11. Elements of Homological Algebra

The present chapter has been written for the English edition. The aim of this extension is to present an introduction to homological methods, which play an increasingly important role in the theory of algebras, and in this way to make the book more suitable as a textbook. Besides the fundamental concepts of a complex, resolutions and derived functors, we shall also briefly examine three special topics: homological dimension, almost split sequences and Auslander algebras.

11.1 Complexes and Homology

A *complex* of A-modules $(V_\bullet, d_\bullet)$, or simply $V_\bullet$, is a sequence of A-modules and homomorphisms

$$\cdots \longrightarrow V_2 \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \xrightarrow{d_0} V_{-1} \xrightarrow{d_{-1}} V_{-2} \longrightarrow \cdots$$

such that $d_n d_{n+1} = 0$ for all indices n. Clearly, this means that $\operatorname{Im} d_{n+1} \subset \operatorname{Ker} d_n$. Thus, one can define the *homology modules* $\mathrm{H}_n(V_\bullet) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$.

The set of the maps $d_\bullet = \{d_n\}$ is called the *differential* of the given complex. In what follows, we shall write often dx instead of $d_n x$ for $x \in V_n$ (and use, without mentioning it, other similar simplifications by omitting subscripts). The coset ("homology coset") $x + \operatorname{Im} d_{n+1}$, where $x \in \operatorname{Ker} d_n$, will be denoted by $[x]$.

If $(V'_\bullet, d'_\bullet)$ is another complex, a complex homomorphism $f_\bullet : V_\bullet \to V'_\bullet$ is a family of homomorphisms $f_n : V_n \to V'_n$ "commuting with the differential", i.e. such that $f_{n-1} d_n = d'_n f_n$ for all n. Evidently, such a family induces homology maps

$$\mathrm{H}_n(f_\bullet) : \mathrm{H}_n(V_\bullet) \to \mathrm{H}_n(V'_\bullet)$$

by $\mathrm{H}_n(f_\bullet)[x] = [f_n(x)]$ for all n (it is easy to see that for $dx = 0$, also $d'f(x) = 0$ and $\big[f(x + dy)\big] = \big[f(x)\big]$). In this way, we can consider the *category of complexes* of A-modules com-A and the family of the functors $\mathrm{H}_n : \text{com-}A \to \text{mod-}A$.

Two homomorphisms $f_\bullet$ and $g_\bullet : V_\bullet \to V'_\bullet$ are said to be *homological* if $\mathrm{H}_n(f_\bullet) = \mathrm{H}_n(g_\bullet)$ for all n; we shall denote this fact by $f_\bullet \equiv g_\bullet$. An important example of homological homomorphisms is the case of homotopic homomorphisms in the following sense. Two homomorphisms $f_\bullet$ and $g_\bullet$ are called *homotopic*: $f_\bullet \sim g_\bullet$ if there are homomorphisms $s_n : V_n \to V'_{n+1}$ such that $f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n$ for all n (the sequence $s_\bullet = \{s_n\}$ is called a *homotopy* between $f_\bullet$ and $g_\bullet$).

Proposition 11.1.1. *Homotopic homomorphisms are homological.*

Proof. For every homology class $[x]$,

$$\begin{aligned} \mathrm{H}_n(f_\bullet)[x] = \big[f(x)\big] &= \big[g(x) + d's(x) + s(dx)\big] = \\ &= \big[g(x) + d's(x)\big] = \big[g(x)\big] = \mathrm{H}_n(g_\bullet)[x] \end{aligned}$$

because $dx = 0$. □

Two complexes $V_\bullet$ and $V'_\bullet$ are called *homotopic* if there are homomorphisms $f_\bullet : V_\bullet \to V'_\bullet$ and $f'_\bullet : V'_\bullet \to V_\bullet$ such that $f_\bullet f'_\bullet \sim 1$ and $f'_\bullet f_\bullet \sim 1$. In this case, we shall write $V_\bullet \sim V'_\bullet$.

Corollary 11.1.2. *If $V_\bullet$ and $V'_\bullet$ are homotopic, then $\mathrm{H}_n(V_\bullet) \simeq \mathrm{H}_n(V'_\bullet)$ for all n.*

Remark. The converse of Proposition 11.1.1 and of Corollary 11.1.2 does not hold in general: $f_\bullet \equiv g_\bullet$ does not imply $f_\bullet \sim g_\bullet$ and $\mathrm{H}_n(V_\bullet) \simeq \mathrm{H}_n(V'_\bullet)$ for all n does not imply $V_\bullet \sim V'_\bullet$ (see Exercise 1 and 2).

Along with complexes of the above type ("chain complexes") it is often convenient to consider "cochain complexes" $(V^\bullet, d^\bullet)$ of the form

$$\cdots \longrightarrow V^{-1} \xrightarrow{d^{-1}} V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} V^2 \longrightarrow \cdots$$

with the condition $d^n d^{n-1} = 0$. In this case, we obtain the *cohomology modules* $\mathrm{H}^n(V^\bullet) = \operatorname{Ker} d^n/\operatorname{Im} d^{n-1}$. Obviously, one can pass from chain to cochain complexes simply by changing the indices, i. e. putting $V^n = V_{-n}$ and $d^n = d_{-n}$; hereby, H_n becomes H^{-n}. One can usually use the "chain" terminology if the complex is bounded from the right, i. e. there is a number n_0 so that $V_n = 0$ for $n < n_0$ and "cochain" terminology if $V_\bullet$ is bounded from the left, i. e. if there is a number n_0 so that $V_n = 0$ for $n > n_0$.

If F : mod-$A \to$ mod-B is a functor, then F induces a functor $F_\bullet$: com-$A \to$ com-B assigning to a complex $V_\bullet = \{V_n, d_n\}$ the complex $F_\bullet(V_\bullet) = \{F(V_n), F(d_n)\}$. For example, considering the functor h_M : mod-$A \to$ Vect for a fixed A-module M (see Example 1 in Sect. 8.1), we obtain the functor com-$A \to$ com-K assigning to a complex $V_\bullet$ the complex $\operatorname{Hom}_A(M, V_\bullet) = \{\operatorname{Hom}_A(M, V_n)\}$. Similarly, for a left A-module N, we have the functor $- \otimes_A N$ assigning to a complex $V_\bullet$ the complex $V_\bullet \otimes_A N = \{V_n \otimes_A N\}$. A contravariant functor from mod-A to mod-B, i. e. a functor $G : (\text{mod-}A)^o \to$ mod-B defines a functor $G^\bullet : (\text{com-}A)^o \to$ com-B, but it is more convenient in this case to consider $G^\bullet(V_\bullet)$ as a cochain complex with the nth component equal to $G(V_n)$. For instance, if $G = h_M^o$ (see Example 6 in Sect. 8.1), we obtain a contravariant functor mapping a chain complex $\{V_n\}$ into a cochain compl $\{\mathrm{Hom}_A(V_n, M)\}$.

It is evident that every such functor maps homotopic homomorphis (and complexes) into homotopic ones; however, again, $f_\bullet \equiv g_\bullet$ does not imp $F_\bullet(f_\bullet) \equiv F_\bullet(g_\bullet)$ (see Exercise 3).

Let $f_\bullet : V_\bullet \to V'_\bullet$ be a complex homomorphism. Then, obvious $d'_n(\mathrm{Im}\, f_n) \subset \mathrm{Im}\, f_{n-1}$ and $d_n(\mathrm{Ker}\, f_n) \subset \mathrm{Ker}\, f_{n-1}$ for all n, and thus we g the complexes $\mathrm{Im}\, f_\bullet = \{\mathrm{Im}\, f_n\}$ and $\mathrm{Ker}\, f_\bullet = \{\mathrm{Ker}\, f_n\}$. Therefore, one c define *exact sequences* of complexes just the same way as exact sequences modules in Sect. 8.2. The following theorem seems to play a fundamental ro in homological algebra.

Theorem 11.1.3. *Let $0 \to V'_\bullet \xrightarrow{f_\bullet} V_\bullet \xrightarrow{g_\bullet} V''_\bullet \to 0$ be an exact sequence of com- plexes. Then, for each n, there is a homomorphism $\partial_n : \mathrm{H}_n(V''_\bullet) \to \mathrm{H}_{n-1}(V$ such that the following sequence is exact:*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{H}_{n+1}(V''_\bullet) & \xrightarrow{\partial_{n+1}} & \mathrm{H}_n(V'_\bullet) & \xrightarrow{\mathrm{H}_n(f_\bullet)} & \mathrm{H}_n(V_\bullet) & \xrightarrow{\mathrm{H}_n(g_\bullet)} \\ \xrightarrow{\mathrm{H}_n(g_\bullet)} & \mathrm{H}_n(V''_\bullet) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(V'_\bullet) & \xrightarrow{\mathrm{H}_{n-1}(f_\bullet)} & \mathrm{H}_{n-1}(V_\bullet) & \longrightarrow \cdots \end{array}$$

Proof. (We shall use the same letter d for differentials in all complexes a omit subscripts.) Let $[x]$ be a homology coset of $\mathrm{H}_n(V''_\bullet)$. Since g_n is an ep morphism, $x = g(y)$ for some $y \in V_n$. Now, $g(dy) = dg(y) = dx = 0$ a thus, in view of the exactness, $dy = f(z)$ for some $z \in V'_{n-1}$. Furthermo $f(dz) = df(z) = d^2y = 0$ and therefore $dz = 0$ because f is a monomorphis

Let us verify that the coset $[z] \in \mathrm{H}_{n-1}(V'_\bullet)$ depends neither on the choice y nor on the choice of x in the homology coset $[x]$. Indeed, if $g(y') = g(y)$, th $g(y'-y) = 0$ and $y'-y = f(u)$ for some u; thus $dy' = dy + df(u) = f(z+du)$ a $[z+du] = [z]$. Furthermore, let $[x'] = [x]$, i. e. $x' = x + dv$ for some $v \in V''_{n+1}$ Then there is $w \in V_{n+1}$ such that $v = g(w)$ and therefore $x' = g(y + d$ Since $d(y+dw) = dy$, the choice of x' does not effect the coset $[z]$.

