we obtain an isomorphism

$$f_*\mathcal{H}om_\mathbb{X}(\mathcal{F}, \omega_\mathbb{X}) \cong \mathcal{H}om_\mathbb{E}(f_*\mathcal{F}, \mathcal{O}_\mathbb{E}) \cong (f_*\mathcal{F})^\vee.$$

To show (i), we consider $\mathbb{X} = \mathbb{E}_n$ and $f = \pi_n$. The claim follows now from $\omega_{\mathbb{E}_n} \cong \mathcal{O}_{\mathbb{E}_n}$, $\mathcal{F}_m^\vee \cong \mathcal{F}_m$ and $\mathcal{L}(\boldsymbol{d}, \lambda)^\vee \cong \mathcal{L}(-\boldsymbol{d}, \lambda^{-1})$ on $\mathbb{E}_n$.

For the proof of (ii) we let $\mathbb{X} = \mathbb{I}_k$ and $f = p_k$. Now $\omega_{\mathbb{I}_k} \cong \mathcal{L}(\boldsymbol{\kappa})$ and the result follows from $\mathcal{L}(\boldsymbol{d})^\vee \cong \mathcal{L}(-\boldsymbol{d})$ on $\mathbb{I}_k$.

Using the description of indecomposable vector bundles via étale coverings it is not difficult to compute their cohomology.

**Lemma 6 ([DGK03]).** *There is the following formula for the cohomology of indecomposable vector bundles*

$$dim_{\mathbf{k}} H^0(\mathcal{B}(\boldsymbol{d}, m, \lambda)) = m\left(\sum_{i=1}^{rn}(d_i + 1)^+ - \theta(\boldsymbol{d})\right) + \delta(\boldsymbol{d}, \lambda)$$

*and*

$$dim_{\mathbf{k}} H^1(\mathcal{B}(\boldsymbol{d}, m, \lambda)) = rm - dim_{\mathbf{k}} H^0(\mathcal{B}(\boldsymbol{d}, m, \lambda)),$$

*where $\delta(\boldsymbol{d}, \lambda) = 1$ if $\mathbf{d} = (0, \ldots, 0)$, $\lambda = 1$ and $0$ otherwise; $k^+ = k$ if $k > 0$ and zero otherwise. The number $\theta(\boldsymbol{d})$ is defined as follows: call a subsequence $\mathbf{p} = (d_{k+1}, \ldots, d_{k+l})$, where $0 \le k < rn$ and $1 \le l \le rn$ a positive part of $\boldsymbol{d}$ if all $d_{k+j} \ge 0$ and either $l = rn$ or both $d_k < 0$ and $d_{k+l+1} < 0$. For such a positive part put $\theta(\boldsymbol{p}) = l$ if either $l = rs$ or $\mathbf{p} = (0, \ldots, 0)$ and $\theta(\boldsymbol{p}) = l + 1$ otherwise. Then $\theta(\boldsymbol{d}) = \sum \theta(\boldsymbol{p})$, where we take a sum over all positive subparts of $\boldsymbol{d}$.*

In order to compute the tensor product of two indecomposable vector bundles we shall need the following lemma.

**Lemma 7.** *Let $\mathbb{E}$ be a cycle of projective lines, $\pi_i : \mathbb{E}_i \longrightarrow \mathbb{E}$ two étale coverings $i = 1, 2$ and $\mathcal{E}_i$ a vector bundle on $\mathbb{E}_i$. Let $\mathbb{E}'$ be the fiber product of $\mathbb{E}_1$ and $\mathbb{E}_2$ over $\mathbb{E}$:*

$$
\begin{array}{ccc}
\mathbb{E}' & \xrightarrow{\ p_1\ } & \mathbb{E}_1 \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi_1} \\
\mathbb{E}_2 & \xrightarrow{\ \pi_2\ } & \mathbb{E}.
\end{array}
$$

*Denote bys $\widetilde{\pi} : \mathbb{E}' \longrightarrow \mathbb{E}$ the composition $\pi_1 p_1$, then*

$$\pi_{1*}(\mathcal{E}_1) \otimes \pi_{2*}(\mathcal{E}_2) \cong \widetilde{\pi}_*\big(p_1^*(\mathcal{E}_1) \otimes p_2^*(\mathcal{E}_2)\big).$$

*Proof.* By the base change and projection formula $\widetilde{\pi}_*\big(p_1^*(\mathcal{E}_1) \otimes p_2^*(\mathcal{E}_2)\big) \cong \pi_{2*} p_{2*}\big(p_1^*(\mathcal{E}_1) \otimes p_2^*(\mathcal{E}_2)\big) \cong \pi_{2*}\big(p_{2*} p_1^*(\mathcal{E}_1) \otimes \mathcal{E}_2\big) \cong \pi_{2*}\big(\pi_2^* \pi_{1*}(\mathcal{E}_1) \otimes \mathcal{E}_2\big) \cong \pi_{1*}(\mathcal{E}_1) \otimes \pi_{2*}(\mathcal{E}_2).$

The following proposition describes the fiber product of two étale coverings of a given cycle of projective lines.

**Proposition 6 ([Bur03]).** *Let $\mathbb{E}_n$ be a cycle of $n$ projective lines and $\pi_i : \mathbb{E}_{d_i n} \longrightarrow \mathbb{E}_n$ two étale covering of degree $d_i$, $i = 1, 2$. Choose a labeling of the irreducible components of each cycle $\mathbb{E}_{d_1 n}$ and $\mathbb{E}_{d_2 n}$ by consecutive nonnegative integers $0, 1, 2, \ldots$, such that $\pi_i$ maps the zero component into a zero component, $i = 1, 2$. Let $d = gcd(d_1, d_2)$ be the greatest common divisor and $D = [d_1, d_2]$ the smallest common multiple of $d_1$ and $d_2$. Then the fiber product is $\widetilde{\mathbb{E}} = \coprod_{i=1}^{d} \mathbb{E}_{Dn}^{(i)}$ and $p_1 : \mathbb{E}_{Dn}^{(i)} \longrightarrow \mathbb{E}_{d_1 n}$ is the étale covering determined by the*

*assumption that it maps the i-th component of $\mathbb{E}_{Dn}^{(i)}$ to the 0-th component of $\mathbb{E}_{d_{1_n}}$. The second morphism $p_2 : \mathbb{E}_{Dn}^{(i)} \longrightarrow \mathbb{E}_{d_{2_n}}$ is the étale covering, mapping the zero component to the zero component.*

These properties allow to describe a decomposition of the tensor product of any two indecomposable vector bundles into a direct sum of indecomposable ones. In particular, in the case of a nodal Weierstraß curve we get the following concrete algorithm, obtained for the first time in [Yud01].