Consequently, setting $\partial_n[x] = [z]$ gives a well-defined homomorphis $\partial_n : \mathrm{H}_n(V''_\bullet) \to \mathrm{H}_{n-1}(V'_\bullet)$. It remains to prove that the long sequence is exac

We are going to show that $\mathrm{Ker}\,\mathrm{H}_n(f_\bullet) \subset \mathrm{Im}\,\partial_{n+1}$ and $\mathrm{Ker}\,\partial_n \subset \mathrm{Im}\,\mathrm{H}_n($ and leave the other (rather easy) verifications to the reader. Let $\mathrm{H}_n(f_\bullet)[x] =$ Thus $f(x) = dy$ for some $y \in V_{n+1}$. Put $z = g(y)$. Then $dz = g(dy)$ $gf(x) = 0$ and we get $[z] \in \mathrm{H}_{n+1}(V''_\bullet)$ satisfying $\partial[z] = [x]$ according to t definition of ∂.

Now, let $\partial_n[x] = 0$. By the definition of ∂, this means that if $x = g($ and $dy = f(z)$, then $z = du$ for some $u \in V'_n$. Hence, $x = g(y - f(u))$ a $d(y - f(u)) = dy - f(du) = 0$, which gives that $[x] = \mathrm{H}_n(g_\bullet)[y - f(u)]$, required.

A complex $V_\bullet$ is called *acyclic in dimension n* if $\mathrm{H}_n(V_\bullet) = 0$ and *acyclic* it is acyclic in all dimensions (trivially, it means that $V_\bullet$ is an exact sequence

Corollary 11.1.4. *Let $0 \to V'_\bullet \xrightarrow{f_\bullet} V_\bullet \xrightarrow{g_\bullet} V''_\bullet \to 0$ be an exact sequence of complexes. Then*

1) *$V_\bullet$ is acyclic in dimension n if and only if ∂_n is a monomorphism and ∂_{n+1} is an epimorphism.*
2) *$V'_\bullet$ is acyclic in dimension n if and only if $\mathrm{H}_n(g_\bullet)$ is a monomorphism and $\mathrm{H}_{n+1}(g_\bullet)$ is an epimorphism.*
3) *$V''_\bullet$ is acyclic in dimension n if and only if $\mathrm{H}_{n-1}(f_\bullet)$ is a monomorphism and $\mathrm{H}_n(f_\bullet)$ an epimorphism.*

Corollary 11.1.5. *Let $0 \to V'_\bullet \to V_\bullet \to V''_\bullet \to 0$ be an exact sequence of complexes.*

1) *If $V'_\bullet$ and $V''_\bullet$ are acyclic in dimension n, then $V_\bullet$ is acyclic in dimension n.*
2) *If $V_\bullet$ is acyclic in dimension n and $V'_\bullet$ in dimension $n-1$, then $V''_\bullet$ is acyclic in dimension n.*
3) *If $V_\bullet$ is acyclic in dimension n and $V''_\bullet$ in dimension $n+1$, then $V'_\bullet$ is acyclic in dimension n.*

The construction of the connecting homomorphisms ∂_n also yields the following statement, whose proof is left to the reader.

Proposition 11.1.6. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V'_\bullet & \longrightarrow & V_\bullet & \longrightarrow & V''_\bullet & \longrightarrow & 0 \\ & & {\scriptstyle\alpha_\bullet}\downarrow & & {\scriptstyle\beta_\bullet}\downarrow & & {\scriptstyle\gamma_\bullet}\downarrow & & \\ 0 & \longrightarrow & W'_\bullet & \longrightarrow & W_\bullet & \longrightarrow & W''_\bullet & \longrightarrow & 0 \end{array}$$

be a commutative diagram of complexes with exact rows. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{H}_n(V''_\bullet) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(V'_\bullet) \\ {\scriptstyle \mathrm{H}_n(\gamma_\bullet)}\downarrow & & \downarrow{\scriptstyle \mathrm{H}_{n-1}(\alpha_\bullet)} \\ \mathrm{H}_n(W''_\bullet) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(W'_\bullet). \end{array}$$

11.2 Resolutions and Derived Functors

Let M be an A-module. A *projective resolution* of M is a complex of A-modules $P_\bullet$ in which $P_n = 0$ for $n < 0$, all P_n are projective, and $P_\bullet$ is acyclic in every dimension $n \neq 0$, while $\mathrm{H}_0(P_\bullet) \simeq M$ is a fixed isomorphism. Observe that $\mathrm{Ker}\, d_0 = P_0$ and thus $\mathrm{H}_0(P_0) = P_0/\mathrm{Im}\, d_1$; hence, we have a fixed epimorphism $\pi : P_0 \to M$ whose kernel is $\mathrm{Im}\, d_1$. Therefore a projective resolution is often considered in the form of an exact sequence

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \longrightarrow 0\,.$$

However, in what follows, we want to underline the fact that M is not included in its projective resolution: the last non-zero term of its resolution is P_0.

In a dual way, one defines an *injective resolution* of an A-module M as a cochain complex $Q^\bullet$ in which $Q^n = 0$ for $n < 0$, all A-modules Q^n are injective and such that $Q^\bullet$ is acyclic in all dimensions $n \neq 0$, while $M \simeq \mathrm{H}^0(Q^\bullet) = \operatorname{Ker} d^0$ is a fixed isomorphism. Such a resolution can be identified with an exact sequence

$$0 \longrightarrow M \xrightarrow{\epsilon} Q^0 \xrightarrow{d^0} Q^1 \xrightarrow{d^1} Q^2 \longrightarrow \cdots$$

Generally speaking, we will deal with projective resolutions, leaving the corresponding formulations (and proofs) for injective resolutions to the reader.

Let $P_\bullet$ be a projective resolution of a module M and $P'_\bullet$ a projective resolution of M'. Then every complex morphism $f_\bullet : P_\bullet \to P'_\bullet$ induces a module homomorphism $\varphi : M \to M'$. The morphism $f_\bullet$ is said to be an *extension of φ to the resolutions $P_\bullet$* and $P'_\bullet$. In other words, an extension of φ to the resolutions is a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & {\scriptstyle \varphi}\downarrow & & \\
\cdots \longrightarrow & P'_2 & \xrightarrow{d'_2} & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\pi'} & M' & \longrightarrow & 0
\end{array}$$

Theorem 11.2.1. 1) *Every A-module M has a projective resolution.*

2) *Any two projective resolutions of a module M are homotopic.*

3) *Every homomorphism $\varphi : M \to M'$ can be extended to the resolutions $P_\bullet$ and $P'_\bullet$ of the modules M and M', respectively.*

4) *Any two extensions of φ to a given pair of resolutions are homotopic.*

Proof. 1) For every A-module M, there is an epimorphism $\varphi : P_0 \to M$ with a projective module P_0 (Corollary 3.3.4). Write $M_1 = \operatorname{Ker}\pi$ and construct an epimorphism $\pi_1 : P_1 \to M_1$, where P_1 is again projective. This epimorphism can be interpreted as a homomorphism $d_1 : P_1 \to P_0$ with $\operatorname{Im} d_1 = \operatorname{Ker}\pi$. Applying the same construction to $M_2 = \operatorname{Ker} d_1$, we obtain $d_2 : P_2 \to P_1$ with $\operatorname{Im} d_2 = \operatorname{Ker} d_1$. Continuing this process, we get a projective resolution $P_\bullet$ of the module M.

3) Let $P'_\bullet$ be a projective resolution of M'. Consider the homomorphism $\varphi\pi : P_0 \to M'$. Since P_0 is projective and $\pi' : P'_0 \to M'$ is an epimorphism, there is a homomorphism $f_0 : P_0 \to P'_0$ such that $\pi' f_0 = \varphi\pi$. From here, $\pi' f_0 d_1 = \varphi\pi d_1 = 0$ and thus $\operatorname{Im} f_0 d_1 \subset \operatorname{Ker}\pi'$. However, $\operatorname{Im} d'_1 = \operatorname{Ker}\pi'$, and P_1 is projective, so there is $f_1 : P_1 \to P'_1$ such that $f_0 d_1 = d'_1 f_1$. In particular, $d'_1 f_1 d_2 = f_0 d_1 d_2 = 0$ and therefore $\operatorname{Im} f_1 d_2 \subset \operatorname{Ker} d'_1$; hence there is $f_2 : P_2 \to P'_2$ such that $f_1 d_2 = d'_2 f_2$. Continuing this procedure, we construct an extension $f_\bullet : P_\bullet \to P'_\bullet$ of the homomorphism φ.

4) If $g_\bullet : P_\bullet \to P'_\bullet$ is another extension of φ, then $f_\bullet - g_\bullet$ is an extension of the zero homomorphism. Hence, it is sufficient to show that $f_\bullet \sim 0$ for any

extension $f_\bullet$ of the zero homomorphism. In such a case we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_3 \arrow[r, "d_3"] \arrow[d, "f_3"'] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"'] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"'] & P_0 \arrow[r] \arrow[d, "f_0"'] & 0 \\
\cdots \arrow[r] & P_3' \arrow[r, "d_3'"] & P_2' \arrow[r, "d_2'"] & P_1' \arrow[r, "d_1'"] & P_0' \arrow[r] & 0
\end{tikzcd}$$

with $\operatorname{Im} f_0 \subset \operatorname{Im} d_1'$ (since $\mathrm{H}_0(f_\bullet) = 0$).