**Theorem 20 ([Yud01, Bur03]).** *Let $\mathbb{E}$ be a Weierstraß nodal curve over an algebraically closed field $\mathbf{k}$ of characteristics zero, $\mathcal{B}(\boldsymbol{d}, 1, \lambda)$ and $\mathcal{B}(\boldsymbol{e}, 1, \mu)$ two vector bundles on $\mathbb{E}$ of rank $k$ and $l$ respectively, $\boldsymbol{d} = d_1 d_2 \ldots d_k$ and $\boldsymbol{e} = e_1 e_2 \ldots e_l$. Let $D$ be the smallest common multiple and $d$ the greatest common divisor of $k$ and $l$. Consider $d$ sequences*

$$
\begin{aligned}
\boldsymbol{f}_1 &= d_1 + e_1, d_2 + e_2, \ldots, d_k + e_l, \\
\boldsymbol{f}_2 &= d_1 + e_2, d_2 + e_3, \ldots, d_k + e_1, \\
&\vdots \\
\boldsymbol{f}_d &= d_1 + e_d, d_2 + e_{d+1}, \ldots, d_k + e_{d-1},
\end{aligned}
$$

*of length $D$. Then the following decomposition holds:*

$$
\mathcal{B}(\boldsymbol{d}, 1, \lambda) \otimes \mathcal{B}(\boldsymbol{e}, 1, \mu) \cong \bigoplus_{i=1}^{d} \mathcal{B}(\boldsymbol{f_i}, 1, \lambda^{\frac{l}{d}} \mu^{\frac{k}{d}}).
$$

*If some $\boldsymbol{f_i}$ is periodic, then we use the isomorphism*

$$
\mathcal{B}(\boldsymbol{g}^l, 1, \lambda) = \bigoplus_{i=1}^{l} \mathcal{B}(\boldsymbol{g}, 1, \xi^i \sqrt[l]{\lambda}),
$$

*where $\boldsymbol{g}^l = \underbrace{\boldsymbol{g} \boldsymbol{g} \ldots \boldsymbol{g}}_{l}$ and $\xi$ a primitive $l$–th root of $1$.*

Even possessing a complete classification of indecomposable torsion free sheaves on a Weierstraß nodal curve, an exact description of stable vector bundles is a non-trivial problem. It can be shown by many methods that for a pair of coprime integers $(r, d) \in \mathbb{Z}^2$, $r > 0$ the moduli space of stable vector bundles of rank $r$ and degree $d$ is $\mathbf{k}^*$, see for example [BK3]. However, for applications it is important to have a description of stable vector bundles via étale coverings. In order to get such a classification note the following useful fact.

**Lemma 8 ([Bur03]).** *Let $\mathbb{E}$ be an irreducible Weierstraß curve. Then a coherent sheaf $\mathcal{F}$ on $\mathbb{E}$ is stable if and only if it is simple i.e. $\mathsf{End}(\mathcal{F}) = \mathbf{k}$.*

In general, for irreducible curves stability implies simplicity, but in the case of irreducible curves of arithmetic genus one both conditions are equivalent. Then one can prove the following theorem:

**Theorem 21 ([Bur03]).** *Let $\mathbb{E}$ be a nodal Weierstraß curve and $\mathcal{E}$ a stable vector bundle on $\mathbb{E}$ of rank $r$ and degree $d$, $0 < d < r$. Then $gcd(r, d) = 1$, $\mathcal{E} \cong \mathcal{B}(\boldsymbol{d}, 1, \lambda)$ and $\boldsymbol{d}$ can be obtained by the following algorithm.*

1. *Let $y = min(d, r-d)$, $x = max(d, r-d)$. If $x = y$, then $\boldsymbol{d} = (0, 1)$. Assume now $x > y$. Consider the triple $(x, y, x+y)$ and write $x+y = (k+1)y+s$, where $0 < s < y$ and $k \geq 1$. If $s > y - s$ then replace $(x, y, x + y)$ by $(s, y - s, y)$ and say say that $(x, y, x + y)$ is obtained from $(s, y - s, y)$ by the blow-up of type $(A, k)$. If $s < y - s$ then replace $(x, y, x + y)$ by $(y - s, s, y)$ and say that $(x, y, x + y)$ is obtained from $(y - s, s, y)$ by the blow-up of type $(B, k)$.*
2. *Repeat this algorithm until we get the triple $(p, 1, p + 1)$. Consider the sequence of reductions $(x, y, x + y) = (x_0, y_0, x_0 + y_0) \xrightarrow{(C_1, k_1)} (x_1, y_1, x_1 + y_1) \longrightarrow \cdots \xrightarrow{(C_n, k_n)} (x_n, y_n, x_n + y_n) = (p, 1, p+1)$, where $C_i \in \{A, B\}$ and $k_i \geq 1$ for $1 \leq i \leq n$.*

   *Now we can recover the vector $\boldsymbol{d}$ :*

1. *Start with sequence $\underbrace{\alpha, \alpha, \ldots, \alpha}_{p\,times}, \beta$, which corresponds to the triple $(x_n, y_n, x_n + y_n) = (p, 1, p + 1)$. If $C_n = A$ then replace each letter $\alpha$ by the block $\underbrace{\alpha, \alpha, \ldots, \alpha}_{k_n + 1}$ and each $\beta$ by the block $\underbrace{\alpha, \alpha, \ldots, \alpha}_{k_n}$. Between these new blocks insert the letter $\beta$. If $C_n = B$ then replace each letter $\alpha$ by the block $\underbrace{\alpha, \alpha, \ldots, \alpha}_{k_n}$ and each letter $\beta$ by the block $\underbrace{\alpha, \alpha, \ldots, \alpha}_{k_n + 1}$. Between these new blocks insert the letter $\beta$ again. We have got a new sequence of letters $\alpha$ and $\beta$ of total length $x_{n-1} + y_{n-1}$ with $x_{n-1}$ letters $\alpha$ and $y_{n-1}$ letters $\beta$.*
2. *Proceed inductively until we get a sequence of length $r$ with $max(d, r - d)$ letters $\alpha$ and $min(d, r - d)$ letters $\beta$.*
3. *If $d > r - d$ then replace each letter $\alpha$ by 1 and each letter $\beta$ by 0. In case $d \leq r - d$ replace $\alpha$ by 0 and $\beta$ by 1. The resulted sequence is the vector $\boldsymbol{d}$ we are looking for.*

*Example 3.* Let rank $r = 19$ and degree 11. The sequence of reductions is

$$(11, 8, 19) \xrightarrow{(B,1)} (5, 3, 8) \xrightarrow{(A,1)} (2, 1, 3).$$

Using the algorithm we get the sequence of blowing-ups

$$\alpha, \alpha, \beta \xrightarrow{(A,1)} \alpha, \alpha, \beta, \alpha, \alpha, \beta, \alpha, \beta$$

$$\alpha, \alpha, \beta, \alpha, \alpha, \beta, \alpha, \beta \xrightarrow{(B,1)} \alpha, \beta, \alpha, \beta, \alpha, \alpha, \beta, \alpha, \beta, \alpha, \beta, \alpha, \alpha, \beta, \alpha, \beta, \alpha, \alpha, \beta$$

and hence
$$\boldsymbol{d} = (1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0).$$

This result was generalized by Mozgovoy [Moz] to get a recursive description of semi-stable torsion free sheaves of arbitrary slope.

Note the following important difference between smooth and singular curves of arithmetic genus one. In the smooth case any indecomposable coherent sheaf is either locally free or torsion free and is automatically semi-stable. This is no longer true for singular curves, in particular, in that case there are indecomposable coherent sheaves which are neither torsion nor torsion free.

*Example 4.* Let $\mathbb{E}$ be a nodal Weierstraß curve, $s$ its singular point, $n : \mathbb{P}^1 \longrightarrow \mathbb{E}$ the normalization map. Then

$$\mathsf{Ext}^1(n_*(\mathcal{O}_{\mathbb{P}^1}), \mathbf{k}(s)) = H^0(\mathcal{E}xt^1(n_*(\mathcal{O}_{\mathbb{P}^1}), \mathbf{k}(s))) = \mathbf{k}^2.$$

Let $w \in \mathsf{Ext}^1(n_*(\mathcal{O}_{\mathbb{P}^1}), \mathbf{k}(s))$ be a non-zero element and

$$0 \longrightarrow \mathbf{k}(s) \xrightarrow{i} \mathcal{F} \xrightarrow{p} n_*(\mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0$$

the corresponding extension. Then $\mathcal{F}$ is an indecomposable coherent sheaf which is neither torsion nor torsion free. To see that $\mathcal{F}$ is indecomposable assume $\mathcal{F} \cong \mathcal{F}' \oplus \mathcal{F}''$. Then one of its direct summands, say $\mathcal{F}'$ is a torsion sheaf. Since $\mathsf{Hom}(\mathcal{F}', n_*(\mathcal{O}_{\mathbb{P}^1})) = 0$, $\mathcal{F}'$ belongs to the kernel $ker(p)$ and hence is isomorphic to $\mathbf{k}(s)$. Therefore the map $i$ has a left inverse and hence $w = 0$, a contradiction.

**Proposition 7 ([BK3]).** *Let $\mathbb{E}$ be a singular Weierstraß curve and $\mathcal{F} \in \mathsf{Coh}(\mathbb{E})$ an indecomposable coherent sheaf which is not semi-stable. Then, all Harder-Narasimhan factors of $\mathcal{F}$ are direct sums of semi-stable sheaves of infinite homological dimension.*

*Proof.* Let $0 \subset \mathcal{F}_n \subset \ldots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{F}$ be the Harder-Narasimhan filtration of $\mathcal{F}$ with semi-stable factors $\mathcal{A}_\nu := \mathcal{F}_\nu / \mathcal{F}_{\nu+1}$ of decreasing slopes $\mu(\mathcal{A}_n) > \mu(\mathcal{A}_{n-1}) > \ldots > \mu(\mathcal{A}_0)$.

Assume $\mathcal{A}_\nu \cong \mathcal{A}'_\nu \oplus \mathcal{A}''_\nu$ and $\mathcal{A}'_\nu$ has finite global dimension. Since $\mathcal{F}_{\nu+1}$ is filtered by semi-stable sheaves $\mathcal{F}_\mu$ for $\mu > \nu$ and $\mathsf{Hom}(\mathcal{A}_\mu, \mathcal{A}'_\nu) = 0$, we get $\mathsf{Ext}^1(\mathcal{A}'_\nu, \mathcal{F}_{\nu+1}) \cong \mathsf{Hom}(\mathcal{F}_{\nu+1}, \mathcal{A}'_\nu)^* = 0$. Therefore $\mathcal{F}_\nu$ contains $\mathcal{A}'_\nu$ as a direct summand: $\mathcal{F}_\nu \cong \mathcal{F}'_\nu \oplus \mathcal{A}_\nu$. From the exact sequence

$$0 \longrightarrow \mathcal{F}'_\nu \oplus \mathcal{A}'_\nu \longrightarrow \mathcal{F}_{\nu-1} \longrightarrow \mathcal{A}_{\nu-1} \longrightarrow 0$$

and the isomorphism $\mathsf{Ext}^1(\mathcal{A}_{\nu-1}, \mathcal{A}'_\nu) \cong \mathsf{Hom}(\mathcal{A}'_\nu, \mathcal{A}_{\nu-1})^* = 0$ we conclude that $\mathcal{F}_{\nu-1}$ contains $\mathcal{A}'_\nu$ as a direct summand as well. Proceeding inductively we obtain that $\mathcal{F}$ itself contains $\mathcal{A}'_\nu$ as a direct summand, a contradiction.

We see that a difference between the combinatorics of indecomposable coherent sheaves on smooth and singular Weierstraß curves is due to the existence of semi-stable sheaves of infinite global dimension together with the failure of the Serre duality on singular curves. In order to classify indecomposable coherent sheaves it is convenient to consider a more general problem: the description of indecomposable objects of the derived category $D^-(\mathsf{Coh}(\mathbb{E}))$. It turns out that the last problem is again tame and can be solved using the technique of representations of bunches of chains, see [BD04] for the details.