Since P_0 is projective, $f_0 = d_1' s_0$ for some $s_0 : P_0 \to P_1'$; thus $f_0 = d_1' s_0 + s_{-1} d_0$ (because $d_0 = 0$). Consider $\bar{f}_1 = f_1 - s_0 d_1$. Then $d_1' \bar{f}_1 = d_1' f_1 - d_1' s_0 d_1 = d_1' f_1 - f_0 d_1 = 0$ and therefore $\operatorname{Im} \bar{f}_1 \subset \operatorname{Ker} d_1' = \operatorname{Im} d_2'$ in view of $\mathrm{H}_1(P_\bullet') = 0$. Since P_1 is projective, there exists $s_1 : P_1 \to P_2'$ such that $\bar{f}_1 = d_2' s_1$, i.e. $f_1 = s_0 d_1 + d_2' s_1$. Now, take $\bar{f}_2 = f_2 - s_1 d_2$; again $d_2' \bar{f}_2 = d_2' f_2 - d_2' s_1 d_2 = d_2' f_2 - \bar{f}_1 d_2 + s_0 d_1 d_2 = 0$ and subsequently $\bar{f}_2 = d_3' s_2$, i.e. $f_2 = s_1 d_2 + d_3' s_2$ for some $s_2 : P_2 \to P_3'$. Again, by induction, $f_\bullet \sim 0$.

2) Let $P_\bullet$ and $P_\bullet'$ be two projective resolutions of a module M. There are extensions $f_\bullet : P_\bullet \to P_\bullet'$ and $f_\bullet' : P_\bullet' \to P_\bullet$ of the identity homomorphism $1 : M \to M$. But then $f_\bullet f_\bullet'$ and $f_\bullet' f_\bullet$ also extend $1 : M \to M$. Since the identity morphisms $1_\bullet : P_\bullet \to P_\bullet$ and $1_\bullet : P_\bullet' \to P_\bullet'$ extend $1 : M \to M$, as well, 4) implies that $f_\bullet f_\bullet' \sim 1$ and $f_\bullet' f_\bullet \sim 1$. Therefore $P_\bullet \sim P_\bullet'$ and the theorem is proved. □

Taking into account the fact that every functor $F : \text{mod-}A \to \text{mod-}B$ translates homotopic complexes and homomorphisms into homotopic ones, and applying Proposition 11.1.1 and Corollary 11.1.2, we get the following consequence.

Corollary 11.2.2. 1) *Let $F : \text{mod-}A \to \text{mod-}B$ be a functor and $P_\bullet$ a projective resolution of an A-module M. Then the homology $\mathrm{H}_n(F(P_\bullet))$ is independent of the choice of the resolution $P_\bullet$.*

2) *If $P_\bullet'$ is a projective resolution of M' and $f_\bullet : P_\bullet \to P_\bullet'$ an extension of a homomorphism $\varphi : M \to M'$, then $\mathrm{H}_n(F_\bullet(f_\bullet))$ is independent of the choice of the extension $f_\bullet$.*

In the situation described in Corollary 11.2.2, we shall write $L_nF(M) = \mathrm{H}_n(F_\bullet(P_\bullet))$ and $L_nF(\varphi) = \mathrm{H}_n(F_\bullet(f_\bullet))$. If $f_\bullet$ is an extension of φ and $g_\bullet$ an extension of $\psi : M' \to M''$, then $g_\bullet f_\bullet$ is an extension of $\psi\varphi$ and thus $L_nF(\psi\varphi) = L_nF(\psi)L_nF(\varphi)$, i.e. L_nF is a functor $\text{mod-}A \to \text{mod-}B$, which is called the *$n$-th left derived functor* of the functor F. Similarly, replacing projective resolutions by injective ones, one can define *right derived functors* R^nF. The definitions of left and right derived functors of a contravariant functor G can be given dually, using injective resolutions for L_nG and projective resolutions for R^nG. All further arguments apply to right derived, as well as contravariant functors.

Proposition 11.2.3. *A right (left) exact functor F satisfies $L_0F \simeq F$ (respectively, $R^0F \simeq F$).*

Proof. If $P_\bullet$ is a projective resolution of M, then $P_1 \xrightarrow{d_1} P_0 \to M \to 0$ is an exact sequence, and thus $F(P_1) \xrightarrow{F(d_1)} F(P_0) \to F(M) \to 0$ is exact, as well. Therefore, $L_0F(M) = H_0(F_\bullet(P_\bullet)) = F(P_0)/\operatorname{Im} F(d_1) \simeq F(M)$. □

The importance of derived functors stems in many respects from the existence of "long exact sequences". Their construction is based on Theorem 11.1.3 and the following lemmas.

Lemma 11.2.4. *For every exact sequence of modules*

$$0 \longrightarrow M' \xrightarrow{\ \varphi\ } M \xrightarrow{\ \psi\ } M'' \longrightarrow 0\,,$$

there are projective resolutions $P'_\bullet$, $P_\bullet$ and $P''_\bullet$ and an exact sequence

$$0 \longrightarrow P'_\bullet \xrightarrow{\ f_\bullet\ } P_\bullet \xrightarrow{\ g_\bullet\ } P''_\bullet \longrightarrow 0\,,$$

in which $f_\bullet$ extends φ and $g_\bullet$ extends ψ.

Proof. Let $\pi' : P'_0 \to M'$ and $\pi'' : P''_0 \to M''$ be epimorphisms. Put $P_0 = P'_0 \oplus P''_0$ and consider a homomorphism $\pi = (\pi', \eta) : P_0 \to M$, where η is a homomorphism $P''_0 \to M$ such that $\psi\eta = \pi''$. It is easy to verify that π is also an epimorphism and that we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M'_1 \arrow[r, "\varphi_1"] \arrow[d] & M_1 \arrow[r, "\psi_1"] \arrow[d] & M''_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_0 \arrow[r, "f_0"] \arrow[d, "\pi'"'] & P_0 \arrow[r, "g_0"] \arrow[d, "\pi"'] & P''_0 \arrow[r] \arrow[d, "\pi''"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "\varphi"] \arrow[d] & M \arrow[r, "\psi"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which all columns and the two lower rows are exact; here $M'_1 = \operatorname{Ker}\pi'$, $M_1 = \operatorname{Ker}\pi$, $M''_1 = \operatorname{Ker}\pi''$. According to part 3) of Corollary 11.1.5 (see also Exercise 3 to Chapter 8) the first row is also exact, and thus we may apply to it the same construction. By repeating this procedure, we obtain a required exact sequence of resolutions. □

Lemma 11.2.5. *If $0 \to V'_\bullet \to V_\bullet \to V''_\bullet \to 0$ is an exact sequence of complexes, where all modules V''_n are projective, then the sequence $0 \to F_\bullet(V'_\bullet) \to F_\bullet(V_\bullet) \to F_\bullet(V''_\bullet) \to 0$ is exact for every functor F.*

Proof. Since every sequence $0 \to V'_n \to V_n \to V''_n \to 0$ splits, the sequence $0 \to F(V'_n) \to F(V_n) \to F(V''_n) \to 0$ also splits. □

Now we apply the preceding lemmas and Theorem 11.1.3 in order to get a long exact sequence for arbitrary functors.

Corollary 11.2.6. *Let $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ be an exact sequence of modules. Then for any functor F, there exist connecting homomorphisms $\partial_n : L_nF(M'') \to L_{n-1}F(M')$ so that the following sequence is exact*

$$\begin{array}{ccccccc} \cdots \to & L_{n+1}F(M'') & \xrightarrow{\partial_{n+1}} & L_nF(M') & \xrightarrow{L_nF(\varphi)} & L_nF(M) & \xrightarrow{L_nF(\psi)} \\ \xrightarrow{L_nF(\psi)} & L_nF(M'') & \xrightarrow{\partial_n} & L_{n-1}F(M') & \xrightarrow{L_{n-1}F(\varphi)} & L_{n-1}F(M) & \to \cdots \end{array}$$

Observe that, by definition, $L_nF = 0$ for $n < 0$ and thus, Corollary 11.2.6 implies that L_0F is always right exact. In particular, if F itself is right exact, then in view of Proposition 11.2.3, the end of the long exact sequence has the following form:

$$\cdots \longrightarrow L_1F(M'') \xrightarrow{\partial_1} F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0\,.$$

Corollary 11.2.7. 1) *A functor F is right (left) exact if and only if $F \simeq L_0F$ (respectively, $F \simeq R^0F$).*

2) *A right (left) exact functor F is exact if and only if $L_1F = 0$ (respectively, $R^1F = 0$).*

Observe that, for an exact F, both $L_nF = 0$ and $R^nF = 0$ for all $n > 0$.

If a module P is projective, then its projective resolution has a very simple form: $P_0 = P$ and $P_n = 0$ for $n > 0$. In particular, $L_nF(P) = 0$ for all $n > 0$.

This trivial observation indicates how to characterize derived functors "axiomatically", in the following way.

Theorem 11.2.8. *Let F be a right exact functor and $\{\Phi_n \mid n \geq 0\}$ a family of functors satisfying the following properties:*

1) *$\Phi_0 \simeq F$ (as functors);*
2) *$\Phi_n(P) = 0$ for all $n > 0$ and all projective P;*
3) *If $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ is an exact sequence of modules, then there are homomorphisms $\Delta_n : \Phi_n(M'') \to \Phi_{n-1}(M')$, $n \geq 0$, so that the following sequence is exact:*

$$\begin{array}{ccccccc} \cdots \to & \Phi_{n+1}(M'') & \xrightarrow{\Delta_{n+1}} & \Phi_n(M') & \xrightarrow{\Phi_n(\varphi)} & \Phi_n(M) & \xrightarrow{\Phi_n(\psi)} \\ \xrightarrow{\Phi_n(\psi)} & \Phi_n(M'') & \xrightarrow{\Delta_n} & \Phi_{n-1}(M') & \xrightarrow{\Phi_{n-1}(\varphi)} & \Phi_{n-1}(M) & \to \cdots \end{array}$$

Then $\Phi_n(M) \simeq L_nF(M)$ for all $n \geq 0$ and all modules M.