### 3.3 Vector bundles on a cuspidal cubic curve

As we have mentioned in the introduction, the category of vector bundles on a curve of arithmetic genus one, different from a cycle of projective lines, is vector bundle wild, see also Corollary 2. Nevertheless, if we restrict ourselves to the subcategory of simple vector bundles $\mathsf{VB}_s$, or even to the subcategory of simple torsion free sheaves $\mathsf{TF}_s$, then the classification problem becomes tame again and, moreover, the combinatorics of the answer resembles the case of smooth and nodal Weierstraß curves (see Theorem 3 and Theorem 21.)

**Theorem 22.** *Let $\mathbb{E}$ be a cuspidal cubic curve over an algebraically closed field $\mathbf{k}$. Then*

1. *the rank $r$ and the degree $d$ of a stable torsion free sheaf $\mathcal{F}$ over $\mathbb{E}$ are coprime;*
2. *for every pair $(r, d)$ of coprime integers with positive $r$, the isomorphism classes of simple vector bundles $\mathcal{E} \in \mathsf{VB}_s(r, d)$ can be parametrized by $\mathbb{A}^1$;*
3. *there is a unique torsion free and not locally free sheaf $\mathcal{F}$ of rank $r$ and degree $d$.*

   Note that $\mathbb{A}^1 \cong \mathbb{E}_{reg}$, is isomorphic to the Picard group $\mathrm{Pic}^\circ(\mathbb{E})$.

   It can be shown that for a pair of coprime integers $r > 0$ and $d$ the moduli space of $\mathsf{TF}_s(r, d)$ is isomorph to $\mathbb{E}$, moreover, vector bundles $\mathcal{E}$ correspond to nonsingular points of $\mathbb{E}$ and the unique torsion free but not locally free sheaf $\mathcal{F}$ corresponds to the singular point $s$.

*Sketch of proof.* Let $\mathbb{E}$ be a cuspidal cubic curve given by the equation $x^3 - y^2 z = 0$. Choose coordinates $(z_0 : z_1)$ on the normalization $\widetilde{\mathbb{E}} \cong \mathbb{P}^1 \xrightarrow{p} \mathbb{E}$ such that the preimage of the singular point $s = (0 : 0 : 1)$ of $\mathbb{E}$ is $(0 : 1)$. Let $U = \{(z_0 : z_1) | z_1 \neq 0\}$ be an affine neighborhood of $(0 : 1)$ and $z = z_0/z_1$. In the notations of Section 3.1 we have: $\mathcal{A} = \mathbf{k}(s)$ and $\widetilde{\mathcal{A}} = \big(\mathbf{k}[\varepsilon]/\varepsilon^2\big)(s)$.

Let $\mathcal{F}$ be a torsion free sheaf of rank $r$ on $\mathbb{E}$ and $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{\imath})$ be the corresponding triple. Then, as in the case of a nodal rational curve, we have

- a splitting $\widetilde{\mathcal{F}} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(n)^{r_n}$, with $\sum_{n \in \mathbb{Z}} r_n = r$;
- an isomorphism $\mathcal{M} \cong \mathcal{A}^t$, for some $t \geq r$, and $t = r$ if and only if $\mathcal{F}$ is a vector bundle;

- an epimorphism of $\widetilde{\mathcal{A}}$–modules $\tilde{i} : \widetilde{\mathcal{M}} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}}$, which is an isomorphism if and only if $\mathcal{F}$ is a vector bundle.

In order to write $\tilde{i}$ in matrix form remember that we identify $\widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{A}}$ with $p_*\big(p^*(\mathcal{F}) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}/\mathcal{I}\big)$, where $\mathcal{I} = \mathcal{I}^2_{(0:1)}$ is the ideal sheaf of the scheme-theoretic preimage of $s$. We choose a basis of $\mathcal{M} \cong \mathbf{k}(s)^r$ and fix the trivializations

$$\mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\mathbb{P}^1}/\mathcal{I} \longrightarrow \big(\mathbf{k}[\varepsilon]/\varepsilon^2\big)(s)$$

given by the map $\zeta \otimes 1 \mapsto pr(\frac{\zeta}{z_1^n})$ for a local section $\zeta$ of $\mathcal{O}(n)$ on an open set $V$ containing 0. Let $pr : \mathbf{k}[V] \longrightarrow \mathbf{k}[\varepsilon]/\varepsilon^2$ be the map induced by the map $\mathbf{k}[z] \longrightarrow \mathbf{k}[\varepsilon]/\varepsilon^2$, $z \mapsto \varepsilon$. Using these choices we may write $\tilde{i} = i(0) + \varepsilon i_\varepsilon(0)$, where both $i(0)$ and $i_\varepsilon(0)$ are square $r \times r$ matrices. Since by Theorem 16 the isomorphism classes of triples are in bijection with isomorphism classes of vector bundles, we have to study the action of automorphisms of $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$ on the matrices $i(0)$ and $i_\varepsilon(0)$. The condition for $\tilde{i}$ to be surjective is equivalent to the surjectivity of $i(0)$. Similarly, for vector bundles we have that $\tilde{i}$ is invertible if and only if $i(0)$ is invertible.

If we have a morphism $\mathcal{O}(n) \longrightarrow \mathcal{O}(m)$ given by a homogeneous form $Q(z_0, z_1)$ of degree $m-n$ then the induced map $\mathcal{O}(n) \otimes \mathcal{O}/\mathcal{I} \longrightarrow \mathcal{O}(m) \otimes \mathcal{O}/\mathcal{I}$ is given by the map $pr(Q(z_0, z_1)/z_1^{m-n}) = Q(0:1) + \varepsilon \frac{dQ}{dz_0}(0:1)$.

Moreover, for any endomorphism $(F, f)$ of the triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$ the induced map $\bar{F} : \widetilde{\mathcal{F}} \otimes \mathcal{O}/\mathcal{I} \longrightarrow \widetilde{\mathcal{F}} \otimes \mathcal{O}/\mathcal{I}$ has the form $\bar{F} = F(0) + \varepsilon F_{z_0}(0)$. If $(F, f)$ is an automorphism then $\bar{F} \in GL_r(\mathbf{k}[\varepsilon]/\varepsilon^2)$ and the transformation rule $\bar{F}\tilde{i} = \tilde{i}'\bar{f}$ in matrix form reads

$$\begin{aligned} i_0' &= F(0)i(0)f^{-1} \\ i_\varepsilon' &= F_{z_0}(0)i_\varepsilon(0)f^{-1} + F(0)i(0)f^{-1}. \end{aligned}$$

As a result, the matrix problem is as follows: we have two matrices $i(0)$ and $i_\varepsilon(0)$ with $r$ rows and $t$ columns, and $rank(i(0)) = r$. In the case of a vector bundle $i(0)$ and $i_\varepsilon(0)$ are square matrices and $i(0)$ is invertible. The matrices $i(0)$ and $i_\varepsilon(0)$ are divided into horizontal blocks labelled by integers called weights. Any two blocks of $i(0)$ and $i_\varepsilon(0)$ marked by the same label are called *conjugated* and have the same number of rows.



$$i(0) \qquad\qquad\qquad i_\varepsilon(0)$$
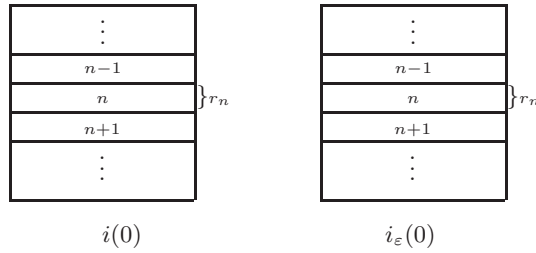
The permitted transformations are listed below:

1. We can simultaneously do any elementary transformations of columns of $i(0)$ and of $i_\varepsilon(0)$.
2. We can simultaneously perform any invertible elementary transformations of rows of $i(0)$ and $i_\varepsilon(0)$ inside of any two conjugated h blocks.
3. We can add a scalar multiple of any row with lower weight to any row with higher weight simultaneously in $i(0)$ and $i_\varepsilon(0)$.
4. We can add a row of $i(0)$ with a lower weight to any row of $i_\varepsilon(0)$ with a higher weight.

This matrix problem turns to be wild, see corollary 2. However, the simplicity condition of a triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$ implies additional restrictions, which make the problem tame. First note that if $\widetilde{\mathcal{F}}$ contains $\widetilde{\mathcal{O}}(c) \oplus \widetilde{\mathcal{O}}(d)$ with $d > c + 1$ as a direct summand, then the pair $(F, f) := (z_0^{d-c}, 0)$ defines a non-scalar endomorphism of the triple $(\widetilde{\mathcal{F}}, \mathcal{M}, \tilde{i})$, which, therefore, can not be simple. Thus, for a simple torsion free sheaf $\mathcal{F}$ we may assume $\widetilde{\mathcal{F}} \cong \widetilde{\mathcal{O}}(c)^{r_1} \oplus \widetilde{\mathcal{O}}(c+1)^{r_2}$ for some $c \in \mathbb{Z}$ and the matrix $\tilde{i}$ consists of two horizontal blocks.

We consider the case of vector bundles only, since in the case of torsion free and not locally free sheaves, the problem is similar, but it should be considered separately.

As was mentioned above, if $\mathcal{F}$ is a vector bundle then $\tilde{i}$ is an isomorphism and by transformations 1 and 2 the matrix $i(0)$ can be reduced to the identity matrix. Moreover, by applying transformation 4 we can make the left lower block of $i_\varepsilon(0)$ zero, as indicated below:

$$i(0) = \boxed{\begin{array}{cc} I_{r_1} & \\ & I_{r_2} \end{array}} \quad \text{and} \quad i_\varepsilon(0) = \boxed{\begin{array}{cc} B_1 & B_{12} \\ & B_2 \end{array}}. \tag{1}$$

Here $I_n$ denotes the identity matrix of size $n$, an empty space denotes a zero block and $B_1, B_{12}, B_2$ denote nonreduced blocks.

Thus we can assume that $i(0)$ is the identity matrix and concentrate on the matrix $i_\varepsilon(0)$, taking into account only those transformations, which leave $i(0)$ unchanged. Then we obtain the category of block matrices

$$\mathsf{BM} = \bigcup_{(r_1, r_2)} \mathsf{BM}(r_1, r_2).$$

Objects of $\mathsf{BM}(r_1, r_2)$ are matrices of the form $i_\varepsilon(0)$ in (1), i.e. upper triangular block matrices $B$ consisting of the blocks $(B_1, B_{12}, B_2)$ with $(B_1, B_2)$ square matrices with sizes $(r_1, r_2)$. Morphisms $C : B \to B'$ are lower triangular block matrices:

$$C = \boxed{\begin{array}{cc} C_1 & \\ C_{21} & C_2 \end{array}}$$

with block sizes $(r_1, r_2)$ and satisfying equations $CB = B'C$. In term of blocks this equation can be written as:

$$
\begin{aligned}
C_1 B_1 &= B_1' C_1 + B_{12}' C_{21}, \\
C_1 B_{12} &= B_{12}' C_2, \\
C_2 B_2 + C_{21} B_{12} &= B_2' C_2.
\end{aligned}
\tag{2}
$$

Two matrices $B$ and $B'$ are called *equivalent* (i.e. correspond to isomorphic vector bundles) if there is a non-degenerate morphism $C : B \to B'$, i.e. if $B' = CBC^{-1}$. In terms of transformations this means: we can add a row $k$ with lower weight to a row $j$ with higher weight and simultaneously add the column $j$ to the column $k$. A matrix $B \in \mathsf{BM}(r_1, r_2)$ is called *simple* if any endomorphism $C : B \to B$ is scalar. Obviously, simplicity is a property defined on equivalence classes. The subcategory of simple $B$ is denoted by $\mathsf{BM}_s(r_1, r_2)$.

Note that, if a block $B_{12}$ has a zero-row $k$ and a zero-column $j$, then by adding column $j$ to column $k$ and row $k$ to row $j$ we construct a nonscalar endomorphism. Moreover, if $r_1 = r_2$ then $B_{12}$ is a square matrix and can be reduced to the identity matrix $I$. Then using $B_{12}$, we can reduce one of matrices $B_1$ and $B_2$, let us say $B_1$, to zero and the other one $B_2$ to its Jordan normal form. If $r_2 = 1$ then $B_2 = \boxed{\lambda}$, $\lambda \in \mathbf{k}$, is simple but for $r_2 > 1$ the Jordan normal form has an endomorphism, which can be extended to an endomorphism of $B$. Therefore, if $B$ is simple then $B_{12}$ can be reduced to one of the following forms

$$
B_{12} =
\begin{cases}
\boxed{\begin{matrix} 0 \\ I_{r_2} \end{matrix}} & \text{if} \quad r_1 > r_2, \\[2ex]
\boxed{I_{r_1}\,0} & \text{if} \quad r_2 > r_1, \\[2ex]
\boxed{1} & \text{if } r_1 = r_2 = 1,
\end{cases}
$$

From the system of equations (2) we get that in case $r_1 > r_2$ block $B_2$ can be reduced to the zero matrix and block $B_1$ to the upper triangular block-matrix formed by three nonzero subblocks $(B_{1.1}, B_{1.12}, B_{1.2})$. The long but straightforward calculations shows that the transformations of $B$ which preserve already reduced blocks are uniquely determined by the automorphisms of $B_1$ in the category $\mathsf{BM}_s$. Moreover, $\mathsf{End}_{\mathsf{BM}_s}(B_1) = \mathsf{End}_{\mathsf{BM}_s}(B)$.