Proof. The exact sequence $0 \to L \xrightarrow{\alpha} P \to M \to 0$ with a projective module P induces a long exact sequence for the functors Φ_n. For $n = 1$, we get the exact sequence

$$\Phi_1(P) = 0 \longrightarrow \Phi_1(M) \xrightarrow{\Delta_1} \Phi_0(L) \xrightarrow{\Phi_0(\alpha)} \Phi_0(P)\,,$$

from where $\Phi_1(M) \simeq \operatorname{Ker}\Phi_0(\alpha) = \operatorname{Ker} F(\alpha) \simeq L_1F(M)$ by the condition 1). For $n > 1$, the exact sequence has the form

$$\Phi_n(P) = 0 \longrightarrow \Phi_n(M) \xrightarrow{\Delta_n} \Phi_{n-1}(L) \longrightarrow \Phi_{n-1}(P) = 0\,,$$

thus Δ_n is an isomorphism and the theorem follows by induction. □

Remark. In fact, in Theorem 11.2.8, $\Phi_n \simeq L_nF$ as functors; however, we will not use this result.

From Proposition 11.1.6, we get also the following consequence.

Corollary 11.2.9. *Let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram with exact rows. Then the following diagram is commutative:

$$\begin{array}{ccc}
L_nF(M'') & \xrightarrow{\partial_n} & L_{n-1}F(M') \\
{\scriptstyle L_nF(\gamma)}\big\downarrow & & \big\downarrow{\scriptstyle L_{n-1}F(\alpha)} \\
L_nF(N'') & \xrightarrow{\partial_n} & L_{n-1}F(N')\,.
\end{array}$$

11.3 Ext and Tor. Extensions

The construction of derived functors applies, in particular, to the functors Hom and $\otimes$ (more precisely, to the functors h_M , h°_N, $X \otimes_A -$ and $- \otimes_A Y$). Since Hom is left exact, it is natural to consider right derived functors $R^n h_M$ (constructed by means of injective resolutions) and $R^n h^\circ_N$ (constructed by means of projective resolutions, since h°_N is contravariant), which coincide for $n = 0$ with h_M and h°_N. It is a remarkable fact that these constructions produce the same result.

Theorem 11.3.1. *For all A-modules M, N and each $n \geq 0$,*

$$R^n h_M(N) \simeq R^n h^\circ_N(M)\,.$$

Proof. Fix a module M and put $\Phi_n(N) = R^n h^\circ_N(M)$. If $\varphi : N \to L$, then φ induces a functor morphism $h^\circ_N \to h^\circ_L$ assigning to a homomorphism $\alpha : M \to N$ the homomorphism $\varphi\alpha : M \to L$, and thus also a derived functor morphism $\Phi_n(\varphi) : \Phi_n(N) \to \Phi_n(L)$. Note that if N is injective, then, in accordance with the definition of injectivity (see Theorem 9.1.4), the functor h_N° is exact and therefore $\Phi_n(N) = 0$ for $n > 0$. In addition, $\Phi_0(N) = R^0 h_N^\circ(M) \simeq h_N^\circ(M) = h_M(N)$ by Proposition 11.2.3. Clearly, this isomorphism is functorial in N, and thus $\Phi_0 \simeq h_M$.

Now, let $0 \to N' \xrightarrow{\varphi} N \xrightarrow{\psi} N'' \to 0$ be an exact sequence. Then, for any complex $P_\bullet$ consisting of projective modules, the sequence of complexes

$$0 \to \mathrm{Hom}_A(P_\bullet, N') \to \mathrm{Hom}_A(P_\bullet, N) \to \mathrm{Hom}_A(P_\bullet, N'') \to 0$$

is exact. Taking for $P_\bullet$ a projective resolution of the module M, we get, according to Theorem 11.1.3, just a long exact cohomology sequence similar to that which appears in the formulation of Theorem 11.2.8 (condition 3)). Thus, all the conditions of this theorem are satisfied, and therefore $\Phi_n(N) \simeq R^n h_M(N)$. The proof of the theorem is completed. □

The common value $R^n h_M(N) \simeq R^n h_N^\circ(M)$ is denoted by $Ext^n_A(M, N)$.

An analogous result holds for the functors $t_M = M \otimes_A -$ and $t_N = - \otimes_A N$, where M is a right and N is a left A-module.

Theorem 11.3.2. *For any right A-module M and any left A-module N, and each $n \geq 0$,*

$$L_n t_M(N) = L_n t_N(M).$$

The proof is (quite similar to the proof of Theorem 3.1) left to the reader.

The common value of these functors is denoted by $\mathrm{Tor}_n^A(M, N)$. Let us point out that $\mathrm{Ext}_A^0(M, N) \simeq \mathrm{Hom}(M, N)$ and $\mathrm{Tor}_0^A(M, N) \simeq M \otimes_A N$.

The functor $\mathrm{Ext}_A^1(M, N)$ is closely related to the module extensions. Referring to Sect. 1.5, let us reformulate the definition of an extension of a module M with kernel N as an exact sequence ζ of the form

$$\zeta : 0 \to N \xrightarrow{\alpha} X \xrightarrow{\beta} M \to 0.$$

Two extensions ζ and ζ', where

$$\zeta' : 0 \to N \xrightarrow{\alpha'} X' \xrightarrow{\beta'} M \to 0$$

are said to be equivalent (which is denoted by $\zeta \simeq \zeta'$) if there is a homomorphism $\gamma : X \to X'$ such that the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "1_N"'] & X \arrow[r, "\beta"] \arrow[d, "\gamma"'] & M \arrow[r] \arrow[d, "1_M"'] & 0 \\ 0 \arrow[r] & N \arrow[r, "\alpha'"] & X' \arrow[r, "\beta'"] & M \arrow[r] & 0. \end{tikzcd}$$

By Lemma 8.2.1 (Five lemma), γ is an isomorphism. Denote by $\mathrm{Ex}(M, N)$ the set of all equivalence classes of extensions of M with kernel N.

By Corollary 11.2.6, an exact sequence ζ induces a connecting homomorphism $\partial_\zeta : \mathrm{Hom}_A(M, M) \to \mathrm{Ext}^1_A(M, N)$. The element $\delta(\zeta) = \partial_\zeta(1_M)$ is called the *characteristic class* of the extension ζ. If $\zeta \simeq \zeta'$, then the diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(M, M) \arrow[r, "\partial_\zeta"] \arrow[d] & \mathrm{Ext}^1_A(M, N) \arrow[d] \\
\mathrm{Hom}_A(M, M) \arrow[r, "\partial_{\zeta'}"] & \mathrm{Ext}^1_A(M, N)
\end{tikzcd}$$

is, by Corollary 11.2.9, commutative (with the vertical maps being identity morphisms). From here, $\delta(\zeta) = \delta(\zeta')$, and therefore we get a well defined map $\delta : \mathrm{Ex}(M, N) \to Ext^1_A(M, N)$.

Theorem 11.3.3. *The map δ is one-to-one.*

Proof. We are going to construct an inverse map ω. To this end, fix an exact sequence $0 \to N \xrightarrow{\varepsilon} Q \xrightarrow{\sigma} L \to 0$ with an injective module Q. By Corollary 11.2.6, the sequence

$$\mathrm{Hom}_A(M, Q) \xrightarrow{h_M(\sigma)} \mathrm{Hom}(M, L) \xrightarrow{\partial} \mathrm{Ext}^1_A(M, N) \longrightarrow 0$$

is exact (since $\mathrm{Ext}^1_A(M, Q) = 0$). In particular, every element $u \in \mathrm{Ext}^1_A(M, N)$ is of the form $u = \partial(\varphi)$ for some $\varphi : M \to L$. Consider a lifting of the given exact sequence along φ (see Exercise 5 to Chap. 8), i. e. the exact sequence

$$\xi : 0 \to N \xrightarrow{f} Z \xrightarrow{g} M \to 0,$$

where Z is a submodule of $Q \oplus M$ consisting of the pairs (q, m) such that $\sigma(q) = \varphi(m)$, and f and g are defined by the rules $f(n) = (\varepsilon(n), 0)$ and $g(q, m) = m$. If φ' is another homomorphism satisfying $\partial(\varphi') = u$, then $\varphi' = \varphi + \sigma\eta$ for some $\eta : M \to Q$. *Then an equivalence of the extensions* ξ *and* $\xi' : 0 \to N \to Z' \to M \to 0$ constructed as a lifting along φ', is given by a homomorphism $\gamma : Z \to Z'$ sending (q, m) into $(q + \eta(m), m)$ (the simple verification is left to the reader). Consequently, by defining $\omega(u) = \xi$, we get a map $\mathrm{Ext}^1_A(M, N) \to \mathrm{Ex}(M, N)$. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "1_N"'] & Z \arrow[r, "g"] \arrow[d, "\psi"'] & M \arrow[r] \arrow[d, "\varphi"'] & 0 \\
0 \arrow[r] & N \arrow[r, "\varepsilon"] & Q \arrow[r, "\sigma"] & L \arrow[r] & 0,
\end{tikzcd}$$

where $\psi(q, m) = q$, yields, in view of Corollary 11.2.9, a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_A(M, M) \arrow[r, "\partial_\xi"] \arrow[d, "h_M(\varphi)"'] & \mathrm{Ext}^1_A(M, N) \arrow[d, "1"] \\
\mathrm{Hom}_A(M, L) \arrow[r, "\partial"] & \mathrm{Ext}^1_A(M, N),
\end{tikzcd}$$

and thus $\delta\omega(u) = \partial_\xi(1_M) = \partial(\varphi) = u$.