In the same way the matrix $B$ can be reduced in case $r_2 > r_1$. Thus the problem $\mathsf{BM}_s(r_1, r_2)$ is *self-reproducing,* which means that, we get a bijection between $\mathsf{BM}_s(r_1, r_2)$ and $\mathsf{BM}_s(r_1 - r_2, r_2)$ if $r_1 > r_2$, between $\mathsf{BM}_s(r_1, r_2)$ and $\mathsf{BM}_s(r_1, r_2 - r_1)$ if $r_2 > r_1$, and if $r_1 = r_2 > 1$ then $\mathsf{BM}_s(r_1, r_1)$ is empty. In this reduction one can easily recognize the Euclidian algorithm. Moreover, the reduction terminates after finitely many steps with the condition $r_1 = r_2 = 1$. Without loss of generality the matrix $B \in \mathsf{BM}_s(1, 1)$ has the form

$$B = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline & \lambda \\ \hline \end{array}. \tag{3}$$

(Note that this form is equivalent to the form $\begin{array}{|c|c|} \hline \lambda & 1 \\ \hline & 0 \\ \hline \end{array}$.)

Objects of $\mathsf{BM}_s(1,1)$ are parametrized by $\lambda \in \mathbf{k}$, thus the same holds for $\mathsf{BM}_s(r_1, r_2)$ with coprime $r_1$ and $r_2$.

Let $\mathcal{E}$ be a vector bundle of rank $r$ and degree $d$ with normalization $\widetilde{\mathcal{E}} = \widetilde{\mathcal{O}}(c)^{r_1} \oplus \widetilde{\mathcal{O}}(c+1)^{r_2}$. Taking into account that by Riemann-Roch theorem (Theorem 9) $r_2 = d \bmod r$ and $r_1 + r_2 = r$, we obtain statements 1,2 of the theorem. Moreover, if coprime integers $r > 0$ and $d$ are given then for $\lambda \in \mathbf{k}$ the matrix $B(\lambda) \in \mathsf{BM}_s(r_1, r_2)$, and hence, the unique vector bundle $\mathcal{E}(r, d, \lambda)$, can be constructed by reversing the reduction procedure described above:

**Algorithm 2.** Let $(r, d) \in \mathbb{Z}^2$ be coprime with positive $r$, and $\lambda \in \mathbf{k}$.

- First, by the Euclidean algorithm we find integers $c$, $r_1$ and $r_2$, $0 < r_1 \leq r$, $0 \leq r_2 < r$ such that $cr + r_2 = d$ and $r_1 + r_2 = r$. Thus we recover the normalization sheaf $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}(c)^{r_1} \oplus \widetilde{\mathcal{O}}(c+1)^{r_2}$.
- If $r_1 = r_2 = 1$ the matrix $B(\lambda)$ has form 3.

Using this input data we construct the matrix $B(\lambda) \in \mathsf{BM}_s(r_1, r_2)$ inductively:

- Let $r_1 + r_2 > 2$ and $r_1 > r_2$. Assume we have the matrix $B_1(\lambda) \in \mathsf{BM}_s(r_1 - r_2, r_2)$, then $B(\lambda) \in \mathsf{BM}_s(r_1, r_2)$ has form

$$B(\lambda) = \begin{array}{|c|c|} \hline B_1(\lambda) & 0 \\ \cline{2-2} & I_{r_2} \\ \hline & 0 \\ \hline \end{array}.$$

- Let $r_1 + r_2 > 2$ and $r_1 < r_2$. Assume that we have the matrix $B_2(\lambda) \in \mathsf{BM}_s(r_1, r_2 - r_1)$, then $B(\lambda) \in \mathsf{BM}_s(r_1, r_2)$ has form

$$B(\lambda) = \begin{array}{|c|c|} \hline 0 & I_{r_1} \; 0 \\ \hline & B_2(\lambda) \\ \hline \end{array}.$$

- Finally, we get the matrix $\tilde{i} = i(0) + \varepsilon i_\varepsilon(0) = I_r + \varepsilon B(\lambda)$.
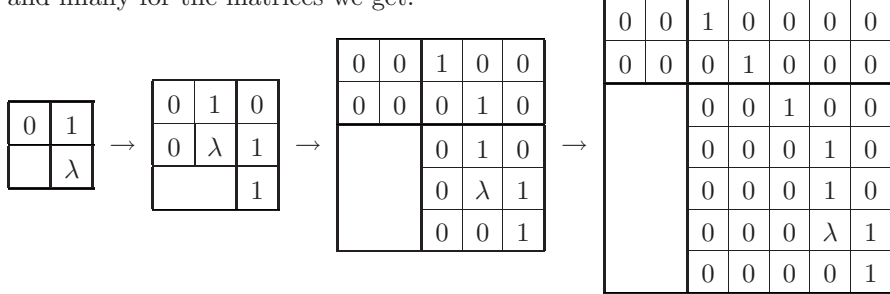
Let us illustrate this with a small example:

*Example 5.* Let $\mathcal{E} \in \mathsf{VB}_s(7, 12)$. To obtain the matrix $i$ we calculate the normalization sheaf $\widetilde{\mathcal{E}}$ first: $\widetilde{\mathcal{E}} = \widetilde{\mathcal{O}}(1)^2 \oplus \widetilde{\mathcal{O}}(2)^5$. Thus, in our notations $r_1 = 2$ and $r_2 = 5$. The Euclidian algorithm of the pair $(2, 5)$ gives:

$$(2, 5) \to (2, 3) \to (2, 1) \to (1, 1).$$

Reversing this sequence, by the above reducing procedure, we obtain a sequence of equivalent matrix problems:

$$\mathsf{BM}_s(1,1) \xrightarrow{\sim} \mathsf{BM}_s(2,1) \xrightarrow{\sim} \mathsf{BM}_s(2,3) \xrightarrow{\sim} \mathsf{BM}_s(2,5),$$

and finally for the matrices we get:

$$
\begin{bmatrix} 0 & 1 \\ & \lambda \end{bmatrix}
\rightarrow
\begin{bmatrix} 0 & 1 & 0 \\ 0 & \lambda & 1 \\ & & 1 \end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
 & & 0 & 1 & 0 \\
 & & 0 & \lambda & 1 \\
 & & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & 0 & 0 & 1 & 0 & 0 \\
 & & 0 & 0 & 0 & 1 & 0 \\
 & & 0 & 0 & 0 & 1 & 0 \\
 & & 0 & 0 & 0 & \lambda & 1 \\
 & & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

The reduction for torsion free but not locally free sheaves can be done by a similar way. The only difference is that $i(0)$ and $i_\varepsilon(0)$ are no longer square:

$$
i(0) = \begin{bmatrix} I_{r_1} & \\ & I_{r_2} \end{bmatrix} \quad \text{and} \quad i_\varepsilon(0) = \begin{bmatrix} B_1 & B_{12} & B_{13} \\ & B_2 & B_{23} \end{bmatrix}.
$$

The matrix $i_\varepsilon(0)$ has two additional blocks $B_{13}$ and $B_{23}$ with a new column size $r_3 > 0$. Investigating such matrices inductively, we get $r_3 = 1$. Moreover, if $r_1+1$ and $r_2+1$ are coprime then there is a unique simple matrix $\tilde{i}$, and there is no simple matrices otherwise. This unique simple matrix $\tilde{i}$ correspond to the unique torsion free but not locally free sheaf $\mathcal{F}$, which can be considered as a *compactifying* object of the family $\mathsf{VB}_s(r,d)$. Let us illustrate this for small ranks:

*Example 6.* Vector bundles $\mathcal{E}$ of $\mathsf{VB}_s(1,0)$ have $\widetilde{\mathcal{E}} = \widetilde{\mathcal{O}}$ as the normalization sheaf and the corresponding matrices $\tilde{i}$ are $\boxed{1} + \varepsilon \boxed{\lambda}$, $\lambda \in \mathbf{k}$. For the unique torsion free but not locally free sheaf $\mathcal{F}$ of rank 1 and degree 0, one computes that $deg(\widetilde{\mathcal{F}}) = deg(\mathcal{F}) - 1 = -1$, thus $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}(-1)$ and the corresponding matrix $\tilde{i}$ is $\boxed{1 \quad 0} + \varepsilon \cdot \boxed{0 \quad 1}$.

*Example 7.* Vector bundles $\mathcal{E}$ from $\mathsf{VB}_s(2,1)$ have as normalization sheaf $\widetilde{\mathcal{E}} = \widetilde{\mathcal{O}} \oplus \widetilde{\mathcal{O}}(1)$ thus the corresponding matrices are

$$
\tilde{i} = \begin{array}{c} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \begin{matrix} 0 \\ 1 \end{matrix} \end{array} + \varepsilon \cdot \begin{array}{c} \boxed{\begin{matrix} 0 & 1 \\ 0 & \lambda \end{matrix}} \begin{matrix} 0 \\ 1 \end{matrix} \end{array},
$$

where $\lambda \in \mathbf{k}$. The normalization sheaf $\widetilde{\mathcal{F}}$ of the torsion free but not locally free sheaf $\mathcal{F} \in \mathsf{TF}_s(2,1)$ has degree $deg(\widetilde{\mathcal{F}}) = deg(\mathcal{F}) - 1 = 0$ thus $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}^2$ and the corresponding matrix is

$$\tilde{i} = \begin{array}{|ccc|} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}_0 + \varepsilon \cdot \begin{array}{|ccc|} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}_0.$$

### 3.4 Coherent sheaves on degenerations of elliptic curves and Fourier-Mukai transforms

The technique of Fourier-Mukai transforms on elliptic curves led to a classification of indecomposable coherent sheaves. This can be generalized to the case of singular Weierstraß cubic curves.

**Theorem 23 ([BK1]).** *Let $\mathbb{E}$ be an irreducible projective curve of arithmetic genus one over an algebraically closed field $\mathbf{k}$, $p_0 \in \mathbb{E}$ a smooth point and $\mathcal{P} = \mathcal{I}_\Delta \otimes \pi_1^*(\mathcal{O}(p_0)) \otimes \pi_2^*(\mathcal{O}(p_0))$, where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal $\Delta \subset \mathbb{E} \times \mathbb{E}$. We have the following properties of the Fourier-Mukai transform $\Phi = \Phi_\mathcal{P}$:*

1. *$\Phi$ is an exact equivalence and $\Phi \circ \Phi \cong i^*[-1]$, where $i : \mathbb{E} \longrightarrow \mathbb{E}$ is an involution of $\mathbb{E}$.*
2. *$\Phi$ transforms semi-stable sheaves to semi-stable ones and stable sheaves to stable ones.*
3. *In particular, $\Phi$ induces an equivalence between the abelian categories $\mathsf{Coh}^\nu(\mathbb{E})$ and $\mathsf{Coh}^{-\frac{1}{\nu}}(\mathbb{E})$, where $\nu \in \mathbb{Q} \cup \{\infty\}$.*
4. *Let $\mathcal{F}$ be a semi-stable sheaf of degree zero. Then the sequence*

$$0 \longrightarrow H^0(\mathcal{E}(p_0)) \otimes \mathcal{O} \xrightarrow{ev} \mathcal{E}(p_0) \longrightarrow coker(ev) \longrightarrow 0$$

*is exact. Moreover, the functor $\Phi(\mathcal{E}) \cong coker(ev)$ establishes an equivalence between the category $\mathsf{Coh}^0(\mathbb{E})$ of semi-stable torsion-free sheaves of degree zero and the category of torsion sheaves $\mathsf{Coh}^\infty(\mathbb{E})$.*

From this theorem follows that for any pair of coprime integers $(r, d) \in \mathbb{Z}^2, r > 0$ the moduli space $M_\mathbb{E}(r, d)$ of stable sheaves of rank $r$ and degree $d$ is isomorphic to $\mathbb{E}$. The unique singular point of $M_\mathbb{E}(r, d)$ corresponds to the stable sheaf which is not locally free.

Let $\mathcal{T}$ be an indecomposable torsion sheaf on $\mathbb{E}$. If $\mathcal{T}$ has support at a smooth point $p \in \mathbb{E}$ than $\mathcal{T} \cong \mathcal{O}_{\mathbb{E},p}/\mathfrak{m}_p^n$ for some $n > 0$.

The structure of torsion sheaves supported at the singular point $s \in \mathbb{E}$ is much more complicated. First of all note that the categories of finite-dimensional modules over $\mathcal{O}_{\mathbb{E},s}$ and $\widehat{\mathcal{O}_{\mathbb{E},s}}$ are equivalent. So, in order to understand semi-stable sheaves on singular Weierstraß curves we have to analyze the structure of finite-dimensional representations of $\mathbf{k}[[x, y]]/(xy)$ and $\mathbf{k}[[x, y]]/(y^2 - x^3)$ first.

Let $\mathbf{R} = \mathbf{k}[[x, y]]/(xy)$, then it is easy to show that all indecomposable finite length $\mathbf{R}$-modules generated by one element are $\mathcal{M}((n, m), 1, \lambda) =$

$\mathbf{R}/(x^n + \lambda y^m)$ for $n, m \geq 1$, $\lambda \in \mathbf{k}^*$ and $\mathcal{N}(0, (n, m), 0) = \mathbf{R}/(x^{n+1}, y^{m+1})$ for $n, m \geq 0$. A classification of all indecomposable $\mathbf{R}$–modules was obtained by Gelfand and Ponomarev [GP68] and independently by Nazarova and Roiter [NR69], see also [BD04] for a description via derived categories. We identify an indecomposable torsion module $\mathcal{T}$ supported at $s$ with the corresponding $\mathbf{k}[[x, y]]/(xy)$–module.

**Theorem 24 ([BK1]).** *The Fourier-Mukai transform $\Phi_{\mathcal{P}}$ maps the torsion module $\mathcal{M}((n, m), 1, \lambda)$ to the degree zero semi-stable vector bundle*

$$\mathcal{B}((\overbrace{1, 0, \ldots, 0}^{m}, \overbrace{-1, 0, \ldots, 0}^{n}), 1, (-1)^{\lambda(n+m)})$$

*and $\mathcal{N}(0, (n, m), 0)$ to the semi-stable torsion free sheaf*

$$\mathcal{S}(\overbrace{0, \ldots, 0}^{m}, -1, \overbrace{0, \ldots, 0}^{n}).$$

In [BK1] a complete correspondence between torsion sheaves and semi-stable sheaves of degree zero was described. Using the technique of relative Fourier-Mukai transforms one gets a powerful tool to construct interesting examples of relatively stable and semi-stable sheaves on elliptically fibered varieties, see [FMW99, BK2].

In a similar way, if $\mathbf{R} = \mathbf{k}[[x, y]]/(y^2 - x^3) = \mathbf{k}[[t^2, t^3]]$, then a finite length $\mathbf{R}$–module is given by a finite dimensional vector space $V$ over $\mathbf{k}$ and two endomorphisms $X, Y : V \to V$ which satisfy $Y^2 - X^3 = 0$. It is again very easy to classify all $\mathbf{R}$–modules of the form $\mathbf{R}/I$, where $I$ is an ideal in $\mathbf{R}$: there are one-parameter families of modules of projective dimension one, $\mathbf{R}/(t^n + \lambda t^{n+1})$, $n \geq 2$ and $\lambda \in \mathbf{k}$ and discrete series of modules of infinite projective dimension, $\mathbf{R}/(t^n, t^{n+1})$, where $n \geq 2$.

However, there is an essential difference between the rings $\mathbf{k}[[x, y]]/(xy)$ and $\mathbf{R} = \mathbf{k}[[x, y]]/(y^2 - x^3)$: the first has tame representation type [GP68, NR69] whereas the second is wild.

**Proposition 8 ([Dro72]).** *The category of finite length modules over the ring $\mathbf{k}[[x, y]]/(y^2 - x^3)$ is wild.*

*Proof.* We have a fully-faithful exact functor $\mathbf{k}\langle z_1, z_2 \rangle \longrightarrow \mathsf{Rep}_\Gamma$, where $\Gamma$ is the quiver $a \circlearrowright \bullet \xrightarrow{b} \bullet$ . This functor maps a $\mathbf{k}\langle z_1, z_2 \rangle$–module $(Z_1, Z_2, \mathbf{k}^n)$ to the representation of $\Gamma$ given by

$$A = \begin{pmatrix} Z_1 & Z_2 \\ I_n & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I_n \end{pmatrix}.$$

Moreover, we have another fully-faithful exact functor

$$\mathsf{Rep}_\Gamma \longrightarrow \mathbf{k}[[x, y]]/(y^2 - x^3),$$

mapping a representation $(A, B, \mathbf{k}^n, \mathbf{k}^m)$ to the $\mathbf{k}[[x,y]]/(y^2 - x^3)$–module given by the matrices

$$Y = \begin{pmatrix} 0_{3n} & 0 & I \\ 0 & 0_m & 0 \\ 0 & 0 & 0_{3n} \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & 0 & X_2 \\ 0 & 0 & X_3 \\ 0 & 0 & X_1 \end{pmatrix},$$

where

$$X_1 = \begin{pmatrix} 0_n & 0 & I \\ 0 & 0_n & 0 \\ 0 & 0 & 0_n \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0_n & 0 & 0 \\ I & 0_n & 0 \\ 0 & A & 0_n \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0_{m \times n} & B_{m \times n} & 0_{m \times n} \end{pmatrix}.$$

Taking the composition of these two functors we see that the category of finite dimensional $\mathbf{k}[[x,y]]/(y^2 - x^3)$–modules is wild.

Since via an appropriate Fourier-Mukai transform the category of torsion modules over the ring $\widehat{\mathcal{O}_{\mathbb{E},s}}$ is equivalent to the category of semi-stable torsion free sheaves of a given slope $\nu$ with non-locally free Jordan-Hölder quotient, we obtain the following corollary.

**Corollary 2.** *Let $\mathbb{E}$ be a cuspidal Weierstraß curve and $\nu \in \mathbb{Q} \cup \{\infty\}$, then the category $\mathsf{Coh}^\nu(\mathbb{E})$ of semi-stable torsion free sheaves of slope $\nu$ on $\mathbb{E}$ has wild representation type.*

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