It remains to show that $\omega\delta(\zeta) \simeq \zeta$ holds for an arbitrary extension $\zeta : 0 \to N \xrightarrow{\alpha} X \xrightarrow{\beta} M \to 0$. Let $\delta(\zeta) = u$. Since Q is injective, the homomorphism $\varepsilon : N \to Q$ extends to $\mu : X \to Q$ such that $\mu\alpha = \varepsilon$, and yields a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_N} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & N & \xrightarrow{\varepsilon} & Q & \xrightarrow{\sigma} & L & \longrightarrow & 0\,.
\end{array}$$

Therefore the following square is commutative:

$$\begin{array}{ccc}
\mathrm{Hom}_A(M,M) & \xrightarrow{\partial_\zeta} & \mathrm{Ext}^1_A(M,N) \\
\downarrow{\scriptstyle h_M(\varphi)} & & \downarrow{\scriptstyle 1} \\
\mathrm{Hom}_A(M,L) & \xrightarrow{\partial} & \mathrm{Ext}^1_A(M,N)\,,
\end{array}$$

and $u = \partial(\varphi)$. Using this φ in constructing $\omega(u)$ as above, we get a sequence $\xi : 0 \to N \to Z \to M \to 0$. But then the homomorphism $\gamma : X \to Z$ given by $\gamma(x) = (\mu(x), \beta(x))$ establishes the equivalence of ζ and $\xi = \omega(u)$. The theorem is proved. $\square$

In the sequel, we shall identify the elements of $\mathrm{Ext}^1_A(M,N)$ and the respective extensions. Since, for a fixed M, $\mathrm{Ext}^1(M,N)$ is a covariant functor of N (and, for a fixed N, a contravariant functor of M), a homomorphism $\varphi : N \to N'$ (a homomorphism $\psi : M' \to M$) induces a map $\varphi_e : \mathrm{Ext}^1_A(M,N) \to \mathrm{Ext}^1_A(M,N')$ (respectively, a map $\psi^e : \mathrm{Ext}^1_A(M,N) \to Ext^1_A(M',N)$). From the explicit form of the one-to-one correspondence $\omega : \mathrm{Ext}^1_A(M,N) \to \mathrm{Ex}(M,N)$ constructed above, we get immediately the following corollary.

Corollary 11.3.4. 1) *The extension $\omega(\psi^e(u))$ is equivalent to the lifting of $\omega(u)$ along ψ.*
2) *The extension $\omega(\varphi_e(u))$ is equivalent to the descent of $\omega(u)$ along φ.*

(A lifting of an exact sequence has been already defined above. A descent of an extension $0 \to N \xrightarrow{f} Z \xrightarrow{g} M \to 0$ along $\varphi : N \to N'$ is, by definition, the exact sequence $0 \to N' \xrightarrow{f'} Z' \xrightarrow{g'} M \to 0$, where $Z' = (N' \oplus Z)/Y$ with $Y = \{(-\varphi(n), f(n)) \mid n \in N\}$ and $f'(n') = [n', 0]$, $g'([n', z]) = g(z)$. Here $[n', z]$ denotes the coset $(n', z) + Y$.)

Using the preceding Corollary 11.3.4, we shall write $\psi^e(\zeta) = \omega(\psi^e(u))$ and $\varphi_e(\zeta) = \omega(\varphi_e(u))$ for $\zeta = \omega(u)$.

Corollary 11.3.5. *The following conditions are equivalent:*

1) *The module M is projective (injective).*
2) $\mathrm{Ext}^1_A(M,N) = 0$ *(respectively, $Ext^1_A(N,M) = 0$) for every module N.*

3) $\mathrm{Ext}^1_A(M,N) = 0$ *(respectively, $Ext^1_A(N,M) = 0$) for every simple module N.*
4) $\mathrm{Ext}^n_A(M,N) = 0$ *(respectively, $Ext^n_A(N,M) = 0$) for each $n > 0$ and every module N.*

Proof. The implications 1) $\Rightarrow$ 4) $\Rightarrow$ 2) are trivial and 2) $\Rightarrow$ 1) follows in view of Theorem 11.3.3 and Theorem 3.3.5 (or Theorem 9.1.4 for injectivity). Also, 2) $\Rightarrow$ 3) is trivial, while 3) $\Rightarrow$ 2) can be proved by induction on the length of N, using the long exact sequence. □

It is remarkable that, for modules over finite dimensional algebras, the following statement also holds.

Proposition 11.3.6. *The following conditions are equivalent:*

1) *The module M is projective.*
2) $\mathrm{Tor}^A_1(M,N) = 0$ *for every module N.*
3) $\mathrm{Tor}^A_1(M,N) = 0$ *for every simple module N.*
4) $\mathrm{Tor}^A_n(M,N) = 0$ *for every module N and each $n > 0$.*

Proof. Again, 1) $\Rightarrow$ 4) $\Rightarrow$ 2) $\Rightarrow$ 3) are trivial. We are going to prove 3) $\Rightarrow$ 1). Consider an exact sequence $0 \to L \to P \xrightarrow{\pi} M \to 0$, where $\pi : P \to M$ is a projective cover of M. Write $\bar{A} = A/R$ with $R = \mathrm{rad}\, A$ and note that $\mathrm{Tor}^A_1(M, \bar{A}) = 0$ because $\bar{A}$ is a direct sum of simple modules. Therefore, in view of Corollary 11.2.6, $0 \longrightarrow L \otimes_A \bar{A} \longrightarrow P \otimes_A \bar{A} \xrightarrow{\pi\otimes 1} M \otimes_A \bar{A} \longrightarrow 0$ is an exact sequence. Now, one can see easily that $M \otimes_A \bar{A} \simeq M/MR$ (an isomorphism can be defined by $x + MR \mapsto x \otimes 1$). Since $\pi : P \to M$ is a projective cover, $\pi \otimes 1$ defines an isomorphism $P/PR \simeq M/MR$. Thus, $L/LR = 0$ and, by Nakayama's lemma, $L = 0$. Hence, $\pi : P \to M$ is an isomorphism and M is projective. □

11.4 Homological Dimensions

The functor mod-$A \to$ Vect assigning to X the space $\mathrm{Ext}^n_A(M,X)$ will be denoted by h^n_M. Notice that if M is a B-A-bimodule then h^n_M can be considered as a functor mod-$A \to$ mod-B. The *projective dimension* of an A-module M is said to be n: *proj.dim*$_A\, M = n$ if $h^n_M \neq 0$ and $h^m_M = 0$ for all $m > n$; if no such number exists, define *proj.dim*$_A\, M = \infty$. Dually, considering the functors $h^{o\,n}_M : X \mapsto \mathrm{Ext}^n_A(X, M)$, we define the *injective dimension inj.dim*$_A\, M$ to be n, if $h^{o\,n}_M \neq 0$ but $h^{o\,m}_M = 0$ for all $m > n$, and *inj.dim*$_A\, M = \infty$ if no such number n exists.

In accordance with Corollary 11.3.5, *proj.dim*$_A\, M = 0$ means that M is projective and *inj.dim*$_A = 0$ that M is injective. Furthermore, Corollary 11.2.6 provides an inductive way for computing these dimensions.

Proposition 11.4.1. *Let* $0 \to L \to P \to M \to 0$ *and* $0 \to M \to Q \to N \to 0$ *be exact sequences with a projective module* P *and an injective module* Q. *If* M *is not projective (not injective), then* $proj.dim_A\, M = proj.dim_A\, L + 1$ *(respectively,* $inj.dim_A\, M = inj.dim_A\, N + 1$*).*

Proposition 11.4.2. *Let* $0 \to L \to P_{k-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *and* $0 \to M \to Q_0 \to Q_1 \to \cdots \to Q_{k-1} \to N \to 0$ *be exact sequences with projective modules* $P_0, P_1, \dots, P_{k-1}$ *and injective modules* $Q_o, Q_1, \dots, Q_{k-1}$. *If* $proj.dim_A\, M \geq k$ *(*$inj.dim_A\, M \geq k$*), then* $proj.dim_A\, M = proj.dim_A\, L + k$ *(respectively,* $inj.dim_A\, M = inj.dim_A\, N + k$*).*

Proposition 11.4.3. *Let* $(P_\bullet, d_\bullet)$ *(respectively,* $(Q^\bullet, d^\bullet)$*) be a projective (injective) resolution of a module* M. *If* M *is not projective (not injective), then* $proj.dim_A\, M = \min\{n \mid \operatorname{Ker} d_{n-1}$ *is projective*$\}$ *(respectively,* $inj.dim_A\, M = \min\{n \mid \operatorname{Coker} d^{n-1}$ *is injective*$\}$*).*

Taking into account Proposition 11.3.6, we obtain also a definition of projective dimension in terms of Tor.

Corollary 11.4.4. $proj.dim_A\, M$ *is equal to* n *if and only if* $\operatorname{Tor}^A_{n+1}(M, N) = 0$ *for all* N *and* $\operatorname{Tor}^A_n(M, N) \neq 0$ *for some module* N *(*$proj.dim_A\, M = \infty$ *if no such* n *exists).*

Let $\bar{A} = A/R$ where $R = \operatorname{rad} A$. In view of condition 3) of Corollary 11.3.5 and Proposition 11.3.6, we get the following result.

Corollary 11.4.5.

$$\begin{aligned} proj.dim_A\, M &= \sup\{n \mid \operatorname{Ext}^n_A(M, \bar{A}) \neq 0\} = \\ &= \sup\{n \mid \operatorname{Tor}^A_n(M, \bar{A}) \neq 0\}\,; \\ inj.dim_A\, M &= \sup\{n \mid \operatorname{Ext}^n_A(\bar{A}, M) \neq 0\}\,. \end{aligned}$$

Corollary 11.4.6. *The following values coincide for any finite dimensional algebra* A:

$\sup\{proj.dim_A\, M \mid M$ *a right* A*-module*$\}$;
$\sup\{inj.dim_A\, M \mid M$ *a right* A*-module*$\}$;
$\sup\{proj.dim_A\, M \mid M$ *a left* A*-module*$\}$;
$\sup\{inj.dim_A\, M \mid M$ *a left* A*-module*$\}$;
$proj.dim_A\, \bar{A}$;
$inj.dim_A\, \bar{A}$;
$\sup\{n \mid \operatorname{Ext}^n_A(\bar{A}, \bar{A}) \neq 0\}$;
$\sup\{n \mid \operatorname{Tor}^n_A(\bar{A}, \bar{A}) \neq 0\}$.

(Here, $\bar{A}$ *can always be considered either as a right or as a left* A*-module.)*

This common value is called the *global dimension* of the algebra A and is denoted by $gl.dim\, A$.

Obviously, $gl.dim\, A = 0$ if and only if A is semisimple. In view of Proposition 11.4.1, if A is not semisimple, then $gl.dim\, A = proj.dim_A\, R + 1$. In particular, $gl.dim\, A = 1$ if and only if R is projective, i. e. if and only if A is hereditary (see Theorem 3.7.1). Later we shall also use the following criterion resulting from Proposition 11.4.3.

Corollary 11.4.7. *The following conditions are equivalent:*

1) $gl.dim\ A \leq 2$;
2) *the kernel of a homomorphism between projective A-modules is projective;*
3) *the cokernel of a homomorphism between injective A-modules is injective.*

11.5 Duality

Given a complex $(V_\bullet, d_\bullet)$ of right (left) A-modules, one can construct a dual complex $(V_\bullet^*, d_\bullet^*)$:

$$\cdots \longrightarrow V_{-2}^* \xrightarrow{d_{-1}^*} V_{-1}^* \xrightarrow{d_0^*} V_0^* \xrightarrow{d_1^*} V_1^* \xrightarrow{d_2^*} V_2^* \longrightarrow \cdots$$

of left (right) A-modules (in view of indexing, it is natural to consider it as a cochain complex). In order to compute its cohomology, we shall recall (without proofs) some well-known facts from linear algebra.

Proposition 11.5.1. *Let $U \supset W$ be subspaces of a vector space V. Then there is a canonical isomorphism $(U/W)^* \simeq W^\perp/U^\perp$.*

Proposition 11.5.2. *For any linear transformation $f : V \to W$, $(\operatorname{Im} f)^\perp = \operatorname{Ker} f^*$ and $(\operatorname{Ker} f)^\perp = \operatorname{Im} f^*$.*

As a result, we get immediately the following statements.

Corollary 11.5.3. $\mathrm{H}^n(V_\bullet^*) \simeq \mathrm{H}_n(V_\bullet)^*$.

Corollary 11.5.4. *For any right A-module M and any left A-module N,* $\operatorname{Ext}_A^n(M, N^*) \simeq \operatorname{Tor}_n^A(M, N)^*$.

Proof. Consider a projective resolution $P_\bullet$ of the left module N: $\cdots \to P_2 \to P_1 \to P_0 \to N \to 0$. Passing to the dual right modules, we get an injective resolution $P_\bullet^*$ of the module N^*: $0 \to N^* \to P_0^* \to P_1^* \to P_2^* \to \cdots$. It follows from the adjoint isomorphism formula (Proposition 8.3.4) that

$$\operatorname{Hom}_A(M, P_\bullet^*) \simeq \operatorname{Hom}_A(M, \operatorname{Hom}_K(P_\bullet, K)) \simeq$$
$$\simeq \operatorname{Hom}_K(M \otimes_A P_\bullet, K) = (M \otimes_A P_\bullet)^*,$$

and thus, by Corollary 11.5.3, the cohomology $\mathrm{Ext}_A^n(M, N^*)$ of the complex $\mathrm{H}_A(M, P_\bullet^*)$ is dual to the homology $\mathrm{Tor}_n^A(M, N)$ of the complex $M \otimes_A P_\bullet$. □

In the sequel, we shall find useful another kind of duality defined by the functor $M \mapsto M\hat{} = \mathrm{Hom}_A(M, A)$. As the "usual" duality, this is a contravariant functor, or more precisely, a pair of contravariant functors mod-A $\to$ A-mod and A-mod $\to$ mod-A. However, these functors are not exact (in fact, they are only left exact) and not reciprocal. Nevertheless, there is a canonical map $\sigma_M : M \to M\hat{}\hat{}$, sending $m \in M$ into $\sigma_M(m) : M\hat{} \to A$ such that $\sigma_M(m)(f) = f(m)$ for all $f : M \to A$.

If M, N are two right modules, then there is a unique map $\lambda = \lambda(M, N) : N \otimes_A M\hat{} \to \mathrm{Hom}_A(M, N)$ such that $\lambda(n \otimes f)(m) = nf(m)$ for all $m \in M$, $n \in N$ and $f \in M\hat{}$.

Proposition 11.5.5. 1) *If M is a projective module, then σ_M is an isomorphism.*

2) *A homomorphism $\varphi : M \to N$ belongs to the image of $\lambda(M, N)$ if and only if it can be factored into a product $\varphi = \beta\alpha$, where $\alpha : M \to P$ and $\beta : P \to N$ with a projective module P.*

Proof. 1) Obviously, σ_A is an isomorphism and therefore also σ_{nA} is an isomorphism. Thus, in view of Theorem 3.3.5, the statement follows.

2) Similarly to 1), if P is a projective module, we can immediately see that $\lambda(P, N)$ is an isomorphism. Now, let $\alpha : M \to P$ with a projective P. Then the following diagram commutes:

$$\begin{tikzcd} N \otimes_A P\hat{} \arrow[r, "1\otimes\alpha\hat{}"] \arrow[d, "\lambda(P,N)"'] & N \otimes_A M\hat{} \arrow[d, "\lambda(M,N)"'] \\ \mathrm{Hom}_A(P, N) \arrow[r, "h_N^o(\alpha)"] & \mathrm{Hom}_A(M, N)\,, \end{tikzcd} \tag{11.5.1}$$

and we get that $\mathrm{Im}\, h_N^o(\alpha) = \{\beta\alpha \mid \beta : P \to N\} \subset \mathrm{Im}\,\lambda(M, N)$.

In order to complete the proof, we shall need the following obvious lemma.

Lemma 11.5.6. *For a right B-module M, a left A-module N and an A-B-bimodule L, there is an isomorphism*

$$\mathrm{Hom}_B(M, \mathrm{Hom}_A(N, L)) \simeq \mathrm{Hom}_A(N, \mathrm{Hom}_B(M, L))$$

assigning to a homomorphism $f : M \to \mathrm{Hom}_A(N, L)$ the homomorphism $f' : N \to \mathrm{Hom}_B(M, L)$ such that $f'(n)(m) = f(m)(n)$ for all $m \in M$ and $n \in N$.

If, in particular, P is a projective module, then

$$\begin{aligned} \mathrm{Hom}_A(P\hat{}, M\hat{}) &= \mathrm{Hom}_A(P\hat{}, \mathrm{Hom}_A(M, A)) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_A(P\hat{}, A)) = \\ &= \mathrm{Hom}_A(M, P\hat{}\hat{}) \simeq \mathrm{Hom}_A(M, P)\,. \end{aligned}$$

Consider now an epimorphism $\psi : P' \to M^\wedge$, where P' is projective. According to 1), we may assume that $P' = P^\wedge$ and $\psi = \alpha^\wedge$ for a projective module P and $\alpha : M \to P$. Then the homomorphism $1 \otimes \alpha^\wedge$ of (11.5.1) is an epimorphism by Proposition 8.3.6. Consequently $\operatorname{Im} \lambda(M, N) = \operatorname{Im} h_N^\circ(\alpha)$ and the proof of 2) is completed. □

In what follows, we shall write $\operatorname{Pr}_A(M, N) = \operatorname{Im} \lambda(M, N)$ and call the homomorphisms from $\operatorname{Pr}_A(M, N)$ the *projective homomorphisms*. Let us also introduce the following notation: $\underline{\operatorname{Hom}}_A(M, N) = \operatorname{Hom}_A(M, N)/\operatorname{Pr}_A(M, N)$.

11.6 Almost Split Sequences

In this section, we are going to prove a theorem which plays a fundamental role in the contemporary investigations of representations and structure of finite dimensional algebras. It is related to the concept of almost split sequences, often called Auslander-Reiten sequences.

Proposition 11.6.1. *Let $\zeta : 0 \to N \xrightarrow{f} X \xrightarrow{g} M \to 0$ be a non-split exact sequence with indecomposable modules M and N. Then the following conditions are equivalent:*

1) *For every $\varphi : M' \to M$, where M' is indecomposable and φ is not an isomorphism, the lifting $\varphi^e(\zeta)$ splits.*

1′) *For every $\varphi : M' \to M$, where M' is indecomposable and φ is not an isomorphism, there is a factorization $\varphi = g\alpha$ for some $\alpha : M' \to X$.*

2) *For every $\psi : N \to N'$, where N' is indecomposable and ψ is not an isomorphism, the descent $\psi_e(\zeta)$ splits.*

2′) *For every $\psi : N \to N'$, where N' is indecomposable and ψ is not an isomorphism, there is a factorization $\psi = \beta f$ for some $\beta : X \to N'$.*

Proof. 1) ⇒ 1′). Consider the commutative diagram involving the lifting $\varphi^e(\zeta)$:

$$\begin{array}{ccccccccccc}
\varphi^e(\zeta): & 0 & \longrightarrow & N & \xrightarrow{f'} & X' & \xrightarrow{g'} & M' & \longrightarrow & 0 \\
& & & \big\downarrow{\scriptstyle 1_N} & & \big\downarrow{\scriptstyle \varphi'} & & \big\downarrow{\scriptstyle \varphi} & & \\
\zeta: & 0 & \longrightarrow & N & \xrightarrow{f} & X & \xrightarrow{g} & M & \longrightarrow & 0.
\end{array}$$

Since $\varphi^e(\zeta)$ is split, there is a homomorphism $\gamma : M' \to X'$ for which $g'\gamma = 1$. But then $\varphi = \varphi g'\gamma = g\varphi'\gamma$, as required.

1′) ⇒ 1). If $\varphi = g\alpha$, then the homomorphism $\gamma : M' \to X'$ given by the formula $\gamma(m') = (\alpha(m'), m')$ defines a splitting of $\varphi^e(\zeta)$. (Recall that, in the construction of lifting, $X' = \{(x, m') \mid g(x) = \varphi(m')\} \subset X \oplus M'$, and $g'(x, m') = m'$.)

$1') \Rightarrow 2)$. Consider the commutative diagram involving the descent $\psi_e(\zeta)$:

$$\begin{tikzcd}
\zeta : & 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "\psi"'] & X \arrow[r, "g"] \arrow[d, "\psi'"'] & M \arrow[r] \arrow[d, "1_M"'] & 0 \\
\psi_e(\zeta) : & 0 \arrow[r] & N' \arrow[r, "f'"] & X' \arrow[r, "g'"] & M \arrow[r] & 0\,.
\end{tikzcd}$$

Let $X' = X_1 \oplus X_2 \oplus \ldots \oplus X_m$ be a direct decomposition into indecomposable summands X_i and g_i the restrictions of g' to X_i. If any of g_i is invertible, i. e. $g_i h = 1_M$ for some $h : M \to X_i$, then the sequence $\psi_e(\zeta)$ splits due to the homomorphism $\gamma : M \to X'$ defined by $\gamma(m) = (0, \ldots, 0, h(m), 0, \ldots, 0)$ with $h(m)$ at the ith position. Thus, assume that none of g_i is invertible. Then in view of the condition $1')$, $g_i = g\alpha_i$ for some $\alpha_i : X_i \to X$ and hence $g' = g\eta$, where $\eta(x_1, x_2, \ldots, x_m) = \sum\limits_i \alpha_i(x_i)$.

Since $g\eta f' = g'f' = 0$, $\operatorname{Im} \eta f' \subset \operatorname{Ker} g = \operatorname{Im} f$, and thus $\eta f' = f\theta$ for some $\theta : N' \to N$. Similarly, since $g(1 - \eta\psi') = g - g'\psi' = 0$, we have a factorization $1 - \eta\psi' = fu$ for some $u : X \to N$. Furthermore, multiplying the equality $1 = \eta\psi' + fu$ by f we get $f = \eta\psi' f + fuf = \eta f'\psi + fuf = f\theta\psi + fuf$. Since f is a monomorphism, this equality yields $1_N = \theta\psi + uf$. Now, N is indecomposable and thus the algebra $E_A(N)$ is local. Consequently, $\theta\psi$ or uf is invertible. However, if $\theta\psi$ is invertible, so is ψ (since N' is also indecomposable) and if uf is invertible, then ζ is split. This contradiction completes the proof.

The assertions $2) \Leftrightarrow 2')$ and $2') \Rightarrow 1)$ can be proved similarly, or follow by duality. $\square$

A sequence ζ possessing the properties listed in Proposition 11.6.1 is called an *almost split sequence with end M and beginning N*.

It is clear that in order that such an almost split sequence exists, it is necessary that M is not projective and N is not injective. It is rather remarkable that this condition is also sufficient.

Theorem 11.6.2 (Auslander-Reiten). 1) *For any indecomposable module M which is not projective, there is an almost split sequence with end M.*
2) *For any indecomposable module N which is not injective, there is an almost split sequence with beginning N.*

Proof. 1) Theorem 3.3.7 implies that there is an epimorphism $\pi : P_0 \to M$ such that P_0 is projective and $\operatorname{Ker} \pi \subset \operatorname{rad} P_0$. Repeating the same procedure for $\operatorname{Ker} \pi$, we get an exact sequence $P_1 \xrightarrow{\theta} P_0 \xrightarrow{\pi} M \to 0$ for which $\operatorname{Im} \theta = \operatorname{Ker} \pi \subset \operatorname{rad} P_0$ and $\operatorname{Ker} \theta \subset \operatorname{rad} P_1$. Now, apply the functor $\hat{} = h_A^o$ (see Sect. 11.5) and put $T = \operatorname{Tr} M = \operatorname{Coker}(\theta\hat{})$. We obtain the following exact sequence:

$$0 \longrightarrow M\hat{} \xrightarrow{\pi\hat{}} P_0\hat{} \xrightarrow{\theta\hat{}} P_1\hat{} \xrightarrow{\sigma} T \longrightarrow 0\,. \tag{11.6.1}$$

We are going to show that T is indecomposable. Indeed, assuming that T is decomposable, we get from Corollary 3.3.8 that $P_1\hat{} = Y_1 \oplus Y_2$ and

$P_0^{\wedge} = Z_1 \oplus Z_2$ such that $\theta^{\wedge}(Z_1) \subset Y_1$ and $\theta^{\wedge}(Z_2) \subset Y_2$. But then, taking into account part 1) of Proposition 11.5.5, we see that $P_1 = Y_1^{\wedge} \oplus Y_2^{\wedge}$, $P_0 = Z_1^{\wedge} \oplus Z_2^{\wedge}$ with $\theta(Y_1^{\wedge}) \subset Z_1^{\wedge}$ and $\theta(Y_2^{\wedge}) \subset Z_2^{\wedge}$. From here, $M \simeq Z_1^{\wedge}/\theta(Y_1^{\wedge}) \oplus Z_2^{\wedge}/\theta(Y_2^{\wedge})$ and, in view of the fact that $\operatorname{Im}\theta \subset \operatorname{rad} P_0$, both summands are non-zero. This contradiction shows that T is indecomposable. Put $N = T^*$.

According to Corollary 11.5.4, for any module L, there is an isomorphism $\operatorname{Ext}^1_A(L,N) \simeq \operatorname{Tor}^A_1(L,T)^*$. To compute $\operatorname{Tor}^A_1(L,T)$, we will use the exact sequence (11.6.1): It turns out that $\operatorname{Tor}^A_1(L,T)$ is isomorphic to the factor space $\operatorname{Ker} t_L(\theta^{\wedge})/\operatorname{Im} t_L(\pi^{\wedge})$ (here t_L is the functor $L \otimes_A -$). Making use of part 2) of Proposition 11.5.5 we obtain $L \otimes_A P_i^{\wedge} \simeq \operatorname{Hom}_A(P_i, L)$, and hence $\operatorname{Ker} t_L(\theta^{\wedge}) \simeq \operatorname{Ker} h^o_L(\theta) \simeq \operatorname{Hom}_A(M,L)$, since the sequence $0 \to \operatorname{Hom}_A(M,L) \to \operatorname{Hom}_A(P_0,L) \to \operatorname{Hom}_A(P_1,L)$ is exact. Moreover, $\operatorname{Im} t_L(\pi^{\wedge})$ is mapped in this isomorphism into $\operatorname{Im}\lambda(M,L) = \operatorname{Pr}_A(M,L)$. Consequently, $\operatorname{Tor}^A_1(L,T) \simeq \underline{\operatorname{Hom}}_A(M,L)$ and $\operatorname{Ext}^1_A(L,N) \simeq \underline{\operatorname{Hom}}_A(M,L)^*$. In particular, $\operatorname{Ext}^1_A(M,N) \simeq \underline{\operatorname{Hom}}_A(M,M)^*$. However, $H = \underline{\operatorname{Hom}}_A(M,M)$ is a quotient algebra of $E_A(M)$ and thus it is a local algebra. Denote by R its radical and consider a non-zero linear functional $\zeta \in H^*$ such that $\zeta(R) = 0$. Let M' be an indecomposable A-module. For any $\varphi : M' \to M$ which is not an isomorphism, the induced map $\operatorname{Hom}_A(M,M') \to \operatorname{Hom}_A(M,M)$ assigns to a homomorphism $f : M \to M'$ the non-invertible endomorphism φf. Thus, denoting by $\bar{f}$ the coset of f in $\underline{\operatorname{Hom}}_A(M,M')$, we get that $\varphi^e(\zeta)(\bar{f}) = \zeta(\varphi\bar{f}) = 0$, which means that the extension of M by kernel N corresponding to the element ζ is an almost split sequence.

The assertion 2) follows from 1) by duality (or can be proved similarly). Let us point out that our computations yield also isomorphisms $M \simeq \operatorname{Tr} N^*$ and $\operatorname{Ext}^1_A(M,L) \simeq \overline{\operatorname{Hom}}_A(L,N)^*$ for every module L: here $\overline{\operatorname{Hom}}_A(L,N)$ denotes the factor space of $\operatorname{Hom}_A(L,N)$ by the subspace $\operatorname{In}_A(L,N)$ consisting of those homomorphisms which factor through an injective module. □

11.7 Auslander Algebras

In conclusion, we will give a homological characterization of an important class of algebras. We call an algebra A an *Auslander algebra* if there is an algebra B possessing only a finite number of non-isomorphic indecomposable modules $M_1, M_2, \dots, M_n$, so that $A \simeq E_B(M)$, where $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$ (more precisely, A is called the Auslander algebra of the algebra B). By definition, such an algebra is always basic. Obviously, a basic semisimple algebra is always an Auslander algebra.

Theorem 11.7.1 (Auslander). *A basic algebra A is an Auslander algebra if and only if gl.dim $A \le 2$ and there is an exact sequence $0 \to A \to I_0 \to I_1$ in which the A-modules I_0 and I_1 are bijective.*

The necessity of the statement will be based on the following lemma.

Lemma 11.7.2. *Let $A = E_B(M)$ be an Auslander algebra. Then:*

1) *M is a projective left A-module.*
2) *The functors $F : N \mapsto \mathrm{Hom}_B(M, N)$ and $G : P \mapsto P \otimes_A M$ establish an equivalence between the category* mod-B *and the category* pr-A *of the projective A-modules.*

Proof. 1) Since M is a direct sum of all indecomposable B-modules, $mM \simeq B \oplus L$ for some L, and thus $mA \simeq \mathrm{Hom}_B(mM, M) \simeq \mathrm{Hom}_B(B, M) \oplus \mathrm{Hom}_B(L, M)$. Therefore, $M \simeq \mathrm{Hom}_B(B, M)$ is a projective A-module.

2) The fact that $F(N)$ is always projective can be verified the same way as the first statement 1). The natural transformation of functors (see Sect. 8.4) $\varphi : 1_{\text{pr-}A} \to FG$ and $\psi : GF \to 1_{\text{mod-}B}$ are isomorphisms on A_A and M_B, respectively, and therefore on all their direct summands. Hence φ and ψ are isomorphisms, respectively, on all projective A-modules and all B-modules, as required. □

Proof of necessity in Theorem 11.7.1. Let $A = E_B(M)$ be the Auslander algebra of an algebra B and $g : P_0 \to P_1$ a homomorphism of projective A-modules. In view of Lemma 11.7.2, we may assume that $P_i = F(N_i)$ and $g = F(f)$ for some B-module homomorphism $f : N_0 \to N_1$. Since F is left exact, $\mathrm{Ker}\, g \simeq F(\mathrm{Ker}\, f)$ is a projective A-module and $gl.dim\, A \leq 2$ by Corollary 11.4.7.

Now, construct an exact sequence $0 \to M \to Q_0 \to Q_1$ with injective B-modules Q_0, Q_1. Applying the functor F, we obtain an exact sequence $0 \to A \to F(Q_0) \to F(Q_1)$. It remains to show that $F(Q_i)$ are injective A-modules. In view of Theorem 9.1.4, it is sufficient to know that $F(B^*)$ is an injective A-module. However, $F(B^*) = \mathrm{Hom}_B\big(M, \mathrm{Hom}_K(B, K)\big) \simeq \mathrm{Hom}_K(M \otimes_B B, K) \simeq M^*$ is injective by part 1) of Lemma 11.7.2.

Proof of sufficiency. Assume that $gl.dim\, A \leq 2$ and that there is an exact sequence $0 \to A \to I_0 \to I_1$ with bijective A-modules I_0 and I_1. Denote by I the direct sum of all indecomposable bijective A-modules, $B = E_A(I)$ and consider the contravariant functors $F' : N \mapsto \mathrm{Hom}_B(N, I)$ and $G' : P \mapsto \mathrm{Hom}_A(P, I)$. For a left B-module N, a projective resolution $P_1 \to P_0 \to N \to 0$ translates to the exact sequence $0 \to F'(N) \to F'(P_0) \to F'(P_1)$. However $F'(B) \simeq I$ and therefore $F'(P_i)$ are projective (even bijective) A-modules. By Corollary 11.4.7, $F'(N)$ is also projective, and thus F' can be viewed as a functor $(B\text{-mod})^o \to \text{pr-}A$.

Consider the natural transformations $\varphi' : 1_{\text{pr-}A} \to F'G'$ and $\psi' : 1_{\text{mod-}B} \to G'F'$ (they act the same way: $\varphi'(P)$ assigns to an element $x \in P$ the B-homomorphism $\mathrm{Hom}_A(P, I) \to I$ sending f into $f(x)$; $\psi'(N)$ acts similarly). Clearly, $\varphi'(I)$ and $\psi'(B)$ are isomorphisms. Thus, if P is bijective and N is projective, also $\varphi'(P)$ and $\psi'(N)$ are isomorphisms. Besides, the functor $F'G'$ is left exact and $G'F'$ is right exact, since I is an injective A-module and thus G' is exact. Therefore the exact sequence $0 \to A \to I_0 \to I_1$ can be extended to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \longrightarrow & I_0 & \longrightarrow & I_1 \\
 & & \Big\downarrow{\scriptstyle \varphi'(A)} & & \Big\downarrow{\scriptstyle \varphi'(I_0)} & & \Big\downarrow{\scriptstyle \varphi'(I_1)} \\
0 & \longrightarrow & F'G'(A) & \longrightarrow & F'G'(I_0) & \longrightarrow & F'G'(I_1).
\end{array}$$

As a consequence, $\varphi'(A)$ is an isomorphism and thus $\varphi'(P)$ is an isomorphism for every projective P. Similarly, $\psi'(N)$ is an isomorphism for every N and we conclude that F' and G' establish an equivalence of the categories $(B\text{-mod})^o$ and pr-A. In particular, since $G'(A) = I$, the algebra A is anti-isomorphic to $\mathrm{End}_B(I)$. Furthermore, A is basic, and thus is a direct sum of non-isomorphic principal A-modules; therefore I is a direct sum of all non-isomorphic indecomposable left B-modules. It follows that I^* is a direct sum of all non-isomorphic indecomposable right B-modules and $E_B(I^*) \simeq E_B(I)^o \simeq A$, so A is an Auslander algebra. $\square$

Exercises to Chapter 11

1. Verify that for a complex $V_\bullet$ which is a short exact sequence $0 \to M \to N \to L \to 0$, $V_\bullet \sim 0$ if and only if the sequence splits. (Clearly, $\mathrm{H}_n(V_\bullet) = 0$ for all n.)

2. Let $A = K[a]$, where $a^2 = 0$, $M = A/aA$ and $\pi : A \to M$ the canonical projection. Furthermore, let $\varepsilon : M \to A$ be the embedding sending $x + aA$ into ax and $f_\bullet : V_\bullet \to V'_\bullet$ the complex homomorphism defined by the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M \oplus M & \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)} & M \oplus M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (\varepsilon\ 0)} & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)} & & \\
0 & \longrightarrow & A & \xrightarrow{\left(\begin{smallmatrix}\pi\\ 0\end{smallmatrix}\right)} & M \oplus M & \longrightarrow & 0.
\end{array}$$

Show that $f_\bullet \equiv 0$, but $f_\bullet \not\sim 0$.

3. Give an example of a complex $V_\bullet$ and a functor F such that $\mathrm{H}_n(V_\bullet) = 0$ for all n, but $\mathrm{H}_n\big(F(V_\bullet)\big) \neq 0$ for some n.

4. Let $V_\bullet$ and $V'_\bullet$ be complexes of projective modules over a hereditary algebra, bounded from the right, and $f_\bullet$ and $g_\bullet$ two homomorphisms $V_\bullet \to V'_\bullet$. Prove that $f_\bullet \equiv g_\bullet$ implies $f_\bullet \sim g_\bullet$.

5. Prove that for every module M there exists a projective resolution $(P_\bullet, d_\bullet)$ satisfying $\mathrm{Im}\, d^n \subset \mathrm{rad}\, P_{n-1}$ for all n, and that any two such resolutions are isomorphic. (Resolutions satisfying this property are called minimal projective resolutions of the module M and are denoted by $P_\bullet(M)$.) Formulate and prove an analogous result for injective resolutions.

6. Let $0 \to N \xrightarrow{\varphi} P_{k-1} \to \cdots \to P_0 \to M \to 0$ be an exact sequence with projective modules $P_0, P_1, \ldots, P_{k-1}$. Let F be a right exact functor. Prove that $L^n F(M) \simeq L^{n-k} F(N)$ for $n > k$ and $L^k F(M) \simeq \mathrm{Ker}\, F(\varphi)$. Formulate and prove similar statements for right derived functors and contravariant functors.

7. Let $P_\bullet(M) = (P_\bullet, d_\bullet)$ be a minimal projective resolution of a right A-module M (see Exercise 5). Prove that, for any simple right A-module V (simple left A-module W), $\operatorname{Ext}^n_A(M,V) \simeq \operatorname{Hom}_A(P_n, V)$ and $\operatorname{Tor}^A_n(M,W) \simeq P_n \otimes_A W$.

8. Let A be a split algebra, $\mathcal{D} = \mathcal{D}(A)$ its diagram and V_i the simple A-module corresponding to the vertex $i \in \mathcal{D}$. Prove that $\operatorname{Ext}^1_A(V_i, V_j) \simeq t_{ij}K$, where (t_{ij}) is the incidence matrix of the diagram $\mathcal{D}$.

9. Construct a one-to-one map $\delta' : \operatorname{Ex}(M,N) \to \operatorname{Ext}^1_A(M,N)$ using the connecting homomorphism with respect to the first variable (and projective resolutions).

10. Prove that $proj.dim_A(\bigoplus_i M_i) = \max_i(proj.dim_A M_i)$ and $inj.dim_A(\bigoplus_i M_i) = \max_i(inj.dim_A M_i)$.

11. Prove that $gl.dim(\prod_i A_i) = \max_i(gl.dim\, A_i)$.

12. Assume that there are no cycles in the diagram $\mathcal{D}(A)$ of an algebra A.
    a) Prove that $gl.dim\, A \le \ell$, where ℓ is the maximal length of paths in $\mathcal{D}(A)$.
    b) If $(\operatorname{rad} A)^2 = 0$, prove that $gl.dim\, A = \ell$.

13. Let L be an extension of the field K. Prove that $gl.dim\, A_L \ge gl.dim\, A$. Prove that the inequality becomes equality if L is a separable extension or if the quotient algebra $A/\operatorname{rad} A$ is separable over K.

14. Prove that $gl.dim\, A \le proj.dim_{A\otimes A^o} A$ and that equality holds if $A/\operatorname{rad} A$ is separable.

15. Prove that any two almost split sequences with a common beginning (or end) are isomorphic.

16. Prove that a hereditary Auslander algebra is semisimple.